Uncertainty quantification in structural mechanics: analysis and identification

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Research Areas

Brief Career: Cambridge (PhD, Postdoc 97-02), Bristol (Lecturer 03-06), Swansea (Chair of Aerospace Eng. 07-), EPSRC Fellowship (04-09).

- Uncertainty Quantification (UQ) in Computational Mechanics
- Bio & Nanomechanics (nanotubes, graphene, cell mechanics)
- Dynamics of Complex Engineering Systems
- Inverse Problems for Linear and Non-linear Structural Dynamics
- Renewable Energy (wind energy, vibration energy harvesting)

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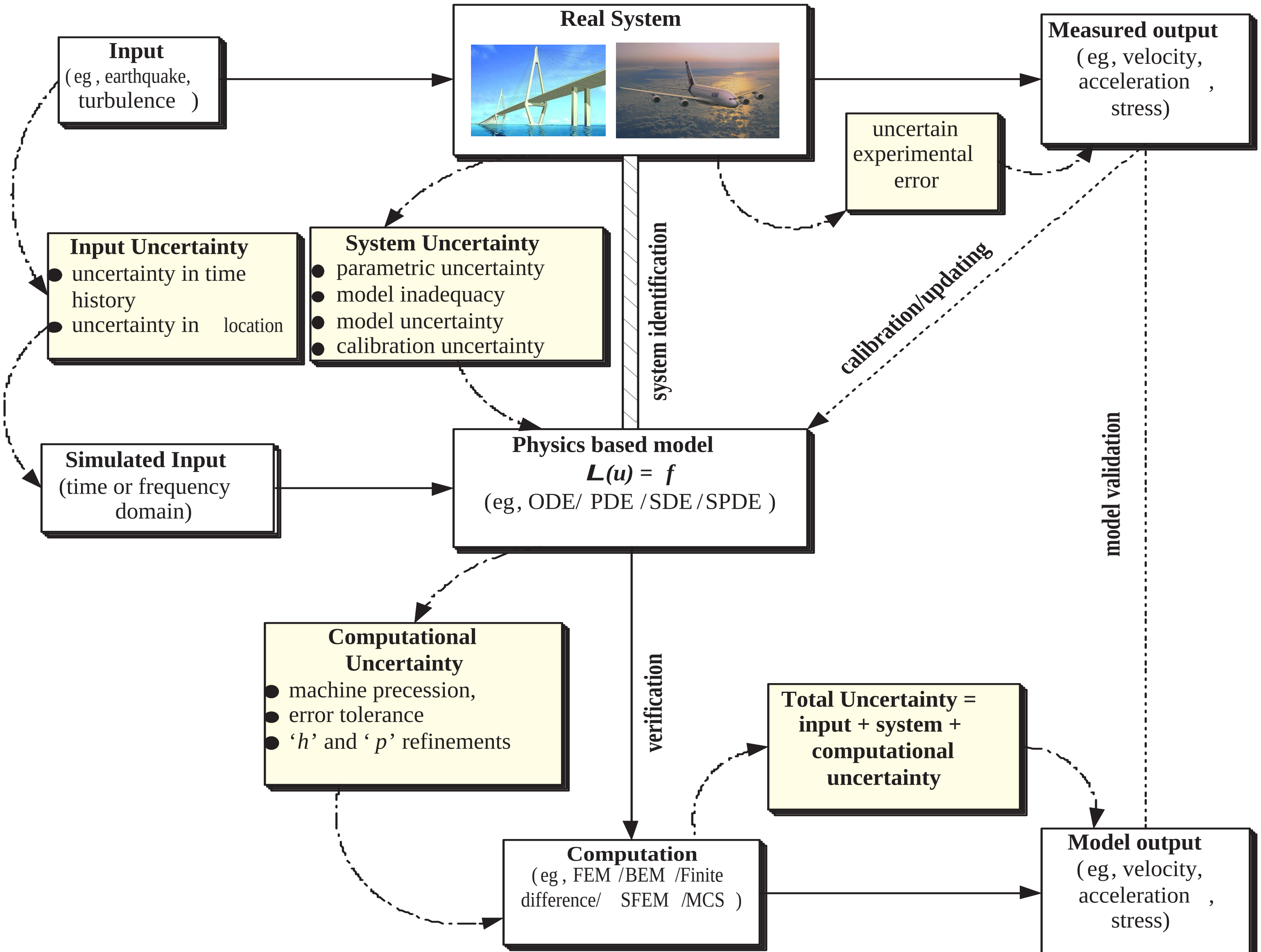
Outline of the presentation

- Uncertainty structural mechanics
- Brief review of parametric approach
  - Stochastic finite element method
- Non-parametric approach: Wishart random matrices
  - Analytical derivation
  - Computational results
- Experimental results
- Identification of random field: Inverse problem
- Stochastic model updating
- Conclusions & future directions

A general overview of computational mechanics



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Ensembles of structural systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems)

A complex structural system

Complex aerospace system can have millions of degrees of freedom and significant 'errors' and/or 'lack of knowledge' in its numerical (Finite Element) model

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Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

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Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM/Finite difference
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating/calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM/RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating/calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random/deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random/deterministic)	Known (random)	Known from experiment and model (random)	*Model validation*	Validation methods
Known (random/deterministic)	Known (random)	Known from different computations (random)	*Model verification*	verification methods

Uncertainty propagation: key challenges

The main difficulties are:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and

Current approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type 'a')
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications (building under earthquake load, steering column vibration in cars)

Current UP approaches - 2

- Nonparametric approaches : Such as the Random Matrix Theory (RMT):
  - aim to characterize nonparametric uncertainty (types ‘b’ - ‘e’)
  - do not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications (eg, noise propagation in vehicles)

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Random continuous dynamical systems

The equation of motion:

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t)}{\partial t^2}+L_1\frac{\partial U(\mathbf{r},t)}{\partial t}+L_2 U(\mathbf{r},t)=p(\mathbf{r},t);\quad \mathbf{r}\in\mathcal{D},t\in[0,T] \tag{1}$$

$U(\mathbf{r},t)$ is the displacement variable, $\mathbf{r}$ is the spatial position vector and t is time.

- $\rho(\mathbf{r},\theta)$ is the random mass distribution of the system, $p(\mathbf{r},t)$ is the distributed time-varying forcing function, L_1 is the random spatial self-adjoint damping operator, L_2 is the random spatial self-adjoint stiffness operator.
- Eq (1) is a Stochastic Partial Differential Equation (SPDE) [ie, the coefficients are random processes].

Stochastic Finite Element Method

Problems of structural dynamics in which the uncertainty in specifying mass and stiffness of the structure is modeled within the framework of random fields can be treated using the Stochastic Finite Element Method (SFEM). The application of SFEM in linear structural dynamics typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the 'discretisation of random fields' is a major step in the analysis.
3. Formulation of the equation of motion of the form $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$ where $\mathbf{D}(\omega)$ is the random dynamic stiffness matrix, $\mathbf{u}$ is the vector of random nodal displacement and $\mathbf{f}$ is the applied forces. In general $\mathbf{D}(\omega)$ is a random symmetric complex matrix.
4. Calculation of the response statistics by either (a) solving the random eigenvalue problem, or (b) solving the set of complex random algebraic equations.

Spectral Decomposition of random fields-2

Suppose $H(\mathbf{r}, \theta)$ is a random field with a covariance function $C_H(\mathbf{r}_1, \mathbf{r}_2)$ defined in a space Ω. Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random process $H(\mathbf{r}, \theta)$ can be expressed in a generalized fourier type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{2}$$

where $\xi_i(\theta)$ are uncorrelated random variables, λ_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_\Omega C_H(\mathbf{r}_1, \mathbf{r}_2)\varphi_i(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \lambda_i\varphi_i(\mathbf{r}_2), \quad \forall\ i = 1, 2, \cdots \tag{3}$$

The spectral decomposition in equation (2) is known as the Karhunen-Loève (KL) expansion. The series in (2) can be ordered in a decreasing series so that it can be truncated after a finite number of terms with a desired accuracy.

Exponential autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{4}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève expansion in the interval $-a \leq x \leq a$ as

$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \tag{5}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions for odd j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \qquad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{c}{\omega_j}, \tag{6}$$

and for even j are given by

$$\lambda_j = \frac{2c}{{\omega_j}^2 + c^2}, \qquad \varphi_j(x) = \frac{\sin(\omega_j x)}{\sqrt{a - \frac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{\omega_j}{-c}. \tag{7}$$

Example: A beam with random properties

The equation of motion of an undamped Euler-Bernoulli beam of length L with random bending stiffness and mass distribution:

$$\frac{\partial^2}{\partial x^2}\left[EI(x,\theta)\frac{\partial^2 Y(x,t)}{\partial x^2}\right] + \rho A(x,\theta)\frac{\partial^2 Y(x,t)}{\partial t^2} = p(x,t). \tag{8}$$

$Y(x,t)$: transverse flexural displacement, $EI(x)$: flexural rigidity, $\rho A(x)$: mass per unit length, and $p(x,t)$: applied forcing. Consider

$$EI(x,\theta) = EI_0\left(1+\epsilon_1 F_1(x,\theta)\right) \tag{9}$$

$$\text{and} \quad \rho A(x,\theta) = \rho A_0\left(1+\epsilon_2 F_2(x,\theta)\right) \tag{10}$$

The subscript 0 indicates the mean values, $0 < \epsilon_i << 1$ (i=1,2) are deterministic constants and the random fields $F_i(x,\theta)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$. Since, $EI(x,\theta)$ and $\rho A(x,\theta)$ are strictly positive, $F_i(x,\theta)$ (i=1,2) are required to satisfy the conditions $P\left[1+\epsilon_i F_i(x,\theta) \leq 0\right] = 0$.

Example: A beam with random properties

We express the shape functions for the finite element analysis of Euler-Bernoulli beams as

$$\mathbf{N}(x) = \mathbf{\Gamma}\, \mathbf{s}(x) \tag{11}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & \frac{-3}{{\ell_e}^2} & \frac{2}{{\ell_e}^3} \\ 0 & 1 & \frac{-2}{{\ell_e}^2} & \frac{1}{{\ell_e}^2} \\ 0 & 0 & \frac{3}{{\ell_e}^2} & \frac{-2}{{\ell_e}^3} \\ 0 & 0 & \frac{-1}{{\ell_e}^2} & \frac{1}{{\ell_e}^2} \end{bmatrix} \quad \text{and} \quad \mathbf{s}(x) = \left[1,\ x,\ x^2,\ x^3\ \right]^T. \tag{12}$$

The element stiffness matrix:

$$\mathbf{K}_e(\theta) = \int_0^{\ell_e} \mathbf{N}''(x) EI(x,\theta) \mathbf{N}''^T(x)\, dx = \int_0^{\ell_e} EI_0\left(1 + \epsilon_1 F_1(x,\theta)\right) \mathbf{N}''(x) \mathbf{N}''^T(x)\, dx. \tag{13}$$

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Example: A beam with random properties

Expanding the random field $F_1(x,\theta)$ in KL expansion

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e0} + \mathbf{\Delta K}_e(\theta) \tag{14}$$

where the deterministic and random parts are

$$\mathbf{K}_{e0} = EI_0 \int_0^{\ell_e} \mathbf{N}^{''}(x)\mathbf{N}^{''T}(x)\, dx \quad \text{and} \quad \mathbf{\Delta K}_e(\theta) = \epsilon_1 \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_{ej}. \tag{15}$$

The constant N_{K} is the number of terms retained in the Karhunen-Loève expansion and $\xi_{\mathrm{K}j}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_{ej}$ can be expressed as

$$\mathbf{K}_{ej} = EI_0 \int_0^{\ell_e} \varphi_{\mathrm{K}j}(x_e + x)\mathbf{N}^{''}(x)\mathbf{N}^{''T}(x)\, dx \tag{16}$$

Example: A beam with random properties

The mass matrix can be obtained as

$$\mathbf{M}_e(\theta) = \mathbf{M}_{e_0} + \mathbf{\Delta M}_e(\theta) \tag{17}$$

The deterministic and random parts is given by

$$\mathbf{M}_{e_0} = \rho A_0 \int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\,dx \quad \text{and} \quad \mathbf{\Delta M}_e(\theta) = \epsilon_2 \sum_{j=1}^{N_\mathrm{M}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_{ej}. \tag{18}$$

The constant N_M is the number of terms retained in Karhunen-Loève expansion and the constant matrices $\mathbf{M}_{ej}$ can be expressed as

$$\mathbf{M}_{ej} = \rho A_0 \int_0^{\ell_e} \varphi_{\mathrm{M}j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\,dx. \tag{19}$$

Example: A beam with random properties

These element matrices can be assembled to form the global random stiffness and mass matrices of the form

$$\mathbf{K}(\theta) = \mathbf{K}_0 + \mathbf{\Delta K}(\theta) \quad \text{and} \quad \mathbf{M}(\theta) = \mathbf{M}_0 + \mathbf{\Delta M}(\theta). \tag{20}$$

Here the deterministic parts $\mathbf{K}_0$ and $\mathbf{M}_0$ are the usual global stiffness and mass matrices obtained form the conventional finite element method. The random parts can be expressed as

$$\mathbf{\Delta K}(\theta) = \epsilon_1 \sum_{j=1}^{N_\mathrm{K}} \xi_{\mathrm{K}j}(\theta) \sqrt{\lambda_{\mathrm{K}j}} \mathbf{K}_j \quad \text{and} \quad \mathbf{\Delta M}(\theta) = \epsilon_2 \sum_{j=1}^{N_\mathrm{M}} \xi_{\mathrm{M}j}(\theta) \sqrt{\lambda_{\mathrm{M}j}} \mathbf{M}_j \tag{21}$$

The element matrices $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ have been assembled into the global matrices $\mathbf{K}_j$ and $\mathbf{M}_j$. The total number of random variables depend on the number of terms used for the truncation of the infinite series. This in turn depends on the respective correlation lengths of the underlying random fields; the smaller the correlation length, the higher the number of terms required and vice versa.

Dynamics of a general linear system

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{22}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{q}$
- Probabilistic solution of this problem is expected to have more credibility compared to a deterministic solution

Random Matrix Method (RMM)

- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ using available information.
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

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Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\boldsymbol{\Sigma}|^{-p/2} |\boldsymbol{\Psi}|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (23)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n\left(\frac{1}{2}p\right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S} \right\} \tag{24}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: If $p = n+1$, then the matrix is non-negative definite.

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$. This ensures that the moments of the response exist for all frequency values.

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{25}$$

and $$\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} \tag{26}$$

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Further constraints

- Suppose that the inverse moments up to order ν of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}} = \left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^T\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln|\mathbf{G}|^{-\nu}\right] < \infty$$

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MEnt distribution - 1

The Lagrangian becomes:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} \, d\mathbf{G} + \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - 1\right) - \nu \int_{\mathbf{G}>0} \ln\left|\mathbf{G}\right| \, p_{\mathbf{G}} \, d\mathbf{G} + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \left[\int_{\mathbf{G}>0} \mathbf{G} \, p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - \overline{\mathbf{G}}\right]\right) \tag{27}$$

Note: ν cannot be obtained uniquely!

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MEnt distribution - 2

Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

or $-\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \mathbf{G}\right) - \ln\left|\mathbf{G}\right|^{\nu}$

or $p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\}\left|\mathbf{G}\right|^{\nu} \mathrm{etr}\left\{-\mathbf{\Lambda}_1 \mathbf{G}\right\}$

MEnt distribution - 3

Using the matrix variate Laplace transform
($\mathbf{T} \in \mathbb{R}_{n,n}, \mathbf{S} \in \mathbb{C}_{n,n}, a > (n+1)/2$)

$$\int_{\mathbf{T}>0} \operatorname{etr}\{-\mathbf{S}\mathbf{T}\} |\mathbf{T}|^{a-(n+1)/2} \, d\mathbf{T} = \Gamma_n(a) |\mathbf{S}|^{-a}$$

and substituting $p_{\mathbf{G}}(\mathbf{G})$ into the constraint equations it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = r^{-nr} \{\Gamma_n(r)\}^{-1} \left|\overline{\mathbf{G}}\right|^{-r} |\mathbf{G}|^{\nu} \operatorname{etr}\left\{-r\overline{\mathbf{G}}^{-1}\mathbf{G}\right\} \qquad (28)$$

where $r = \nu + (n+1)/2$.

MEnt Distribution - 4

Comparing it with the Wishart distribution we have: If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\boldsymbol{\Sigma} = \overline{\mathbf{G}}/(2\nu + n + 1)$, that is $\mathbf{G} \sim W_n\left(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1)\right)$.

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Properties of the distribution

- Covariance tensor of $\mathbf{G}$:

$$\mathrm{cov}\left(G_{ij}, G_{kl}\right) = \frac{1}{2\nu + n + 1}\left(\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}\right)$$

- Normalized standard deviation matrix

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2} = \frac{1}{2\nu + n + 1}\left\{1 + \frac{\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}\right\}$$

- $\sigma_G^2 \leq \dfrac{1+n}{2\nu + n + 1}$ and $\nu \uparrow \Rightarrow \delta_{\mathbf{G}}^2 \downarrow$.

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Wishart random matrix approach

- Suppose we 'know' (e.g, by measurements or stochastic finite element modeling) the mean ($\mathbf{G}_0$) and the (normalized) standard deviation (σ_G) of the system matrices:

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{29}$$

- The parameters of the Wishart distribution can be identified using the expressions derived before.

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Stochastic dynamic response

- Taking the Laplace transform of the equation of motion:

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{30}$$

  The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2\mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{31}$$

  where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- We define the matrices

$$\boldsymbol{\Omega} = \mathrm{diag}\left[\omega_1, \omega_2, \ldots, \omega_n\right] \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right]. \tag{32}$$

$$\text{so that} \quad \boldsymbol{\Phi}^T\mathbf{K}_e\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_n \tag{33}$$

Stochastic dynamic response

- Transforming it into the modal coordinates:

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{34}$$

- Here

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = 2\boldsymbol{\zeta}\mathbf{\Omega}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{35}$$

- When we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ will be a random matrix of dimension n. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{n\times n}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{36}$$

Stochastic dynamic response

- Since Ξ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\mathbf{\Psi}_r$ such that

$$\mathbf{\Psi}_r^T \mathbf{\Xi} \mathbf{\Psi}_r = \mathbf{\Omega}_r^2 \tag{37}$$

  Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix Ξ.

- Recalling that $\mathbf{\Psi}_r^T \mathbf{\Psi}_r = \mathbf{I}_n$ we obtain

$$\bar{\mathbf{q}}' = \left[s^2 \mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2 \right]^{-1} \bar{\mathbf{f}}' \tag{38}$$

$$= \mathbf{\Psi}_r \left[s^2 \mathbf{I}_n + 2s\boldsymbol{\zeta}\mathbf{\Omega}_r + \mathbf{\Omega}_r^2 \right]^{-1} \mathbf{\Psi}_r^T \bar{\mathbf{f}}' \tag{39}$$

Stochastic dynamic response

- The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \boldsymbol{\Phi}\bar{\mathbf{q}}'(s) = \boldsymbol{\Phi}\boldsymbol{\Psi}_r \left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1} (\boldsymbol{\Phi}\boldsymbol{\Psi}_r)^T \bar{\mathbf{f}}(s)$$
$$= \sum_{j=1}^{n} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j}.$$

Here

$$\boldsymbol{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}\right], \quad \mathbf{X}_r = \boldsymbol{\Phi}\boldsymbol{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_n}\right]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.

Parameter-selection of Wishart matrices

Approach 1: M and K are fully correlated Wishart (most complex). For this case $\mathbf{M} \sim W_n(p_1, \mathbf{\Sigma}_1)$, $\mathbf{K} \sim W_n(p_1, \mathbf{\Sigma}_1)$ with $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$ and $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices. Here

$$\mathbf{\Sigma}_1 = \mathbf{M}_0/p_1, p_1 = \frac{\gamma_M + 1}{\delta_M} \tag{40}$$

and $$\mathbf{\Sigma}_2 = \mathbf{K}_0/p_2, p_2 = \frac{\gamma_K + 1}{\delta_K} \tag{41}$$

$$\gamma_G = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right) \tag{42}$$

Parameter-selection of Wishart matrices

Approach 2: Scalar Wishart (most simple) In this case it is assumed that

$$\boldsymbol{\Xi} \sim W_n\left(p, \frac{a^2}{n}\mathbf{I}_n\right) \tag{43}$$

Considering $\mathrm{E}\left[\boldsymbol{\Xi}\right] = \boldsymbol{\Omega}_0^2$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1+\gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \mathrm{Trace}\left(\boldsymbol{\Omega}_0^2\right)/p \tag{44}$$

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Parameter-selection of Wishart matrices

Approach 3: Diagonal Wishart with different entries (something in the middle). For this case $\boldsymbol{\Xi} \sim W_n\left(p, \boldsymbol{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\boldsymbol{\Xi}^{-1}\right] = \boldsymbol{\Omega}_0^{-2}$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem.

The parameters can be obtained as

$$p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1) \tag{45}$$

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Parameter-selection of Wishart matrices

- Defining $\mathbf{H}_0 = \mathbf{M}_0{}^{-1}\mathbf{K}_0$, the constant γ_H:

$$\gamma_H = \frac{\{\mathrm{Trace}\,(\mathbf{H}_0)\}^2}{\mathrm{Trace}\,(\mathbf{H}_0{}^2)} = \frac{\left\{\mathrm{Trace}\,\left(\mathbf{\Omega}_0^2\right)\right\}^2}{\mathrm{Trace}\,\left(\mathbf{\Omega}_0^4\right)} = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{46}$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\delta_H = \frac{\left(p_1{}^2 + (p_2 - 2 - 2\,n)\,p_1 + (-n-1)\,p_2 + n^2 + 1 + 2\,n\right)\gamma_H}{p_2\,(-p_1+n)\,(-p_1+n+3)} + \frac{p_1{}^2 + (p_2 - 2\,n)\,p_1 + (1-n)\,p_2 - 1 + n^2}{p_2\,(-p_1+n)\,(-p_1+n+3)} \tag{47}$$

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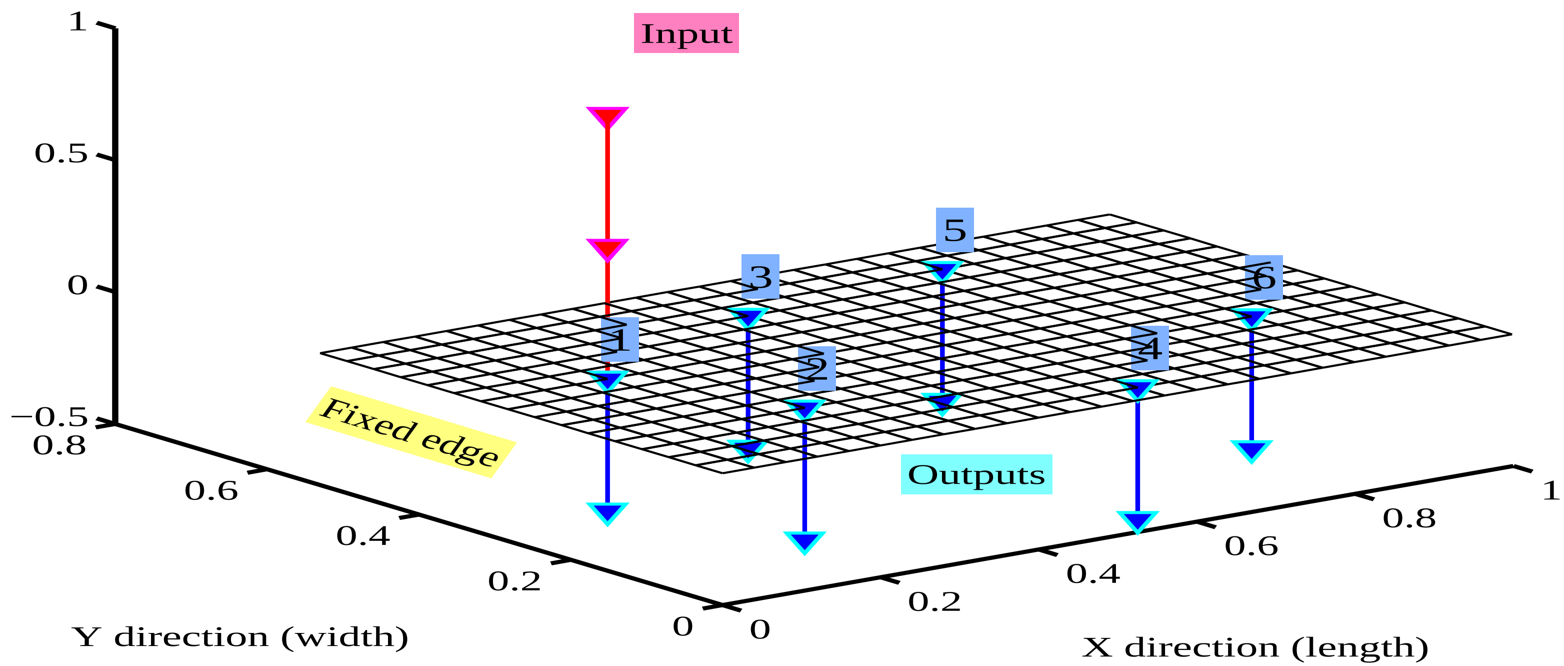
A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment.

Uncertainty type 1: random fields

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \quad (48)$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \quad (49)$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \quad (50)$$

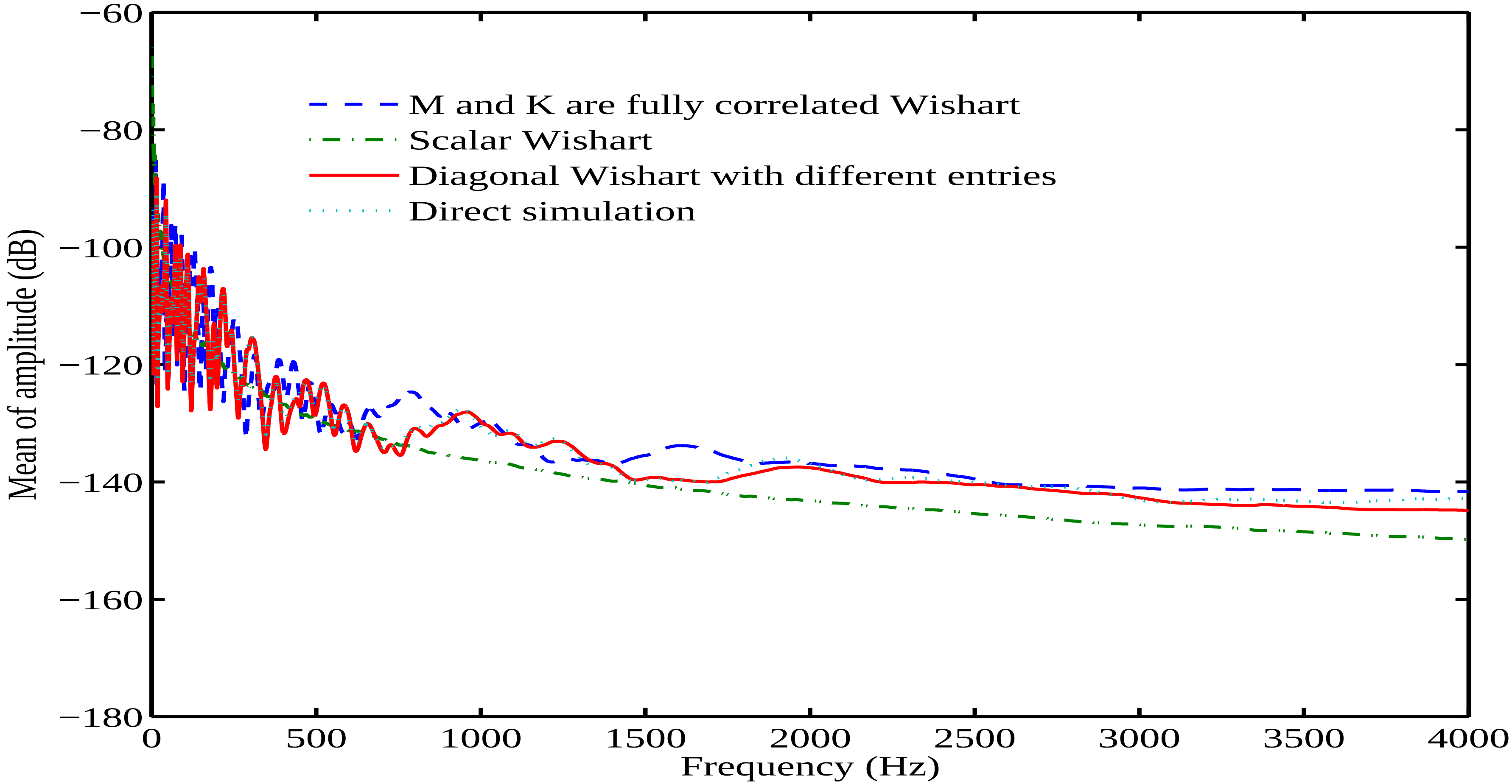
$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \quad (51)$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

Uncertainty type 2: random attached oscillators

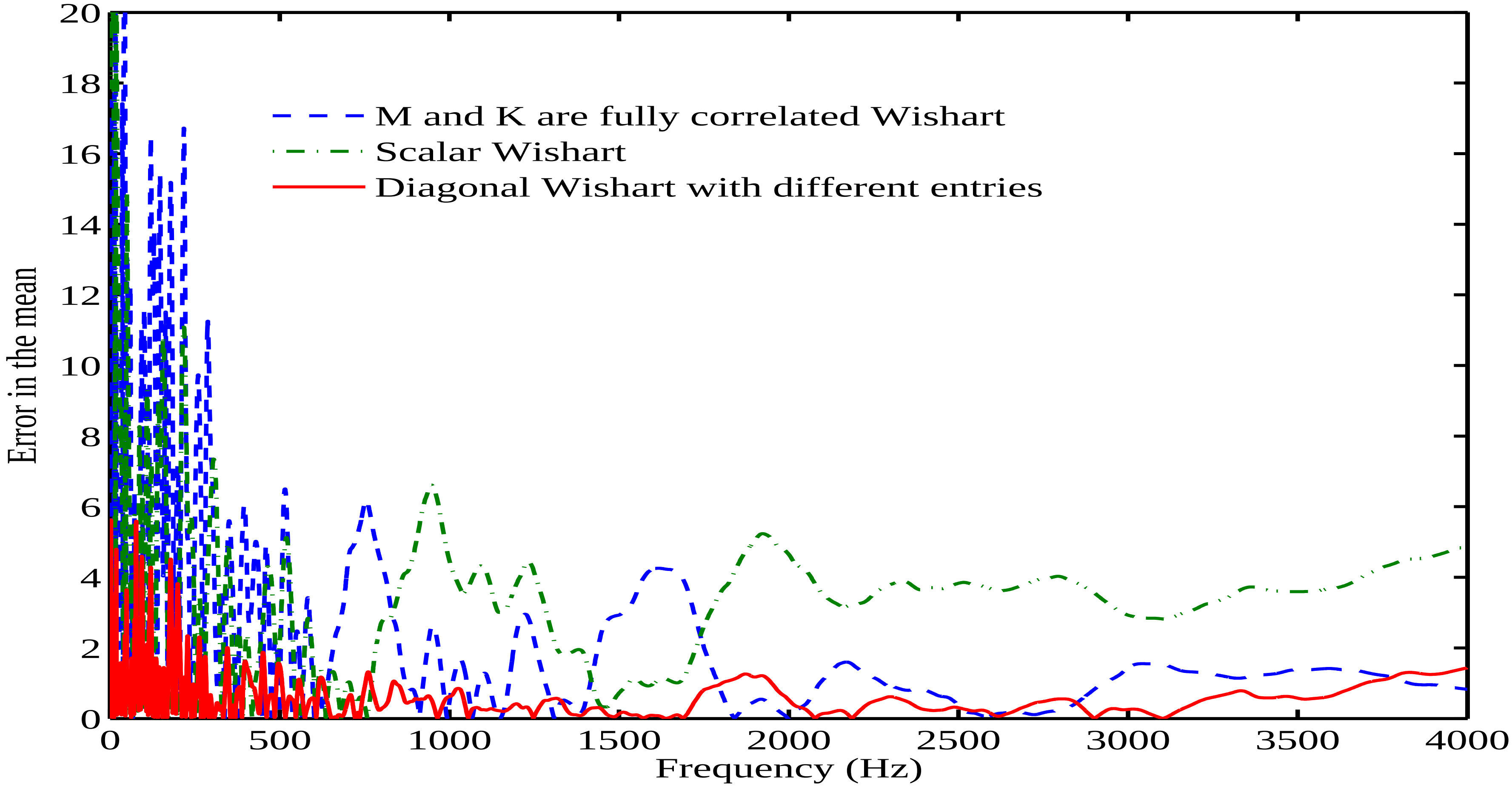
- Here we consider that the baseline plate is 'perturbed' by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty.
- This case will be investigated experimentally.

Mean of cross-FRF



Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the mean of cross-FRF



Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

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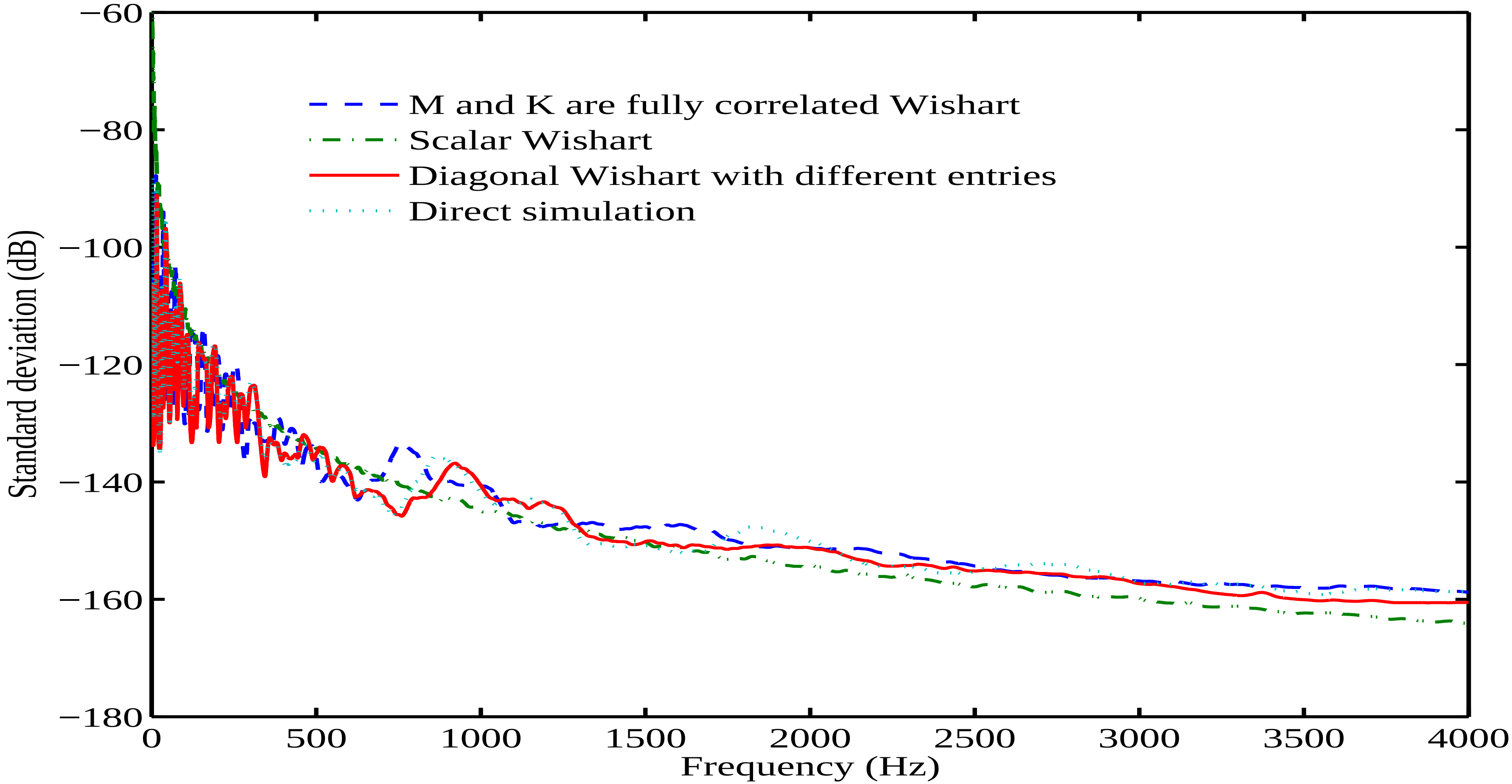
Standard deviation of driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

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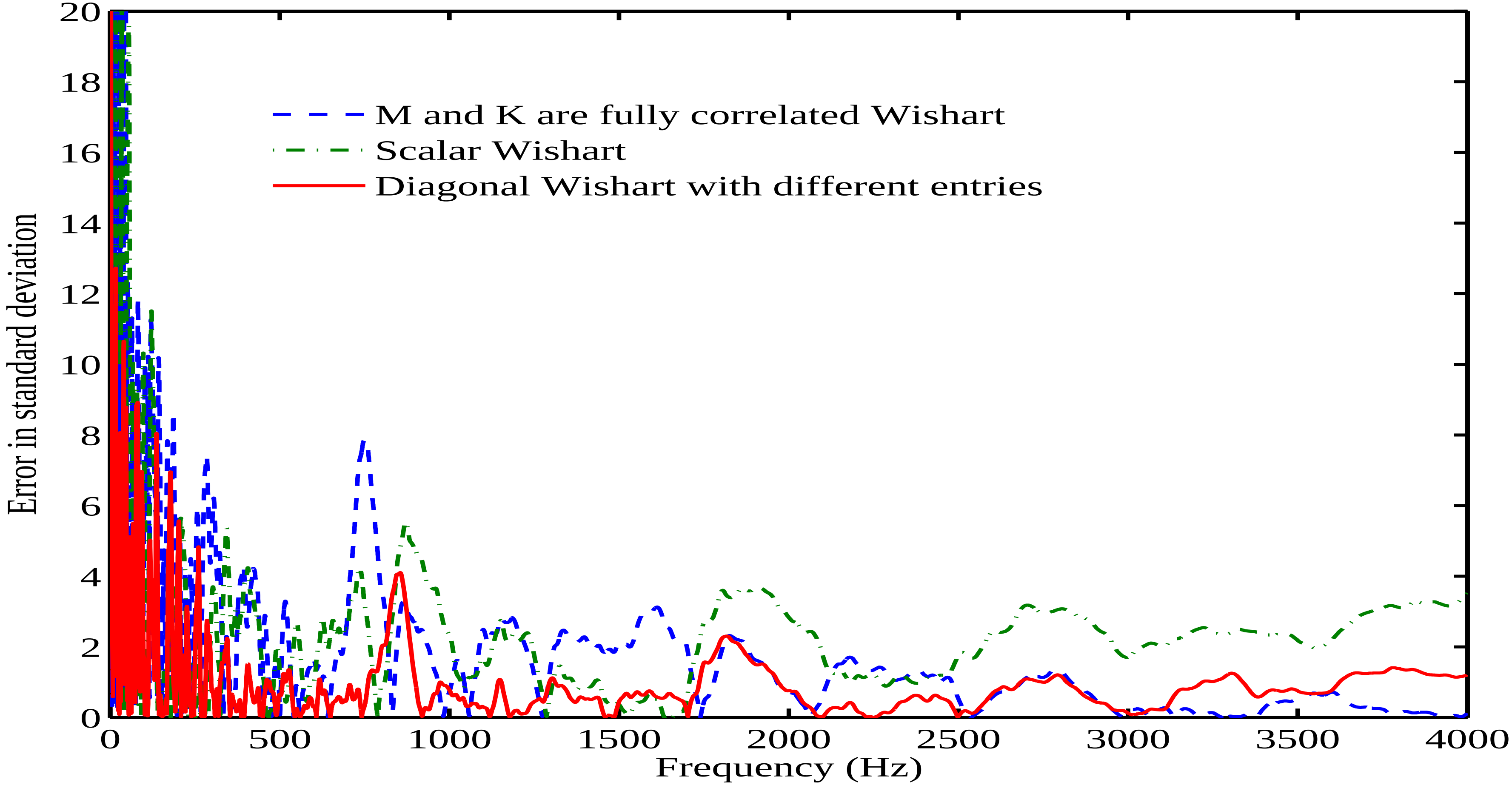
Error in the standard deviation of driving-point-FRF



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

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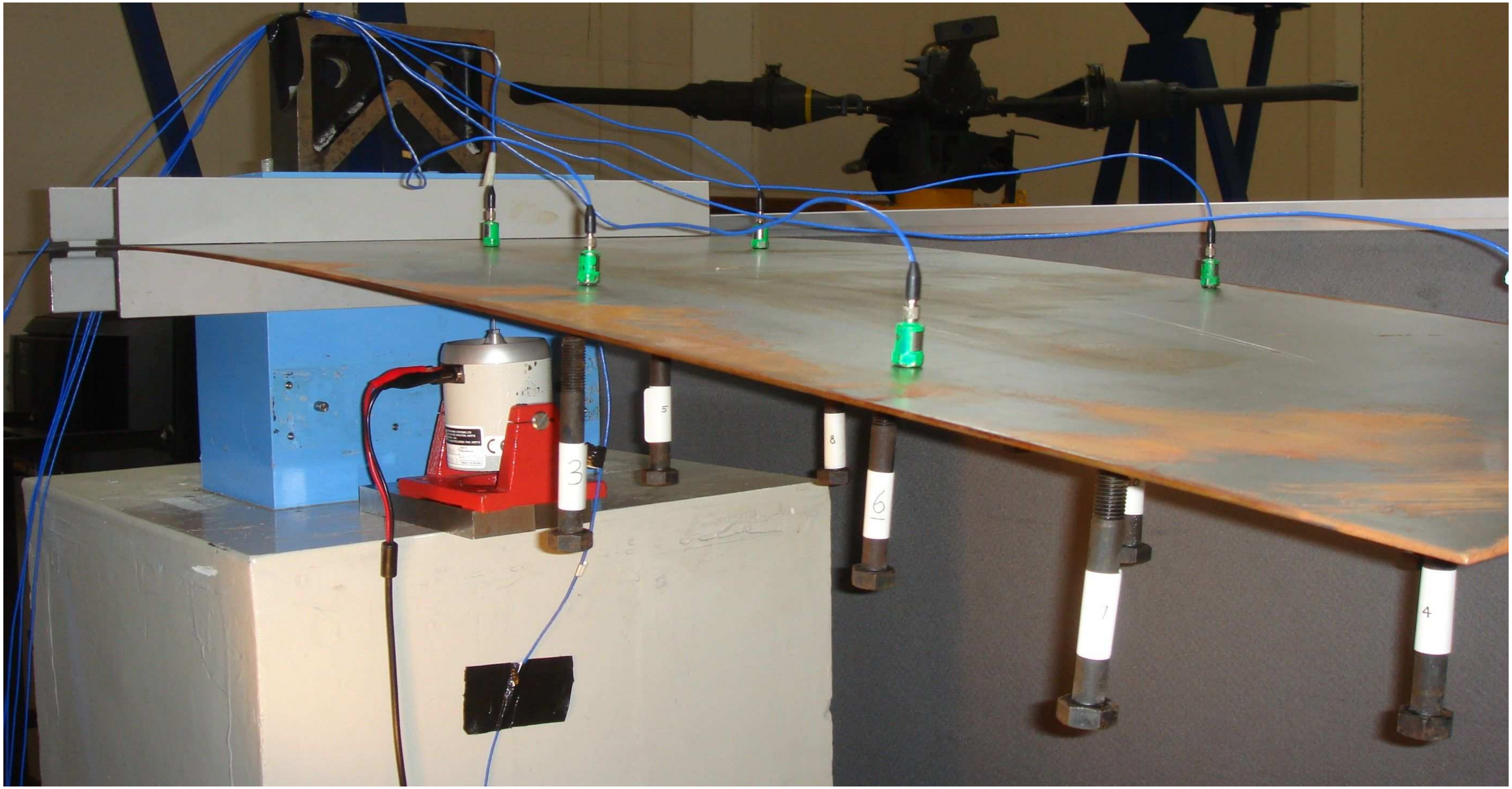
Experimental investigation for uncertainty type 2 (randomly attached oscillators)

A cantilever plate: top view



Experimental setup showing the shaker and ac- celerometer locations.

A cantilever plate: bottom view

Experimental setup showing a realization of the attached oscillators.

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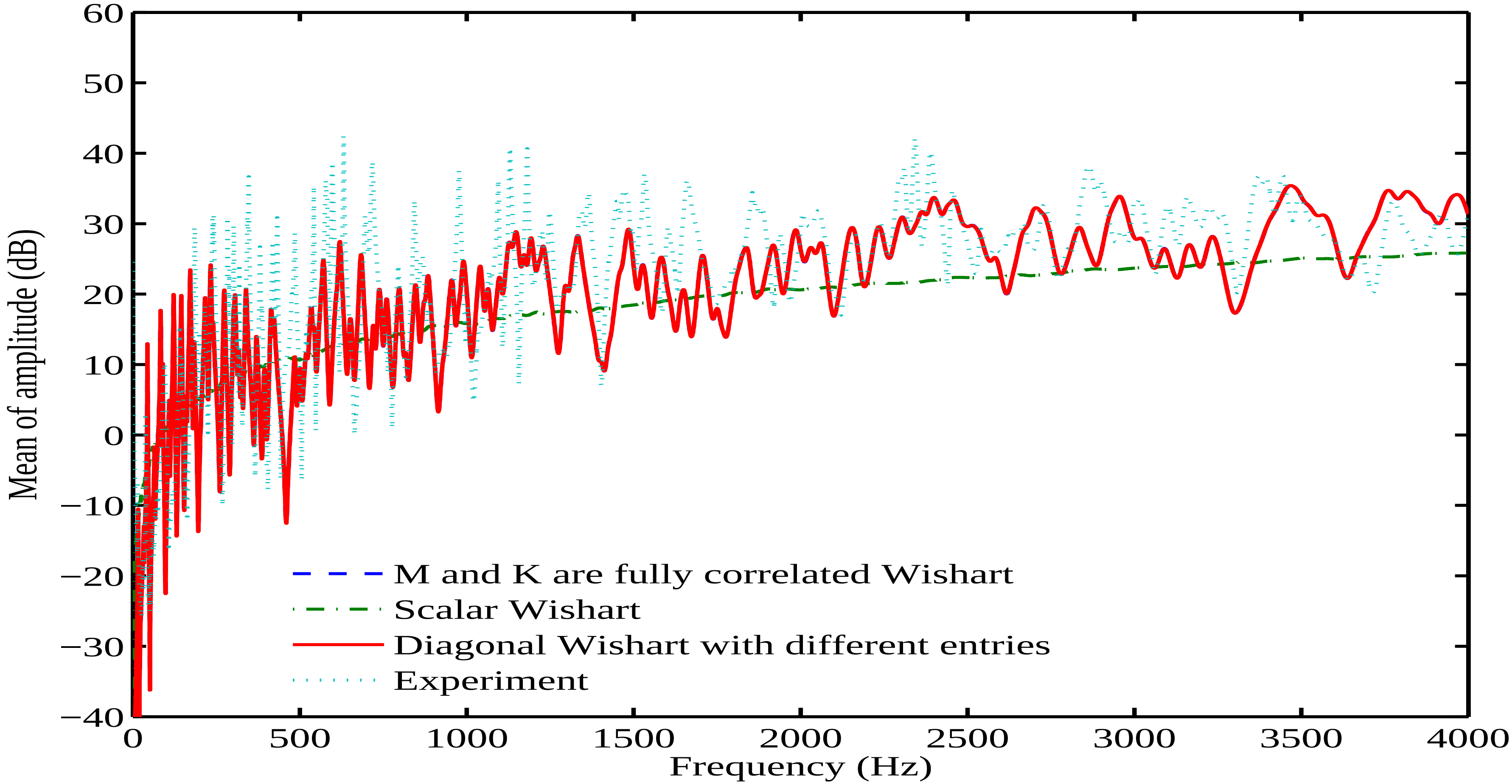
Comparison of driving-point-FRF



Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

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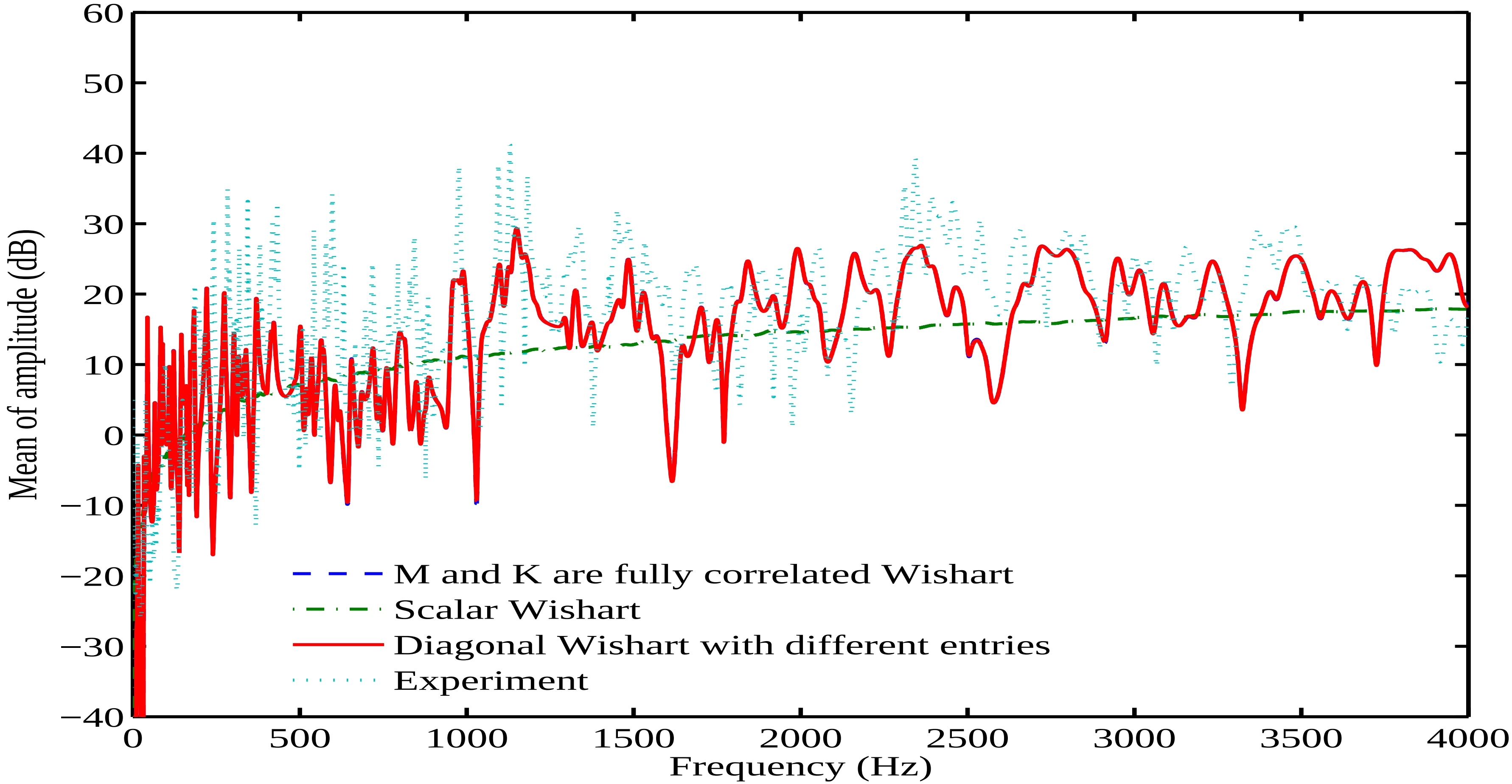
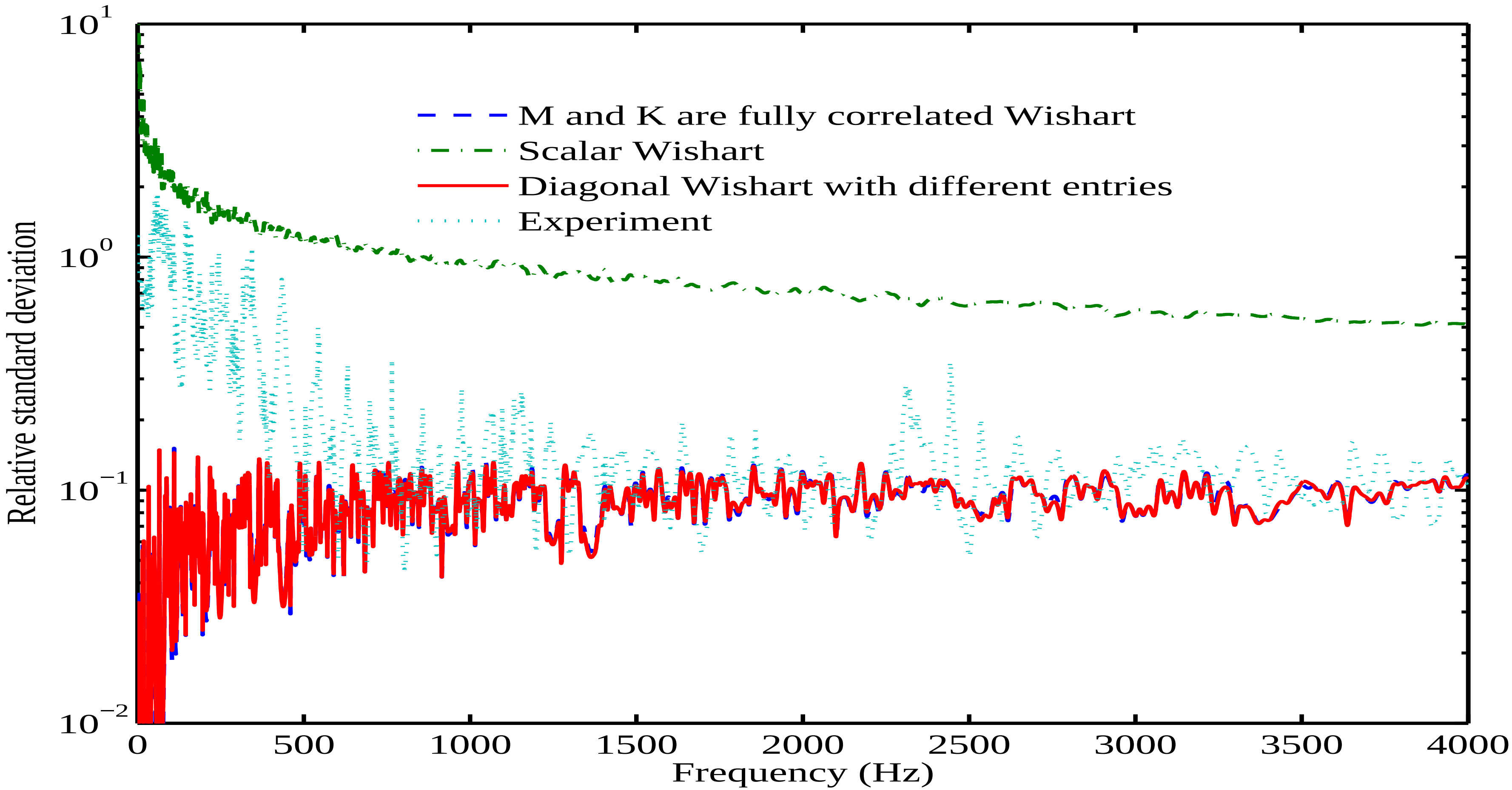
Comparison of Cross-FRF



Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of driving-point-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

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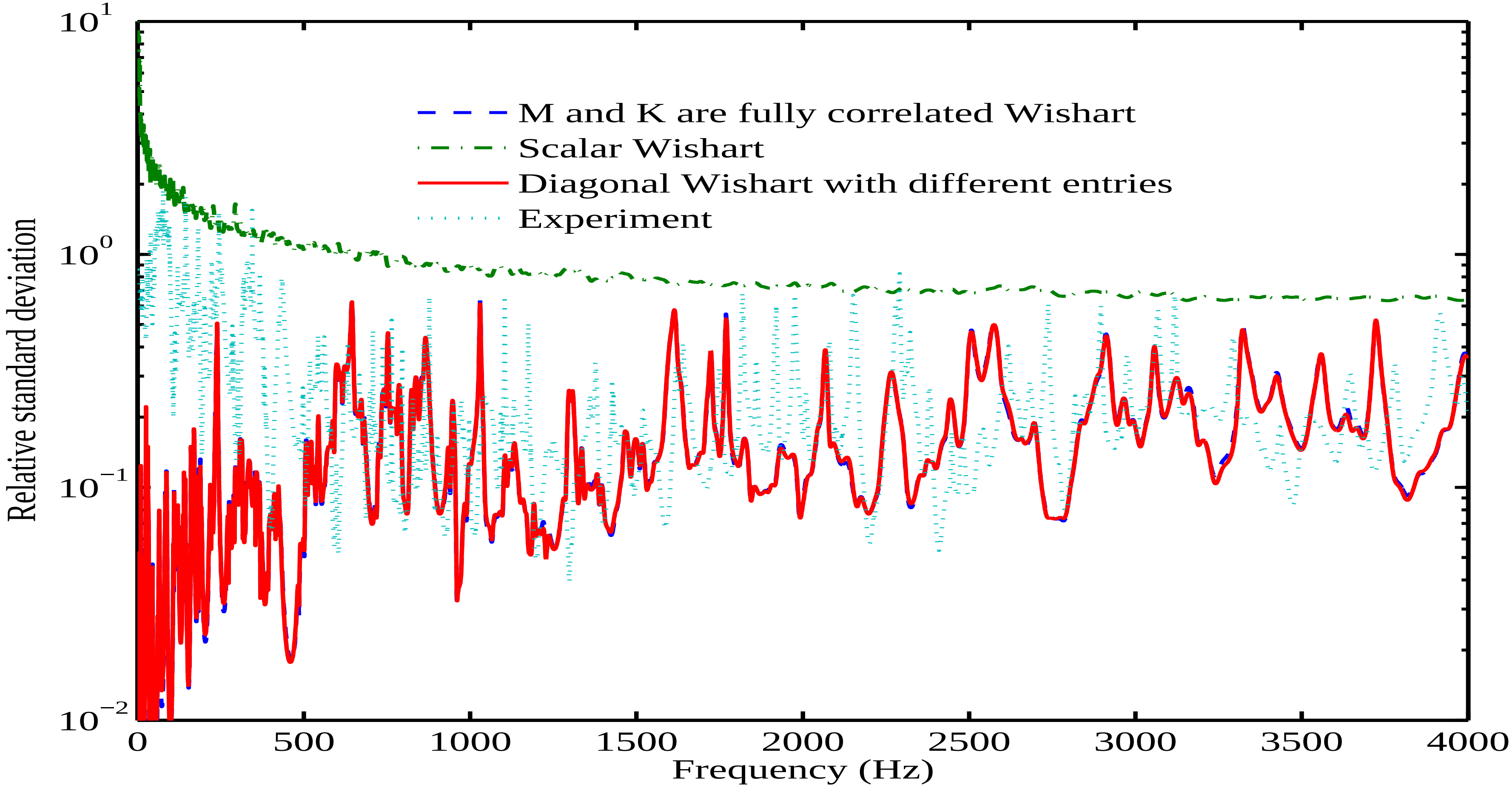
Comparison of Cross-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Identification of uncertain systems (inverse problmes)

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Identification of random field

- How to identify random field corresponding to the system parameters from experimental observations is a major concern for various aero-mechanical systems.
- Suppose we know the nominal values of the system parameters and the 'deviations' are not very large from the nominal values.
- We proposed a simple approach based on sensitivity analysis and KL expansion.

Eigen-sensitivity based approach

- Let us consider the random beam example discussed earlier for illustration .
- The random eigenvalue problem can be expressed as

$$[\mathbf{K}_0 + \mathbf{\Delta K}(\theta)]\,\boldsymbol{\phi}_i = \omega_i^2\,[\mathbf{M}_0 + \mathbf{\Delta M}(\theta)]\,\boldsymbol{\phi}_i. \tag{52}$$

  Recall that $\mathbf{\Delta K}(\theta)$ and $\mathbf{\Delta M}(\theta)$ can be expressed as sums of random variables.
- The eigenvalues ω_i(related to the resonance frequencies of the system) can be obtained from experiments.

Eigen-sensitivity based approach

- Using the Karhunen-Loève expansion of the stiffness and mass matrices and the first-order perturbation method, each eigenvalue can be expressed as

$$\omega_i \approx \omega_{0i} + \sum_{j=1}^{N_\mathrm{K}} \frac{\partial \omega_i}{\partial \xi_{\mathrm{K}j}} \xi_{\mathrm{K}j}(\theta) + \sum_{j=1}^{N_\mathrm{M}} \frac{\partial \omega_i}{\partial \xi_{\mathrm{M}j}} \xi_{\mathrm{M}j}(\theta). \tag{53}$$

- $$\frac{\partial \mathbf{K}}{\partial \xi_{\mathrm{K}j}} = \epsilon_1 \sqrt{\lambda_{\mathrm{K}j}} \mathbf{K}_j \quad \text{and} \quad \frac{\partial \mathbf{M}}{\partial \xi_{\mathrm{M}j}} = \epsilon_2 \sqrt{\lambda_{\mathrm{M}j}} \mathbf{M}_j, \tag{54}$$

- The derivative of the eigenvalues can be obtained as

$$\frac{\partial \omega_i}{\partial \xi_{\mathrm{K}j}} = s_{ij} = \epsilon_1 \sqrt{\lambda_{\mathrm{K}j}} \frac{\boldsymbol{\phi}_{0i}^T \mathbf{K}_j \boldsymbol{\phi}_{0i}}{2\omega_{0i}} \tag{55}$$

and

$$\frac{\partial \omega_i}{\partial \xi_{\mathrm{M}j}} = s_{i(N_\mathrm{K}+j)} = -\epsilon_2 \frac{1}{2} \omega_{0i} \sqrt{\lambda_{\mathrm{M}j}} \boldsymbol{\phi}_{0i}^T \mathbf{M}_j \boldsymbol{\phi}_{0i}. \tag{56}$$

Eigen-sensitivity based approach

- Suppose m number of natural frequencies have been measured. Combining the preceding four equations for all m we can express

$$\boldsymbol{\omega} \approx \boldsymbol{\omega}_0 + \mathbf{S}\,\boldsymbol{\xi} \tag{57}$$

- Here the elements of the $m \times (N_{\rm K} + N_{\rm M})$ sensitivity matrix $\mathbf{S}$ are given before and the $(N_{\rm K} + N_{\rm M})$ dimensional vector of updating parameters $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_{\rm K1} & \xi_{\rm K2} & \ldots & \xi_{{\rm K}N_{\rm K}} & \xi_{\rm M1} & \xi_{\rm M2} & \ldots & \xi_{{\rm M}N_{\rm M}} \end{bmatrix}^T. \tag{58}$$

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Eigen-sensitivity based approach

- This problem may be expressed as the minimization of J, where

$$J(\boldsymbol{\xi}) = \|\boldsymbol{\omega}_{\mathrm{m}} - \boldsymbol{\omega}(\boldsymbol{\xi})\|^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}; \quad \boldsymbol{\varepsilon} = \boldsymbol{\omega}_{\mathrm{m}} - \boldsymbol{\omega}(\boldsymbol{\xi}). \tag{59}$$

- Here $\boldsymbol{\omega}_{\mathrm{m}}$ is the vector of measured natural frequencies corresponding to the predicted natural frequencies $\boldsymbol{\omega}(\boldsymbol{\xi})$, $\boldsymbol{\xi}\mathbb{R}^{n_p}$ is the vector of unknown parameters, and $\boldsymbol{\varepsilon}$ is the modal residual vector.
- The samples of reconstructed random field can be obtained using the truncated series

$$H(\mathbf{r},\theta) = H_0(\mathbf{r}) + \sum_{i=1}^{n_p} \sqrt{\lambda_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{60}$$

Eigen-sensitivity based approach

- Assuming there are more measurements than parameters the updated parameter estimate is obtained using the pseudo inverse as

$$\boldsymbol{\xi} = \left[\mathbf{S}^T\mathbf{S}\right]^{-1}\mathbf{S}^T\left(\boldsymbol{\omega}_{\mathrm{m}} - \boldsymbol{\omega}_0\right). \tag{61}$$

- It is often convenient to weight the measurements to give the penalty function

$$J(\boldsymbol{\xi}) = \boldsymbol{\varepsilon}^T\mathbf{W}\boldsymbol{\varepsilon} \tag{62}$$

where $\mathbf{W}$ is the weighting matrix. Optimizing this penalty function gives the parameter estimate

$$\boldsymbol{\xi} = \left[\mathbf{S}^T\mathbf{W}\mathbf{S}\right]^{-1}\mathbf{S}^T\mathbf{W}\left(\boldsymbol{\omega}_m - \boldsymbol{\omega}_0\right). \tag{63}$$

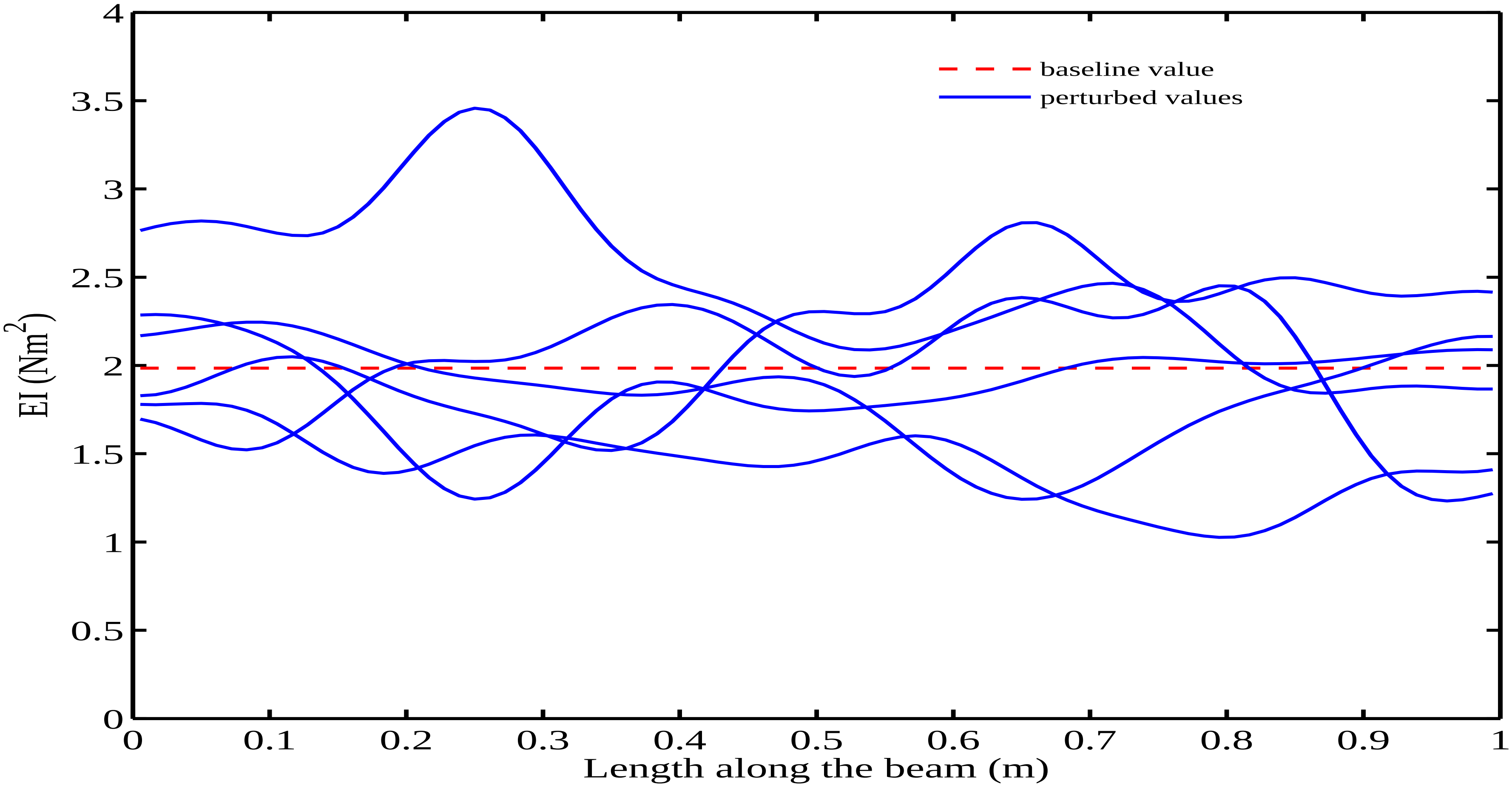
Eigen-sensitivity based approach: summary

The iterative procedure can be computationally implemented using the following steps:

1. Set the counter $r = 0$, select the error tolerance ϵ_e, number of parameters n_p, number of modes m and initialize $\boldsymbol{\xi} = \mathbf{0} \in \mathbb{R}^{n_p}$. For numerical stability $n_p < m$.
2. Increase the counter $r = r + 1$
3. Obtain the system matrices $\mathbf{K}^{(r)}$ and $\mathbf{M}^{(r)}$ using equations KL expansion
4. Solve the undamped eigenvalue problem $\mathbf{K}^{(r)}\boldsymbol{\phi}_i^{(r)} = \omega_i^{(r)^2}\mathbf{M}^{(r)}\boldsymbol{\phi}_i^{(r)}$
5. Obtain the sensitivity matrix $\mathbf{S}^{(r)} \in \mathbb{R}^{m \times n_p}$ with elements $s_{ij}^{(r)} = \epsilon_1 \sqrt{\lambda_{\mathrm{K}j}} \dfrac{\boldsymbol{\phi}_i^{(r)^T}\mathbf{K}_j\boldsymbol{\phi}_i^{(r)}}{2\omega_i^{(r)}}$

   and $s_{i(N_{\mathrm{K}}+j)}^{(r)} = -\epsilon_2 \frac{1}{2}\omega_i^{(r)}\sqrt{\lambda_{\mathrm{M}j}}\boldsymbol{\phi}_i^{(r)^T}\mathbf{M}_j\boldsymbol{\phi}_i^{(r)}, \forall\, i, j$
6. Calculate the updated parameter vector
   $\boldsymbol{\xi}^{(r+1)} = \left[\mathbf{S}^{(r)^T}\mathbf{W}\mathbf{S}^{(r)}\right]^{-1}\mathbf{S}^{(r)^T}\mathbf{W}\left(\boldsymbol{\omega}_m - \boldsymbol{\omega}^{(r)}\right)$
7. Calculate the difference $\epsilon = \left\|\boldsymbol{\xi}^{(r+1)} - \boldsymbol{\xi}^{(r)})\right\|$
8. If $\epsilon \leq \epsilon_e$ then exit, else go back to step 2.

Sample realizations



Some random realizations of the bending rigidity EI of the beam for correlation length $b = L/3$ and strength parameter $\epsilon_1 = 0.2$

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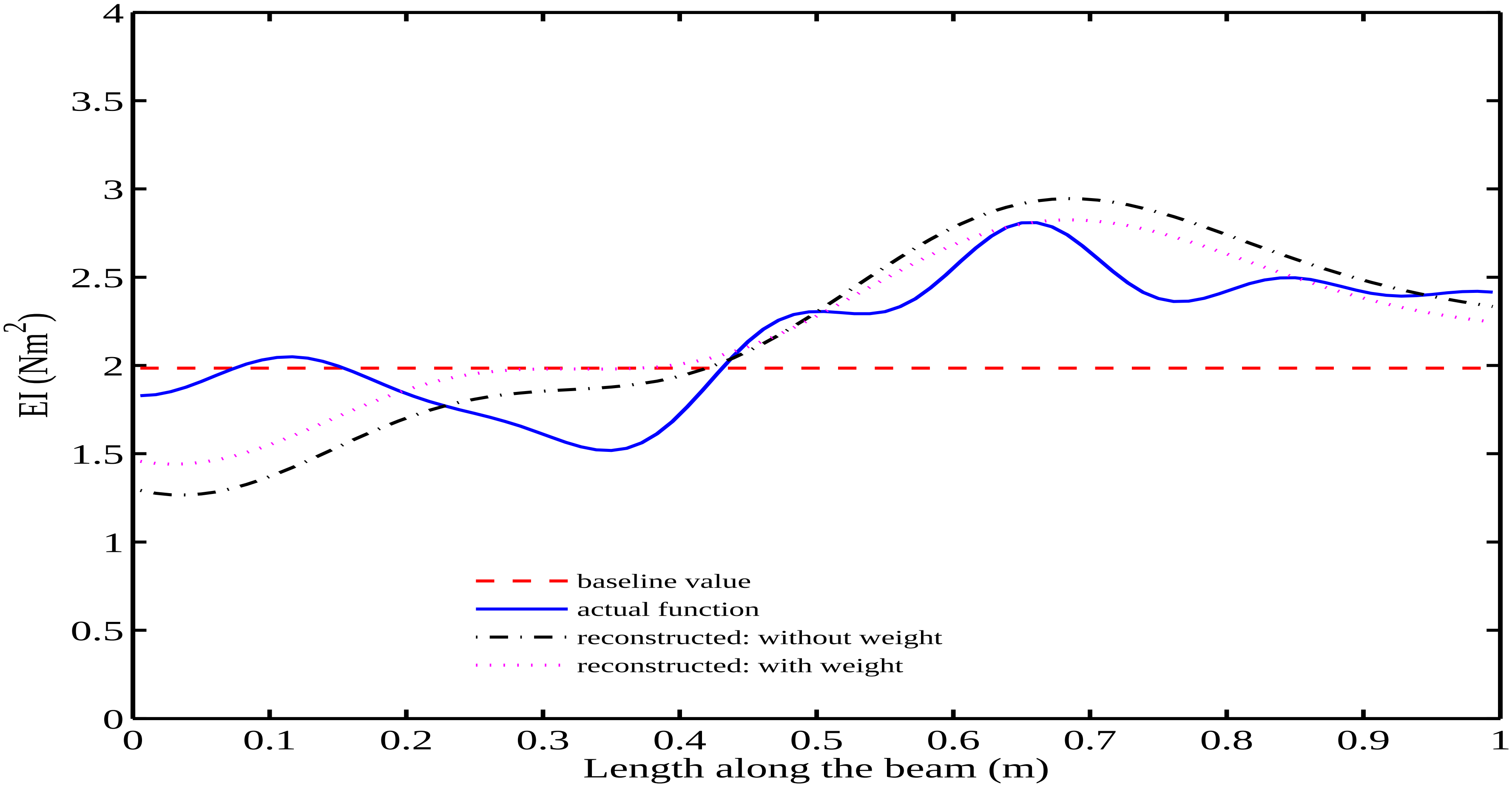
Reconstructed samples



Baseline, actual and reconstructed values of the bending rigidity (EI) along the length of the beam; $m = 26$, $n_p = 6$

Conclusions

- Uncertainties need to be taken into account for credible predictions using computational methods.
- This talk concentrated on uncertainty propagation and identification in structural dynamic problems.
- A general uncertainty propagation approach based on Wishart random matrix is discussed and the results are compared with experimental results.
- Based on numerical and experimental studies, a suitable simple Wishart random matrix model has been identified.
- A sensitivity based method for identification of random field has been proposed.

Summary of research activities

- Dynamics of Complex Engineering Systems
  - Generally damped systems
  - Uncertainty quantification
- Inverse problems and model updating
  - Linear systems (stochastic model updating)
  - Nonlinear systems (kalman filtering)
- Bio & Nanomechanics
  - Carbon nanotube, Graphene sheet
  - Cell mechanics, mechanics of DNA
- Renewable Energy
  - Wind energy quantification
  - Piezoelectric vibration energy harvesting