Perturbation-enhanced extended polynomial-chaos expansion for stochastic finite element problems

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Outline of the talk

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   - Discretisation of Stochastic PDE
2. **Polynomial Chaos expansion**
3. **Motivation behind the hybrid approach**
4. **Construction of the hybrid approach**
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   - PC Projection
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Stochastic elliptic PDE

- We consider the stochastic elliptic partial differential equation (PDE)

$$-\nabla\left[a(\mathbf{r},\theta)\nabla u(\mathbf{r},\theta)\right]=p(\mathbf{r});\quad \mathbf{r} \text{ in } \mathcal{D} \tag{1}$$

  with the associated boundary condition

$$u(\mathbf{r},\theta)=0;\quad \mathbf{r} \text{ on } \partial\mathcal{D} \tag{2}$$

- Here $a:\mathbb{R}^d\times\Omega\to\mathbb{R}$ is a random field, which can be viewed as a set of random variables indexed by $\mathbf{r}\in\mathbb{R}^d$.
- We assume the random field $a(\mathbf{r},\theta)$ to be stationary and square integrable. Based on the physical problem the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities.

Discretized Stochastic PDE

- The random process $a(\mathbf{r},\theta)$ can be expressed in a generalized fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r},\theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{3}$$

Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation
$\int_{\mathcal{D}} C_a(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \nu_j\varphi_j(\mathbf{r}_2), \quad \forall\, j = 1, 2, \cdots$.

- Truncating the series (3) upto the M-th term, substituting $a(\mathbf{r},\theta)$ in the governing PDE (1) and applying the boundary conditions, the discretized equation can be written as

$$\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]\mathbf{u}(\theta) = \mathbf{f} \tag{4}$$

Polynomial Chaos expansion

- After the finite truncation, concisely, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k \tag{5}$$

where $H_k(\xi(\theta))$ are the polynomial chaoses.

- The value of the number of terms P depends on the number of basic random variables M and the order of the PC expansion r as

$$P = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!} \tag{6}$$

Polynomial Chaos expansion

- After the finite truncation, concisely, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k \tag{7}$$

where $H_k(\xi(\theta))$ are the polynomial chaoses and $\mathbf{u}_k \in \mathbb{R}^n$ are deterministic vectors to be determined.

- The value of the number of terms P depends on the number of basic random variables M and the order of the PC expansion r as

$$P = \sum_{j=0}^{r} \frac{(M + j - 1)!}{j!(M - 1)!} \tag{8}$$

Polynomial Chaos expansion

We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k$ for every frequency point:

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{9}$$

or

$$\widetilde{\mathbf{A}}\mathbf{U} = \mathbf{F}$$

P increases exponentially with M:

M	2	3	5	10	20	50	100
2nd order PC	5	9	20	65	230	1325	5150
3rd order PC	9	19	55	285	1770	23425	176850

Polynomial Chaos expansion: Some Observations

- Computational cost increase exponentially with the number of random variables
- Particularly efficient compared to 'local methods' (e.g., perturbation method, Neumann approach) when the coefficients associated with the random variables are large.
- However, there is an ordering of the coefficient matrices $\mathbf{A}_i$ due to the decaying nature of the eigenvalues in the Karhunen-Loève expansion
- Recall that the local methods produce acceptable accuracy when the influence of the randomness is 'less'

Motivation behind the hybrid approach

- The idea is to propagate the random variables associated with ‘higher’ variability by polynomial chaos expansion and the random variables with ‘lower’ variability by perturbation expansion.
- This way the ‘curse’ of dimensionality can be avoided to some extend.
- Often the number of random variables used in a polynomial chaos expansion has to be truncated due to the computational considerations.
- Considering a perturbation expansion in these ‘ignored’ variables would be better that completely ignoring them.

Separation of the random variables

$$\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i$$

- The random variables are divided into two groups

$$\mathbf{x}(\theta) = \{\xi_i(\theta)\}, i = 1, \cdots, M_1$$

  and

$$\mathbf{y}(\theta) = \{\xi_i(\theta)\}, i = M_1 + 1, \cdots, M$$

- Therefore **x** and **y** are vector of random variables of dimensions M_1 and M_2 respectively such that $M_1 + M_2 = M$.
- We construct a polynomial chaos with **x** and perturbation expansion on **y** such that the response can be expressed as

$$\mathbf{u}(\theta) = \sum_{k=1}^{P_1} H_k(\mathbf{x}(\theta))\mathbf{u}_k(\mathbf{y}(\theta)) \tag{10}$$

Separation of the random variables

We rewrite the system matrix as

$$\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{M_1} x_i \mathbf{A}_i + \sum_{j=1}^{M_2} y_j \mathbf{B}_j = \mathbf{A}_y + \sum_{i=1}^{M_1} x_i \mathbf{A}_i$$

Where

$$\mathbf{A}_y = \mathbf{A}_0 + \sum_{j=1}^{M_2} y_j \mathbf{B}_j$$

is the effective 'constant' matrix while considering polynomial chaos expansion with respect to the random variables $x_i, i = 1, 2, \cdots, M_1$

Polynomial chaos expansion

We express the polynomial chaos solution as

$$\mathbf{u}(\theta) = \sum_{k=1}^{P_1} H_k(\mathbf{x}(\theta))\mathbf{u}_{y_k} \quad (11)$$

where $P_1 = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!}$. The $P_1 n$ dimensional coefficient vector $\mathbf{U}_y = \{\mathbf{u}_{y_0}, \mathbf{u}_{y_1}, \cdots, \mathbf{u}_{y_{P_1-1}}\}^T$ can be obtained from the usual $P_1 n \times P_1 n$ matrix equation as

$$\left[\widetilde{\mathbf{A}}_0 + \sum_{j=1}^{M_2} y_j \widetilde{\mathbf{B}}_j\right] \mathbf{U}_y = \mathbf{F} \quad (12)$$

Perturbation expansion

The vector of PC coefficients $\mathbf{U}_y$ can be expanded as

$$\left[\widetilde{\mathbf{A}}_0 + \sum_{j=1}^{M_2} y_j \widetilde{\mathbf{B}}_j\right] \mathbf{U}_y = \mathbf{F} \tag{13}$$

$$\text{or} \quad \mathbf{U}_y = \left[\widetilde{\mathbf{A}}_0 + \sum_{j=1}^{M_2} y_j \widetilde{\mathbf{B}}_j\right]^{-1} \mathbf{F} \tag{14}$$

$$\text{or} \quad \mathbf{U}_y \approx \left[\underbrace{\mathbf{I} - \widetilde{\mathbf{A}}_0^{-1} \sum_{j=1}^{M_2} y_j \widetilde{\mathbf{B}}_j + \left(\widetilde{\mathbf{A}}_0^{-1} \sum_{j=1}^{M_2} y_j \widetilde{\mathbf{B}}_j\right)^2 - \cdots}\right] \mathbf{U}_0 \tag{15}$$

where the classical PC coefficient is given by

$$\mathbf{U}_0 = \left[\widetilde{\mathbf{A}}_0\right]^{-1} \mathbf{F}$$

The hybrid expression

The hybrid PC-Perturbation coefficient vector

$$\mathbf{U}_y(\theta) \approx \Bigg[\mathbf{I} \underbrace{-\widetilde{\mathbf{A}}_0^{-1} \sum_{j=1}^{M_2} y_j(\theta) \widetilde{\mathbf{B}}_j}_{\text{The missing contribution}} \Bigg] \mathbf{U}_0 \tag{16}$$

The complete hybrid PC-Perturbation solution is therefore

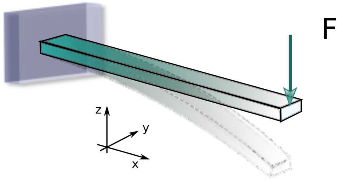
$$\mathbf{u}(\theta) = \sum_{k=1}^{P_1} H_k(\mathbf{x}(\theta)) \mathbf{u}_{y_k}(\theta) \tag{17}$$

The classical PC solution would be

$$\mathbf{u}(\theta) = \sum_{k=1}^{P_1} H_k(\mathbf{x}(\theta)) \mathbf{u}_{0_k} \tag{18}$$

The Euler-Bernoulli beam example

- An Euler-Bernoulli cantilever beam with stochastic bending modulus



- Length : 1.0 m, Nominal EI_0 : 1/3
- We study the deflection of the beam under the action of a point load on the free end.

Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta))$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The autocorrelation function of this random field is assumed to be

$$C_a(x_1,x_2) = \sigma_a^2 e^{-(|x_1-x_2|)/\mu_a}$$

where μ_a is the correlation length and σ_a is the standard deviation.

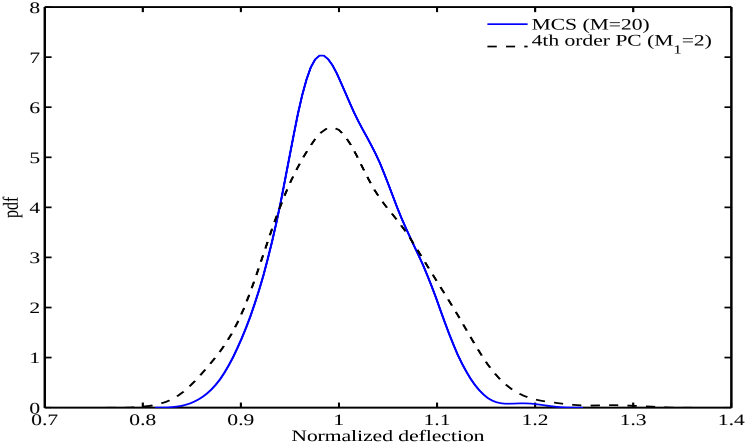
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

Problem details

The random field is Gaussian with correlation length $\mu_a = L/5$. The results are compared with the polynomial chaos expansion with M_1=2.

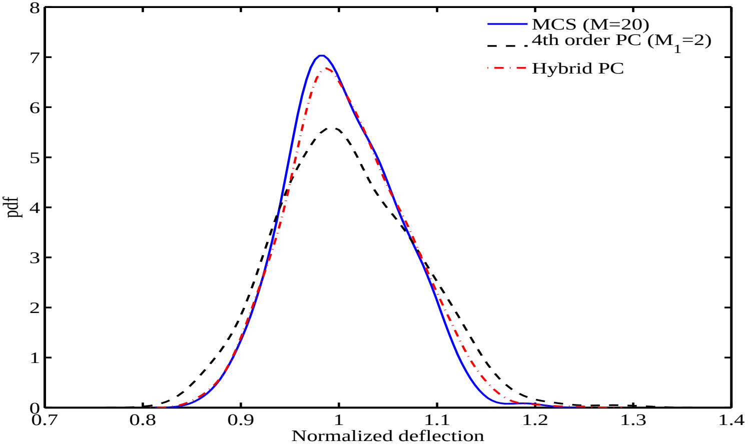
- The number of degrees of freedom of the system is $n = 200$.
- The number of random variables in KL expansion used for discretising the stochastic domain is M =20.
- Simulations have been performed with 10,000 MCS samples with the standard deviation of the random field $\sigma_a = 0.1$.
- Comparison have been made with 4^{th} order Polynomial chaos results.

MCS against truncated PC



The PDf of the tip deflection - comparison between MCS ($M = 20$) and 4th order PC ($M_1 = 2$).

MCS, truncated PC and Hybrid approach



The PDf of the tip deflection - comparison between MCS ($M = 20$), 4th order PC ($M_1 = 2$) and hybrid 4th order PC and 2nd order perturbtion.

Summary and conclusion

- The objective was to propagate the random variables associated with ‘higher’ variability by polynomial chaos expansion and the random variables with ‘lower’ variability by perturbation expansion.
- The hybrid PC-Perturbation coefficient vector

$$\mathbf{U}_y(\theta) \approx \Bigg[\mathbf{I} \underbrace{- \widetilde{\mathbf{A}}_0^{-1} \sum_{j=1}^{M_2} y_j(\theta) \widetilde{\mathbf{B}}_j}_{\text{The missing contribution}} \Bigg] \mathbf{U}_0 \tag{19}$$

- These are the ‘ghost’ terms - they were always there - but invisible so far! No matter how many random variables you have considered in the PC analysis, there is always some you didn’t!
- The complete hybrid PC-Perturbation solution is therefore $\mathbf{u}(\theta) = \sum_{k=1}^{P_1} H_k(\mathbf{x}(\theta)) \mathbf{u}_{y_k}(\theta)$

Summary and conclusion

- How shall we choose the 'higher' and 'lower' variability in the context of the proposed method? Where shall we draw the borderline?
- We can use higher order Neumann expansion combined with (different orders of) PC.
- Given the magnitude of the coefficients, can we optimise the 'partition' of the random variables, the order of the PC and the order of the Neumann expansion?