Identification of Damping Using Proper Orthogonal Decomposition

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Outline

- Motivation
- Brief overview of damping identification
- Independent Component Analysis
- Numerical Validation
- Conclusions

Damping Identification 1

Unlike the inertia and stiffness forcers, in general damping cannot be obtained using ‘first principle’. Two briad approaches are:

- damping identification from modal testing and analysis
- direct damping identification from the forced response measurements in the frequency or time domain

Some References

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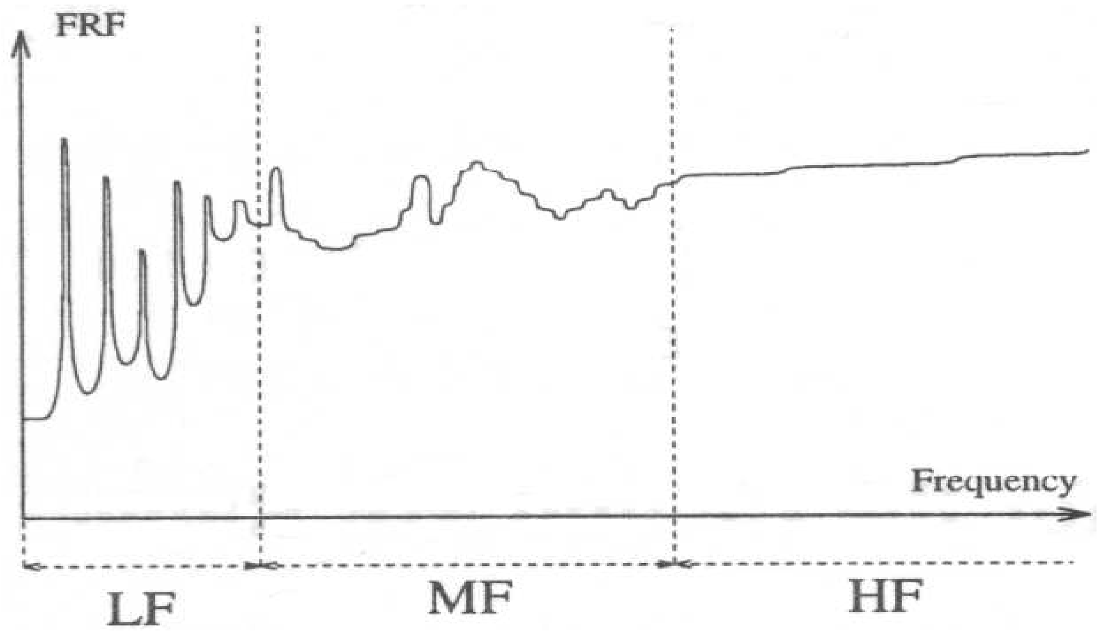
Damping Identification 2

Some shortcomings of the modal analysis based methodologies are:

- Difficult to extend in the mid-frequency range: Relies on the presence of FRF distinct peaks
- Computationally expensive and time-consuming for large systems
- Non-proportional damping leading to complex modes adds to the computational burden

Mid-Frequency Range



- Low-Frequency Range: A uniform low modal density
- High-Frequency Rage: A uniform high modal density
- Mid-Frequency Range: Intermediate band in which modal density varies greatly

Discrete Linear Systems

- The equations describing the forced vibration of a viscously damped linear discrete system with n dof:

$$\mathbf{M}_n \ddot{\mathbf{u}}_n(t) + \mathbf{C}_n \dot{\mathbf{u}}_n(t) + \mathbf{K}_n \mathbf{u}_n(t) = \mathbf{f}_n(t)$$

- $\mathbf{M}_n$ is the mass matrix, $\mathbf{C}_n$ is the damping matrix and $\mathbf{K}_n$ is the stiffness matrix
- $\mathbf{u}_n(t)$ is the displacement vector, and $\mathbf{f}_n(t)$ is the forcing vector at time t
- In the frequency domain, one has

$$\left[-\omega^2 \mathbf{M}_n + \mathrm{i}\omega \mathbf{C}_n + \mathbf{K}_n\right] \mathbf{U}_n(\omega) = \mathbf{F}_n(\omega)$$

Damping Matrix Identification

- Applying Kronecker Algebra and taking the vec operator to the frequency domain representation

$$\left(\mathbf{U}_n(\omega_i)^T \otimes \mathrm{i}\omega_i \mathbf{I}_n\right) \mathrm{vec}\mathbf{C}_n = \mathbf{F}_n(\omega_i) + \omega_i^2 \mathbf{M}_n \mathbf{U}_n(\omega_i) - \mathbf{K}_n \mathbf{U}_n(\omega_i),$$

- For many frequencies, we have

$$\begin{bmatrix} \mathbf{U}_n(\omega_1)^T \otimes \mathrm{i}\omega_1 \mathbf{I}_n \\ \mathbf{U}_n(\omega_2)^T \otimes \mathrm{i}\omega_2 \mathbf{I}_n \\ \vdots \\ \mathbf{U}_n(\omega_J)^T \otimes \mathrm{i}\omega_J \mathbf{I}_n \end{bmatrix} \mathrm{vec}\mathbf{C}_n = \left\{ \begin{array}{c} \mathbf{F}_n(\omega_1) + \omega_1^2 \mathbf{M}_n \mathbf{U}_n(\omega_1) - \mathbf{K}_n \mathbf{U}_n(\omega_1) \\ \mathbf{F}_n(\omega_2) + \omega_2^2 \mathbf{M}_n \mathbf{U}_n(\omega_2) - \mathbf{K}_n \mathbf{U}_n(\omega_2) \\ \vdots \\ \mathbf{F}_n(\omega_J) + \omega_J^2 \mathbf{M}_n \mathbf{U}_n(\omega_J) - \mathbf{K}_n \mathbf{U}_n(\omega_J) \end{array} \right\}.$$

- The above equation can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

Least-Square Approach

- In case the system of equations being overdetermined, $\mathbf{x}$ can be solved in the least-square sense using the least-square inverse of the matrix $\mathbf{A}$, as follows

$$\widehat{\mathbf{x}} = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{y}.$$

- $\widehat{\mathbf{x}}$ is the least-square estimate of $\mathbf{x}$ and $\left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T$ is the Moore-Penrose inverse of $\mathbf{A}$

Physics-Based Tikhonov Regularisation

- In order to satisfy symmetry, for instance, in the damping matrix $\mathbf{C}_m$, we need to have

$$\mathbf{C}_n = \mathbf{C}_n^T$$

- The symmetry condition in the mass matrix gives rise to the constraint equation:

$$\mathbf{L}_C \mathbf{x} = \mathbf{0}_{n^2}$$

- $\mathbf{0}_{m^2}$ is the zero vector of order m^2 and the subscript in $\mathbf{L}_C$ indicates that the constraint is on the damping matrix

Tikhonov Regularisation

- Applying Tikhonov Regularisation to estimate $\mathbf{x}$, we obtain the following solution

$$\widehat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A} + \lambda_C^2 \mathbf{L}_C^T \mathbf{L}_C\right)^{-1} \left(\mathbf{A}^T \mathbf{y}\right).$$

- The above solution depends on the values chosen for the regularisation parameter λ_C
- If λ_C is very large, the constraint enforcing the symmetry condition predominates in the solution of $\mathbf{x}$
- On the other hand, if it is chosen to be small, the symmetry constraint is less satisfied and the solution depends more heavily on the observed data $\mathbf{y}$

The Need for Model Order Reduction

- In the proposed recursive least squares method, we are required to obtain the inverse of a square matrix of order n^2
- If we are trying to estimate the damping matrix of a complex system with large n, this is not feasible, even on high performance computers
- There is a need to reduce the order of the model prior to the system identification step

Proper Orthogonal Decomposition

- Entails the extraction of the dominant eigenspace of the response correlation matrix over a given frequency band
- These dominant eigenvectors span the system response optimally on the prescribed frequency range of interest
- POD is essentially the following eigenvalue problem

$$\mathbf{R}_{uu}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$$

- $\mathbf{R}_{uu}$ is the response correlation matrix given by

$$\mathbf{R}_{uu} = \left\langle \mathbf{u}_n\left(t\right)\mathbf{u}_n\left(t\right)^T \right\rangle \simeq \frac{1}{T}\sum_{t=1}^{T}\mathbf{u}_n\left(t\right)\mathbf{u}_n^T\left(t\right)$$

Spectral Decomposition of R_{uu}

- Using the spectral decomposition of $\mathbf{R}_{uu}$, one obtains

$$\mathbf{R}_{uu} = \sum_{i=1}^{n} \lambda_i \boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T$$

- The eigenvalues are arranged: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$
- The first few modes capture most of the systems energy, i.e. $\mathbf{R}_{uu}$ can be approximated by $\mathbf{R}_{uu} \approx \sum_{i=1}^{m} \lambda_i \boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T$
- m is the number of dominant POD modes, generally much smaller than n

Model Reduction using POD

- The output vector can be approximated by a linear representation involving the first m POD modes using

$$\mathbf{u}_n(t) = \sum_{i=1}^{m} a_i(t)\,\boldsymbol{\varphi}_i = \boldsymbol{\Sigma}\mathbf{a}(t)$$

- $\boldsymbol{\Sigma}$ is the matrix containing the first m dominant POD eigenvectors:

$$\boldsymbol{\Sigma} = [\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m] \in \mathbb{R}^{n\times m}$$

Model Reduction using POD

- Using $\mathbf{\Sigma}$ as a transformation matrix, our reduced order model becomes

$$\mathbf{\Sigma}^T \mathbf{M}_n \mathbf{\Sigma} \ddot{\mathbf{a}}(t) + \mathbf{\Sigma}^T \mathbf{C}_n \mathbf{\Sigma} \dot{\mathbf{a}}(t) + \mathbf{\Sigma}^T \mathbf{K}_n \mathbf{\Sigma} \mathbf{a}(t) = \mathbf{\Sigma}^T \mathbf{f}_n(t)$$

- The system of equations can now be rewritten in the reduced-order dimension as

$$\mathbf{M}_m \ddot{\mathbf{u}}_m(t) + \mathbf{C}_m \dot{\mathbf{u}}_m(t) + \mathbf{K}_m \mathbf{u}_m(t) = \mathbf{f}_m(t)$$

Model Reduction using POD

- The reduced order mass, damping, and stiffness matrices as well as the reduced order displacement and forcing vectors are

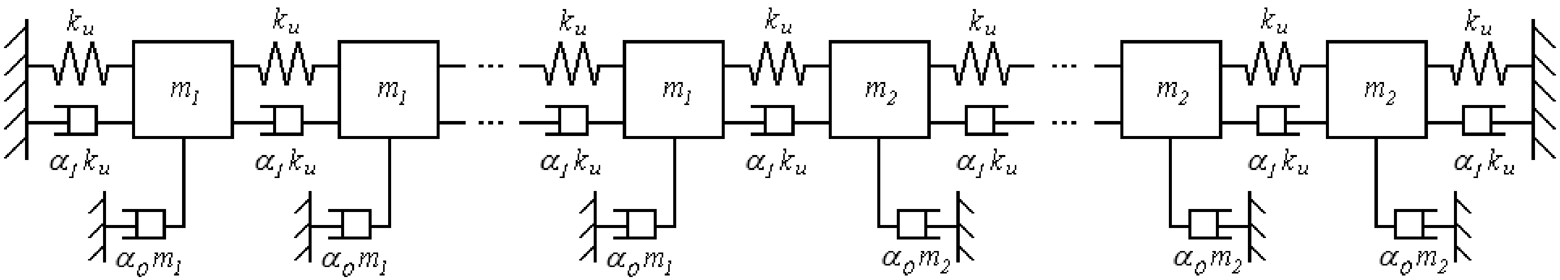
$$
\begin{aligned}
\mathbf{M}_m &= \mathbf{\Sigma}^T \mathbf{M}_n \mathbf{\Sigma} \in \mathbb{R}^{m\times m} \\
\mathbf{C}_m &= \mathbf{\Sigma}^T \mathbf{C}_n \mathbf{\Sigma} \in \mathbb{R}^{m\times m} \\
\mathbf{K}_m &= \mathbf{\Sigma}^T \mathbf{K}_n \mathbf{\Sigma} \in \mathbb{R}^{m\times m} \\
\mathbf{u}_m(t) &= \mathbf{\Sigma}^T \mathbf{u}_n(t) = \mathbf{a}(t) \\
\mathbf{f}_m(t) &= \mathbf{\Sigma}^T \mathbf{f}_f(t)
\end{aligned}
$$

Reduced-Order Model Identification

- Once either the POD transformation is applied, there will be m^2 unknowns to be identified, as opposed to n^2 for our original model, where m is much smaller than n
- The aforementioned least square estimation method can now be used to estimate the reduced order damping matrix
- Once the reduced order damping matrix is estimated, we can carry out system simulations at the lower order dimension m, and project the displacement results back into the original n-dimensional space

Numerical Validation

- A coupled linear array of mass-spring oscillators is considered to be the original system
- A lighter system is coupled with a heavier system
- The lighter system posses higher modal densities compared to the heavier system



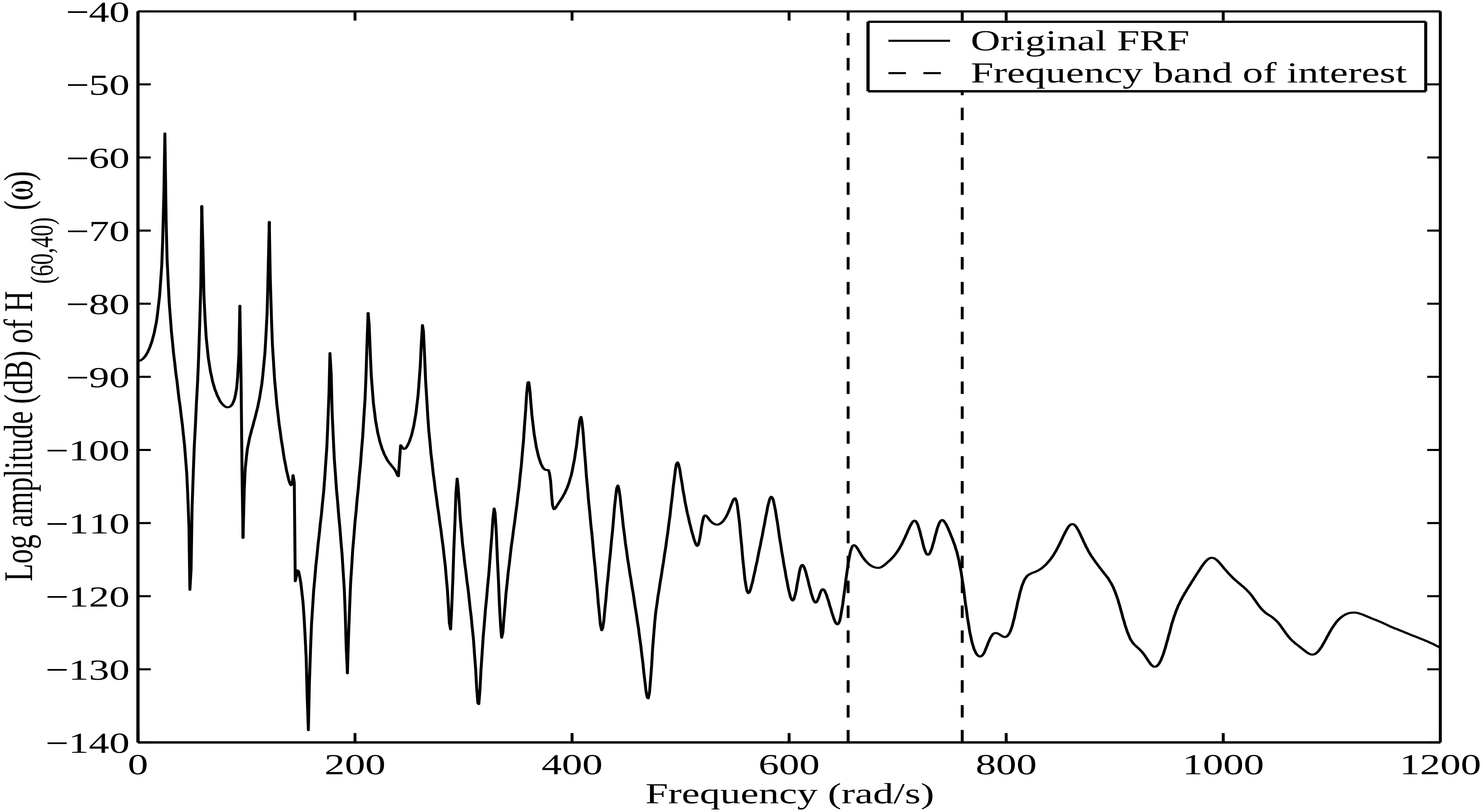
System Description

- The mass and stiffness matrices have the form

$$\mathbf{M}_n = \begin{bmatrix} m_1\mathbf{I}_{n/2} & \mathbf{0}_{n/2} \\ \mathbf{0}_{n/2} & m_2\mathbf{I}_{n/2} \end{bmatrix} \quad \mathbf{K}_n = k_u \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

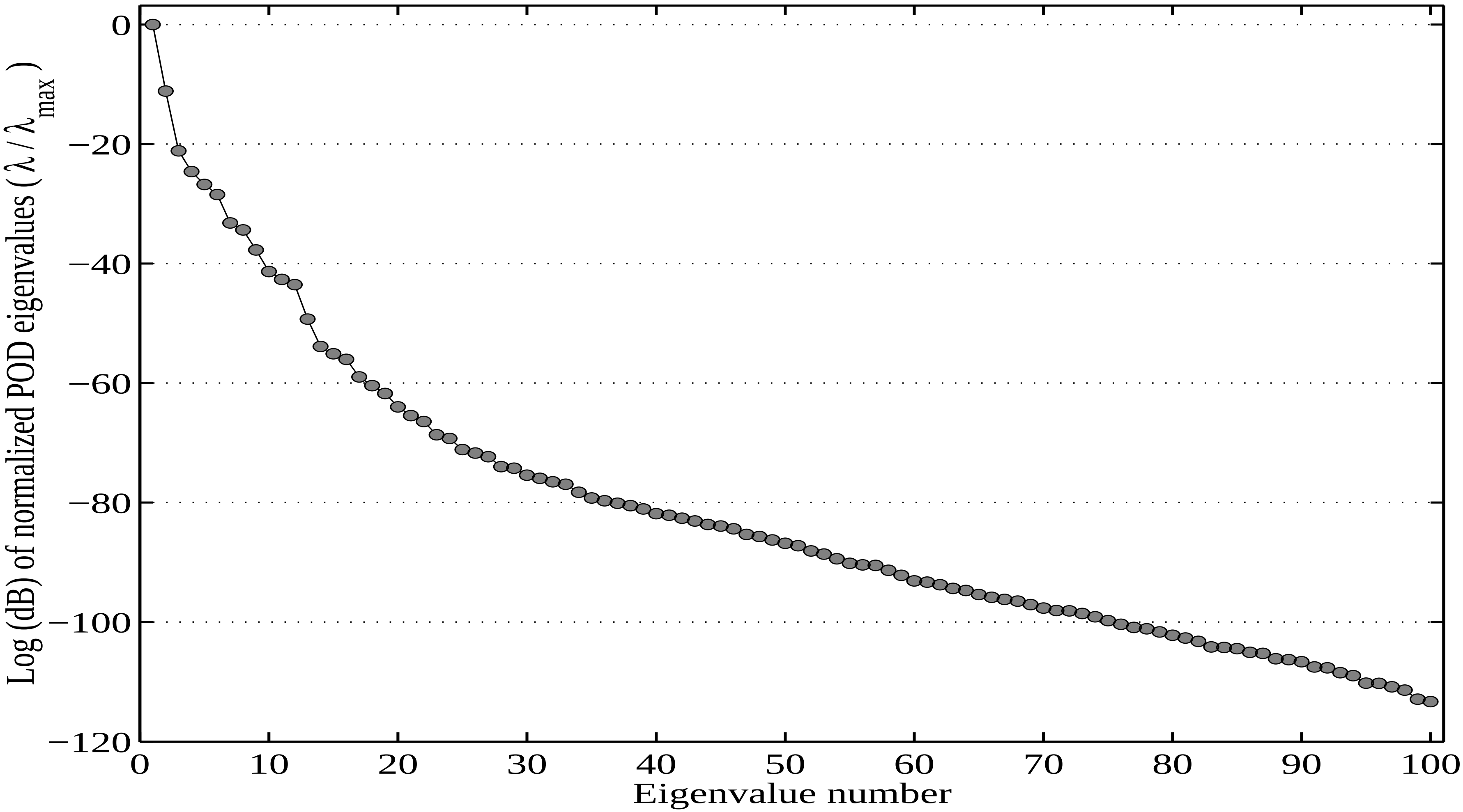
- n, m_1, and m_2 are chosen to be 100 DOFs, 0.1kg, and 1kg and $k_u = 4 \times 10^5$ N/m
- The system is assumed to have Rayleigh damping by $\mathbf{C}_n = \alpha_0 \mathbf{M}_n + \alpha_1 \mathbf{K}_n$, where $\alpha_0 = 0.5$ and $\alpha_1 = 3 \times 10^{-5}$

Typical System Response



The frequency range considered for the construction of the POD is shown

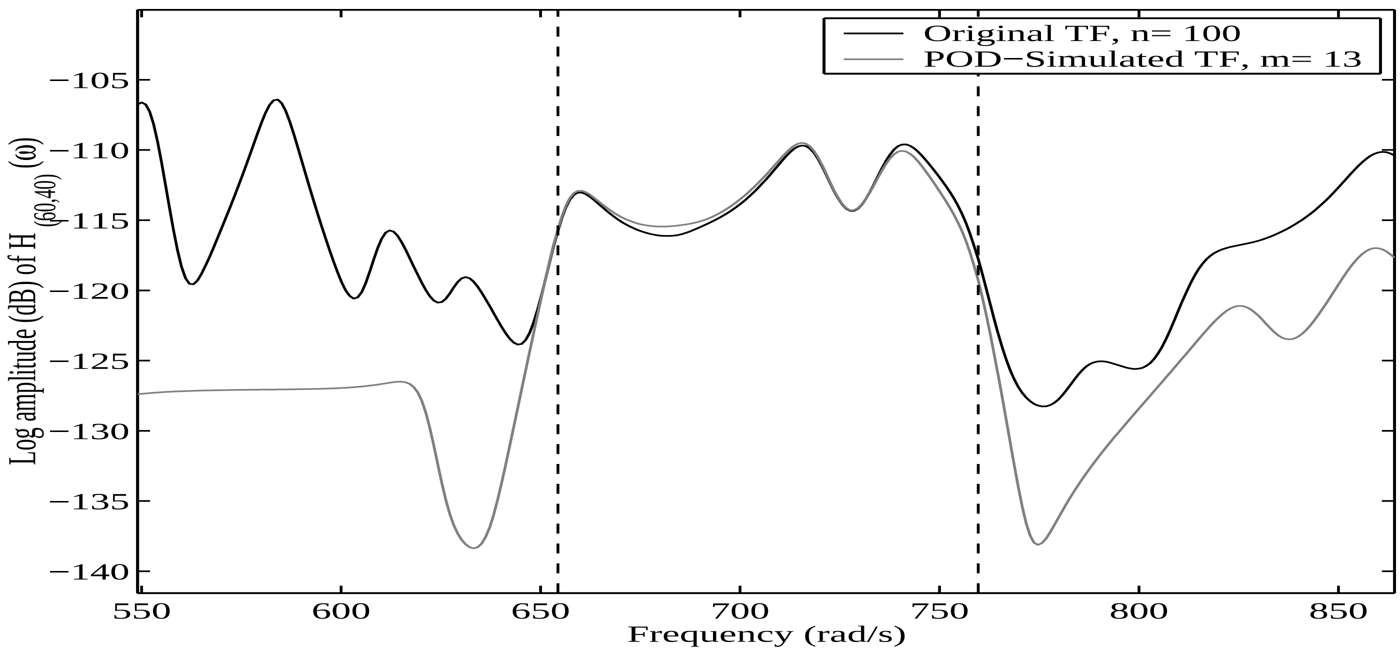
POD Eigenvalues



Normalized eigenvalues of the correlation matrix

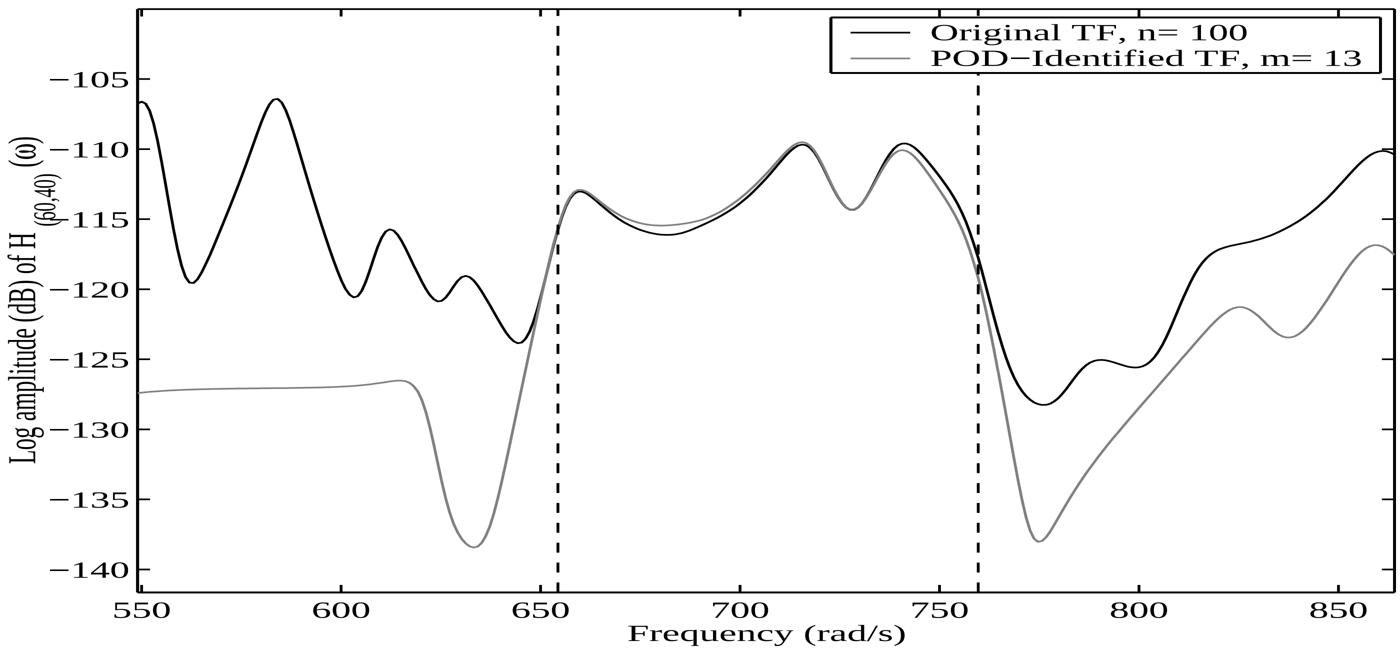
POD-Reduced Model

- A typical FRF of the POD reduced model is compared with the original FRF below
- The reduced order model FRF match reasonably well with the original FRF in the frequency band of interest



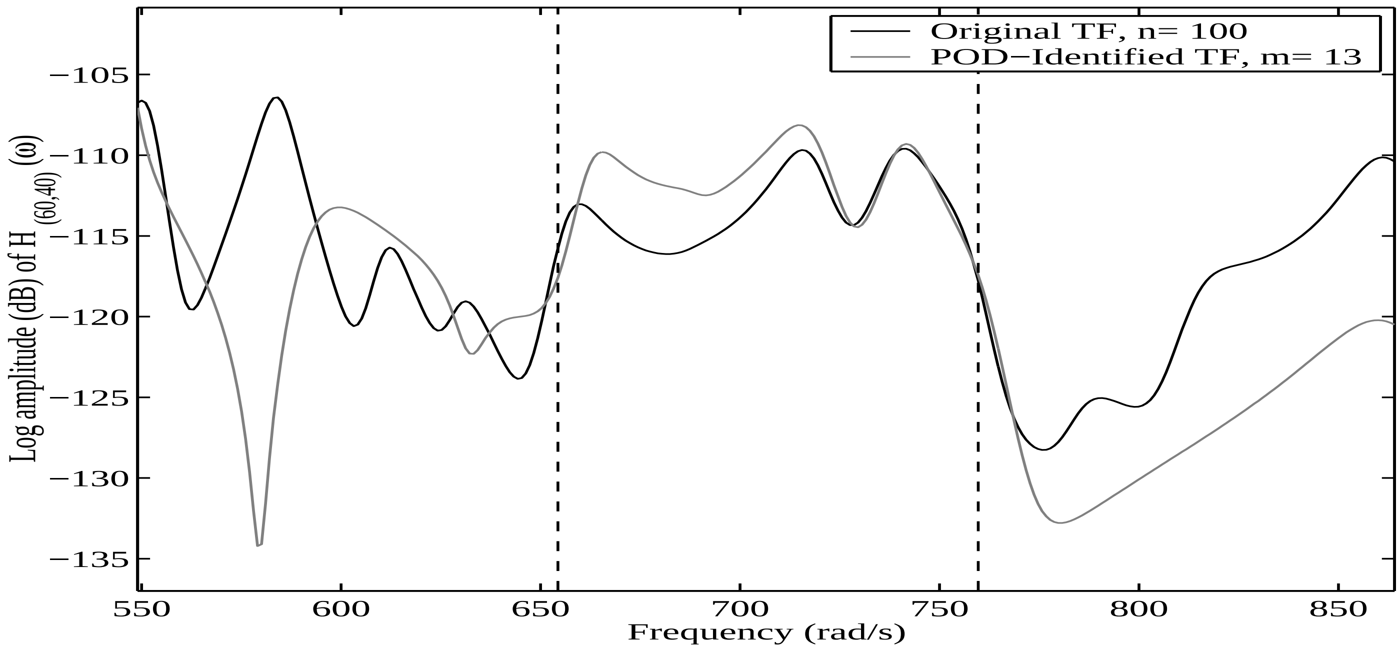
Noise-Free Identification

- In the noise-free case, we obtain the identified POD reduced order damping matrix
- The identified matrix is used to obtain a typical FRF of the system below



Effect of Noise

- The system response is contaminated with noise
- The variance of the noise is ten times smaller than that of the response
- We obtain the identified FRF shown below



Conclusion

The salient features that emerged from the current investigation are:

- POD can be successfully applied for reduced-order modelling
- Kronecker Algebra in conjunction with Tikhonov Regularisation provide an elegant theoretical formulation involving identification of the damping matrix
- Using a noise-sensitivity study, the identification method is demonstrated to be robust in a noisy environment