Probabilistic Structural Dynamics: Parametric vs. Nonparametric Approach

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Research Areas

- Uncertainty Quantification (UQ) in Computational Mechanics
- Bio & Nanomechanics (nanotubes, graphene, cell mechanics)
- Dynamics of Complex Engineering Systems
- Inverse Problems for Linear and Non-linear Structural Dynamics
- Renewable Energy (wind energy, vibration energy harvesting)

Outline of the presentation

- Uncertainty structural mechanics
- Parametric uncertainty quantification
  - Stochastic finite element method
- Non-parametric uncertainty quantification
  - Wishart random matrix approach
- Hybrid parametric and non-parametric uncertainty
  - Both type uncertainties cover the entire domain
  - Each type uncertainty is confined within non-overlapping subdomains
- Conclusions & collaboration opportunities

Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Uncertainty propagation: key challenges

The main difficulties are:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and

Random continuous dynamical systems

The equation of motion:

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t)}{\partial t^2}+L_1\frac{\partial U(\mathbf{r},t)}{\partial t}+L_2U(\mathbf{r},t)=p(\mathbf{r},t);\quad \mathbf{r}\in\mathcal{D},t\in[0,T] \tag{1}$$

$U(\mathbf{r},t)$ is the displacement variable, $\mathbf{r}$ is the spatial position vector and t is time.

- $\rho(\mathbf{r},\theta)$ is the random mass distribution of the system, $p(\mathbf{r},t)$ is the distributed time-varying forcing function, L_1 is the random spatial self-adjoint damping operator, L_2 is the random spatial self-adjoint stiffness operator.
- Eq (1) is a Stochastic Partial Differential Equation (SPDE) [ie, the coefficients are random processes].

Stochastic Finite Element Method

Problems of structural dynamics in which the uncertainty in specifying mass and stiffness of the structure is modeled within the framework of random fields can be treated using the Stochastic Finite Element Method (SFEM). The application of SFEM in linear structural dynamics typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the 'discretisation of random fields' is a major step in the analysis.
3. Formulation of the equation of motion of the form $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$ where $\mathbf{D}(\omega)$ is the random dynamic stiffness matrix, $\mathbf{u}$ is the vector of random nodal displacement and $\mathbf{f}$ is the applied forces. In general $\mathbf{D}(\omega)$ is a random symmetric complex matrix.
4. Calculation of the response statistics by either (a) solving the random eigenvalue problem, or (b) solving the set of complex random algebraic equations.

Spectral Decomposition of random fields-2

Suppose $H(\mathbf{r}, \theta)$ is a random field with a covariance function $C_H(\mathbf{r}_1, \mathbf{r}_2)$ defined in a space Ω. Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random process $H(\mathbf{r}, \theta)$ can be expressed in a generalized fourier type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \tag{2}$$

where $\xi_i(\theta)$ are uncorrelated random variables, λ_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\Omega} C_H(\mathbf{r}_1, \mathbf{r}_2) \varphi_i(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 = \lambda_i \varphi_i(\mathbf{r}_2), \quad \forall\ i = 1, 2, \cdots \tag{3}$$

The spectral decomposition in equation (2) is known as the Karhunen-Loève (KL) expansion. The series in (2) can be ordered in a decreasing series so that it can be truncated after a finite number of terms with a desired accuracy.

Exponential autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{4}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève expansion in the interval $-a \leq x \leq a$ as

$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \tag{5}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions for odd j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \qquad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{c}{\omega_j}, \tag{6}$$

and for even j are given by

$$\lambda_j = \frac{2c}{{\omega_j}^2 + c^2}, \qquad \varphi_j(x) = \frac{\sin(\omega_j x)}{\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{\omega_j}{-c}. \tag{7}$$

Example: A beam with random properties

The equation of motion of an undamped Euler-Bernoulli beam of length L with random bending stiffness and mass distribution:

$$\frac{\partial^2}{\partial x^2}\left[EI(x,\theta)\frac{\partial^2 Y(x,t)}{\partial x^2}\right] + \rho A(x,\theta)\frac{\partial^2 Y(x,t)}{\partial t^2} = p(x,t). \tag{8}$$

$Y(x,t)$: transverse flexural displacement, $EI(x)$: flexural rigidity, $\rho A(x)$: mass per unit length, and $p(x,t)$: applied forcing. Consider

$$EI(x,\theta) = EI_0\left(1+\epsilon_1 F_1(x,\theta)\right) \tag{9}$$

$$\text{and} \quad \rho A(x,\theta) = \rho A_0\left(1+\epsilon_2 F_2(x,\theta)\right) \tag{10}$$

The subscript 0 indicates the mean values, $0 < \epsilon_i << 1$ (i=1,2) are deterministic constants and the random fields $F_i(x,\theta)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$. Since, $EI(x,\theta)$ and $\rho A(x,\theta)$ are strictly positive, $F_i(x,\theta)$ (i=1,2) are required to satisfy the conditions $P\left[1+\epsilon_i F_i(x,\theta) \leq 0\right] = 0$.

Example: A beam with random properties

We express the shape functions for the finite element analysis of Euler-Bernoulli beams as

$$\mathbf{N}(x) = \mathbf{\Gamma}\, \mathbf{s}(x) \tag{11}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & \dfrac{-3}{{\ell_e}^2} & \dfrac{2}{{\ell_e}^3} \\ 0 & 1 & \dfrac{-2}{{\ell_e}^2} & \dfrac{1}{{\ell_e}^2} \\ 0 & 0 & \dfrac{3}{{\ell_e}^2} & \dfrac{-2}{{\ell_e}^3} \\ 0 & 0 & \dfrac{-1}{{\ell_e}^2} & \dfrac{1}{{\ell_e}^2} \end{bmatrix} \quad \text{and} \quad \mathbf{s}(x) = \left[1,\ x,\ x^2,\ x^3\ \right]^T. \tag{12}$$

The element stiffness matrix:

$$\mathbf{K}_e(\theta) = \int_0^{\ell_e} \mathbf{N}^{''}(x) EI(x,\theta) \mathbf{N}^{''T}(x)\, dx = \int_0^{\ell_e} EI_0\left(1 + \epsilon_1 F_1(x,\theta)\right) \mathbf{N}^{''}(x) \mathbf{N}^{''T}(x)\, dx. \tag{13}$$

Example: A beam with random properties

Expanding the random field $F_1(x,\theta)$ in KL expansion

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e0} + \mathbf{\Delta K}_e(\theta) \tag{14}$$

where the deterministic and random parts are

$$\mathbf{K}_{e0} = EI_0 \int_0^{\ell_e} \mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \quad \text{and} \quad \mathbf{\Delta K}_e(\theta) = \epsilon_1 \sum_{j=1}^{N_\mathrm{K}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_{ej}. \tag{15}$$

The constant N_K is the number of terms retained in the Karhunen-Loève expansion and $\xi_{\mathrm{K}j}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_{ej}$ can be expressed as

$$\mathbf{K}_{ej} = EI_0 \int_0^{\ell_e} \varphi_{\mathrm{K}j}(x_e + x)\mathbf{N}''(x)\mathbf{N}''^T(x)\,dx \tag{16}$$

Example: A beam with random properties

The mass matrix can be obtained as

$$\mathbf{M}_e(\theta) = \mathbf{M}_{e_0} + \mathbf{\Delta M}_e(\theta) \tag{17}$$

The deterministic and random parts is given by

$$\mathbf{M}_{e_0} = \rho A_0 \int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\, dx \quad \text{and} \quad \mathbf{\Delta M}_e(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_{ej}. \tag{18}$$

The constant N_{M} is the number of terms retained in Karhunen-Loève expansion and the constant matrices $\mathbf{M}_{ej}$ can be expressed as

$$\mathbf{M}_{ej} = \rho A_0 \int_0^{\ell_e} \varphi_{\mathrm{M}j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\, dx. \tag{19}$$

Example: A beam with random properties

These element matrices can be assembled to form the global random stiffness and mass matrices of the form

$$\mathbf{K}(\theta) = \mathbf{K}_0 + \mathbf{\Delta K}(\theta) \quad \text{and} \quad \mathbf{M}(\theta) = \mathbf{M}_0 + \mathbf{\Delta M}(\theta). \tag{20}$$

Here the deterministic parts $\mathbf{K}_0$ and $\mathbf{M}_0$ are the usual global stiffness and mass matrices obtained form the conventional finite element method. The random parts can be expressed as

$$\mathbf{\Delta K}(\theta) = \epsilon_1 \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K}j}(\theta) \sqrt{\lambda_{\mathrm{K}j}} \mathbf{K}_j \quad \text{and} \quad \mathbf{\Delta M}(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta) \sqrt{\lambda_{\mathrm{M}j}} \mathbf{M}_j \tag{21}$$

The element matrices $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ have been assembled into the global matrices $\mathbf{K}_j$ and $\mathbf{M}_j$. The total number of random variables depend on the number of terms used for the truncation of the infinite series. This in turn depends on the respective correlation lengths of the underlying random fields; the smaller the correlation length, the higher the number of terms required and vice versa.

General form of the system matrices

- Global matrices can be assembled using the element matrices following the usual method. Each global matrix has a general series form involving random variables:

$$\mathbf{G}(\theta) = \mathbf{G}_0 + \sum_{j=1}^{N_{\mathrm{G}}} \xi_{\mathrm{G_j}}(\theta)\mathbf{G}_j$$

- If the original random fields are Gaussian, then the resulting random matrices will be Gaussian random matrices.
- Once the random matrices are formed, efficient computational approaches are needed for the stochastic analysis (uncertainty propagation problem).

Dynamics of general linear systems

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{22}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The system matrices can be expressed in the series form for the case of parametric uncertainty.
- For the nonparametric case, we do not have explicit information about uncertainty of the parameters in the governing partial differential equations.

Nonparametric uncertainty: general approach

- Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ using available (mathematical and physical) information.
- Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \operatorname{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T \right\} \quad (23)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n \left(\frac{1}{2} p \right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr} \left\{ -\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{S} \right\} \tag{24}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$. This ensures that the moments of the response exist for all frequency values.

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{25}$$

and

$$\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} \tag{26}$$

Further constraints

- Suppose that the inverse moments up to order ν of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}} = \left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^{T}\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln\left|\mathbf{G}\right|^{-\nu}\right] < \infty$$

MEnt distribution - 1

The Lagrangian becomes:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\}\, d\mathbf{G} + \\ \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right)\, d\mathbf{G} - 1\right) - \nu \int_{\mathbf{G}>0} \ln\left|\mathbf{G}\right|\, p_{\mathbf{G}}\, d\mathbf{G} \\ + \mathrm{Trace}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}\left(\mathbf{G}\right)\, d\mathbf{G} - \overline{\mathbf{G}}\right]\right) \quad (27)$$

Note: ν cannot be obtained uniquely!

MEnt distribution - 2

Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

$$\text{or } -\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \text{Trace}\left(\mathbf{\Lambda}_1\mathbf{G}\right) - \ln\left|\mathbf{G}\right|^{\nu}$$

$$\text{or } p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\}\left|\mathbf{G}\right|^{\nu}\text{etr}\left\{-\mathbf{\Lambda}_1\mathbf{G}\right\}$$

MEnt distribution - 3

Using the matrix variate Laplace transform
($\mathbf{T} \in \mathbb{R}_{n,n}, \mathbf{S} \in \mathbb{C}_{n,n}, a > (n+1)/2$)

$$\int_{\mathbf{T}>0} \operatorname{etr}\{-\mathbf{S}\mathbf{T}\} \left|\mathbf{T}\right|^{a-(n+1)/2} d\mathbf{T} = \Gamma_n(a) \left|\mathbf{S}\right|^{-a}$$

and substituting $p_{\mathbf{G}}(\mathbf{G})$ into the constraint equations it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = r^{-nr} \left\{\Gamma_n(r)\right\}^{-1} \left|\overline{\mathbf{G}}\right|^{-r} \left|\mathbf{G}\right|^{\nu} \operatorname{etr}\left\{-r\overline{\mathbf{G}}^{-1}\mathbf{G}\right\} \qquad (28)$$

where $r = \nu + (n+1)/2$.

MEnt Distribution - 4

Comparing it with the Wishart distribution we have: If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/(2\nu + n + 1)$, that is $\mathbf{G} \sim W_n\left(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1)\right)$.

Wishart random matrix approach

- Suppose we know (e.g, by measurements or stochastic finite element modeling) the mean ($\mathbf{G}_0$) and the (normalized) variance (δ_G^2) of the system matrices:

$$\delta_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{29}$$

This parameter is known as the dispersion parameter.

- The parametric and non-parametric models can be related via the dispersion parameter as

$$\delta_G^2 = \frac{\sum_j^M \|(\mathbf{G}_j)\|_{\mathrm{F}}^2}{\|\mathbf{G}_0\|_{\mathrm{F}}^2} \tag{30}$$

Parameter-selection of Wishart matrices

Mass and stiffness matrices are Wishart random matrices: We have $\mathbf{M} \sim W_n(p_1, \mathbf{\Sigma}_1)$, $\mathbf{K} \sim W_n(p_1, \mathbf{\Sigma}_1)$ with $\mathrm{E}[\mathbf{M}] = \mathbf{M}_0$ and $\mathrm{E}[\mathbf{M}] = \mathbf{M}_0$. Here

$$\mathbf{\Sigma}_1 = \mathbf{M}_0/p_1, p_1 = \frac{\gamma_M + 1}{\delta_M^2} \tag{31}$$

and $$\mathbf{\Sigma}_2 = \mathbf{K}_0/p_2, p_2 = \frac{\gamma_K + 1}{\delta_K^2} \tag{32}$$

$$\gamma_G = \{\mathrm{Trace}\,(\mathbf{G}_0)\}^2/\mathrm{Trace}\left(\mathbf{G}_0{}^2\right) \tag{33}$$

The simulation of these matrices requires the simulation of Gaussian random numbers.

Uncertainty propagation problems-1

- Real random algebraic equations:

$$\mathbf{A}(\theta)\mathbf{x} = \mathbf{f}$$

here $\mathbf{A} : \Theta \to \mathbb{R}^{n \times n}$ is a real symmetric random matrix and $\mathbf{f} \in \mathbb{R}^n$ is a deterministic vector. This type of problem typically arise in stochastic elliptic problems.

- Complex random algebraic equations:

$$\left[-\omega^2 \mathbf{A}(\theta) + \mathbf{B}(\theta) + \mathrm{i}\omega \mathbf{C}\right] \mathbf{x} = \mathbf{f}$$

here $\mathbf{A}, \mathbf{B}, \mathbf{C} : \Theta \to \mathbb{R}^{n \times n}$ are a real symmetric random matrices and $\omega \in \mathbb{R}$ is a deterministic variable. This type of problem typically arise in stochastic dynamic problems.

Uncertainty propagation problems-2

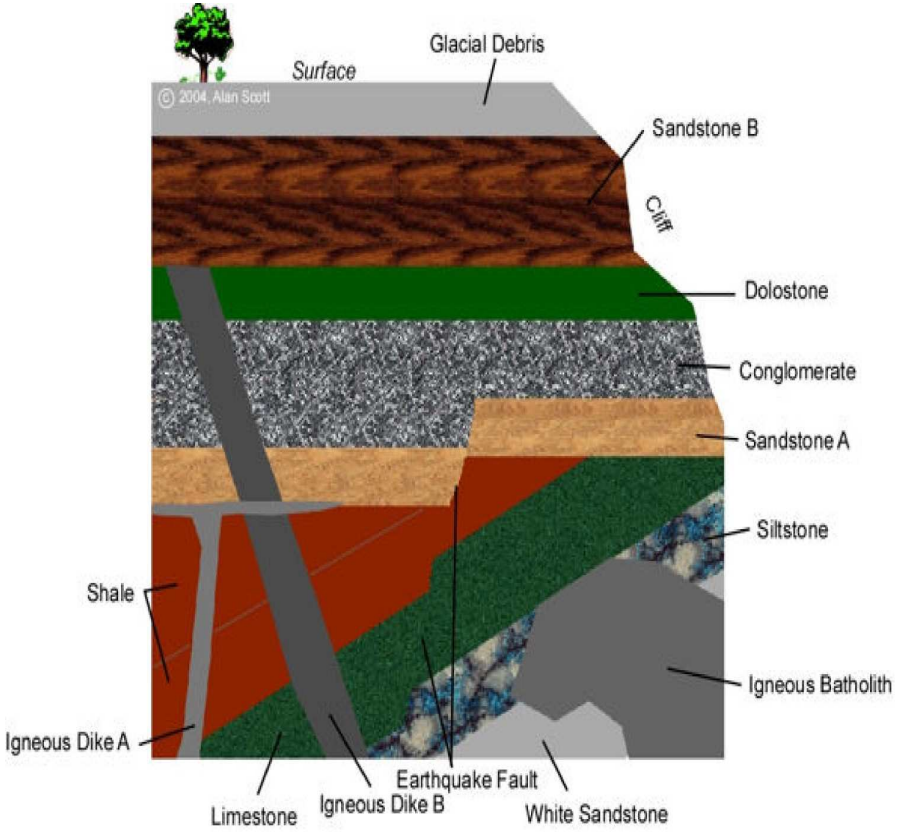
- Random eigenvalue problems:

$$\mathbf{B}(\theta)\boldsymbol{\phi}_j = \lambda_j \mathbf{A}(\theta)\boldsymbol{\phi}_j$$

  here $\lambda_j \in \mathbb{R}$, $\boldsymbol{\phi}_j \in \mathbb{R}^n, j = 1, 2, \cdots, n$ are respectively random eigenvalues and eigenvectors of the the system. This is another approach to solve the dynamic problem.

- To solve these types of problems, we are developing methods based on :
  - Gaussian process emulator (GPE)
  - High dimensional model representation (HDMR)
  - Random matrix theory (RMT)
  - Asymptotic integral method

Hybrid uncertainty problems

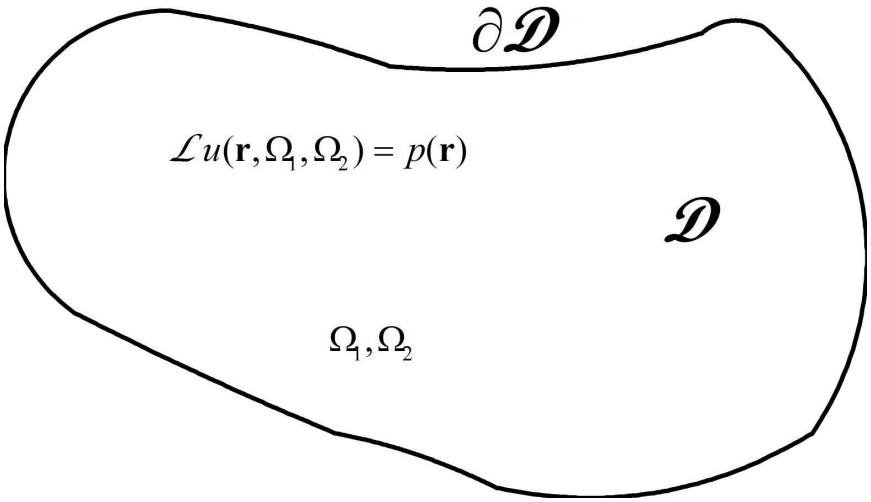


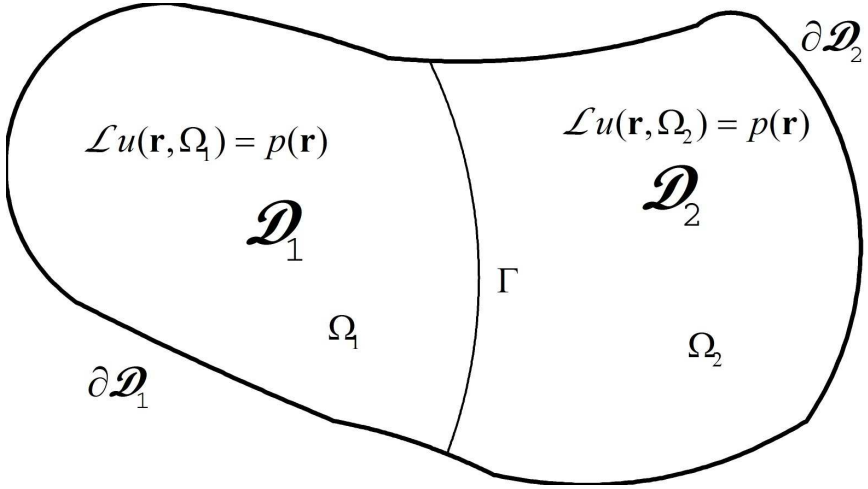
(a) Stratum in soil

(b) Aircraft wing with engine

Two different types of hybrid uncertainty problems arising in computational mechanics.

Uncorrelated sample spaces





We consider a bounded domain $\mathcal{D} \in \mathbb{R}^d$ with piecewise Lipschitz boundary $\partial\mathcal{D}$, where $d \leq 3$ is the spatial dimension. For $j = 1, 2$, consider that $(\Theta_j, \mathcal{F}_j, P_j)$ be a probability space with with $\theta_j \in \Theta_j$ denoting a sampling point in the sampling space Θ_j, $\mathcal{F}_j$ is the complete σ-algebra over the subsets of Θ_j and P_j is the probability measure. Here subscript 1 denotes parametric uncertainty and subscript 2 denotes nonparametric uncertainty. Also consider two subdomains $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{D}$ such that $\mathcal{D}_1 \bigcap \mathcal{D}_2 = \emptyset$.

Discretisation of the stochastic PDEs

- We employ a spectral stochastic finite element method in Θ_1 and maximum entropy based random matrix theory in Θ_2.
- After discretising, the governing stochastic PDE can be expressed as

$$\mathbf{\Lambda}(\theta_1, \theta_2)\mathbf{x}(\theta_1, \theta_2) = \mathbf{f} \tag{34}$$

where $\mathbf{\Lambda} : (\Theta_1 \times \Theta_2) \rightarrow \mathbb{R}^{n \times n}$ and $\mathbf{x} : (\Theta_1 \times \Theta_2) \rightarrow \mathbb{R}^n$ in the unknown random vector.

Nature of the discretised matrices

Theorem 1: For the case of hybrid uncertainty over the entire domain $\mathcal{D}$, $\mathbf{\Lambda}$ is distributed as a Wishart random matrix $W_n(\delta_{\Theta_2}, \mathbf{\Lambda}_{\Theta_1}(\theta_1))$, where δ_{Θ_2} is the dispersion parameter in Θ_2, $\mathbf{\Lambda}_{\Theta_1}(\theta_1) = \mathbf{\Lambda}_0 + \sum_{k=1}^{\infty} \xi_k(\theta_1)\mathbf{\Lambda}_k$, $\mathbf{\Lambda}_0$ is the disretized matrix corresponding to the baseline model and $\xi_k(\theta_1)$ are uncorrelated standard Gaussian random variables.

Theorem 2: For the case of hybrid uncertainty over non-overlapping subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$, $\mathbf{\Lambda}$ can be partitioned as

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_{11_0} + \sum_{k=1}^{\infty} \xi_k(\theta_1)\mathbf{\Lambda}_{11_k} & \mathbf{\Lambda}_{12_0} \\ \mathbf{\Lambda}_{12_0}^T & W_{n_2}(p_{\Theta_2}, \mathbf{\Lambda}_{22_0}) \end{bmatrix}$$ where

$\mathbf{\Lambda}_{11_k} \in \mathbb{R}^{n_1 \times n_1}, k = 0, 1, \cdots$, $\mathbf{\Lambda}_{12_0} \in \mathbb{R}^{n_1 \times n_2}$, $\mathbf{\Lambda}_{22_0} \in \mathbb{R}^{n_2 \times n_2}$ and $n_1 + n_2 = n$.

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Outline of the solution method

- Due to the fact that $\mathbf{\Lambda}$ in Eq. (34) needs to be sampled from $\Theta_1 \times \Theta_2$, the solution for $\mathbf{x}(\theta_1, \theta_2)$ possess significantly new challenges in computational mechanics.
- For example, in the context of Monte Carlo simulation, if s_j is the number of samples in Θ_j, then one needs $s_1 \times s_2$ number of samples to obtain a credible statistical description of $\mathbf{x}$.
- New methods involving polynomial chaos and random matrix theory are currently being investigated.

Conclusions-1

- Uncertainties need to be taken into account for credible predictions using computational methods. They can be broadly categorised into parametric and nonparametric uncertainty.
- Methods for quantification of parametric uncertainty (SFEM) and nonparametric uncertainty (Wishart random matrices) in the context of some stochastic PDEs have been discussed.
- Three types of mathematical problems arising in stochastic mechanics problems (irrespective of what uncertainty model is used) have been outlined.

Conclusions-2

- The concept of hybrid parametric and nonparametric uncertainty quantification has been introduced.
- The nature of the discretised system matrix is illustrated for two practical cases, namely (a) both type uncertainties cover the entire domain, and (b) each type uncertainty is confined within non-overlapping subdomains, are considered.

Collaboration opportunities

- Stochastic methods in fluid mechanics, heat transfer and fluid-structure interaction problems
- Stochastic computational electromagnetics problems (high and low frequency probelms)
- Biological systems?