Characterization of Uncertainty in Damping Modeling

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Outline

- Motivation
- Nonviscous damping models
- Sample space of damping models
- Matrix variate distributions
- Probability density function of the damping matrix
- Numerical example
- Conclusions

Uncertainty in damping modeling

There are two broad types of uncertainties:

- Aleatoric uncertainty: inherent variability in the system *parameters* - irreducible
- epistemic uncertainty or model uncertainty: the lack of knowledge/information or errors - reducible
- Uncertainty in damping mainly belongs to the second type
- This talk addresses such model form uncertainty in damping

Nonviscous damping

- Any model which makes the energy dissipation functional non-negative is a possible candidate for a valid damping model.
- To avoid any 'model biases', we use possibly the most general linear damping model expressed by:

$$\mathbf{F}_d(t) = \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau \tag{1}$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized coordinates with $t \in \mathbb{R}^+$ denotes time, $\boldsymbol{\mathcal{G}}(\hat{t}) \in \mathbb{R}^{n\times n}$ is the kernel function matrix.

Sample space of damping models

- For a physically realistic model of damping we must have $\Re\{G(\mathrm{i}\omega)\} \geq 0$, where $G(\mathrm{i}\omega)$ is the Fourier transform of $\boldsymbol{\mathcal{G}}(\hat{t})$.
- Clearly many functions will satisfy this requirement
- To obtain nominally identical sample space of functions, we consider that the first-moment of the damping functions $\theta_j = \int_0^\infty \hat{t}\, g_{(j)}(\hat{t})\, \mathrm{d}\hat{t}$ are the same.
- Such first-order equivalent damping models are considered to form the complete sample space. Selection of any one function [such as the viscous model] can then be considered as a random event.

Damping Models - 1

MODEL 1: Exponential model

$$g_{(1)}(\hat{t}) = \mu_1 \exp[-\mu_1 \hat{t}]; \quad G_{(1)}(s) = \frac{\mu_1}{s + \mu_1} \tag{2}$$

MODEL 2: Gaussian model

$$g_{(2)}(\hat{t}) = 2\sqrt{\frac{\mu_2}{\pi}} \exp[-\mu_2 \hat{t}^2]; \quad G_{(2)}(s) = e^{s^2/4\mu_2} \left[1 - \operatorname{erf}\left(\frac{s}{2\sqrt{\mu_2}}\right)\right] \tag{3}$$

Damping Models - 2

MODEL 3: Step function model

$$g_{(3)}(\hat{t}) = \begin{cases} 1/\mu_3 & (0 < \hat{t} < \mu_3) \\ 0 & (\hat{t} > \mu_3) \end{cases} \quad ; \quad G_{(3)}(s) = \frac{1 - e^{-s\mu_3}}{s\mu_3} \tag{4}$$

MODEL 4: Cosine model

$$g_{(4)}(\hat{t}) = \begin{cases} \frac{1}{\mu_4}\left[1 + \cos\left(\frac{\pi\hat{t}}{\mu_4}\right)\right] & (0 < \hat{t} < \mu_4) \\ 0 & (\hat{t} > \mu_4) \end{cases} \quad ; \quad G_{(4)}(s) = \frac{1}{s\mu_4}\frac{1 + 2(s\mu_4/\pi)^2 - e^{-s\mu_4}}{1 + 2(s\mu_4/\pi)^2} \tag{5}$$

MODELS 5-8: Multiple exponential model

$$g_{(5,\cdots 8)}(\hat{t}) = \sum_{j=1}^{m} \widetilde{\mu}_j \exp[-\widetilde{\mu}_j \hat{t}]; \quad G_{(5,\cdots 8)}(s) = \sum_{j=1}^{m} \frac{\widetilde{\mu}_j}{s + \widetilde{\mu}_j}; \quad m = 2, 4, 8, 16 \tag{6}$$

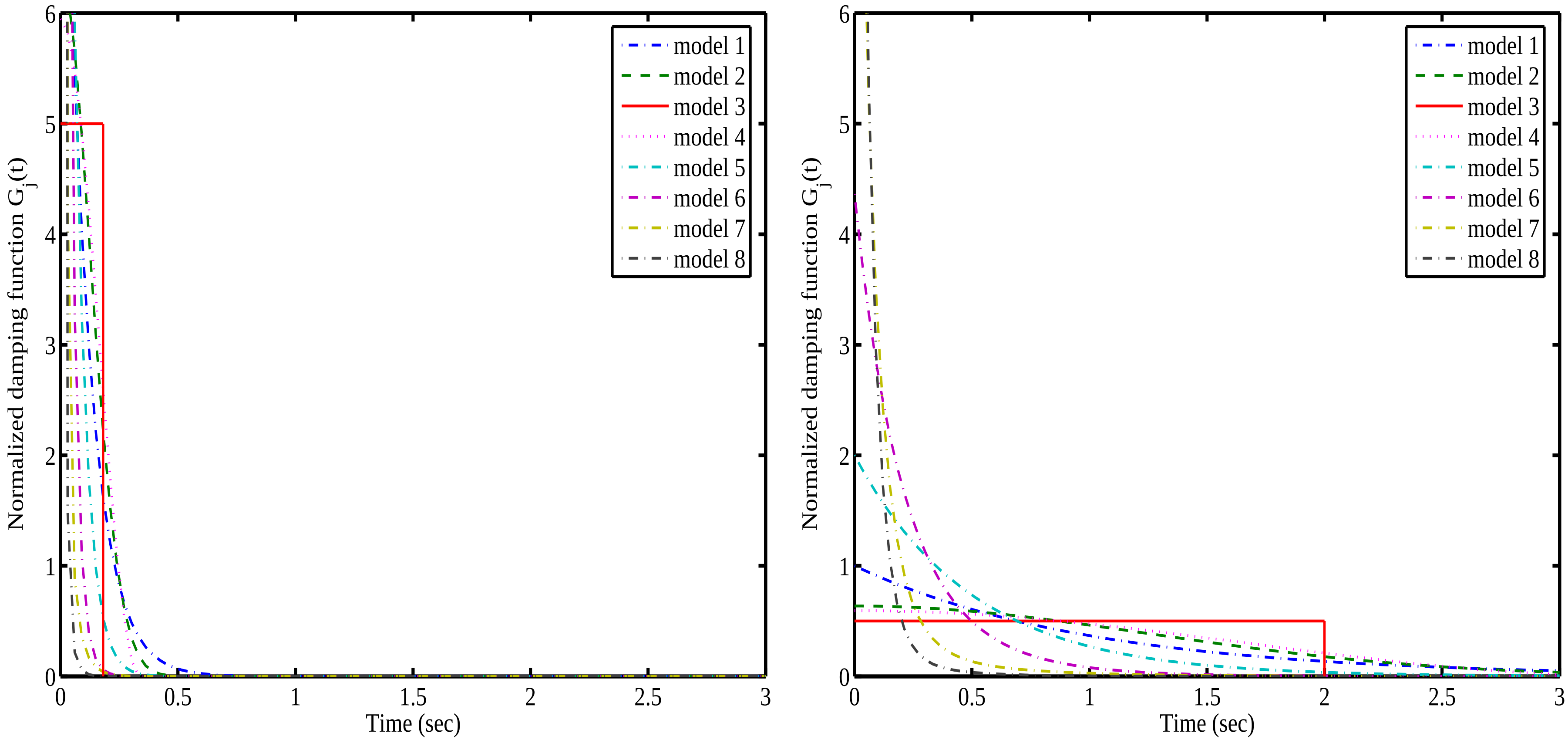
Model parameters - 1

- For the eight models we have considered, the first-order equivalence yields:

$$\theta = \frac{1}{\mu_1} = \frac{1}{\sqrt{\pi \mu_2}} = \frac{\mu_3}{2} = \frac{(\pi^2 - 4)\,\mu_4}{2\pi^2} = \sum_{j=1}^{m} \frac{1}{\widetilde{\mu}_j} \qquad (7)$$

- The characteristic time constant θ gives a convenient measure of 'width': if it is close to zero the damping behaviour will be near-viscous, and vice versa.

Model parameters - 2



Eight models for the damping kernel functions for $\theta_j = 0.1$ and $\theta_j = 1.0$.

Damping uncertainty

- There is a basic difference in emphasis between this study and other related studies on uncertainty in damping reported in the literature.
- Majority assume from the outset that the system is viscously damped and then characterize uncertainty in the dynamic response due to uncertainty in the viscous damping parameters.
- Here we investigate how much one can achieve by considering a random matrix model for the viscous damping matrix when the actual system is non-viscously damped, as one must expect to be the case in general.

Random damping matrix

- The main difficulty in the quantification of model-form uncertainty is that it is not possible to consider any uncertain parameters.
- This is because there is no ‘fixed function’ and consequently no parameters to fit a probability density function.
- In this situation the non-parametric approach provided by the random matrix theory may be useful.
- We explore whether a Wishart random viscous damping matrix can represent the damping model-form uncertainty.

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\boldsymbol{\Sigma}|^{-p/2} |\boldsymbol{\Psi}|^{-n/2} \\ \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (8)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n \left(\frac{1}{2}p \right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \operatorname{etr} \left\{ -\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{S} \right\} \tag{9}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \left|\mathbf{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\} \tag{10}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \tag{11}$$

pdf of the damping matrix

The distribution of the random damping matrix $\mathbf{C}$ should be such that it is

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$
- Recall that the mass and stiffness matrices are assumed to be deterministic

Maximum Entropy distribution

Suppose that the mean value of $\mathbf{C}$ is given by $\overline{\mathbf{C}}$. The matrix variate density function of $\mathbf{C} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{C}}(\mathbf{C}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{C}}(\mathbf{C})$:

$$\int_{\mathbf{C}>0} p_{\mathbf{C}}(\mathbf{C})\, d\mathbf{C} = 1 \quad \text{(normalization)} \tag{12}$$

and
$$\int_{\mathbf{C}>0} \mathbf{C}\, p_{\mathbf{C}}(\mathbf{C})\, d\mathbf{C} = \overline{\mathbf{C}} \quad \text{(the mean matrix)} \tag{13}$$

Wishart Random Damping Matrix

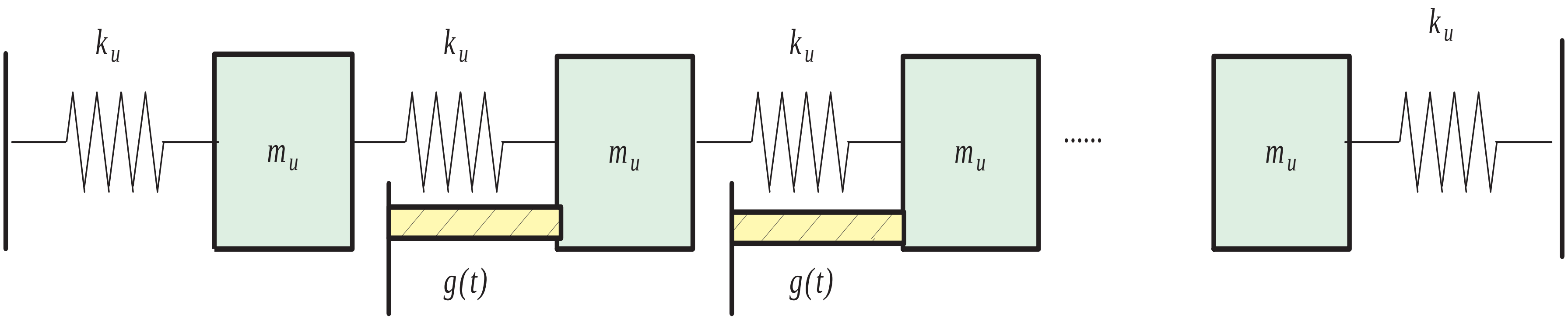
Solving the associated optimization problem using the matrix calculus of variation we have

$$p_{\mathbf{C}}(\mathbf{C}) = r^{nr} \{\Gamma_n(r)\}^{-1} \left|\overline{\mathbf{C}}\right|^{-r} \mathrm{etr}\left\{-r\overline{\mathbf{C}}^{-1}\mathbf{C}\right\} \tag{14}$$

where $r = \frac{1}{2}(n+1)$. Comparing, it can be observed that $\mathbf{C}$ has the Wishart distribution with parameters $p = n+1$ and $\mathbf{\Sigma} = \overline{\mathbf{C}}/(n+1)$.

Theorem 1. *If only the mean of the damping matrix is available, say $\overline{\mathbf{C}}$, then the maximum-entropy pdf of $\mathbf{C}$ follows the Wishart distribution with parameters $(n+1)$ and $\overline{\mathbf{C}}/(n+1)$, that is $\mathbf{C} \sim W_n\left(n+1, \overline{\mathbf{C}}/(n+1)\right)$.*

MDOF oscillators



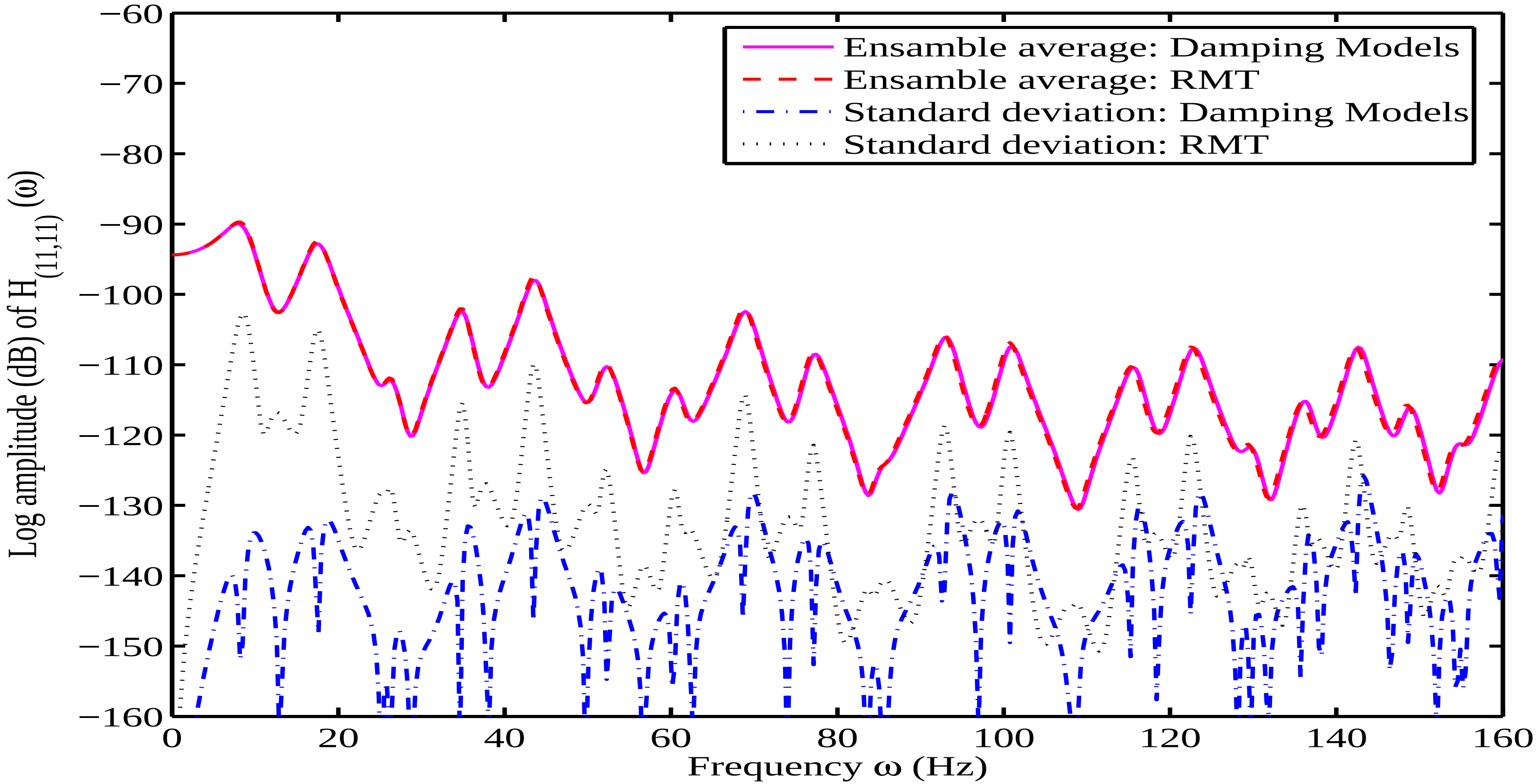
Linear array of n spring-mass oscillators, $n = 35$, $m_u = 1\,Kg$, $k_u = 4 \times 10^3\,N/m$, dampers between 6 and 27 masses with $c = 27$ Ns/m.

- We define the non dimensional measure of non viscous damping γ as:

$$\theta = \gamma T_{min} \tag{15}$$

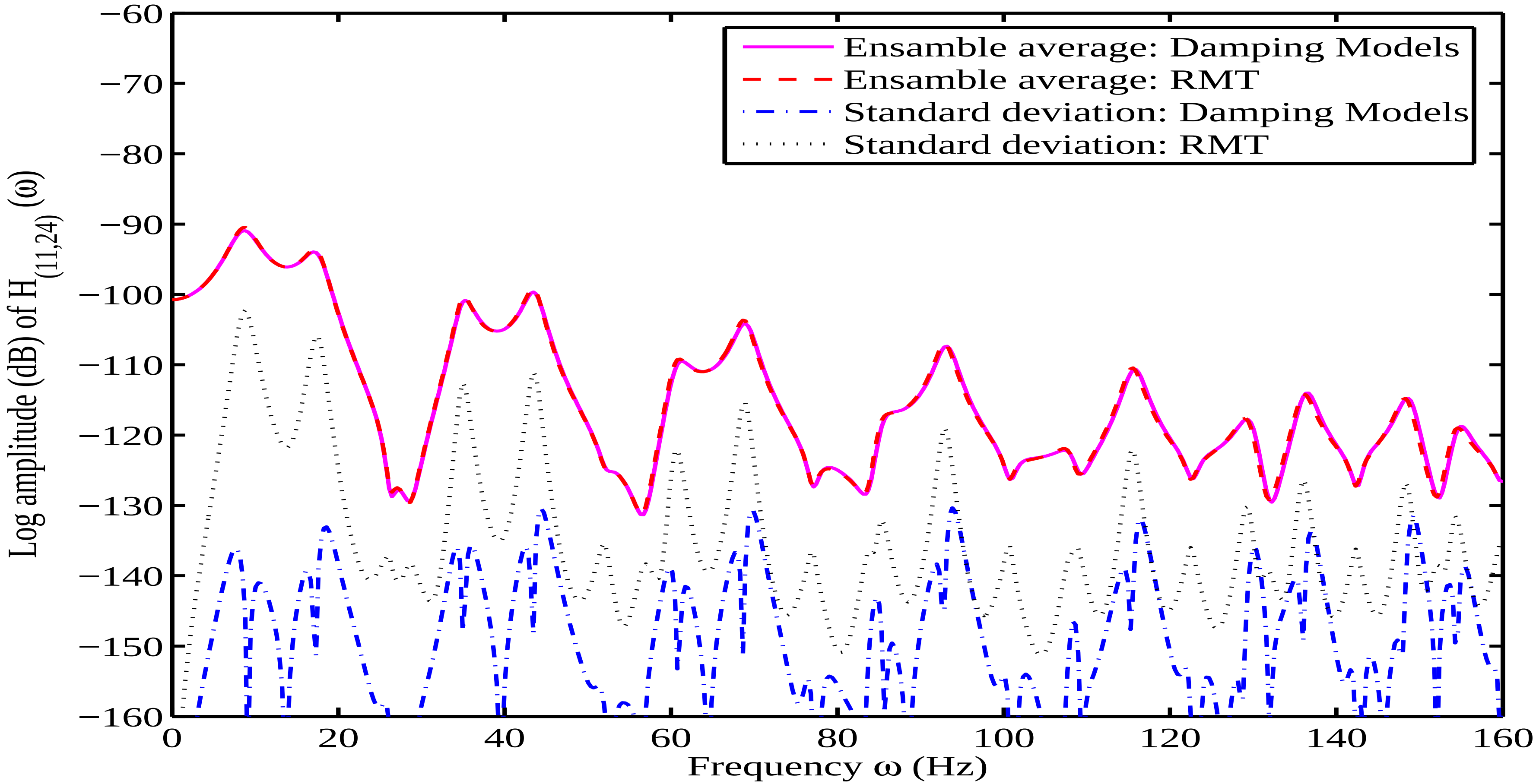
- When γ is small compared with unity the damping behaviour can be expected to be essentially viscous, but when γ is of order unity non-viscous effects should become significant.

Results for small γ



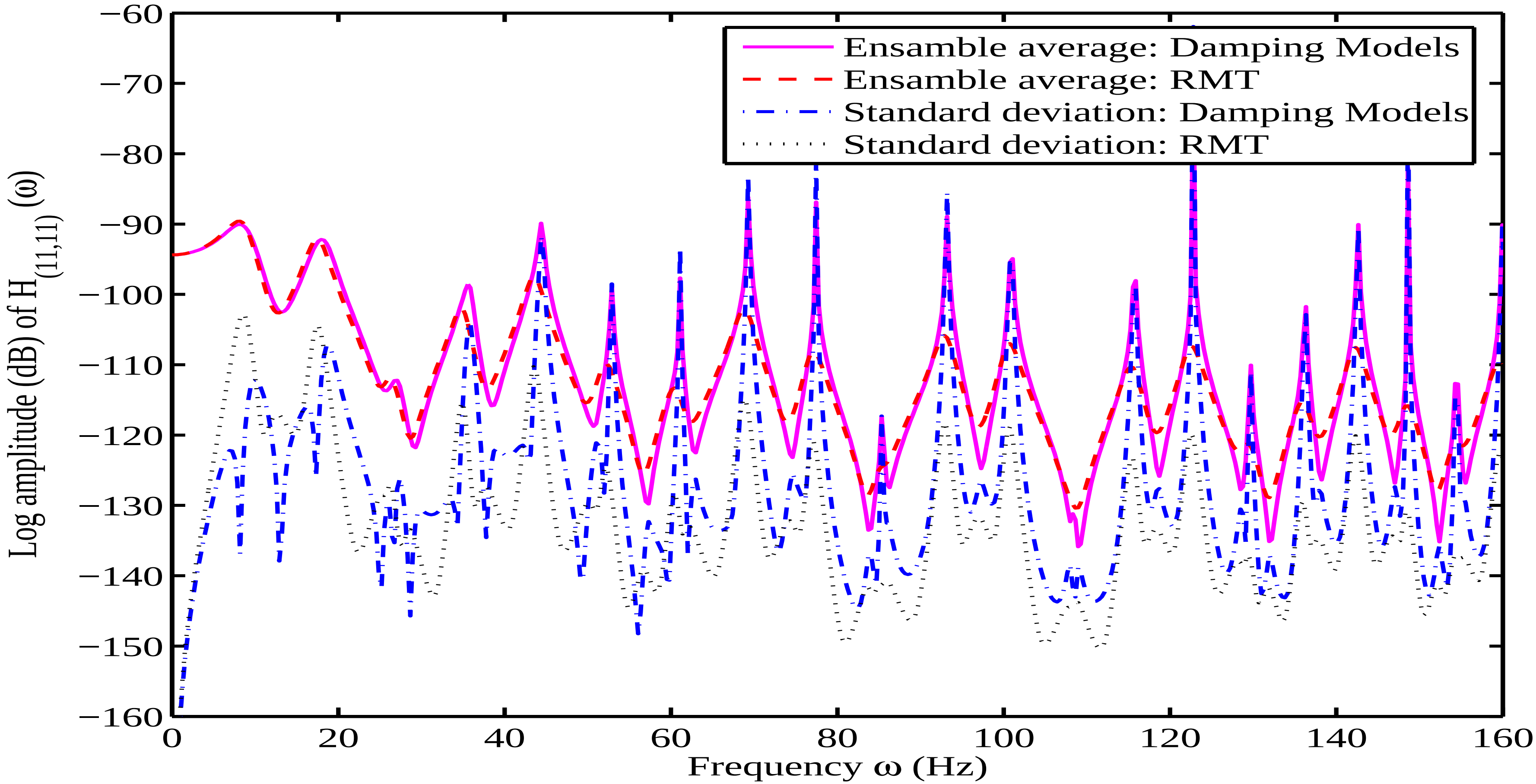
Comparison of the mean and standard deviation of the amplitude of the FRF obtained using eight damping models and proposed Wishart damping matrix; driving-point-FRF, $\gamma = 0.1$.

Results for small γ



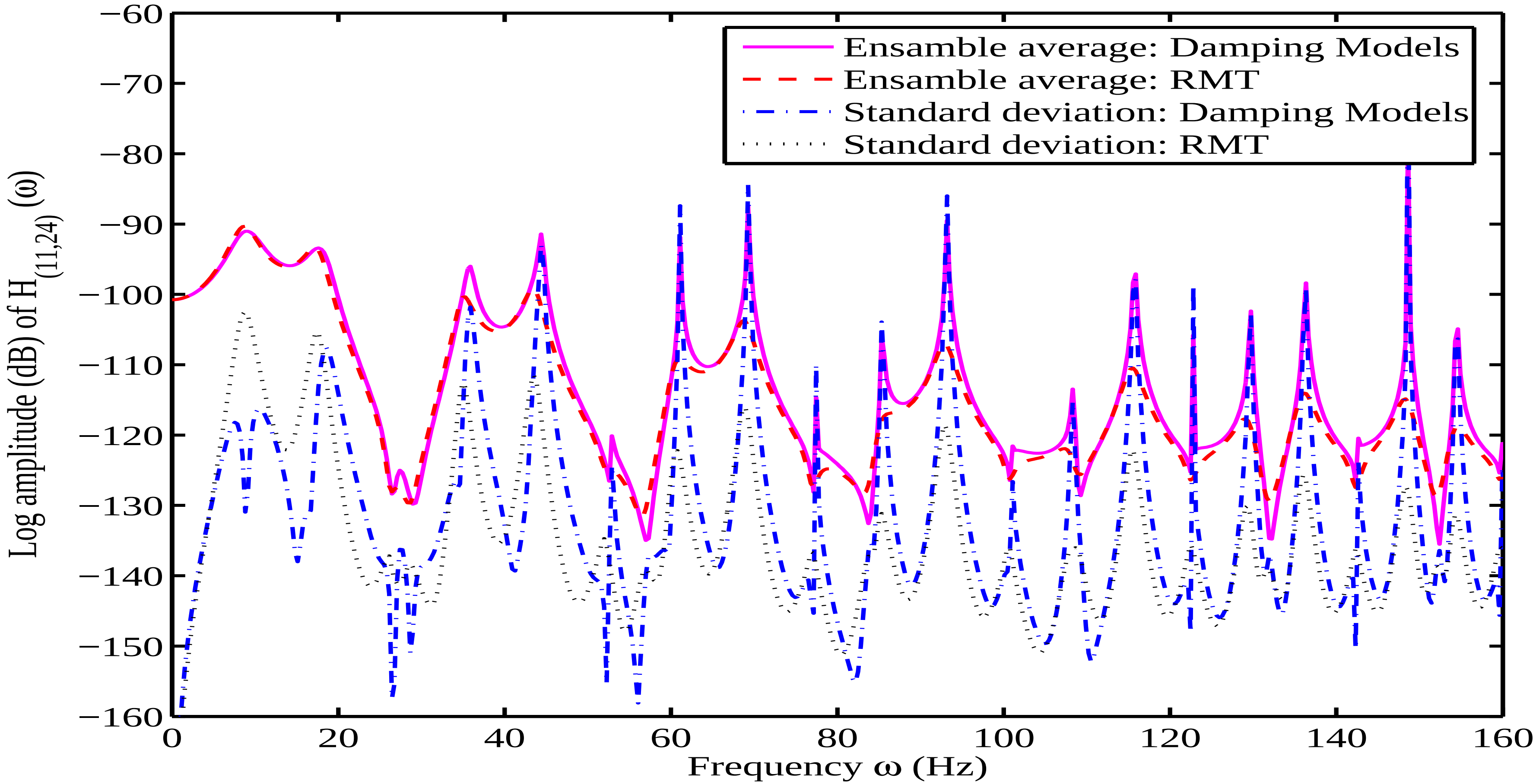
Comparison of the mean and standard deviation of the amplitude of the FRF obtained using eight damping models and proposed Wishart damping matrix; cross-FRF, $\gamma = 0.1$.

Results for large γ



Comparison of the mean and standard deviation of the amplitude of the FRF obtained using eight damping models and proposed Wishart damping matrix; driving-point-FRF, $\gamma = 1.0$.

Results for large γ



Comparison of the mean and standard deviation of the amplitude of the FRF obtained using eight damping models and proposed Wishart damping matrix; cross-FRF, $\gamma = 1.0$.

Summary of results

- For low values of γ(=0.1) the agreement is good in the high frequency range.
- For high values of γ(=1.0) the agreement is good across the frequency range.

Conclusions and outlook - 1

- Two novel approaches to quantify uncertainty arising due to the possibility of different damping models have been proposed.
- The first approach is based on an ensemble of equivalent damping functions and the second approach is based on random matrix theory.

Conclusions and outlook - 2

- In the first approach different equivalent functional forms of are derived and their parameters are selected using a new concept of first-order equivalent damping models. The collection of these different equivalent functional forms are then assumed to form the random sample space so that the selection of any one model (such as the viscous model) can be regarded as a random event in the space of the admissible functions.
- In the second approach the viscous damping matrix is considered to be a random Wishart matrix.

Outstanding issues

- Sample space consisting of eight damping functions are not enough for a reliable statistical analysis.
- More advanced random matrix model may be useful
- Numerical examples involving more complex systems are currently being investigated