Uncertainty Quantification in Structural Dynamics: A Reduced Random Matrix Approach

5th International ASRANet Conference

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Outline of the presentation

- Introduction: current status and challenges
- Uncertainty Propagation (UP) in structural dynamics
  - Parametric uncertainty
  - Nonparametric uncertainty
- Reduced Wishart random matrix model
  - Analytical derivation
  - Parameter estimation
- Computational results
- Experimental results
- Integration with commercial Finite Element code
- Conclusions & future directions

Ensembles of structural dynamical systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems)

A complex structural dynamical system

Complex aerospace system can have millions of degrees of freedom and significant ‘errors’ and/or ‘lack of knowledge’ in its numerical (Finite Element) model

Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Uncertainty propagation: key challenges

The main difficulties are:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and

Uncertainty propagation (1)

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type ‘a’)
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications (building under earthquake load)

Uncertainty propagation (2)

- Nonparametric approaches : Such as the Random matrix theory:
  - aim to characterize nonparametric uncertainty (types 'b' - 'e')
  - does not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications (eg, noise propagation in vehicles)

Dynamics of a general linear system

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices (and consequently in the eigensolutions)
  - to predict the variability in the response vector $\mathbf{q}$

Random matrix model for dynamical system

- Suppose $H(x,\theta)$ is a distributed random field describing a system parameter. This can be expanded using the Karhunen-Loève expansion as

$$H(x,\theta) = H_0(x) + \epsilon \sum_{j=1}^{M} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \tag{2}$$

where $H_0(x)$ is the mean of the random field, ϵ is its standard deviation and M is the number of terms used to truncate the infinite series.

- Substituting this in the equation of motion and following the usual finite element method, and of the system matrix can be expressed as

$$\mathbf{G}(\theta) = \mathbf{G}_0 + \epsilon_G \sum_{j=1}^{M} \xi_{G_j}(\theta) \mathbf{G}_j \tag{3}$$

- Q: how non parametric uncertainties can be taken into account?

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \left|\boldsymbol{\Sigma}\right|^{-p/2} \left|\boldsymbol{\Psi}\right|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (4)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}\left(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}\right)$.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}\left(\mathbf{W}\right) = \left\{\Gamma_n\left(a\right)\left|\mathbf{\Psi}\right|^{-a}\right\}^{-1}\left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)}\mathrm{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\};\ \Re(a) > \frac{1}{2}(n-$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n\left(a\right) = \pi^{\frac{1}{4}n(n-1)}\prod_{k=1}^{n}\Gamma\left[a - \frac{1}{2}(k-1)\right];\ \text{for}\ \Re(a) > (n-1)/2$$

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np}\, \Gamma_n\left(\frac{1}{2}p\right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S} \right\} \quad (5)$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{6}$$

$$\text{and} \quad \int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} \tag{7}$$

Further constraints

- Suppose that the inverse moments up to order ν of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}} = \left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^T\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln |\mathbf{G}|^{-\nu}\right] < \infty$$

The random matrix model

- Following the maximum entropy method it can be shown that the system matrices are distributed as Wishart matrices, i.e., $\mathbf{G} \sim W_n(\mathbf{G}_0, \delta_G^2)$
- Here $\mathbf{G}_0$ is the mean and the dispersion parameter (normalized) standard deviation of the system matrices:

$$\delta_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{8}$$

- This method is computationally expensive as the simulation of two Wishart matrices and the solution of a generaized eigenvalue problem is necessary for each sample.

The dispersion parameter

$$\delta_G^2 = \frac{\mathrm{E}\left[\left\|\epsilon_G \sum_{j=1}^M \xi_{G_j}(\theta)\mathbf{G}_j\right\|_\mathrm{F}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2} \tag{9}$$

Since both trace and expectation operators are linear they can be swaped. Doing this we obtain

$$\delta_G^2 = \frac{\epsilon_G^2 \mathrm{Trace}\left(\mathrm{E}\left[\left(\sum_{j=1}^M \sum_{k=1}^M \xi_{G_j}(\theta)\xi_{G_k}(\theta)\mathbf{G}_j\mathbf{G}_k\right)\right]\right)}{\|\mathbf{G}_0\|_\mathrm{F}^2} \tag{10}$$

Recalling that the matrices $\mathbf{G}_j$ are not random and $\{\xi_{G_1}(\theta), \xi_{G_2}(\theta), \dots\}$ is a set of uncorrelated random variables with zero mean and $\mathrm{E}\left[\xi_{G_j}(\theta)\xi_{G_k}(\theta)\right] = \delta_{jk}$, we have

$$\begin{aligned}\delta_G^2 &= \frac{\epsilon_G^2 \mathrm{Trace}\left(\left(\sum_{j=1}^M \sum_{k=1}^M \mathrm{E}\left[\xi_{G_j}(\theta)\xi_{G_k}(\theta)\right]\mathbf{G}_j\mathbf{G}_k\right)\right)}{\|\mathbf{G}_0\|_\mathrm{F}^2} \\ &= \frac{\epsilon_G^2 \mathrm{Trace}\left(\left(\sum_{j=1}^M \mathbf{G}_j^2\right)\right)}{\|\mathbf{G}_0\|_\mathrm{F}^2} = \epsilon_G^2 \frac{\sum_j^M \|(\mathbf{G}_j)\|_\mathrm{F}^2}{\|\mathbf{G}_0\|_\mathrm{F}^2}\end{aligned} \tag{11}$$

Reduced random matrix approach (1)

- Taking the Laplace transform of the equation of motion:

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{12}$$

  The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2 \mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{13}$$

  where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- Suppose the number of modes to be retained is m. In general $m \ll n$. We form the truncated undamped modal matrices

$$\boldsymbol{\Omega} = \operatorname{diag}\left[\omega_1, \omega_2, \ldots, \omega_m\right] \in \mathbb{R}^{m\times m} \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_m\right] \in \mathbb{R}^{n\times m} \tag{14}$$

  so that $\boldsymbol{\Phi}^T\mathbf{K}_e\boldsymbol{\Phi} = \boldsymbol{\Omega}^2$ and $\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_m$

Reduced random matrix approach (2)

- Transforming it into the modal coordinates:

$$\left[s^2\mathbf{I}_m + s\mathbf{C}' + \mathbf{\Omega}^2\right] \bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{15}$$

- Here

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = 2\zeta\mathbf{\Omega}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{16}$$

- When we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ will be a random matrix of dimension m. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{m\times m}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{17}$$

Reduced random matrix approach (3)

- Since $\boldsymbol{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\boldsymbol{\Psi}_r$ such that

$$\boldsymbol{\Psi}_r^T \boldsymbol{\Xi} \boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \tag{18}$$

  Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\boldsymbol{\Xi}$.

- Recalling that $\boldsymbol{\Psi}_r^T \boldsymbol{\Psi}_r = \mathbf{I}_m$ we obtain

$$\bar{\mathbf{q}}' = \left[s^2 \mathbf{I}_m + s\mathbf{C}' + \boldsymbol{\Omega}^2 \right]^{-1} \bar{\mathbf{f}}' \tag{19}$$

$$= \boldsymbol{\Psi}_r \left[s^2 \mathbf{I}_m + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2 \right]^{-1} \boldsymbol{\Psi}_r^T \bar{\mathbf{f}}' \tag{20}$$

Reduced random matrix approach (4)

- The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \mathbf{\Phi}\bar{\mathbf{q}}'(s) = \mathbf{\Phi}\mathbf{\Psi}_r \left[s^2\mathbf{I}_m + 2s\zeta\mathbf{\Omega}_r + \mathbf{\Omega}_r^2\right]^{-1} (\mathbf{\Phi}\mathbf{\Psi}_r)^T\bar{\mathbf{f}}(s)$$

$$= \sum_{j=1}^{m} \frac{\mathbf{x}_{r_j}^T\bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2}\mathbf{x}_{r_j}.$$

Here

$$\mathbf{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_m}\right], \quad \mathbf{X}_r = \mathbf{\Phi}\mathbf{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}\right]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.

Wishart system matrices

$\mathbf{M}$ and $\mathbf{K}$ are Wishart matrices. For this case $\mathbf{M} \sim W_n(p_1, \mathbf{\Sigma}_1)$, $\mathbf{K} \sim W_n(p_1, \mathbf{\Sigma}_1)$ with $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$ and $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$.
Here

$$\mathbf{\Sigma}_1 = \mathbf{M}_0/p_1, p_1 = \frac{\gamma_M + 1}{\delta_M^2} \tag{21}$$

$$\text{and} \quad \mathbf{\Sigma}_2 = \mathbf{K}_0/p_2, p_2 = \frac{\gamma_K + 1}{\delta_K^2} \tag{22}$$

$$\gamma_G = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right) \tag{23}$$

Parameter-estimation for the reduced matrix (1)

We have $\Xi \sim W_m\left(p, \mathbf{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\Xi^{-1}\right] = \mathbf{\Omega}_0^{-2}$ and $\delta_{\Xi} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem. The parameters can be obtained as

$$p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1 + \gamma_H)}{\delta_H^2} - (n + 1) \tag{24}$$

Parameter-estimation for the reduced matrix (2)

- Defining $\mathbf{H}_0 = {\mathbf{M}_0}^{-1}\mathbf{K}_0$, the constant γ_H:

$$\gamma_H = \frac{\{\mathrm{Trace}\,(\mathbf{H}_0)\}^2}{\mathrm{Trace}\left({\mathbf{H}_0}^2\right)} = \frac{\left\{\mathrm{Trace}\left(\mathbf{\Omega}_0^2\right)\right\}^2}{\mathrm{Trace}\left(\mathbf{\Omega}_0^4\right)} = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{25}$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\delta_H = \frac{\left({p_1}^2 + (p_2 - 2 - 2\,n)\,p_1 + (-n-1)\,p_2 + n^2 + 1 + 2\,n\right)\gamma_H}{p_2\,(-p_1+n)\,(-p_1+n+3)} + \frac{{p_1}^2 + (p_2 - 2\,n)\,p_1 + (1-n)\,p_2 - 1 + n^2}{p_2\,(-p_1+n)\,(-p_1+n+3)} \tag{26}$$

Computational strategy (1)

- Calculate the parameters

$$\theta = \frac{(1+\beta_H)}{\delta_H^2} - (m+1) \quad \text{and} \quad p = [m+1+\theta] \tag{27}$$

where p is approximated to the nearest integer of $m+1+\theta$.

- Create an $m \times p$ matrix $\mathbf{Y}$ such that

$$Y_{ij} = \omega_{0_i} \widehat{Y}_{ij} / \sqrt{\theta}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, p \tag{28}$$

where $\widehat{Y}_{ij}$ are independent and identically distributed (i.i.d.) Gaussian random numbers with zero mean and unit standard deviation.

- Simulate the $m \times m$ Wishart random matrix

$$\boldsymbol{\Xi} = \mathbf{Y}\mathbf{Y}^T \quad \text{or} \quad \Xi_{ij} = \frac{\omega_{0_i}\omega_{0_j}}{\theta} \sum_{k=1}^{p} \widehat{Y}_{ik}\widehat{Y}_{jk}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, m \tag{29}$$

Since $\boldsymbol{\Xi}$ is symmetric, only the upper or lower triangular part need to be simulated.

Computational strategy (2)

- Solve the symmetric eigenvalue problem ($\boldsymbol{\Omega}_r, \boldsymbol{\Psi}_r \in \mathbb{R}^{m \times m}$) for every sample

$$\boldsymbol{\Xi}\boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \boldsymbol{\Psi}_r \tag{30}$$

  and obtain the random eigenvector matrix

$$\mathbf{X}_r = \boldsymbol{\Phi}_0 \boldsymbol{\Psi}_r = [\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}] \in \mathbb{R}^{n \times m} \tag{31}$$

- Finally calculate the dynamic response in the frequency domain as

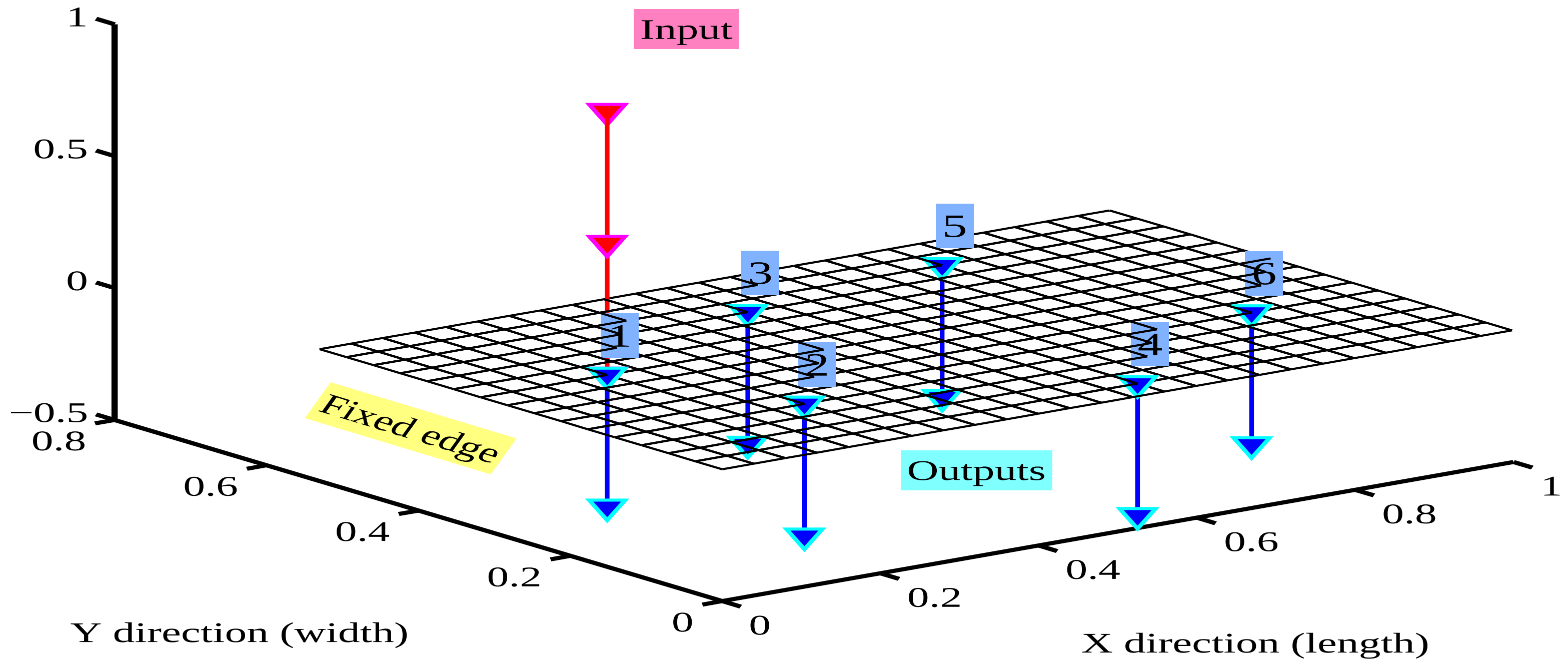
$$\bar{\mathbf{q}}_r(\mathrm{i}\omega) = \sum_{j=1}^{m} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{-\omega^2 + 2\mathrm{i}\omega\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j} \tag{32}$$

- The samples of the response in the time domain can also be obtained from the random eigensolutions as

$$\mathbf{q}_r(t) = \sum_{j=1}^{m} a_{r_j}(t)\mathbf{x}_{r_j}, \ \text{where } a_{r_j}(t) = \frac{1}{\omega_{r_j}} \int_0^t \mathbf{x}_{r_j}^T \mathbf{f}(\tau) e^{-\zeta_j \omega_{r_j}(t-\tau)} \sin\left(\omega_{r_j}(t-\tau)\right) d\tau \tag{33}$$

Numerical Examples

A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/∼adhikaris/uq/.

Uncertainty type 1: random fields

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \tag{34}$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \tag{35}$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \tag{36}$$

$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \tag{37}$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

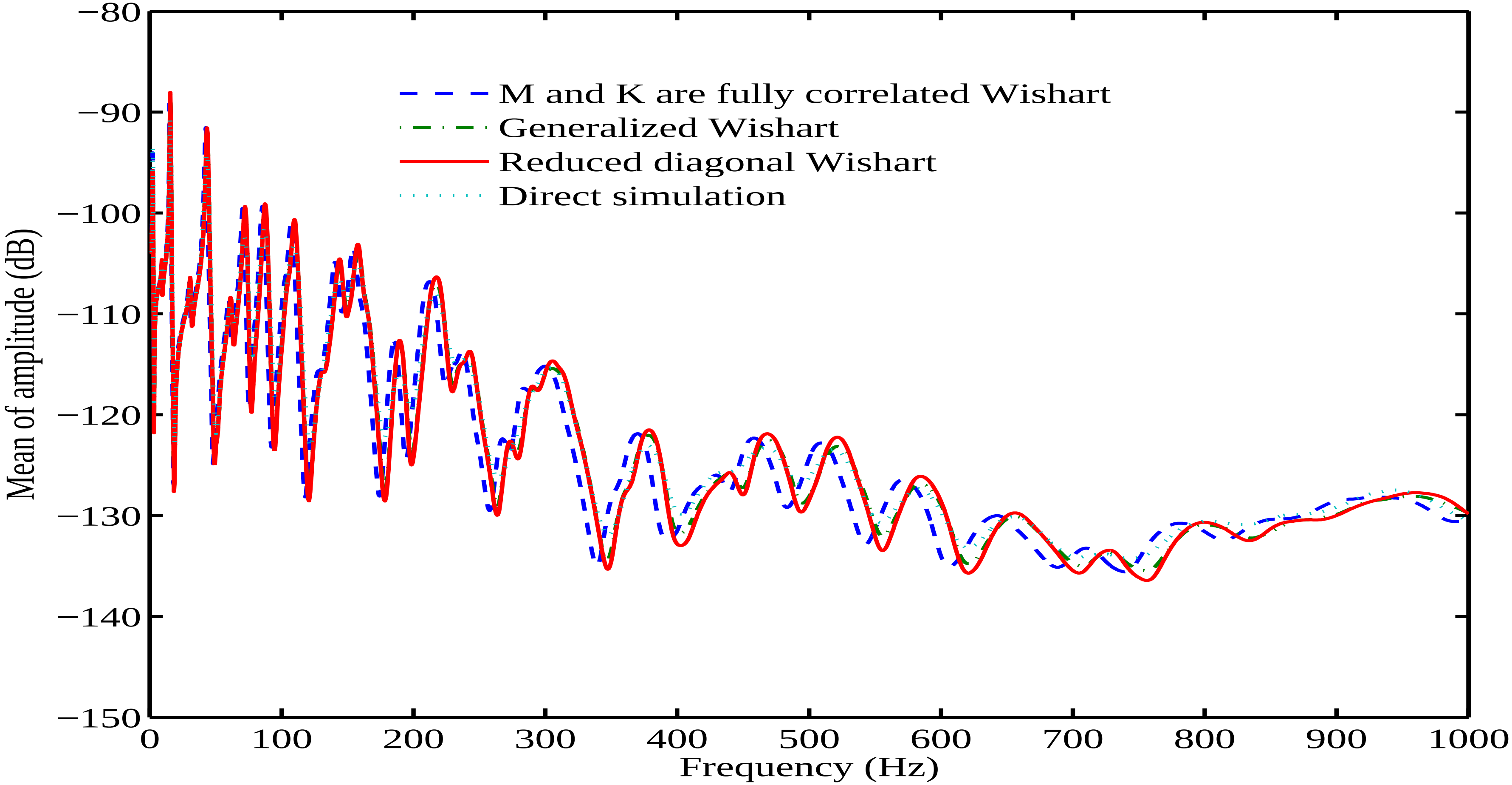
Uncertainty type 2: random attached oscillators

- Here we consider that the baseline plate is ‘perturbed’ by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty.
- This case will be investigated experimentally.

Methodologies compared

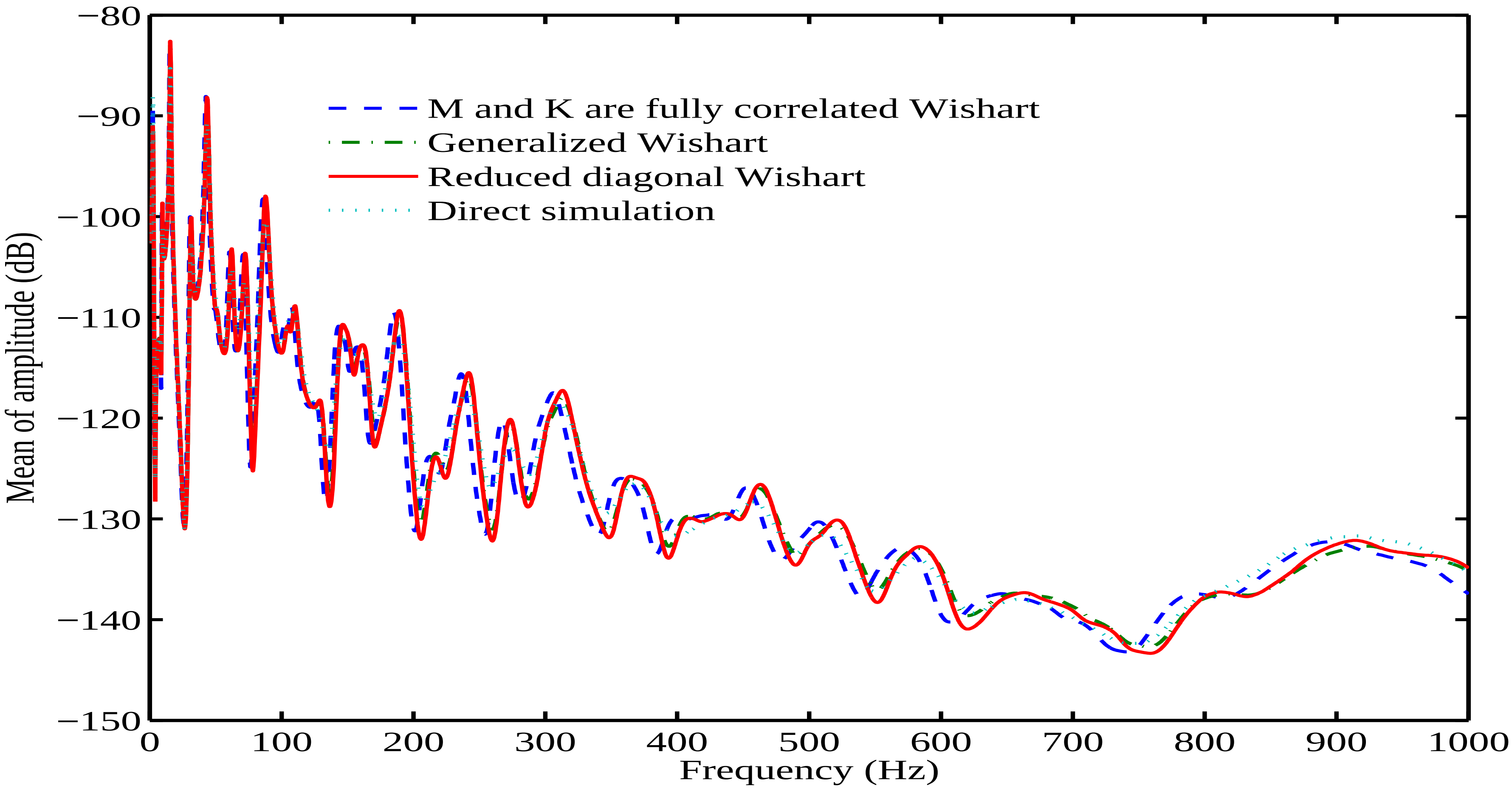
- *Method 1 - Mass and stiffness matrices are fully correlated Wishart matrices:* For this case $\mathbf{M} \sim W_n(p_M, \mathbf{\Sigma}_M)$, $\mathbf{K} \sim W_n(p_K, \mathbf{\Sigma}_K)$ with $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$ and $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices. The computational cost of this approach is $\approx 2O(n^3)$.
- *Method 2 - Generalized Wishart Matrix:* For this case $\mathbf{\Xi} \sim W_n\left(p, \mathbf{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\mathbf{\Xi}^{-1}\right] = \mathbf{\Omega}_0^{-2}$ and $\delta_{\mathbf{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem. The computational cost of this approach is $\approx O(n^3)$.
- *Method 3 - Reduced diagonal Wishart Matrix:* For this case $\widetilde{\mathbf{\Xi}} \sim W_m\left(\widetilde{p}, \widetilde{\mathbf{\Omega}}_0^2/\theta\right)$ with $\mathrm{E}\left[\widetilde{\mathbf{\Xi}}^{-1}\right] = \widetilde{\mathbf{\Omega}}_0^{-2}$ and $\delta_{\widetilde{\mathbf{\Xi}}} = \delta_H$. This requires the simulation of one $m \times m$ uncorrelated Wishart matrix and the solution of a $m \times m$ standard eigenvalue problem. For large complex systems m can be significantly smaller than n. The computational cost of this approach is $\approx O(m^3)$.

Mean of cross-FRF: Utype 1



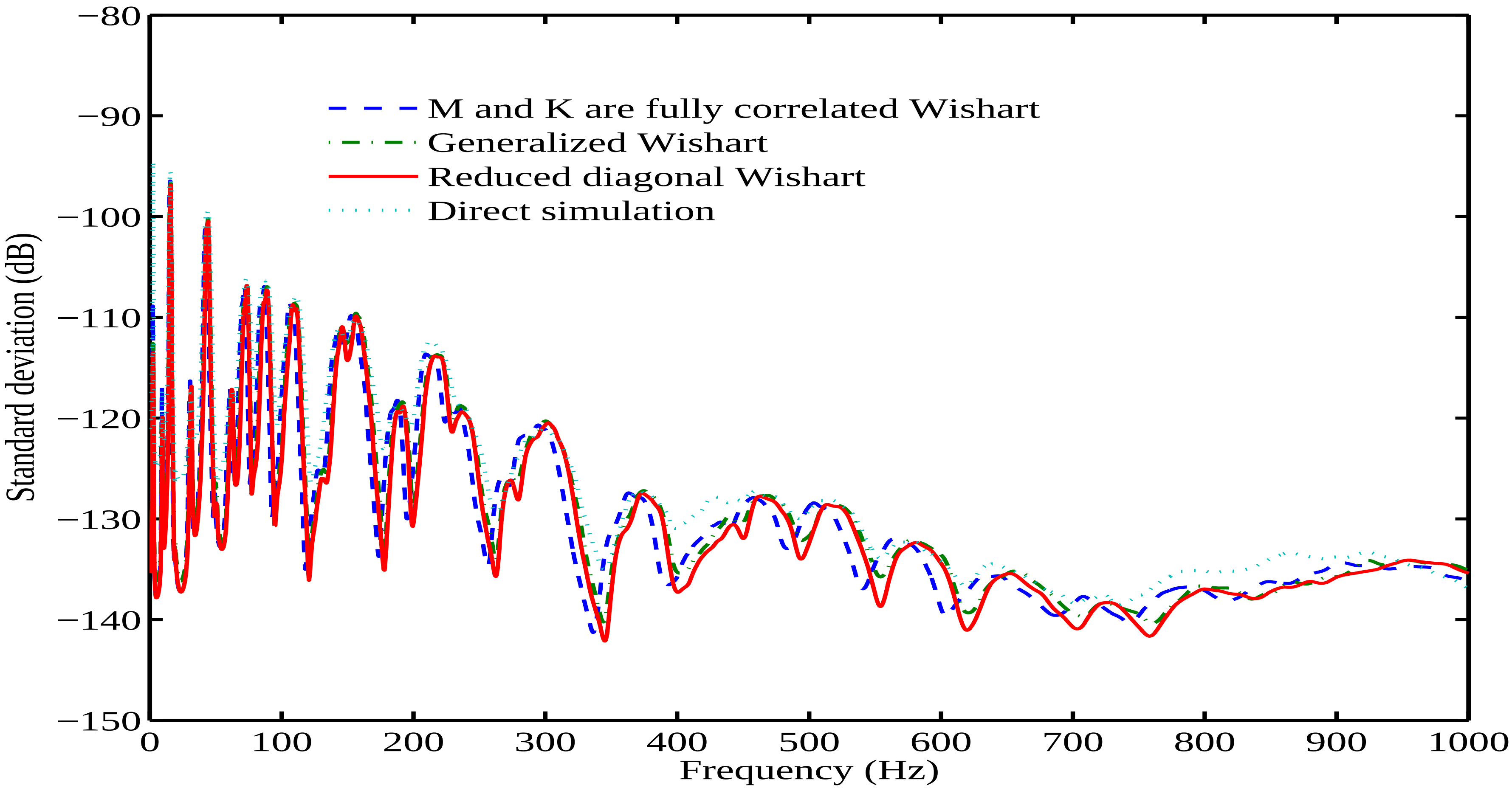
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Mean of driving-point-FRF: Utype 1



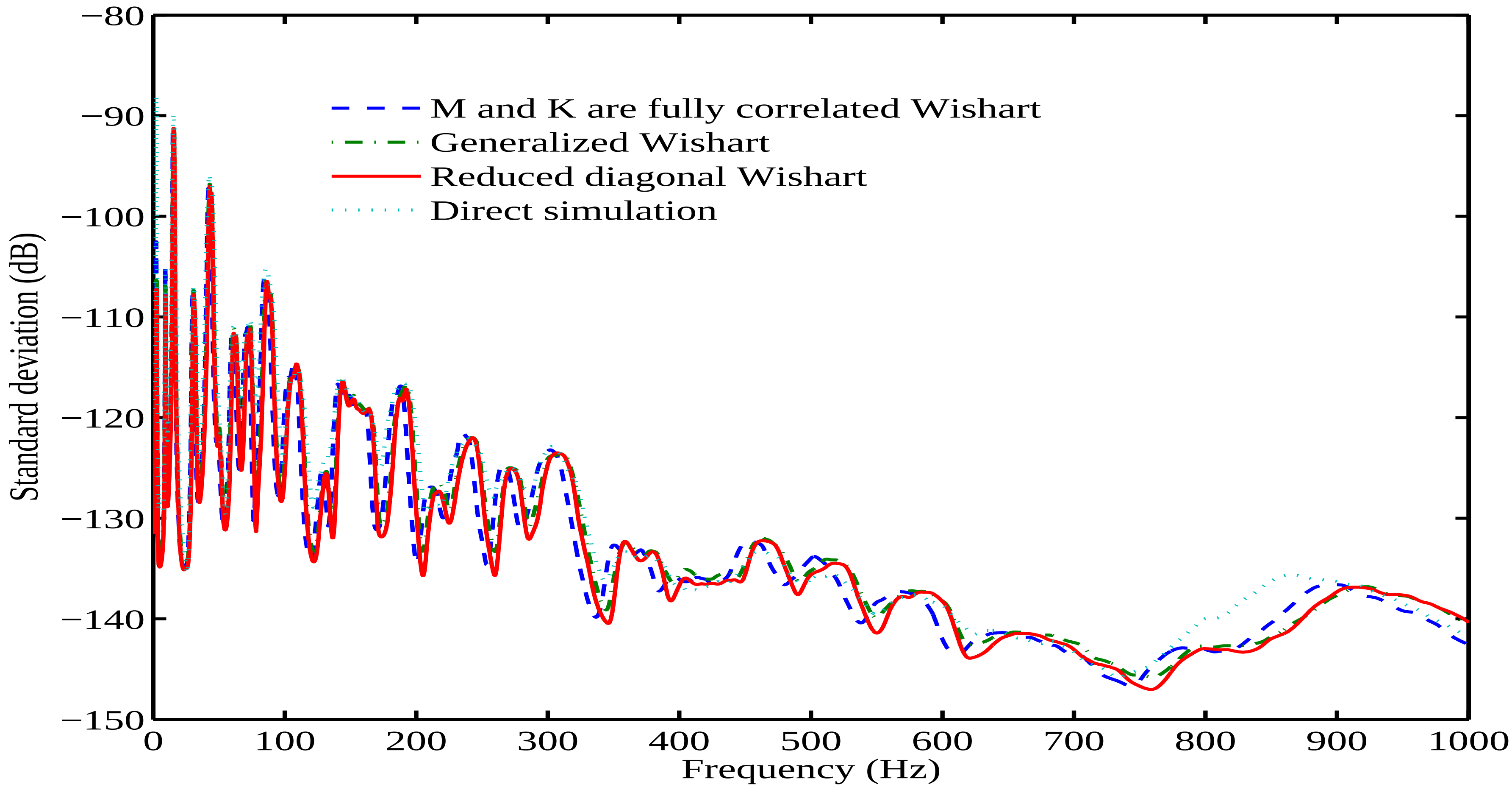
Mean of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Standard deviation of cross-FRF: Utype 1



Standard deviation of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

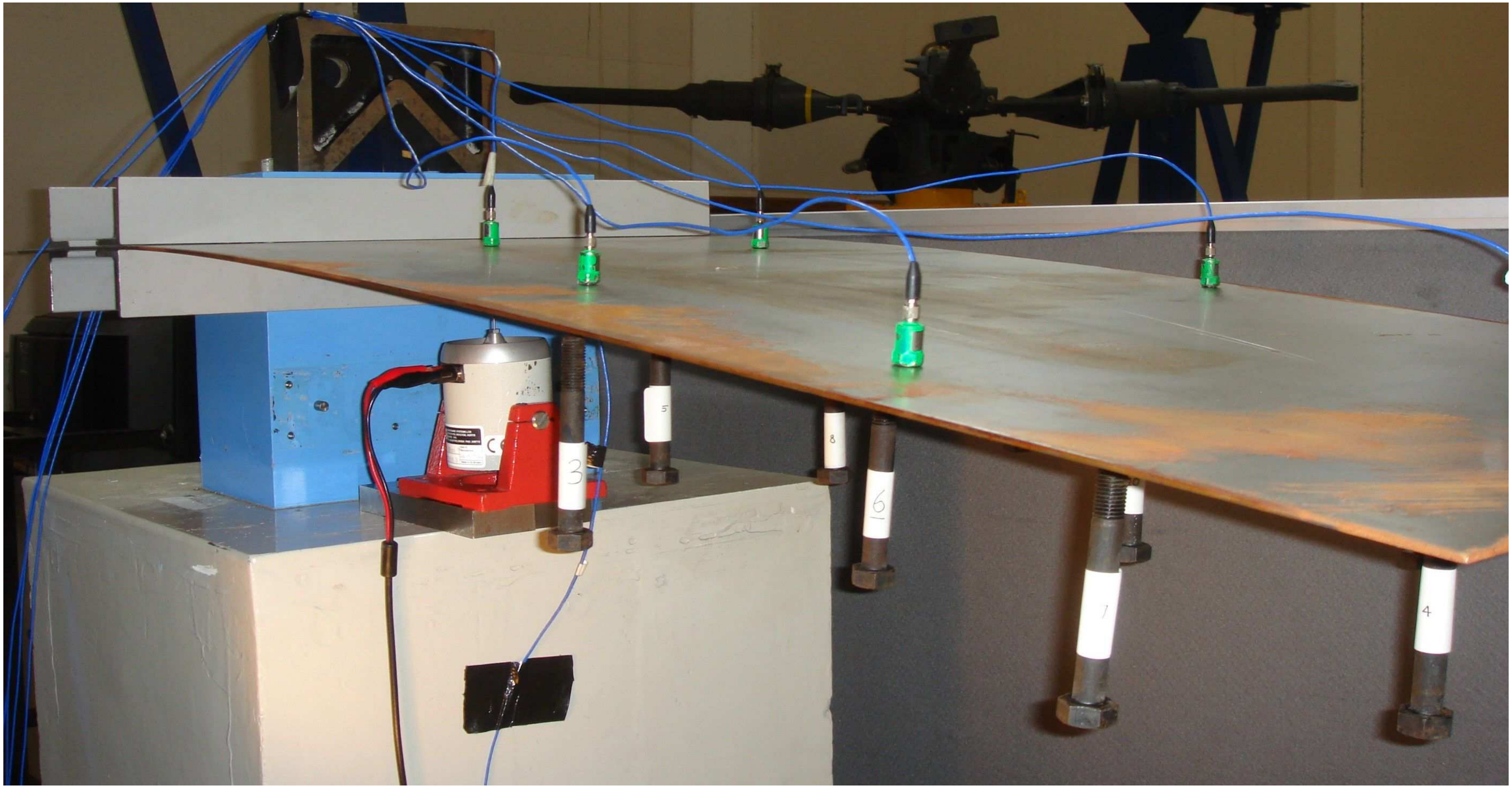
Standard deviation of driving-point-FRF: Utype 1



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Experimental investigation for uncertainty type 2 (randomly attached oscillators)

A cantilever plate: front view

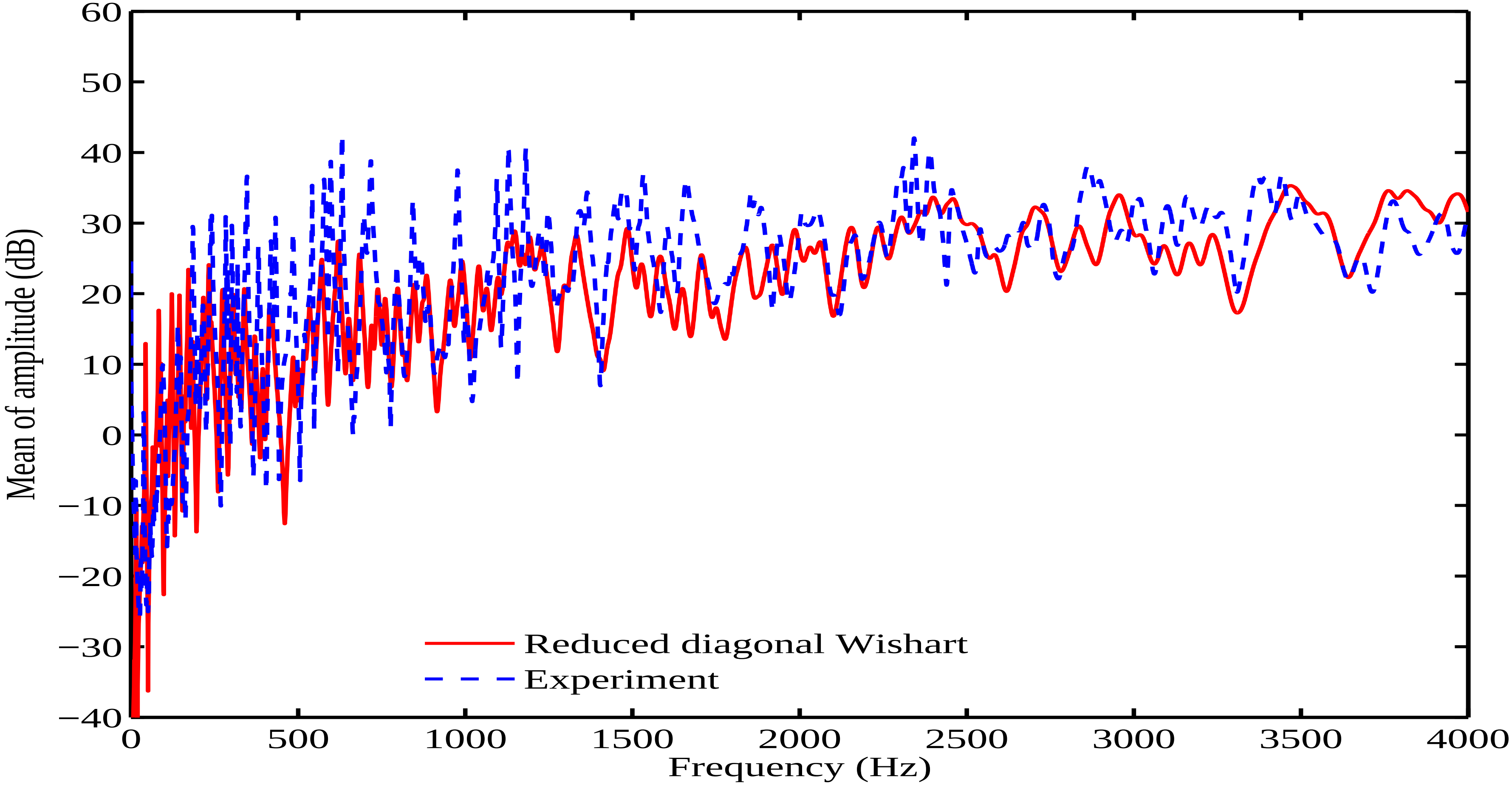


The test rig for the cantilever plate; front view (to appear in Probabilistic Engineering Mechanics).

A cantilever plate: side view

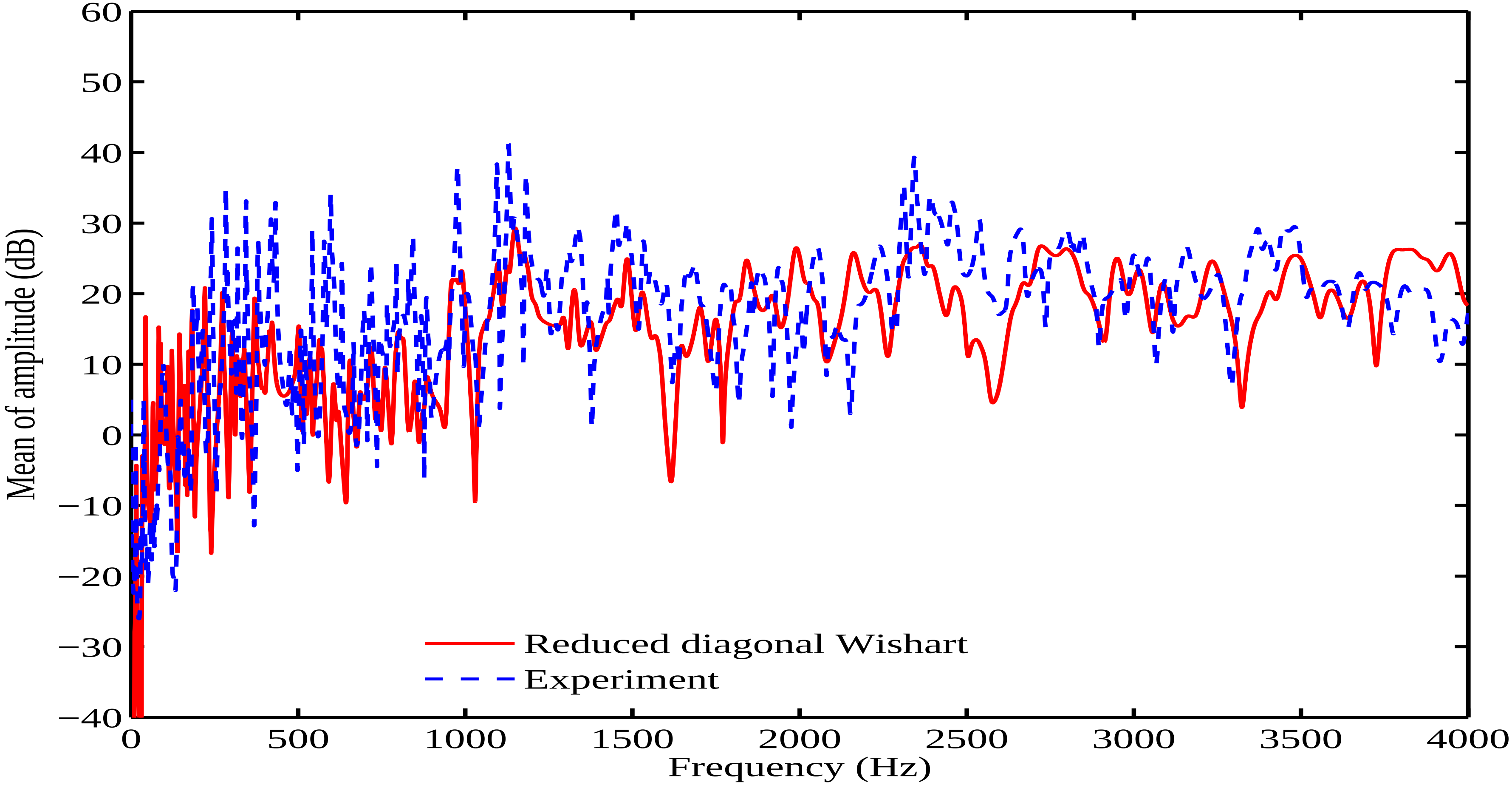
The test rig for the cantilever plate; side view.

Comparison of driving-point-FRF



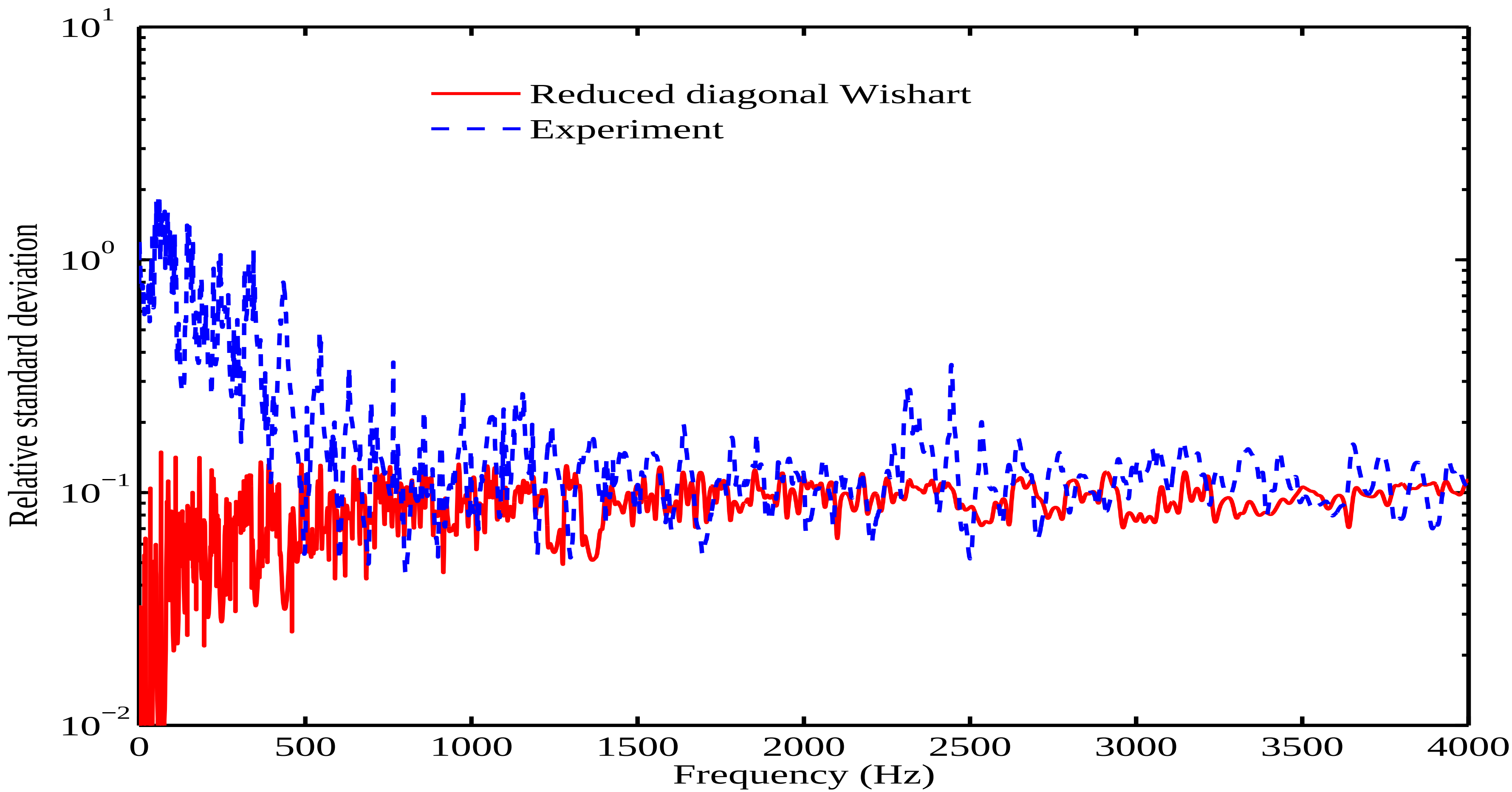
Comparison of the mean of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators

Comparison of Cross-FRF



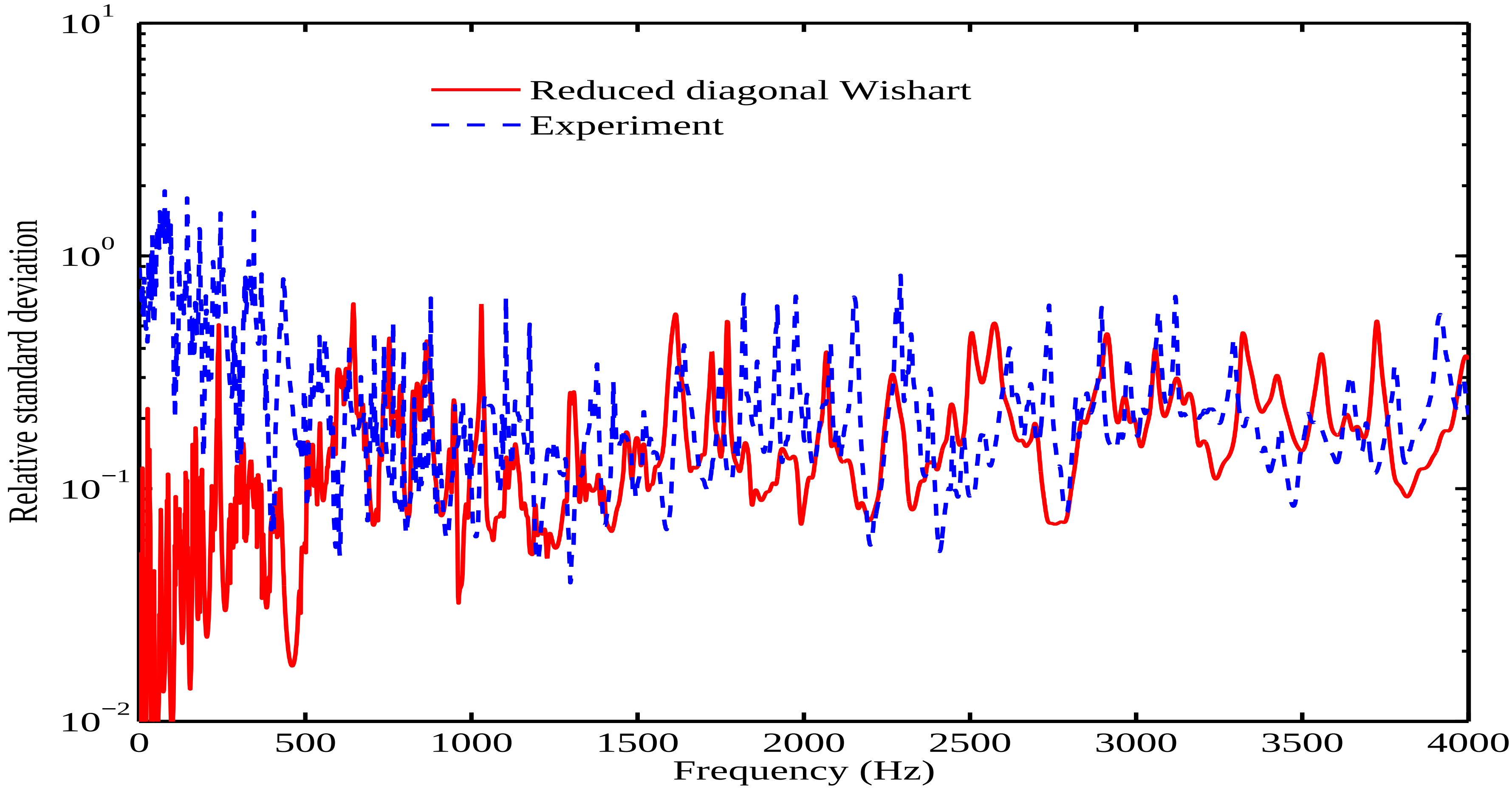
Comparison of the mean of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators

Comparison of driving-point-FRF



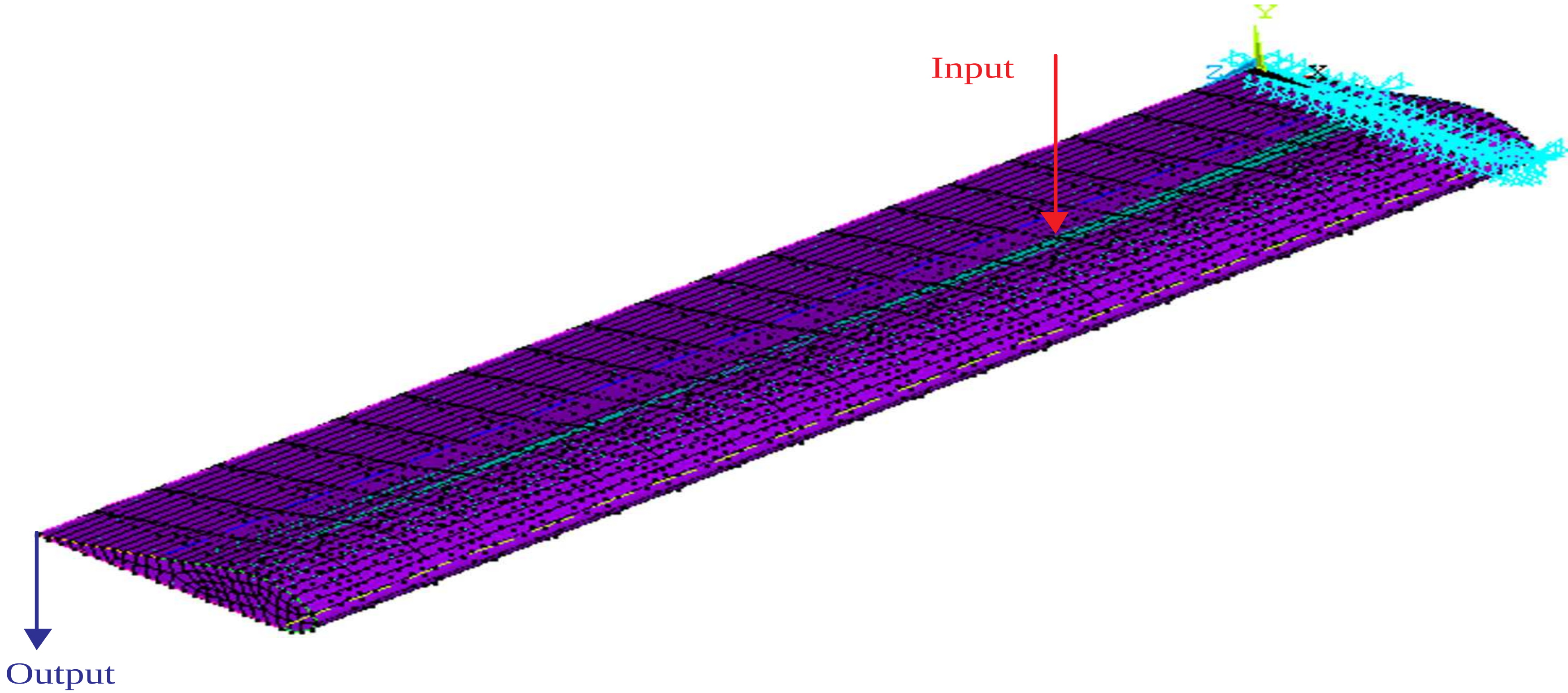
Comparison of relative standard deviation of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators

Comparison of Cross-FRF



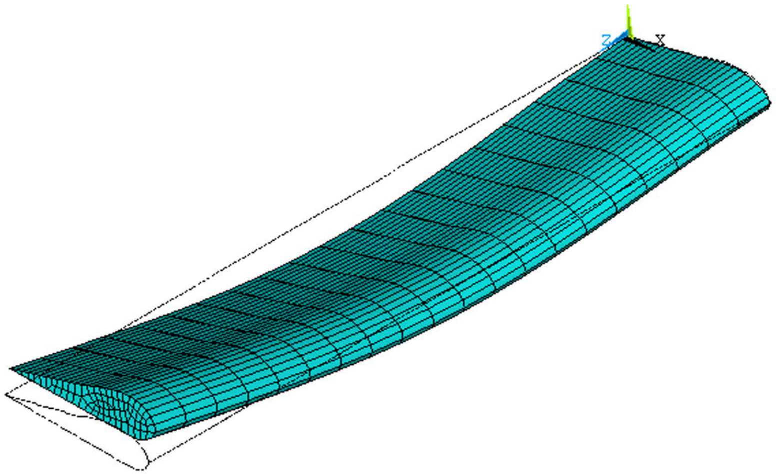
Comparison of relative standard deviation of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators

Integration with ANSYS$^{\mathrm{TM}}$

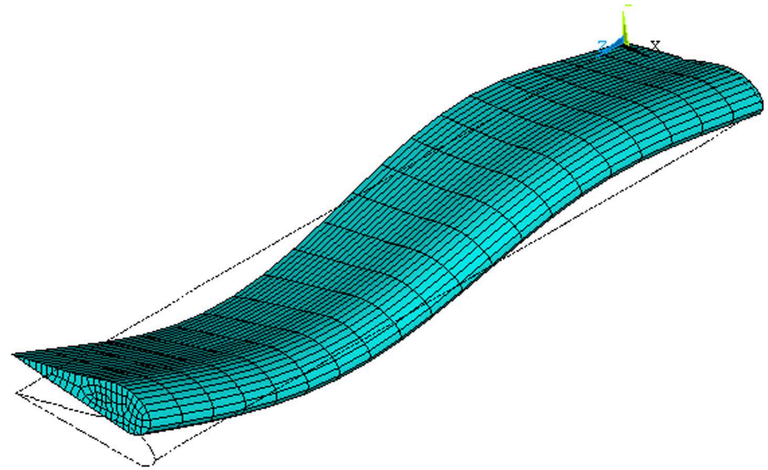


The Finite Element (FE) model of an aircraft wing (5907 degrees-of-freedom). The width is 1.5m, length is 20.0m and the height of the aerofoil section is 0.3m. The material properties are: Young's modulus 262Mpa, Poisson's ratio 0.3 and mass density 888.10kg/m^3. Input node number: 407 and the output node number 96. A 2% modal damping factor is assumed for all modes.

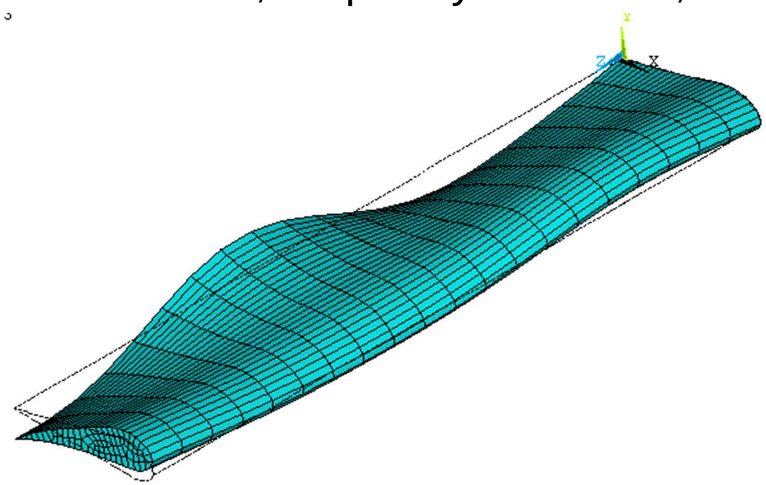
Vibration modes



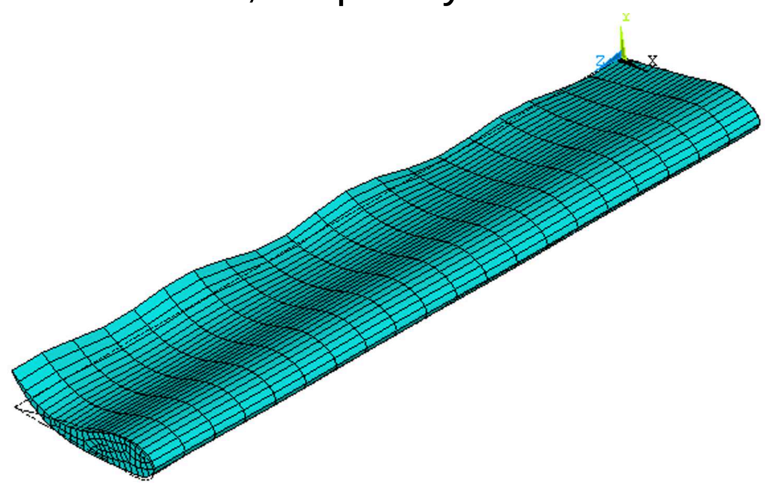
Mode 3, frequency 19.047Hz,



Mode 5, frequency 53.628Hz

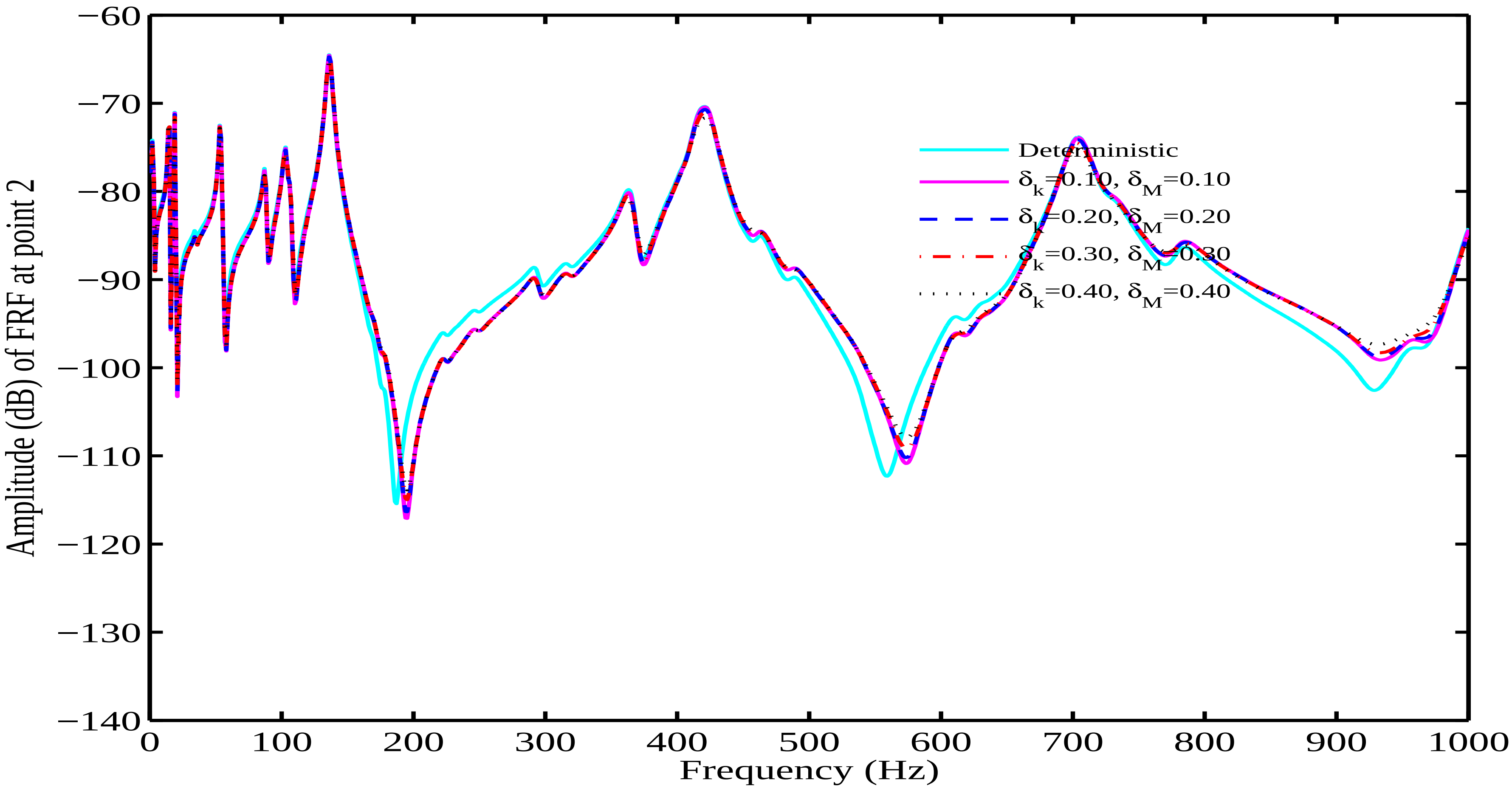


Mode 10, frequency 168.249Hz,



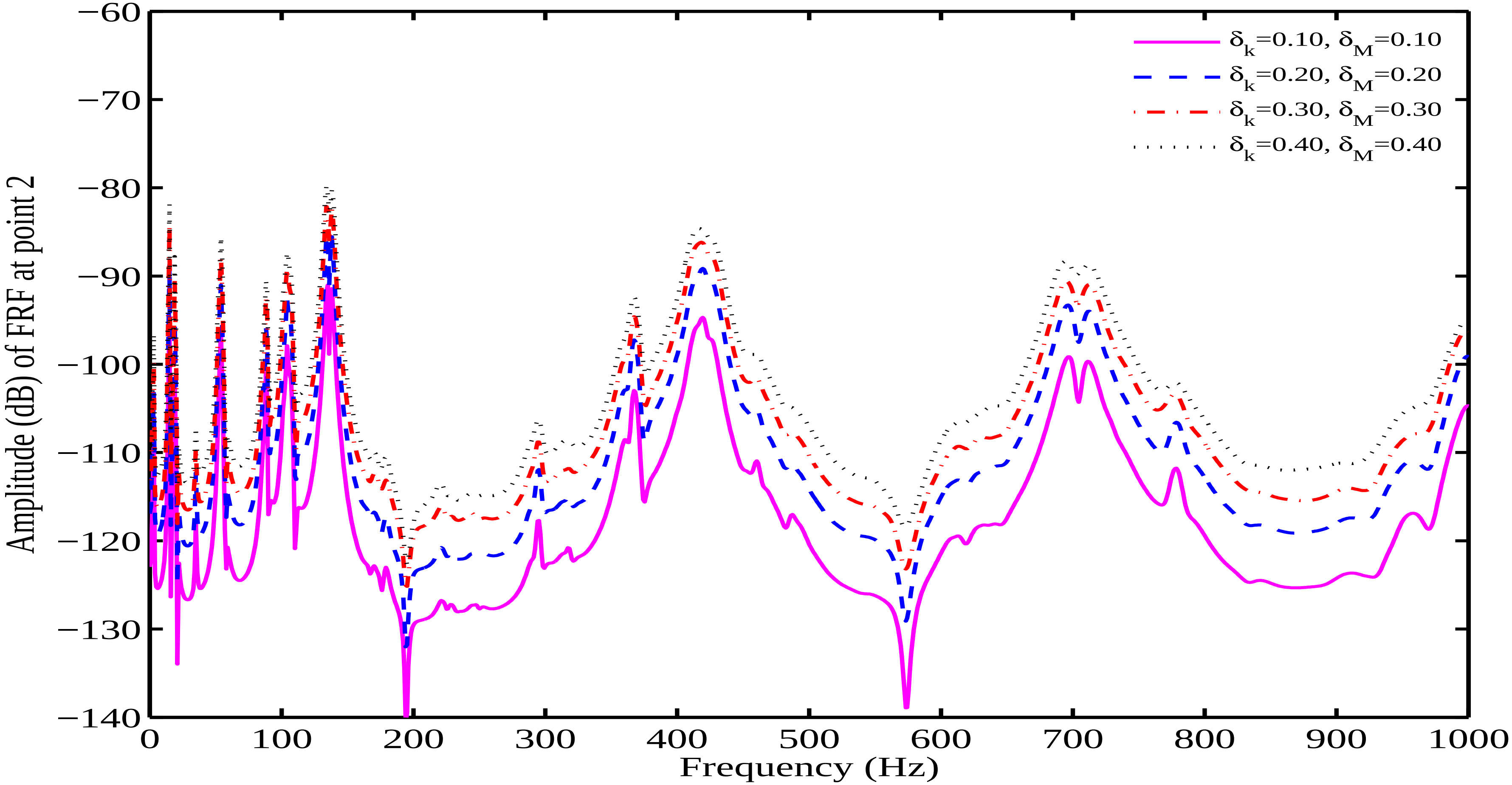
Mode 20, frequency 403.711Hz

Mean of a Cross-FRF



Baseline and mean of the amplitude of a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters

Standard deviation of a Cross-FRF



Standard deviation of the amplitude of a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters

Conclusions

- Linear multiple degrees of freedom dynamic systems with uncertain properties are considered.
- A general uncertain propagation approach based on reduced Wishart random matrix is discussed and the results are compared with experimental results.
- Based on numerical and experimental studies, a suitable simple Wishart random matrix model has been identified and a simulation based computational method has been proposed.
- The proposed reduced approach has been integrated with a commercial FE software.

Acknowledgements

Swansea University
Prifysgol Abertawe

THE ROYAL SOCIETY
CELEBRATING 350 YEARS

newton
international fellowships

The Leverhulme Trust

EPSRC
Pioneering research and skills