Novel reduced Galerkin projection schemes for stochastic dynamical systems

A Kundu and S Adhikari

Civil & Computational Engineering Research Centre (C^{2}EC), Swansea University, Swansea UK

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Maresias, Sao Sebastiao, Sao Paulo, Brazil

Swansea University

Swansea University

Outline of the talk

1. **Introduction**
   - Stochastic Partial Differential Equations for dynamical systems
2. **Spectral decomposition in the modal space**
   - Projection in the modal space
   - Properties of the spectral functions
3. **Error minimization in the Hilbert space**
   - The Galerkin approach
   - Model Reduction
   - Computational method
4. **Numerical illustration**
   - The Euler-Bernoulli beam
5. **Conclusions**

Stochastic PDEs

We consider the stochastic elliptic partial differential equation (PDE)

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t,\theta)}{\partial t^2} + \mathcal{L}_\alpha \frac{\partial U(\mathbf{r},t,\theta)}{\partial t} + \mathcal{L}_\beta U(\mathbf{r},t,\theta) = p(\mathbf{r},t) \tag{1}$$

The stochastic operator $\mathcal{L}_\beta$ can be

- $\mathcal{L}_\beta \equiv \frac{\partial}{\partial x} AE(x,\theta) \frac{\partial}{\partial x}$ axial deformation of rods
- $\mathcal{L}_\beta \equiv \frac{\partial^2}{\partial x^2} EI(x,\theta) \frac{\partial^2}{\partial x^2}$ bending deformation of beams

$\mathcal{L}_\alpha$ denotes the stochastic damping, which is mostly proportional in nature.
Here $\alpha, \beta : \mathbb{R}^d \times \Theta \to \mathbb{R}$ are stationary square integrable random fields, which can be viewed as a set of random variables indexed by $\mathbf{r} \in \mathbb{R}^d$.
Based on the physical problem the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities (e.g., $AE(x,\theta)$, $EI(x,\theta)$).

Discretized Stochastic PDE

- A random process $a(\mathbf{r},\theta)$ can be expressed in a generalized fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r},\theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{2}$$

  Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \nu_j\varphi_j(\mathbf{r}_2), \quad \forall\, j = 1,2,\cdots \tag{3}$$

- For non-Gaussian random fields (e.g. uniform, lognormal), Eq. 2 can represented with a PC type expansion and different sets of orthogonal polynomials from the Weiner-Askey scheme can be utilized to represent the trial basis.

Discrete equation for stochastic mechanics

- The stochastic PDE along with the boundary conditions results in:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t)+\mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t)+\mathbf{K}(\theta)\mathbf{u}(\theta,t)=\mathbf{f}(t) \tag{4}$$

- $\mathbf{M}(\theta)=\mathbf{M}_0+\sum_{i=1}^{p}\mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta)=\mathbf{K}_0+\sum_{i=1}^{p}\nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta)\in\mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$) obtained from discretizing the stochastic field with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta)=\zeta_1\mathbf{M}(\theta)+\zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.

Frequency domain representation

- For the harmonic analysis of the structural system, taking the Fourier transform

$$\left[-\omega^2\mathbf{M}(\theta)+i\omega\mathbf{C}(\theta)+\mathbf{K}(\theta)\right]\widetilde{\mathbf{u}}(\omega,\theta)=\widetilde{\mathbf{f}}(\omega) \tag{5}$$

  where $\widetilde{\mathbf{u}}(\omega,\theta)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}(\omega)$ is the amplitude of the harmonic force.
- For convenience we group the random variables associated with the mass and stiffness matrices as

$$\xi_i(\theta)=\mu_i(\theta) \quad \text{and} \quad \xi_{j+p_1}(\theta)=\nu_j(\theta) \quad \begin{array}{ll} \text{for} & i=1,2,\ldots,p_1 \\ \text{and} & j=1,2,\ldots,p_2 \end{array}$$

Discrete equation for stochastic mechanics

- Using $M = p_1 + p_2$ which we have

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i(\omega) \right) \widetilde{\mathbf{u}}(\omega, \theta) = \widetilde{\mathbf{f}}(\omega) \tag{6}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n \times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble.

- For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1 \right] \mathbf{M}_0 + [i\omega\zeta_2 + 1] \mathbf{K}_0, \tag{7}$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1 \right] \mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{8}$$

$$\text{and} \quad \mathbf{A}_{j+p_1}(\omega) = [i\omega\zeta_2 + 1] \mathbf{K}_j \quad \text{for} \quad j = 1, 2, \ldots, p_2 \, .$$

Time domain representation

If the time steps are fixed to Δt, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{9}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$\left[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{10}$$

$$\text{and,}\quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f\left(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)\right) \tag{11}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t + \Delta t$ which consists of contributions of the system response at the previous time step.

Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0 \left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2 \dot{\mathbf{u}}_t(\theta) - a_3 \ddot{\mathbf{u}}_t(\theta) \tag{12}$$

$$\text{and,} \quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6 \ddot{\mathbf{u}}_t(\theta) + a_7 \ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{13}$$

where the integration constants $a_i,\ i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha \Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha \Delta t}; \qquad a_2 = \frac{1}{\alpha \Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{14}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1-\delta); \qquad a_7 = \delta \Delta t \tag{15}$$

Newmark's method

Following this development, the linear structural system in (10) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{16}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{17}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{18}$$
$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2 \,.$$

General mathemarical representation

Whether time-domain or frequency domain methods were used, in general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_i)\mathbf{A}_i\right)\mathbf{u}(\theta) = \mathbf{f}(\theta) \tag{19}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.

Polynomial Chaos expansion

- After the finite truncation, concisely, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k \tag{20}$$

  where $H_k(\xi(\theta))$ are the polynomial chaoses and $\mathbf{u}_k \in \mathbb{R}^n$ are deterministic vectors to be determined.
- The value of the number of terms P depends on the number of basic random variables M and the order of the PC expansion r as

$$P = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!} \tag{21}$$

Polynomial Chaos expansion

We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k$ for every frequency point:

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{22}$$

P increases exponentially with M:

M	2	3	5	10	20	50	100
2nd order PC	5	9	20	65	230	1325	5150
3rd order PC	9	19	55	285	1770	23425	176850

Polynomial Chaos expansion: Some Observations

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- For an n-dimensional output vector, the number of terms in the projection can be more than n (depends on the number of random variables). This implies that many of the vectors $\mathbf{u}_k$ are linearly dependent.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

Possibilities of solution types

As an example, consider the frequency domain response vector of the stochastic system $\mathbf{u}(\omega, \theta)$ governed by $\left[-\omega^2 \mathbf{M}(\xi(\theta)) + i\omega \mathbf{C}(\xi(\theta)) + \mathbf{K}(\xi(\theta))\right] \mathbf{u}(\omega, \theta) = \mathbf{f}(\omega)$. Some possibilities are

$$
\begin{aligned}
\mathbf{u}(\omega, \theta) &= \sum_{k=1}^{P_1} H_k(\xi(\theta)) \mathbf{u}_k(\omega) \\
\text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\omega, \xi(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_3} a_k(\omega) H_k(\xi(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_4} a_k(\omega) H_k(\xi(\theta)) \mathbf{U}_k(\xi(\theta)) \quad \ldots \text{etc.}
\end{aligned}
\tag{23}
$$

What about classical modal analysis?

For a deterministic system, the response vector $\mathbf{u}(\omega)$ can be expressed as

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} \Gamma_k(\omega)\mathbf{u}_k$$

$$\text{where} \quad \Gamma_k(\omega) = \frac{\phi_k^T \mathbf{f}}{-\omega^2 + 2\mathrm{i}\zeta_k\omega_k\omega + \omega_k^2}$$

$$\mathbf{u}_k = \phi_k \quad \text{and} \quad P \le n \text{ (number of dominant modes)} \tag{24}$$

Can we extend this idea to stochastic systems?

Projection in the modal space

There exist a finite set of complex frequency dependent functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ and a complete basis $\phi_k \in \mathbb{R}^n$ for $k = 1, 2, \ldots, n$ such that the solution of the discretized stochastic finite element equation (4) can be expiressed by the series

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k(\omega, \boldsymbol{\xi}(\theta))\phi_k \tag{25}$$

Outline of the derivation: In the first step a complete basis is generated with the eigenvectors $\phi_k \in \mathbb{R}^n$ of the generalized eigenvalue problem

$$\mathbf{K}_0\phi_k = \lambda_{0_k}\mathbf{M}_0\phi_k; \quad k = 1, 2, \ldots n \tag{26}$$

Projection in the modal space

- We define the matrix of eigenvalues and eigenvectors

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n\times n}; \boldsymbol{\Phi} = \left[\phi_1, \phi_2, \ldots, \phi_n\right] \in \mathbb{R}^{n\times n} \tag{27}$$

Eigenvalues are ordered in the ascending order: $\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$.

- We use the orthogonality property of the modal matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi}^T \mathbf{K}_0 \boldsymbol{\Phi} = \boldsymbol{\lambda}_0, \quad \text{and} \quad \boldsymbol{\Phi}^T \mathbf{M}_0 \boldsymbol{\Phi} = \mathbf{I} \tag{28}$$

- Using these we have

$$\begin{aligned} \boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} &= \boldsymbol{\Phi}^T \left([-\omega^2 + i\omega\zeta_1]\mathbf{M}_0 + [i\omega\zeta_2 + 1]\mathbf{K}_0 \right) \boldsymbol{\Phi} \\ &= \left(-\omega^2 + i\omega\zeta_1 \right) \mathbf{I} + (i\omega\zeta_2 + 1)\, \boldsymbol{\lambda}_0 \end{aligned} \tag{29}$$

This gives $\boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} = \boldsymbol{\Lambda}_0$ and $\mathbf{A}_0 = \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}_0 \boldsymbol{\Phi}^{-1}$, where $\boldsymbol{\Lambda}_0 = \left(-\omega^2 + i\omega\zeta_1\right) \mathbf{I} + (i\omega\zeta_2 + 1)\, \boldsymbol{\lambda}_0$ and $\mathbf{I}$ is the identity matrix.

Projection in the modal space

- Hence, $\mathbf{\Lambda}_0$ can also be written as

$$\mathbf{\Lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{C}^{n\times n} \tag{30}$$

where $\lambda_{0_j} = \left(-\omega^2 + i\omega\zeta_1\right) + \left(i\omega\zeta_2 + 1\right)\lambda_j$ and λ_j is as defined in Eqn. (27). We also introduce the transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{\Phi}^T \mathbf{A}_i \mathbf{\Phi} \in \mathbb{C}^{n\times n}; i = 0, 1, 2, \ldots, M. \tag{31}$$

Note that $\widetilde{\mathbf{A}}_0 = \mathbf{\Lambda}_0$ is a diagonal matrix and

$$\mathbf{A}_i = \mathbf{\Phi}^{-T} \widetilde{\mathbf{A}}_i \mathbf{\Phi}^{-1} \in \mathbb{C}^{n\times n}; i = 1, 2, \ldots, M. \tag{32}$$

Projection in the modal space

Suppose the solution of Eq. (4) is given by

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1} \mathbf{f}(\omega) \tag{33}$$

Using Eqs. (27)–(32) and the mass and stiffness orthogonality of $\mathbf{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{\Phi}^{-T}\mathbf{\Lambda}_0(\omega)\mathbf{\Phi}^{-1} + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Phi}^{-T}\widetilde{\mathbf{A}}_i(\omega)\mathbf{\Phi}^{-1}\right]^{-1} \mathbf{f}(\omega)$$

$$\Rightarrow \quad \hat{\mathbf{u}}(\omega,\theta) = \mathbf{\Phi}\underbrace{\left[\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1}}_{\mathbf{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)} \mathbf{\Phi}^{-T}\mathbf{f}(\omega) \tag{34}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \left\{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\right\}^T.$$

Projection in the modal space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{35}$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n \times n} \tag{36}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix.

$$\mathbf{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \Bigg[\underbrace{\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Lambda}_i(\omega)}_{\mathbf{\Lambda}(\omega, \boldsymbol{\xi}(\theta))} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Delta}_i(\omega)}_{\mathbf{\Delta}(\omega, \boldsymbol{\xi}(\theta))}\Bigg]^{-1} \tag{37}$$

where $\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{R}^{n \times n}$ is a diagonal matrix and $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix.

Projection in the modal space

We rewrite Eq. (37) as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\left[\boldsymbol{\Lambda}\left(\omega,\boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n+\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]\right]^{-1} \tag{38}$$

The above expression can be represented using a Neumann type of matrix series as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\sum_{s=0}^{\infty}(-1)^s\left[\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]^s\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right) \tag{39}$$

Projection in the modal space

Taking an arbitrary r-th element of $\hat{\mathbf{u}}(\omega,\theta)$, Eq. (34) can be rearranged to have

$$\hat{u}_r(\omega,\theta)=\sum_{k=1}^{n}\Phi_{rk}\left(\sum_{j=1}^{n}\Psi_{kj}\left(\omega,\xi(\theta)\right)\left(\phi_j^T\mathbf{f}(\omega)\right)\right) \tag{40}$$

Defining

$$\Gamma_k\left(\omega,\xi(\theta)\right)=\sum_{j=1}^{n}\Psi_{kj}\left(\omega,\xi(\theta)\right)\left(\phi_j^T\mathbf{f}(\omega)\right) \tag{41}$$

and collecting all the elements in Eq. (40) for $r=1,2,\ldots,n$ one has

$$\hat{\mathbf{u}}(\omega,\theta)=\sum_{k=1}^{n}\Gamma_k\left(\omega,\xi(\theta)\right)\phi_k \tag{42}$$

Spectral functions

Definition

The functions $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right), k = 1, 2, \ldots n$ *are the* *frequency-adaptive spectral functions* *as they are expressed in terms of the spectral properties of the coefficient matrices at each frequency of the governing discretized equation.*

- Each of the spectral functions $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right)$ contain infinite number of terms and they are highly nonlinear functions of the random variables $\xi_i(\theta)$.
- For computational purposes, it is necessary to truncate the series after certain number of terms.
- Different order of spectral functions can be obtained by using truncation in the expression of $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right)$

First-order and second order spectral functions

Definition

The different order of spectral functions $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots, n$ *are obtained by retaining as many terms in the series expansion in Eqn. (39).*

Retaining one and two terms in (39) we have

$$\boldsymbol{\Psi}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{43}$$

$$\boldsymbol{\Psi}^{(2)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) - \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{44}$$

which are the first and second order spectral functions respectively.

- From these we find $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \sum_{j=1}^{n} \Psi_{kj}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) \left(\phi_j^T \mathbf{f}(\omega)\right)$ are non-Gaussian random variables even if $\xi_i(\theta)$ are Gaussian random variables.

Summary of the basis functions (frequency-adaptive spectral functions)

The basis functions are:

1. not polynomials in $\xi_i(\theta)$ but ratio of polynomials.
2. independent of the nature of the random variables (i.e. applicable to Gaussian, non-Gaussian or even mixed random variables).
3. not general but specific to a problem as it utilizes the eigenvalues and eigenvectors of the system matrices.
4. such that truncation error depends on the off-diagonal terms of the matrix $\mathbf{\Delta}(\omega, \boldsymbol{\xi}(\theta))$.
5. showing 'peaks' when ω is near to the system natural frequencies

Next we use these frequency-adaptive spectral functions as trial functions within a Galerkin error minimization scheme.

The Galerkin approach

One can obtain constants $c_k \in \mathbb{C}$ such that the error in the following representation

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} c_k(\omega) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta)) \phi_k \tag{45}$$

can be minimised in the least-square sense. It can be shown that the vector $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^T$ satisfies the $n \times n$ complex algebraic equations $\mathbf{S}(\omega)\, \mathbf{c}(\omega) = \mathbf{b}(\omega)$ with

$$S_{jk} = \sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}; \quad \forall\, j, k = 1, 2, \ldots, n; \widetilde{A}_{i_{jk}} = \phi_j^T \mathbf{A}_i \phi_k, \tag{46}$$

$$D_{ijk} = \mathrm{E}\left[\xi_i(\theta) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\right], b_j = \mathrm{E}\left[\phi_j^T \mathbf{f}(\omega)\right]. \tag{47}$$

The Galerkin approach

- The error vector can be obtained as

$$\varepsilon(\omega,\theta)=\left(\sum_{i=0}^{M}\mathbf{A}_i(\omega)\xi_i(\theta)\right)\left(\sum_{k=1}^{n}c_k\widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\phi_k\right)-\mathbf{f}(\omega)\in\mathbb{C}^{N\times N} \tag{48}$$

  The solution is viewed as a projection where $\phi_k \in \mathbb{R}^n$ are the basis functions and c_k are the unknown constants to be determined. This is done for each frequency step.

- The coefficients c_k are evaluated using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon(\omega,\theta)\perp\phi_j \Rrightarrow \left\langle\phi_j,\varepsilon(\omega,\theta)\right\rangle=0\;\forall j=1,2,\ldots,n \tag{49}$$

The Galerkin approach

- Imposing the orthogonality condition and using the expression of the error one has

$$\mathrm{E}\left[\phi_j^T \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i(\theta)\right)\left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\phi_k\right) - \phi_j^T \mathbf{f}\right] = 0, \forall j \tag{50}$$

- Interchanging the $\mathrm{E}[\bullet]$ and summation operations, this can be simplified to

$$\sum_{k=1}^{n}\left(\sum_{i=0}^{M}\left(\phi_j^T \mathbf{A}_i \phi_k\right) \mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right) c_k = \mathrm{E}\left[\phi_j^T \mathbf{f}\right] \tag{51}$$

$$\text{or} \quad \sum_{k=1}^{n}\left(\sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}\right) c_k = b_j \tag{52}$$

Model Reduction by reduced number of basis

- Suppose the eigenvalues of $\mathbf{A}_0$ are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n} \tag{53}$$

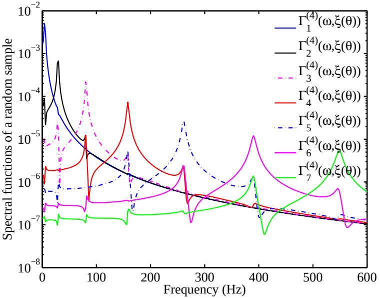
- From the expression of the spectral functions observe that the eigenvalues ($\lambda_{0_k} = \omega_{0_k}^2$) appear in the denominator:

$$\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^M \xi_i(\theta)\Lambda_{i_k}(\omega)} \tag{54}$$

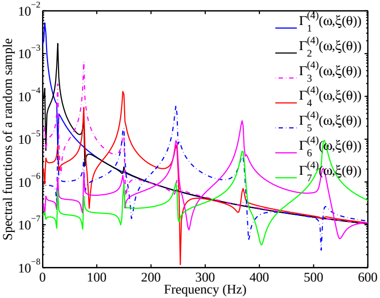
where $\Lambda_{0_k}(\omega) = -\omega^2 + i\omega(\zeta_1 + \zeta_2\omega_{0_k}^2) + \omega_{0_k}^2$

- The series can be truncated based on the magnitude of the eigenvalues relative to the frequency of excitation. Hence for the frequency domain analysis all the eigenvalues that cover almost twice the frequency range under consideration can be chosen.

Nature of the spectral functions



(a) Spectral functions for $\sigma_a = 0.1$.



(b) Spectral functions for $\sigma_a = 0.2$.

The amplitude of first seven spectral functions of order 4 for a particular random sample under applied force. The spectral functions are obtained for two different standard deviation levels of the underlying random field: $\sigma_a = \{0.10, 0.20\}$.

Model Reduction by reduced number of modes

(Stochastic modal reduction) The solution of the discretized stochastic finite element equation (4) can be expressed by the series representation

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{p} c_k \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\phi_k \tag{55}$$

such that the error is minimized in a least-square sense. c_k, $\widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\omega))$ and ϕ_k can be obtained following the procedure described in the previous section by letting the indices j, k upto p in Eqs. (46) and (47).

Computational method

- The mean vector can be obtained as

$$\bar{\mathbf{u}} = \mathrm{E}\left[\hat{\mathbf{u}}(\theta)\right] = \sum_{k=1}^{p} c_k \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right] \boldsymbol{\phi}_k \tag{56}$$

- The covariance of the solution vector can be expressed as

$$\boldsymbol{\Sigma}_u = \mathrm{E}\left[\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)^T\right] = \sum_{k=1}^{p}\sum_{j=1}^{p} c_k c_j \Sigma_{\Gamma_{kj}} \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T \tag{57}$$

where the elements of the covariance matrix of the spectral functions are given by

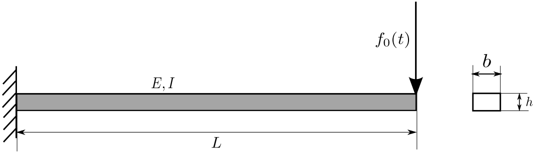
$$\Sigma_{\Gamma_{kj}} = \mathrm{E}\left[\left(\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)\left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\right]\right)\right] \tag{58}$$

Summary of the computational method

1. Solve the generalized eigenvalue problem associated with the mean mass and stiffness matrices to generate the orthonormal basis vectors: $\mathbf{K}_0\mathbf{\Phi} = \mathbf{M}_0\mathbf{\Phi}\boldsymbol{\lambda}_0$
2. Select a number of samples, say N_{samp}. Generate the samples of basic random variables $\xi_i(\theta), i = 1, 2, \ldots, M$.
3. Calculate the spectral basis functions (for example, first-order): $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega)+\sum_{i=1}^{M}\xi_i(\theta)\Lambda_{i_k}(\omega)}$, for $k = 1, \cdots p, p < n$
4. Obtain the coefficient vector: $\mathbf{c}(\omega) = \mathbf{S}^{-1}(\omega)\mathbf{b}(\omega) \in \mathbb{R}^n$, where $\mathbf{b}(\omega) = \widetilde{\mathbf{f}(\omega)} \odot \overline{\mathbf{\Gamma}(\omega)}$, $\mathbf{S}(\omega) = \mathbf{\Lambda}_0(\omega) \odot \mathbf{D}_0(\omega) + \sum_{i=1}^{M} \widetilde{\mathbf{A}}_i(\omega) \odot \mathbf{D}_i(\omega)$ and $\mathbf{D}_i(\omega) = \mathrm{E}\left[\mathbf{\Gamma}(\omega,\theta)\xi_i(\theta)\mathbf{\Gamma}^T(\omega,\theta)\right], \forall\, i = 0, 1, 2, \ldots, M$
5. Obtain the samples of the response from the spectral series: $\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{p} c_k(\omega)\Gamma_k(\boldsymbol{\xi}(\omega,\theta))\phi_k$

The Euler-Bernoulli beam example

- An Euler-Bernoulli cantilever beam with stochastic bending modulus



- Length : 1.0 m, Cross-section : 39 $\times$ 5.93 mm^2, Young's Modulus: 2 $\times$ 10^{11} Pa.
- We study the deflection of the beam under the action of a harmonic point load on the free end.

Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{59}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The autocorrelation function of this random field is assumed to be

$$C_a(x_1,x_2) = \sigma_a^2 e^{-(|x_1-x_2|)/\mu_a} \tag{60}$$

where μ_a is the correlation length and σ_a is the standard deviation.

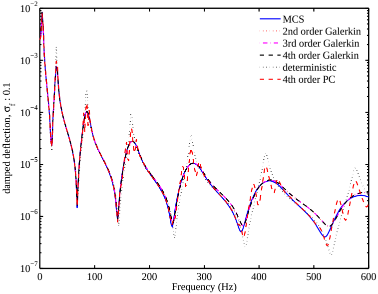
- A correlation length of $\mu_a = L/2$ is considered in the present numerical study.

Problem details

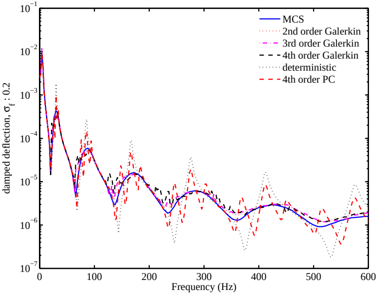
The random field is Gaussian with correlation length $\mu_a = L/2$. The results are compared with the polynomial chaos expansion.

- The number of degrees of freedom of the system is n = 80.
- The number of random variables in KL expansion used for discretizing the stochastic domain is 18 (90% of variability retained).
- Simulations have been performed with 10,000 MCS samples and for two values of standard deviation of the random field, $\sigma_a = 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Frequency range of interest: $0 - 600$ Hz at an interval of 2 Hz.
- Upto 4^{th} order spectral functions have been considered in the present problem. Comparison have been made with 4^{th} order Polynomial choas results.

Frequency domain response of the beam



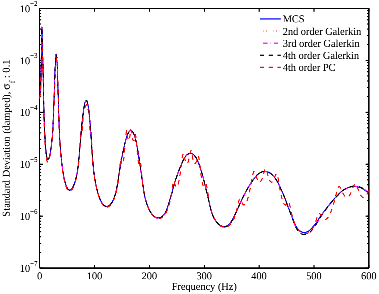
(c) Beam deflection for $\sigma_a = 0.1$.



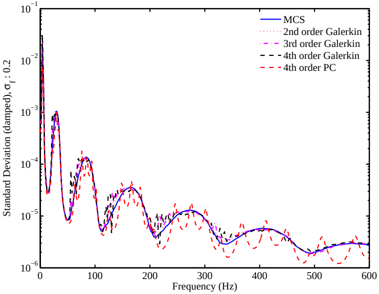
(d) Beam deflection for $\sigma_a = 0.2$.

The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$. The proposed Galerkin approach needs solution of a 14×14 linear system of equations only

Standard deviation of the beam response



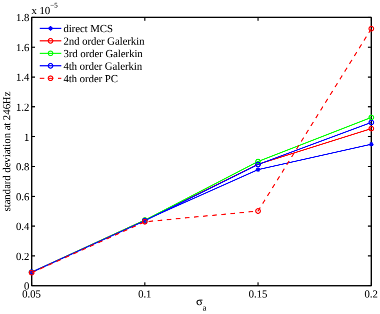
(e) Standard deviation of the response for $\sigma_a = 0.1$.



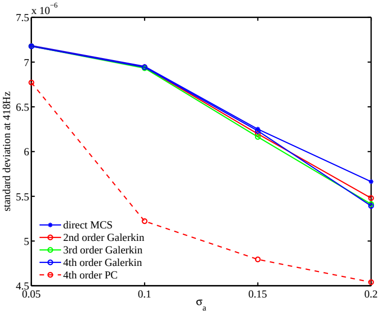
(f) Standard deviation of the response for $\sigma_a = 0.2$.

The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Standard deviation vs field variability



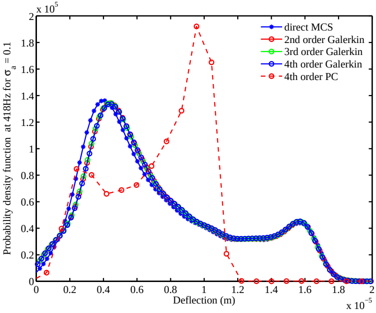
(g) Standard deviation at 246Hz.



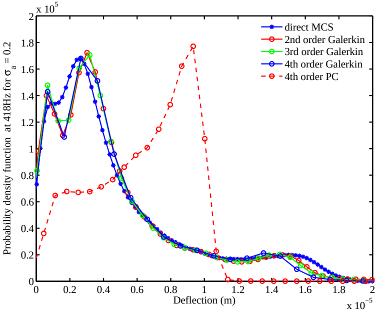
(h) Standard deviation at 418Hz.

The standard deviation of the tip deflection for different values of the variability of the random field (stochastic elastic modulus). The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$. 246 and 418 Hz correspond to the anti-resonance and resonance frequencies of the beam respectively.

Probability density function of the tip deflection



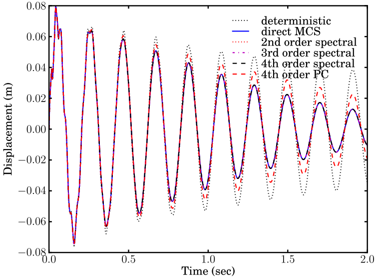
(i) PDF for $\sigma_a = 0.1$.



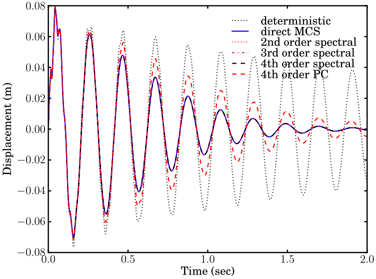
(j) PDF for $\sigma_a = 0.2$.

The probability density function of the deflection of the tip of the beam under a unit amplitude harmonic point load at 418 Hz (resonance frequency). The correlation length of the random field describing the bending rigidity is taken to be $\mu_a = L/2$. The pdfs are obtained with 10,000 sample MCS and two values of $\sigma_a = 0.10, 0.20$.

Mean of the response: time domain



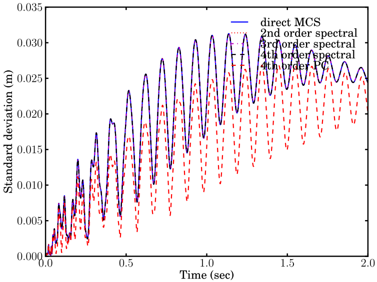
(k) Mean deflection, $\sigma_a = 0.1$.



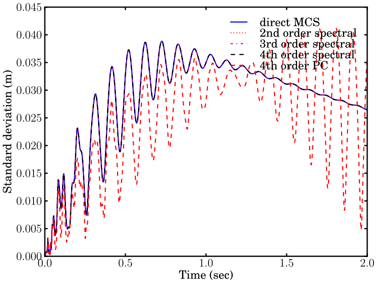
(l) Mean deflection, $\sigma_a = 0.2$.

The mean deflection of the tip of the cantilever beam under an unit impulse load at time $t = 0$ for the duration of $1/800$ seconds. The response of the reduced order spectral function method is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Standard deviation of the response: time domain



(m) Standard deviation of deflection, $\sigma_a = 0.1$.



(n) Standard deviation of deflection, $\sigma_a = 0.2$.

The standard deviation of the deflection of the tip of the cantilever beam under an unit impulse load at time $t = 0$ for the duration of $1/800$ seconds. The response of the reduced order spectral function method is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Conclusions

1. The stochastic partial differential equations for structural dynamics is considered.
2. The solution is projected into the modal basis and the associated stochastic coefficient functions are obtained at each frequency step (or time step).
3. The coefficient functions, called as the spectral functions, are expressed in terms of the spectral properties of the system matrices.
4. If $p < n$ number of orthonormal vectors are used and M is the number of random variables, then the computational complexity grows in $O(Mp^2) + O(p^3)$ for large M and p in the worse case.

Discussions

- The proposed method takes advantage of the fact that for a given maximum frequency only a small number of modes are necessary to represent the dynamic response. This modal reduction leads to a significantly smaller basis. This type of reduction is difficult to incorporate within the scope of PC as no information regarding the system matrices are used in constructing the orthogonal polynomial basis.