Extremely strong convergence of eigenvalue-density of linear stochastic dynamical systems

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Outline of the presentation

- Introduction
- Uncertainty Propagation (UP) in structural dynamics
- Wishart random matrices
  - Parameter selection
  - Density of eigenvalues
- Computational results
- Experimental results
- Conclusions

Dynamics of linear systems

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objective of the ‘forward problem’ is to predict the variability in the response vector $\mathbf{q}$.

There can be two broad possibilities:

- quantify uncertainties in the system matrices first and then obtain uncertainties in the response
- directly quantify uncertainties in the eigenvalues & eigenvectors and then obtain uncertainties in the response

Stochastic dynamic response

- Taking the Laplace transform of the equation of motion:

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{2}$$

  The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2 \mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{3}$$

  where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- We define the matrices

$$\boldsymbol{\Omega} = \operatorname{diag}\left[\omega_1, \omega_2, \ldots, \omega_n\right] \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right]. \tag{4}$$

$$\text{so that} \quad \boldsymbol{\Phi}^T\mathbf{K}_e\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_n \tag{5}$$

Stochastic dynamic response

- Transforming it into the modal coordinates:

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{6}$$

- Here

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = 2\zeta\mathbf{\Omega}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{7}$$

- When we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ will be a random matrix of dimension n. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{n\times n}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{8}$$

Stochastic dynamic response

- Since $\boldsymbol{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\boldsymbol{\Psi}_r$ such that

$$\boldsymbol{\Psi}_r^T \boldsymbol{\Xi} \boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \tag{9}$$

  Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\boldsymbol{\Xi}$.

- Recalling that $\boldsymbol{\Psi}_r^T \boldsymbol{\Psi}_r = \mathbf{I}_n$ we obtain

$$\bar{\mathbf{q}}' = \left[s^2 \mathbf{I}_n + s\mathbf{C}' + \boldsymbol{\Omega}^2 \right]^{-1} \bar{\mathbf{f}}' \tag{10}$$

$$= \boldsymbol{\Psi}_r \left[s^2 \mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2 \right]^{-1} \boldsymbol{\Psi}_r^T \bar{\mathbf{f}}' \tag{11}$$

Stochastic dynamic response

- The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \boldsymbol{\Phi}\bar{\mathbf{q}}'(s) = \boldsymbol{\Phi}\boldsymbol{\Psi}_r \left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1} (\boldsymbol{\Phi}\boldsymbol{\Psi}_r)^T \bar{\mathbf{f}}(s)$$

$$= \sum_{j=1}^{n} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j}.$$

Here

$$\boldsymbol{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}\right], \quad \mathbf{X}_r = \boldsymbol{\Phi}\boldsymbol{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_n}\right]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.

Wishart random matrix approach

- Suppose we 'know' (e.g, by measurements or stochastic finite element modeling) the mean ($\mathbf{G}_0$) and the (normalized) variance (dispersion parameter) (δ_G) of the system matrices:

$$\delta_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{12}$$

- It can be proved that a positive definite symmetric matrix can e expressed by a Wishart matrix $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$ with

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0/\theta \tag{13}$$

where

$$\theta = \frac{1}{\delta_G^2}\left\{1 + \gamma_G\right\} - (n+1) \tag{14}$$

and

$$\gamma_G = \frac{\left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2}{\mathrm{Trace}\left({\mathbf{G}_0}^2\right)} \tag{15}$$

Parameter-selection for structural dynamics

Approach 1: M and K are fully correlated Wishart (most complex). For this case $\mathbf{M} \sim W_n(p_1, \mathbf{\Sigma}_1)$, $\mathbf{K} \sim W_n(p_1, \mathbf{\Sigma}_1)$ with $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$ and $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices. Here

$$\mathbf{\Sigma}_1 = \mathbf{M}_0/p_1, p_1 = \frac{\gamma_M + 1}{\delta_M^2} \tag{16}$$

and
$$\mathbf{\Sigma}_2 = \mathbf{K}_0/p_2, p_2 = \frac{\gamma_K + 1}{\delta_K^2} \tag{17}$$

$$\gamma_G = \{\mathrm{Trace}\left(\mathbf{G}_0\right)\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right) \tag{18}$$

Parameter-selection for structural dynamics

Approach 2: Scalar Wishart (most simple) In this case it is assumed that

$$\mathbf{\Xi} \sim W_n\left(p, \frac{a^2}{n}\mathbf{I}_n\right) \tag{19}$$

Considering $\mathrm{E}\left[\mathbf{\Xi}\right] = \mathbf{\Omega}_0^2$ and $\delta_{\mathbf{\Xi}} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1+\gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \mathrm{Trace}\left(\mathbf{\Omega}_0^2\right)/p \tag{20}$$

Parameter-selection for structural dynamics

Approach 3: Diagonal Wishart with different entries (something in the middle). For this case $\mathbf{\Xi} \sim W_n\left(p, \mathbf{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\mathbf{\Xi}^{-1}\right] = \mathbf{\Omega}_0^{-2}$ and $\delta_{\mathbf{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem.

The parameters can be obtained as

$$p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1) \tag{21}$$

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Parameter-selection for structural dynamics

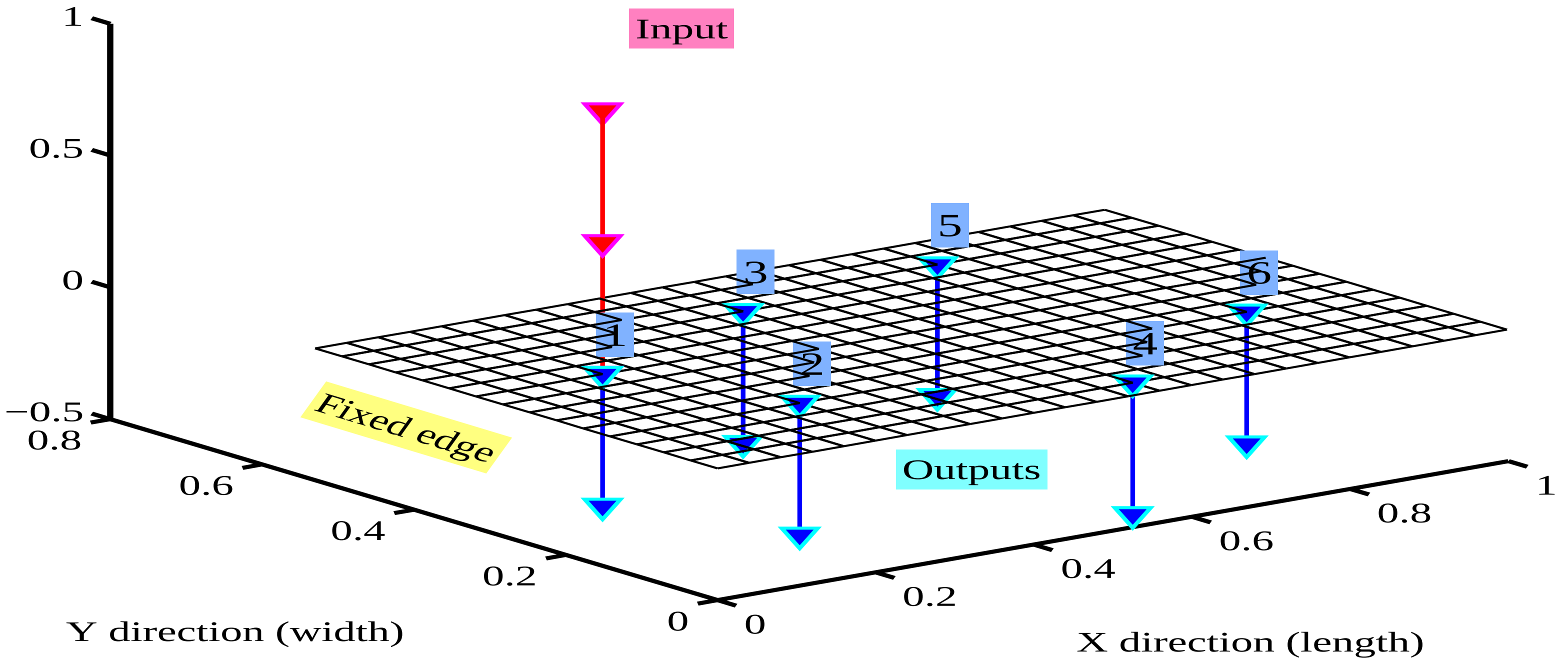
- Defining $\mathbf{H}_0 = \mathbf{M}_0{}^{-1}\mathbf{K}_0$, the constant γ_H:

$$\gamma_H = \frac{\{\text{Trace}\,(\mathbf{H}_0)\}^2}{\text{Trace}\,(\mathbf{H}_0{}^2)} = \frac{\left\{\text{Trace}\,(\boldsymbol{\Omega}_0^2)\right\}^2}{\text{Trace}\,(\boldsymbol{\Omega}_0^4)} = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{22}$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\delta_H = \frac{\left(p_1{}^2 + (p_2 - 2 - 2\,n)\,p_1 + (-n-1)\,p_2 + n^2 + 1 + 2\,n\right)\gamma_H}{p_2\,(-p_1+n)\,(-p_1+n+3)} + \frac{p_1{}^2 + (p_2 - 2\,n)\,p_1 + (1-n)\,p_2 - 1 + n^2}{p_2\,(-p_1+n)\,(-p_1+n+3)} \tag{23}$$

A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Uncertainty modeling by random fields

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \quad (24)$$

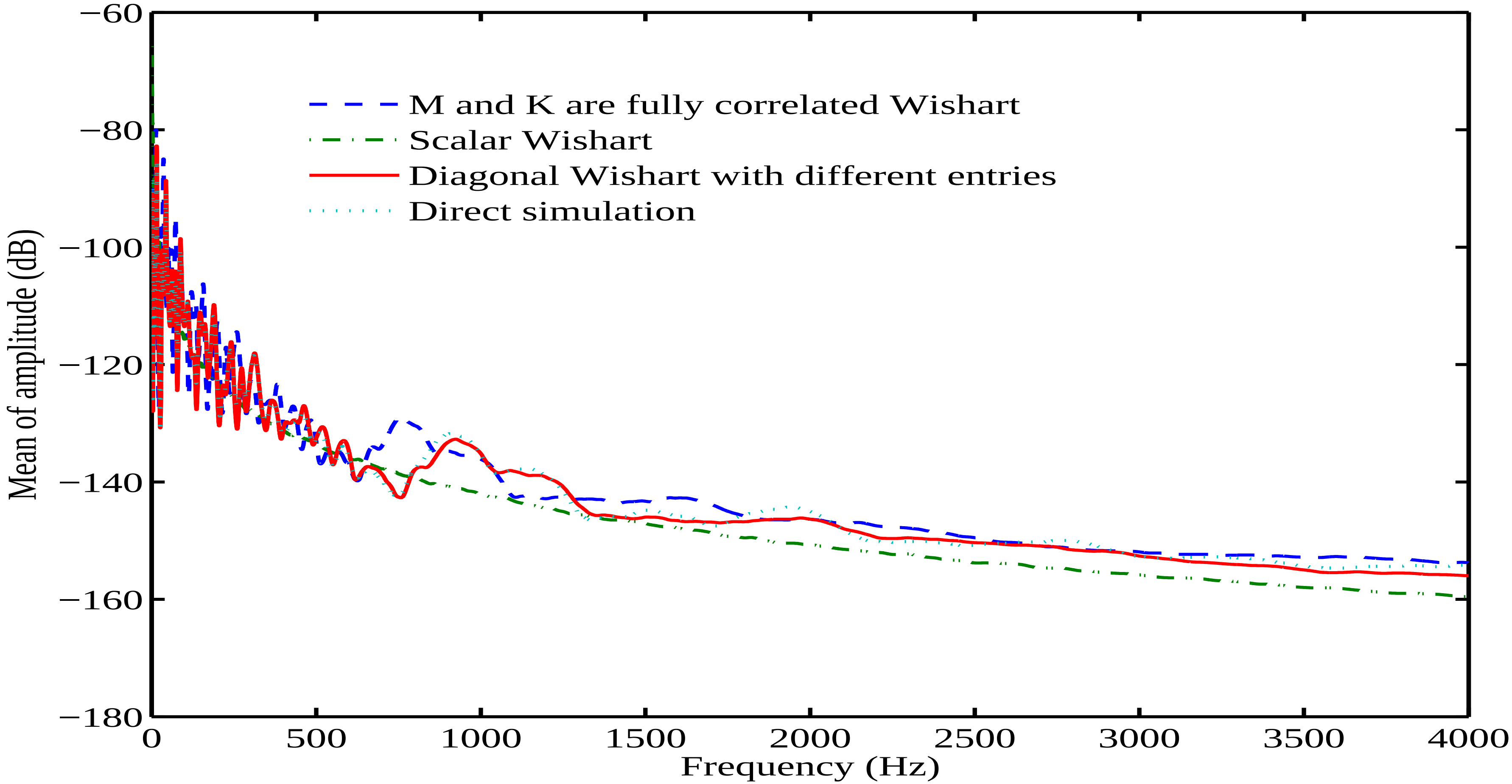
$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \quad (25)$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \quad (26)$$

$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \quad (27)$$

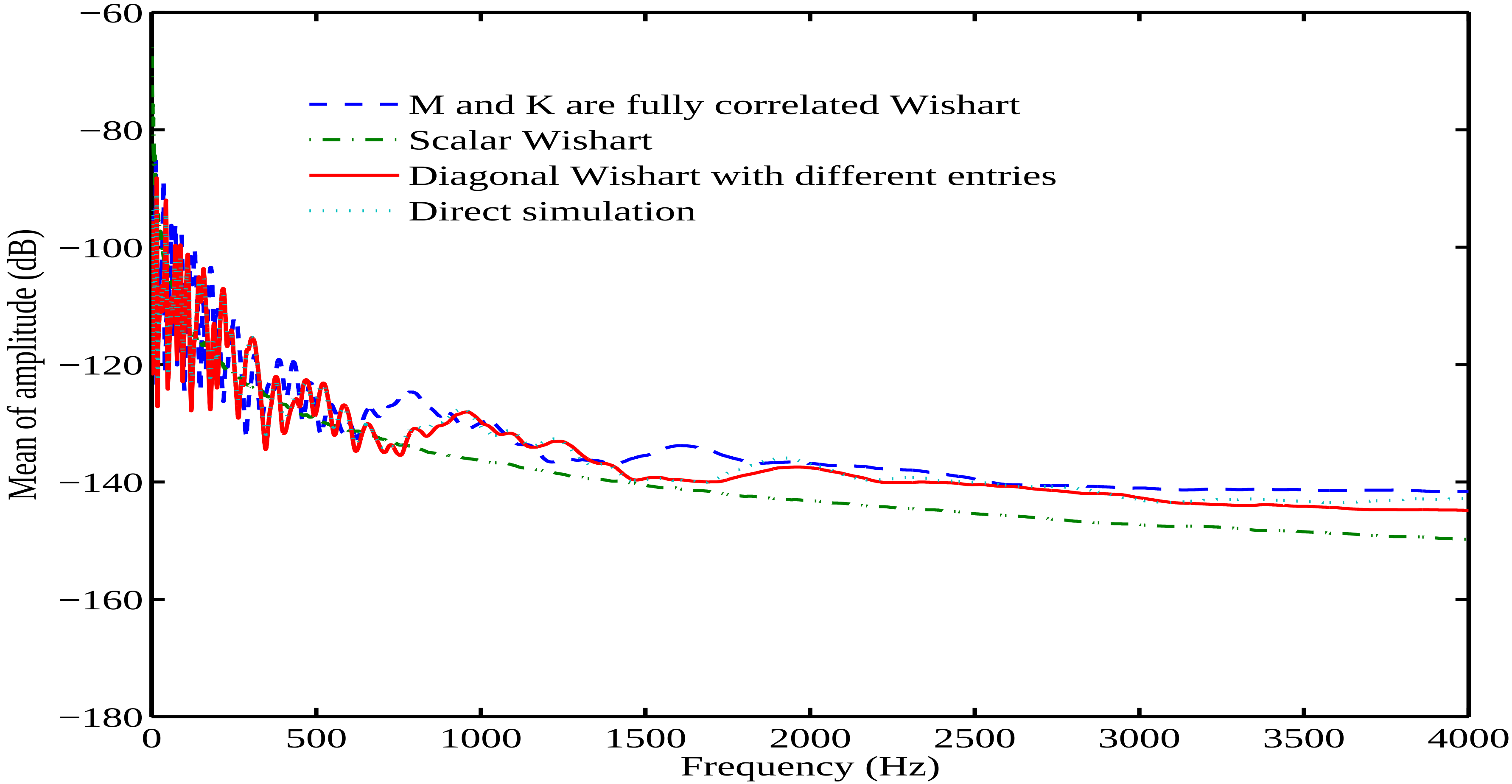
- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

Mean of the driving-FRF



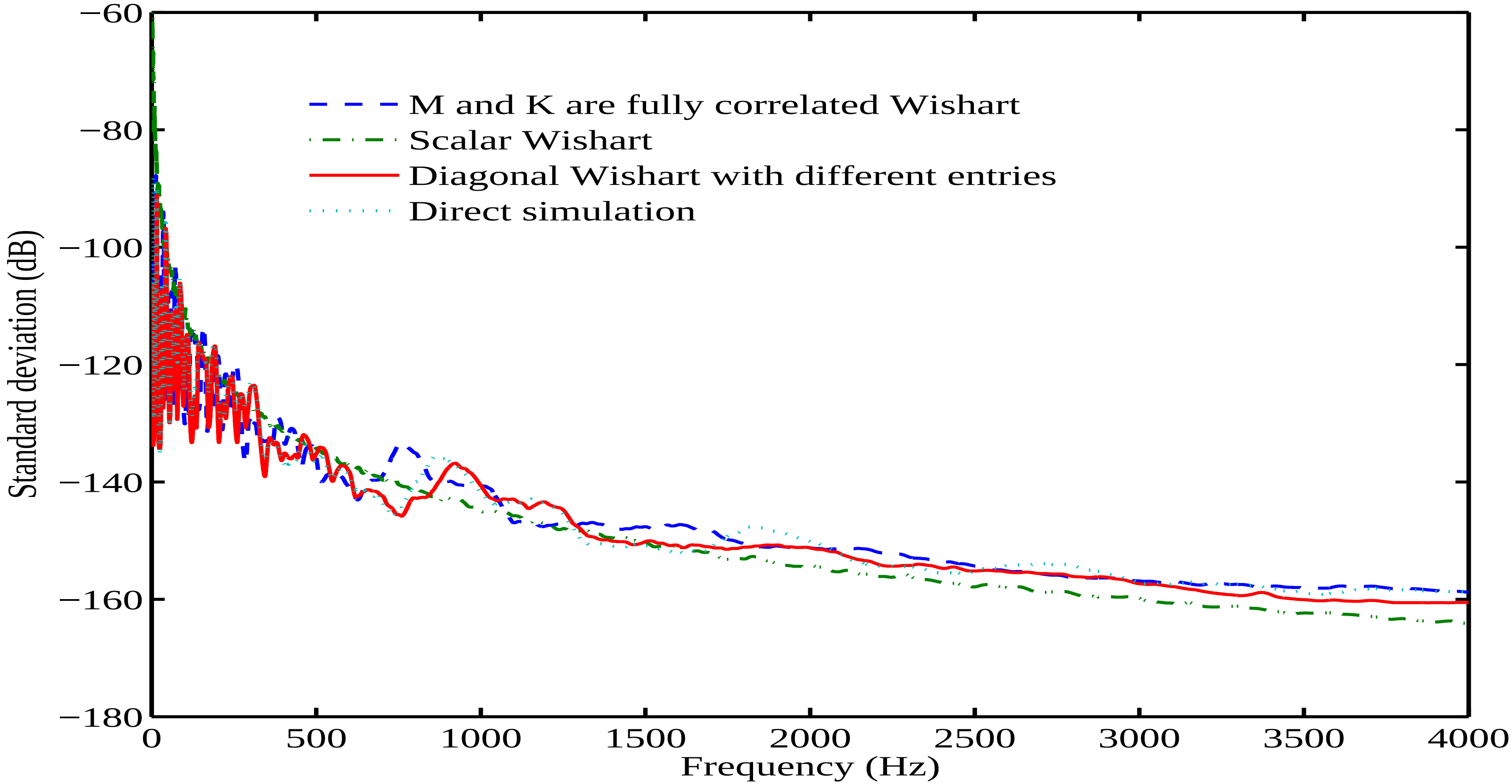
Mean of the amplitude of the response of the driving-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Mean of a cross-FRF



Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Standard deviation of the driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

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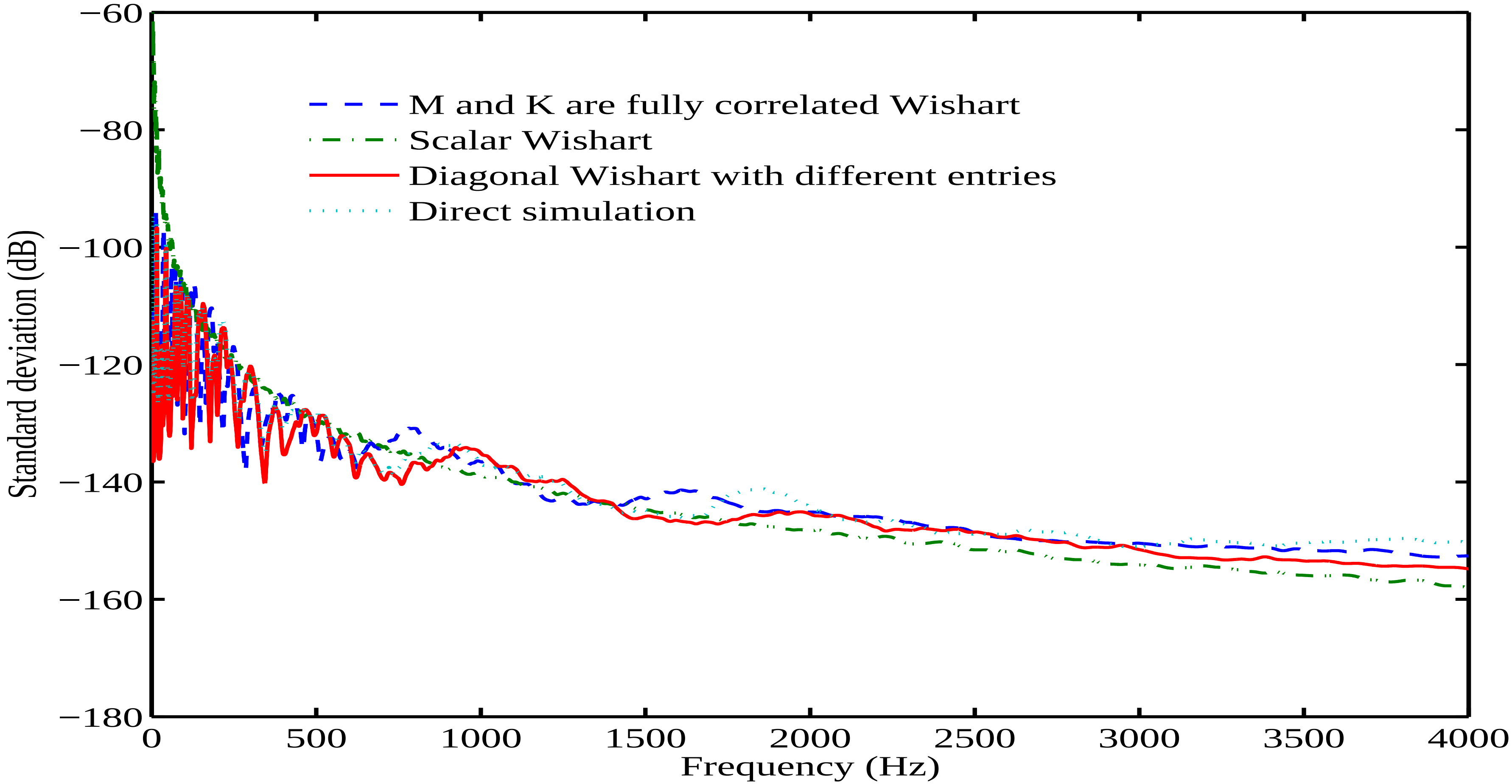
Standard deviation of a cross-point-FRF



Standard deviation of the amplitude of the response of a cross-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Summary so far ...

We can choose a 'surrogate' random matrix model to 'replace' the actual stochastic dynamic system by a Wishart matrix $\boldsymbol{\Xi} \sim W_n\left(p, \boldsymbol{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\boldsymbol{\Xi}^{-1}\right] = \boldsymbol{\Omega}_0^{-2}$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$. Here

$$p = n + 1 + \theta, \;\; \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1), \;\; \gamma_H = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{28}$$

and

$$\delta_H = \frac{\left({p_1}^2 + (p_2 - 2 - 2\,n)\,p_1 + (-n-1)\,p_2 + n^2 + 1 + 2\,n\right)\gamma_H}{p_2\,(-p_1+n)\,(-p_1+n+3)} + \frac{{p_1}^2 + (p_2 - 2\,n)\,p_1 + (1-n)\,p_2 - 1 + n^2}{p_2\,(-p_1+n)\,(-p_1+n+3)} \tag{29}$$

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Eigenvalue density

- Knowing what surrogate model to use, now we want to develop analytical methods to obtain response statistics. The aim is to bypass the Monte Carlo simulation approach shown before (although MCS on the surrogate is more efficient compared to MCS of the actual system).
- Eigenvalue density is a key part for developing analytical approaches.
- Our main result is that the density of the eigenvalues have the ‘self averaging’ property. This implies that the density of the eigenvalues of nominally identical systems have very strong convergence property.

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Linear Eigenvalue Statistic

- Let Ξ be a $n \times n$ random matrix and $\{\lambda_l\}_{l=1}^n$ its eigenvalues. Then the (empirical) eigenvalue density is

$$\rho_n(\lambda) = n^{-1} \sum_{l=1}^{n} \delta(\lambda - \lambda_l), \tag{30}$$

where δ is the Dirac delta-function.

- Without loss of generality we define a *linear eigenvalue statistics* for any sufficiently smooth *test function* φ as

$$N_n[\varphi] = n^{-1} \sum_{l=1}^{n} \varphi(\lambda_l) = \int \varphi(\mu)\rho_n(\mu)d\mu \tag{31}$$

Note that ρ_n in equation (30) correspond formally to $\varphi(\mu) = \delta(\lambda - \mu)$ for a given λ.

Strong convergence

- We can prove that the fluctuations of $N_n[\varphi]$ around its expectation $\mathbf{E}\{N_n[\varphi]\}$ vanish sufficiently fast in the limit

$$n \to \infty \to, \; p \to \infty, \; p/n \to \; c \in (0, \infty) \tag{32}$$

- To this end we obtain a bound for the variance

$$\mathbf{Var}\{N_n[\varphi]\} = \mathbf{E}\{|N_n[\varphi]|^2\} - |\mathbf{E}\{N_n[\varphi]\}^2$$

  of $N_n[\varphi]$. The bound is

$$\mathbf{Var}\{N_n[\varphi]\} \leq \frac{4\sqrt{3}}{n^2 p} \mathrm{Tr}\, \mathbf{\Sigma}^2 (\max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|)^2. \tag{33}$$

  It is valid for real symmetric as well as for hermitian Wishart matrices.

- Considering $\max_{p,n} n^{-1} \mathrm{Tr} \mathbf{\Sigma}^2 \leq C < \infty$. and $\max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| < \infty$, we obtain that

$$\mathbf{Var}\{N_n[\varphi]\} = O(n^{-2}) \tag{34}$$

Marčenko-Pastur density

- We proved that the eigenvalue density of a (large) random system converges to a deterministic limit. But where does it converges to?
- The converged density is NOT universal but depends on the property underlying matrix.
- In the case, where $\mathbf{\Sigma} = \mathbf{I}_n$ and $p/n = c > 1$ we have

$$\rho(\lambda) = \frac{1}{2\pi\lambda} \begin{cases} \sqrt{(a_+ - \lambda)(\lambda - a_-)}, & \lambda \in [a_-, a_+], \\ 0, & \lambda \notin [a_-, a_+], \end{cases} \tag{35}$$

where $a_\pm = (1 \pm \sqrt{c})^2$.

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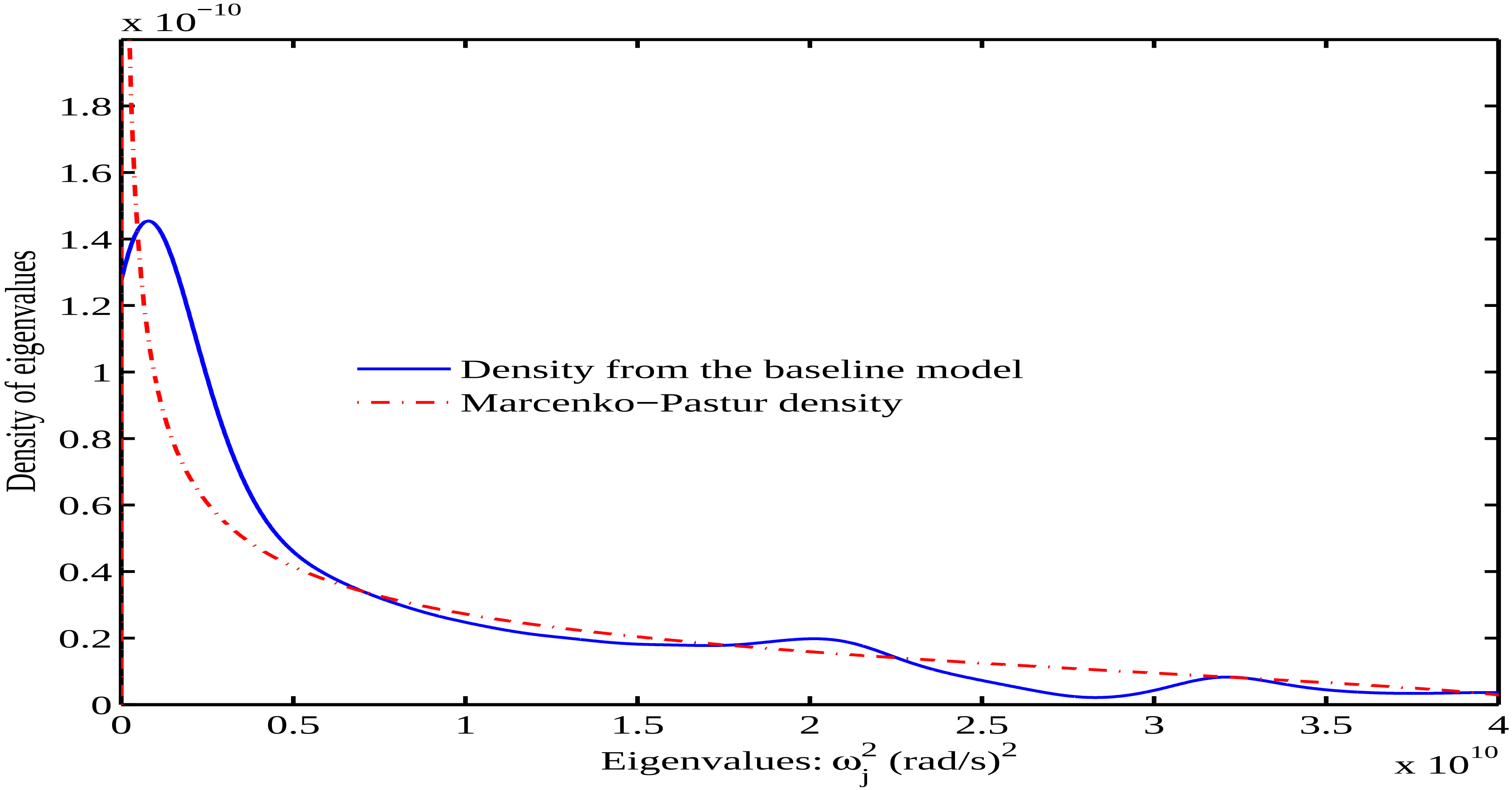
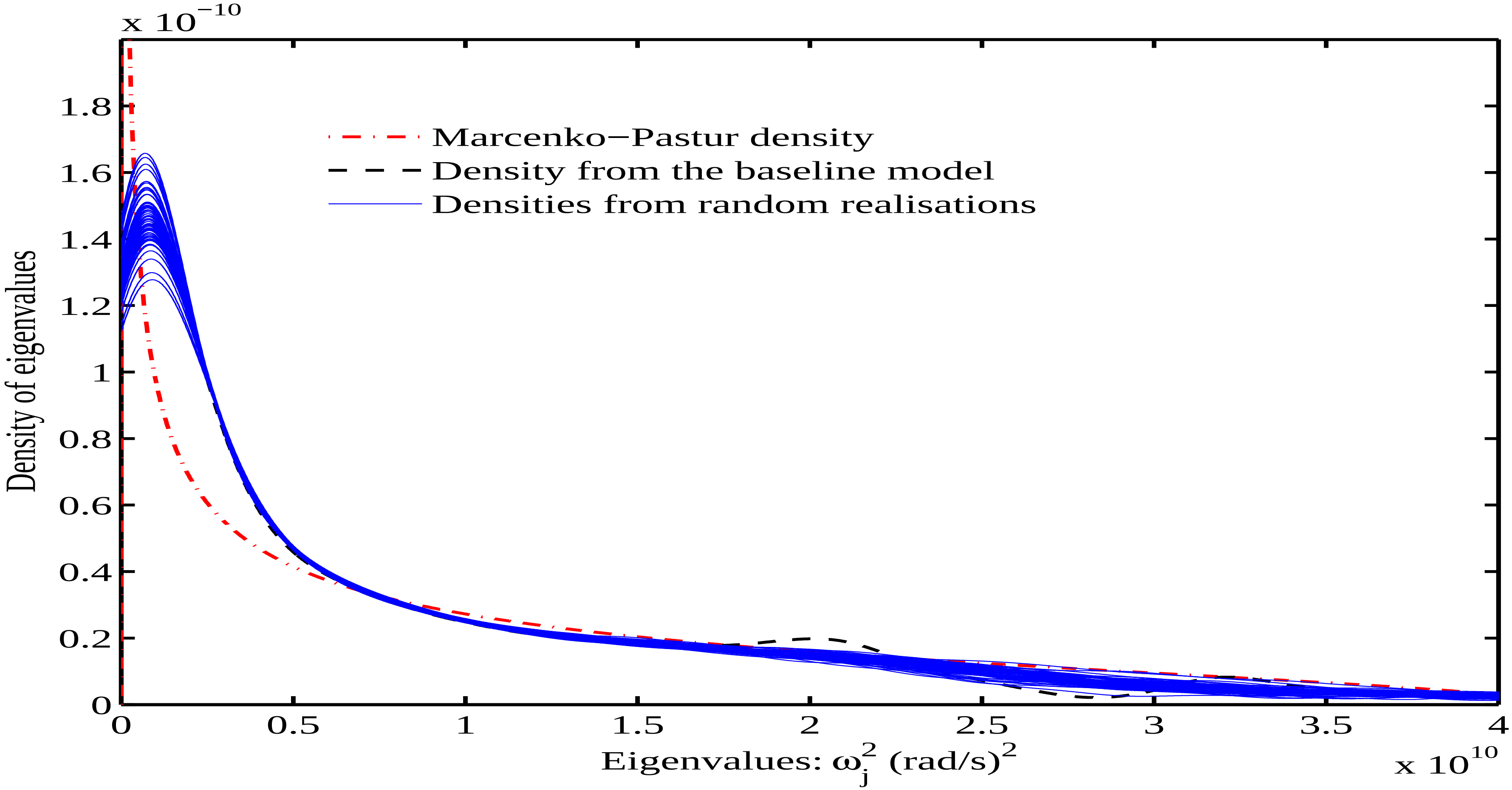
Density of the baseline plate model



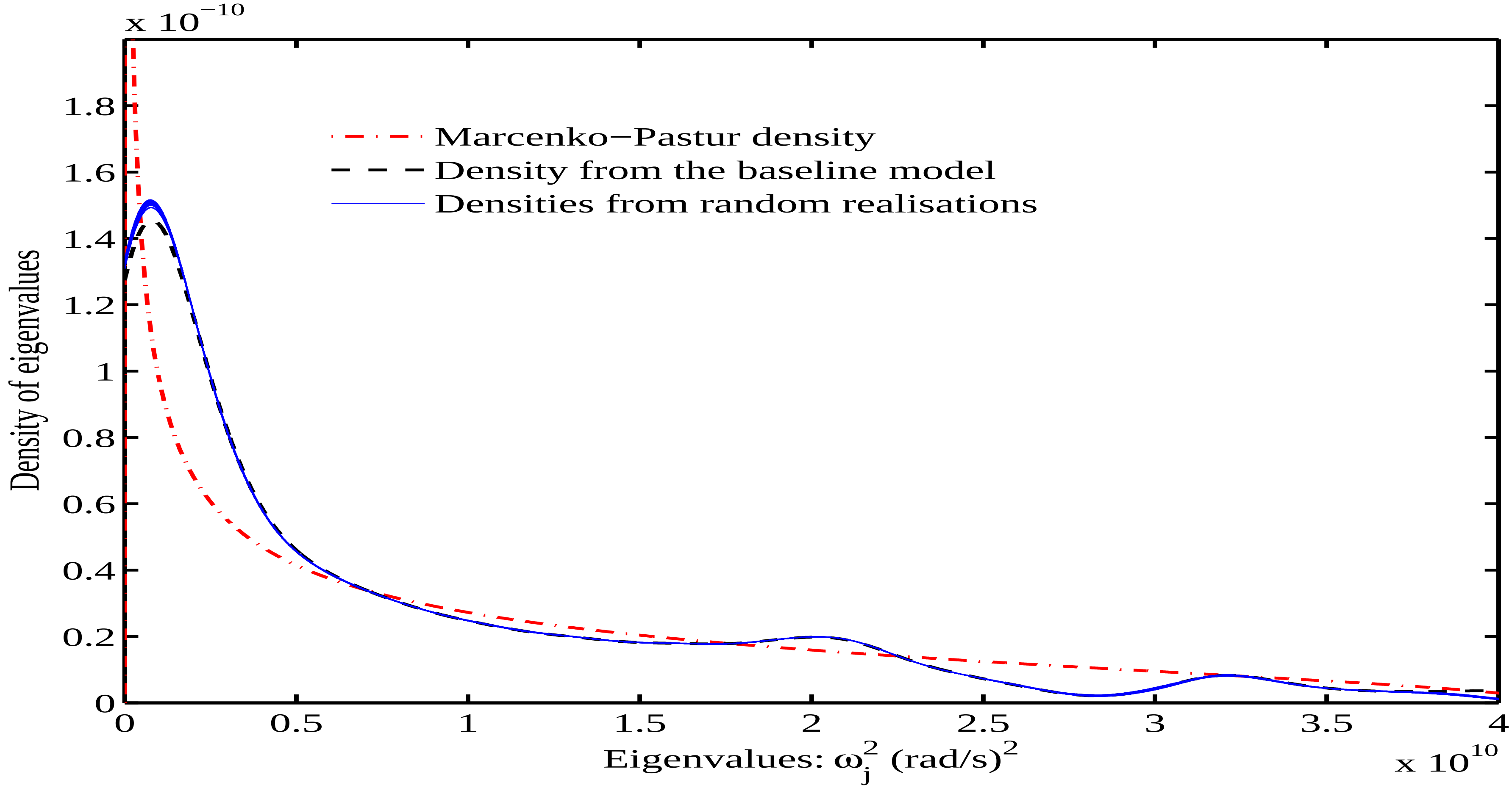
The density of 1200 eigenvalues of the baseline model.

Density of random plate - 1



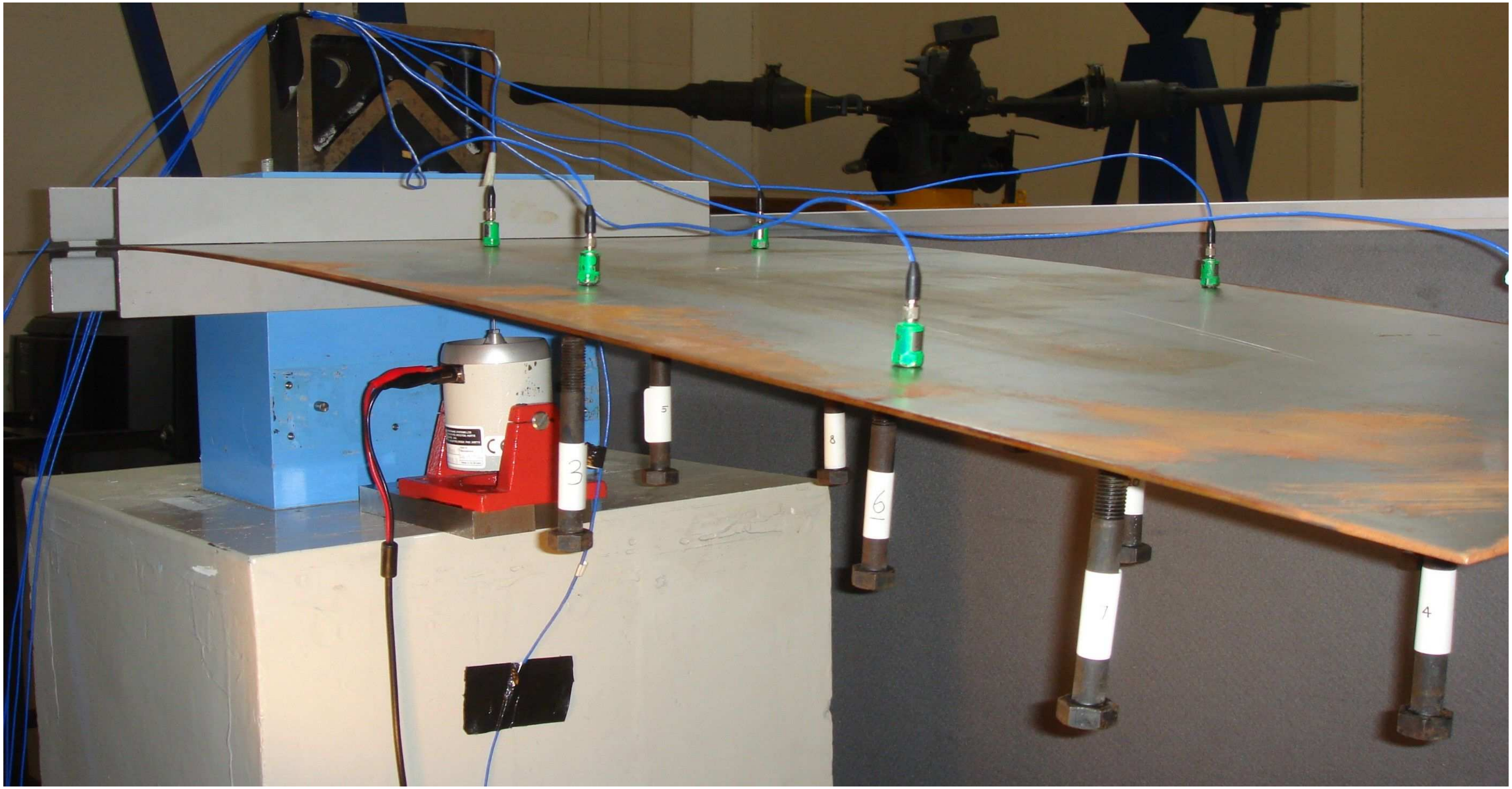
The density of eigenvalues of the plate with randomly inhomogeneous material properties.

Density of random plate - 2



The density of eigenvalues of the plate with randomly attached oscillators.

A cantilever plate: front view

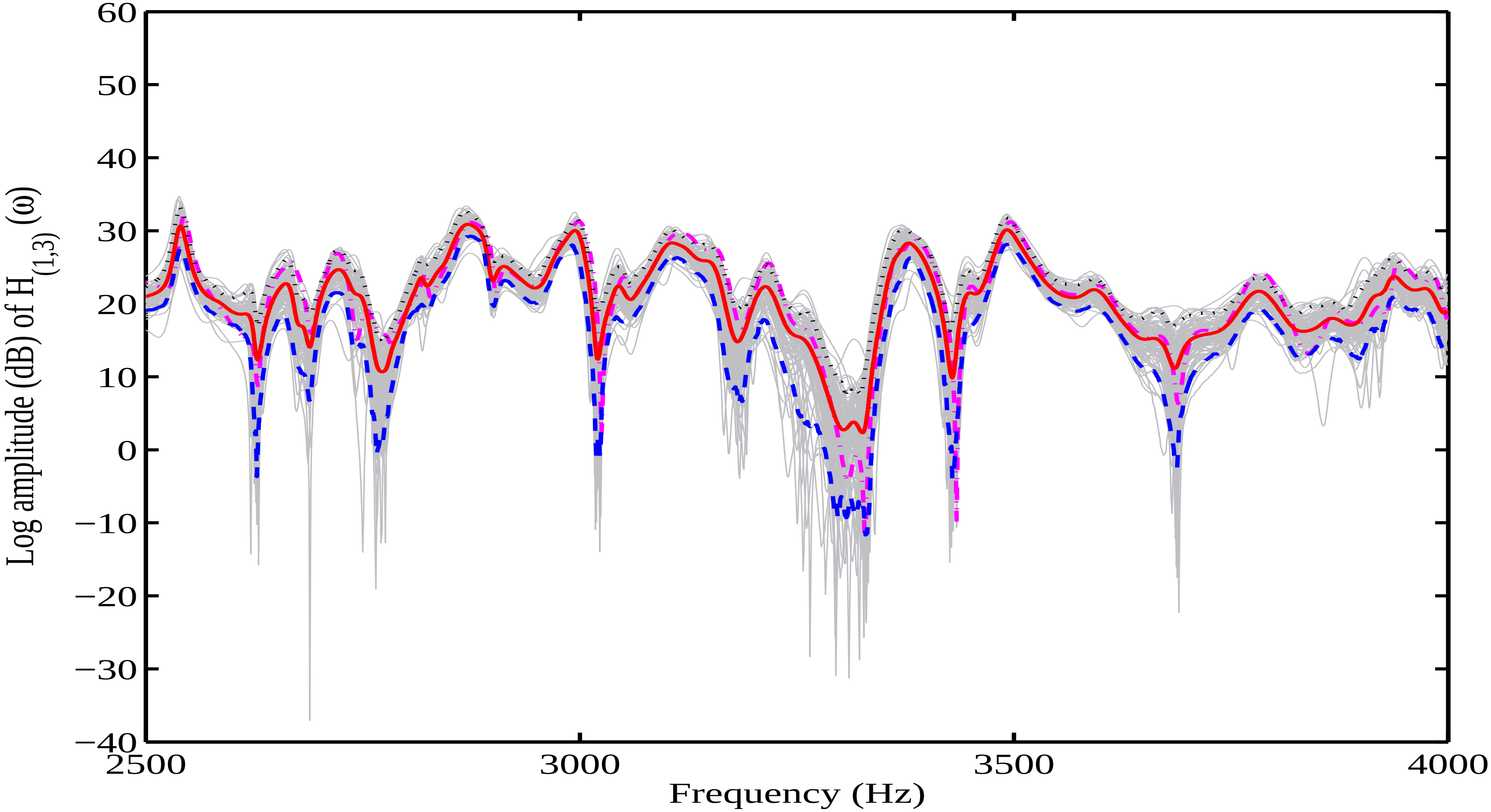


The test rig for the cantilever plate; front view.

A cantilever plate: side view

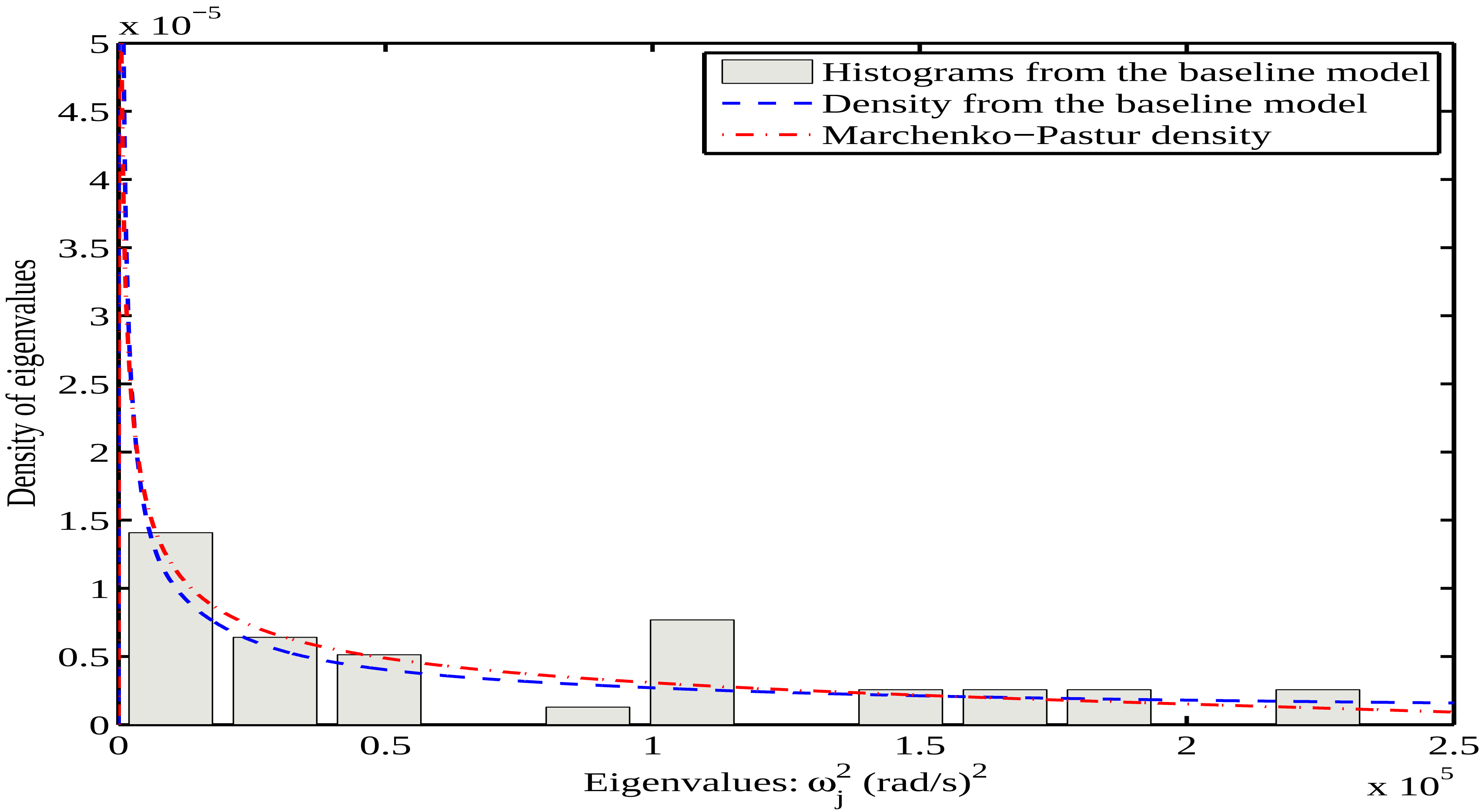
The test rig for the cantilever plate; side view.

Random FRFs



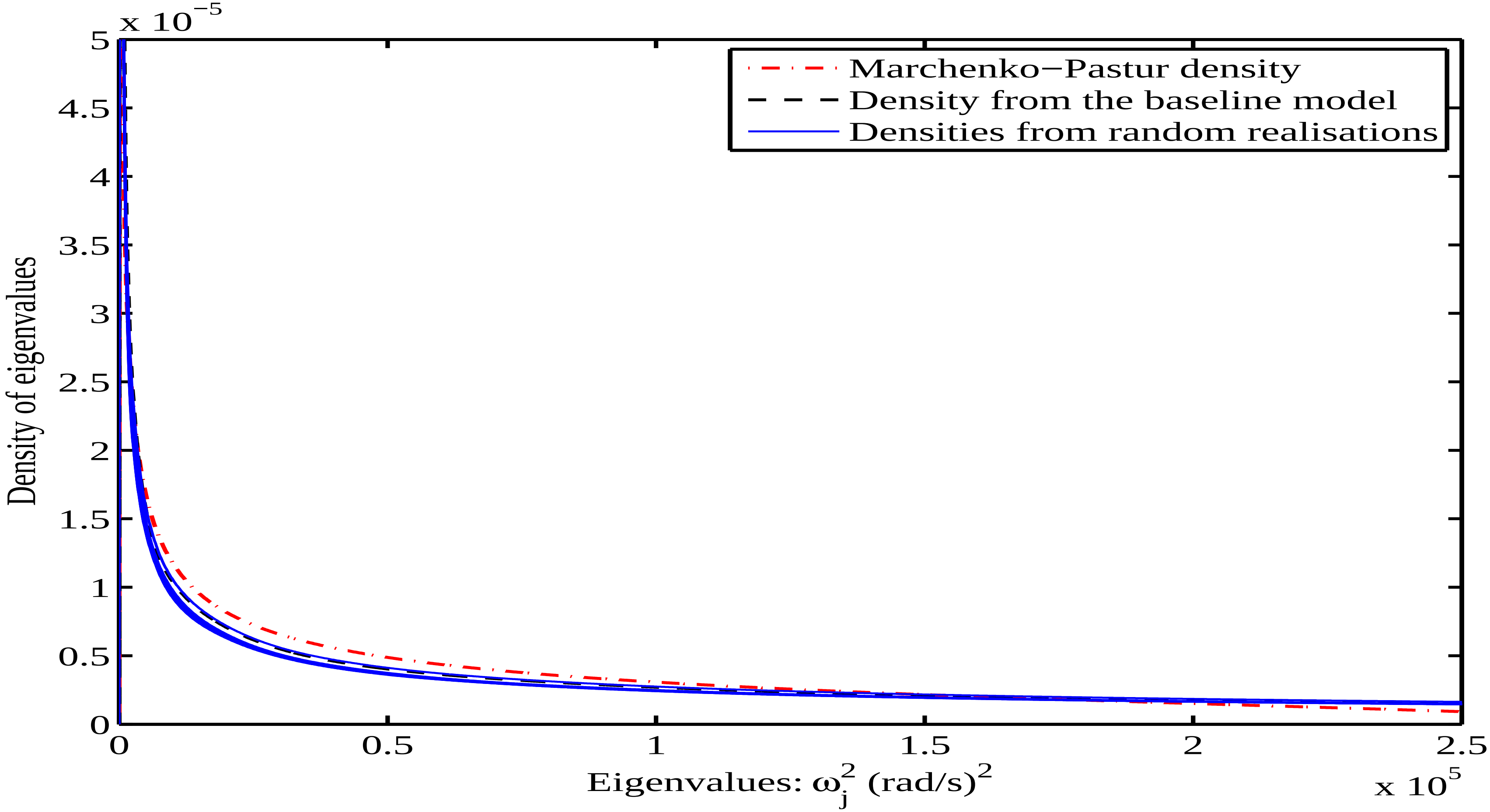
One hundred measured FRF amplitudes with the mean, 95% and 5% probability lines.

Eigenvalue density: baseline model



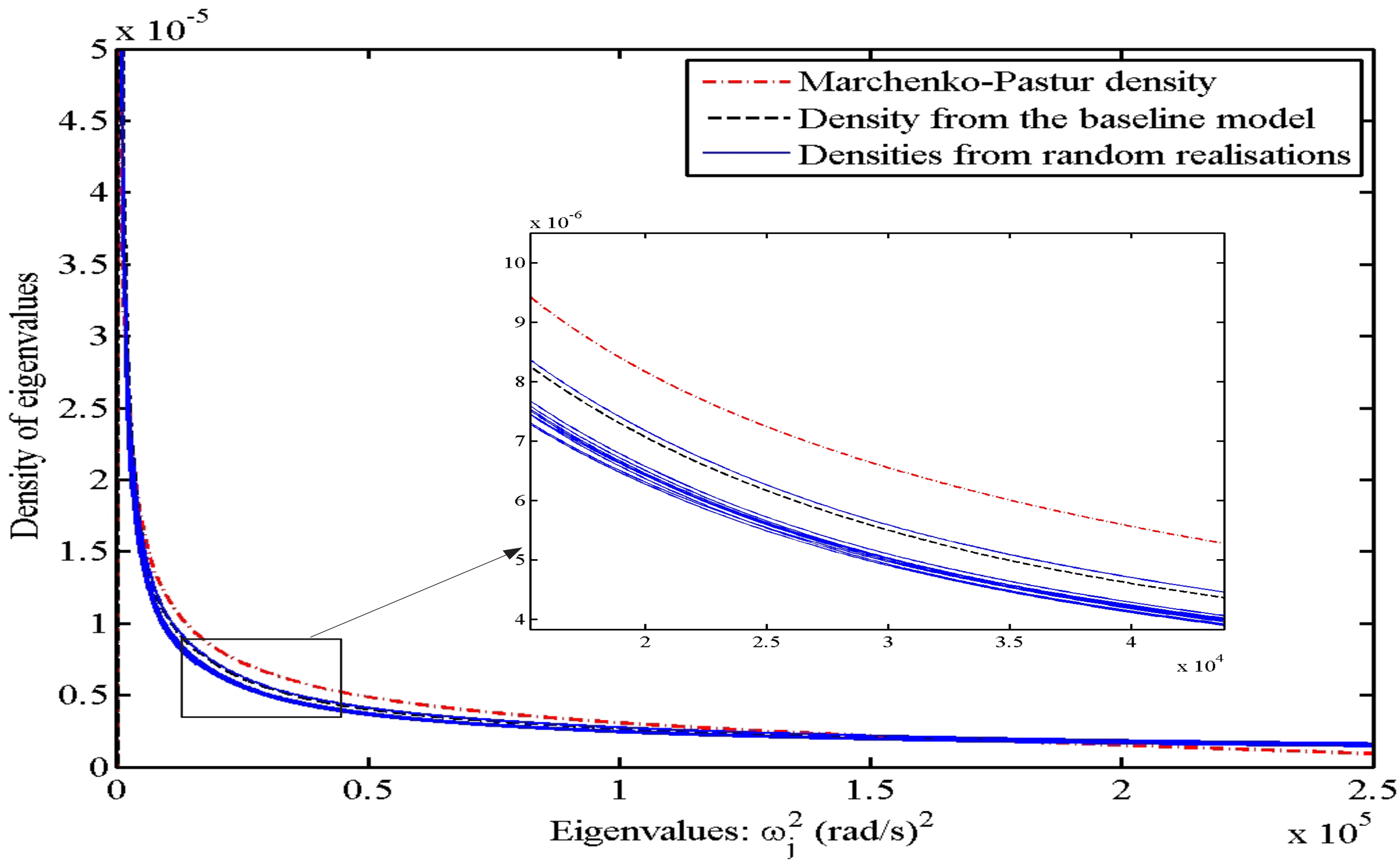
The density of first 40 experimentally measured eigenvalues of the baseline plate.

Eigenvalue density: random system



The density of first 40 experimentally measured eigenvalues of the plate with 10 randomly attached oscillators.

Eigenvalue density: strong convergence



The density of first 40 experimentally measured eigenvalues of the plate with 10 randomly attached oscillators.

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Conclusions

- This talk concentrated on Uncertainty Propagation (UP) in linear structural dynamic problems.
- A general UP approach based on Wishart random matrix is discussed and a suitable simple Wishart random matrix model has been identified.
- Based on analytical, numerical and experimental studies, it was shown that the density of eigenvalues has an extremely strong convergence property [$O\left(n^{-2}\right)$].
- It was shown that the Marčenko-Pastur density fits the experimental and well as numerically obtained density very well.