Uncertainty Quantification for Complex Aero-mechanical Systems

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Outline of the presentation

- Uncertainty Quantification (UQ) in structural dynamics
- Review of current approaches
- Non-parametric approach: Wishart random matrices
  - Parameter selection
  - Computational method
  - Analytical method
- Parametric approach: Gaussian emulator
  - Frequency response function emulation
  - Random field generation
- Experimental results
- Conclusions & future directions

Complex aerospace system

Complex aerospace system can have millions of degrees of freedom and significant uncertainty in its numerical (Finite Element) model

Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Structural dynamics

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of (parametric/nonparametric or both) uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{q}$
- Probabilistic solution of this problem is expected to have more credibility compared to a deterministic solution

Current UQ approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
    - aim to characterize parametric uncertainty (type 'a')
    - assumes that stochastic fields describing parametric uncertainties are known in details
    - suitable for low-frequency dynamic applications

Current UQ approaches - 2

- Nonparametric approaches : Such as the Statistical Energy Analysis (SEA) and Wishart random matrix theory:
  - aim to characterize nonparametric uncertainty (types 'b' - 'e')
  - does not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications
  - extensive works over the past decade $\longrightarrow$ general purpose commercial software is now available

UQ approaches: challenges

The main difficulties are due to:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and
- as the state-of-the art methodology stands now (such as the Stochastic Finite Element Method), only very few highly trained professionals (such as those with PhDs) can even attempt to apply the complex concepts (e.g., random fields) and methodologies to real-life problems.

Main objectives

Our work is aimed at developing methodologies [the 10-10-10 challenge] with the ambition that they should:

- not take more than 10 times the computational time required for the corresponding deterministic approach;
- result a predictive accuracy within 10% of direct Monte Carlo Simulation (MCS);
- use no more than 10 times of input data needed for the corresponding deterministic approach; and
- enable ‘normal’ engineering graduates to perform probabilistic structural dynamic analyses with a reasonable amount of training.

Wishart random matrix approach

- The probability density function of the mass ($\mathbf{M}$), damping ($\mathbf{C}$) and stiffness ($\mathbf{K}$) matrices should be such that they are symmetric and non-negative matrices.
- Wishart random matrix (a non-Gaussian matrix) is the simplest mathematical model which can satisfy these two criteria: $[\mathbf{M}, \mathbf{C}, \mathbf{K}] \equiv \mathbf{G} \sim W_n(p, \mathbf{\Sigma})$.
- Suppose we 'know' (e.g, by measurement or stochastic modeling) the mean (G_0) and the (normalized) standard deviation (σ_G) of the system matrices:

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{2}$$

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Wishart parameter selection - 1

The parameters p and $\mathbf{\Sigma}$ can be obtained based on what criteria we select. We investigate four possible choices.

1. Criteria 1: $\mathrm{E}\left[\mathbf{G}\right] = \mathbf{G}_0$ and $\sigma_G = \widetilde{\sigma}_G$ which results

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0 / p \tag{3}$$

where $\theta = (1+\beta)/\widetilde{\sigma}_G^2 - (n+1)$ and $\beta = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right)$.

2. Criteria 2: $\left\|\mathbf{G}_0 - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}}$ and $\left\|{\mathbf{G}_0}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}}$ are minimum and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0 / \alpha \tag{4}$$

where $\alpha = \sqrt{\theta(n+1+\theta)}$.

Wishart parameter selection - 2

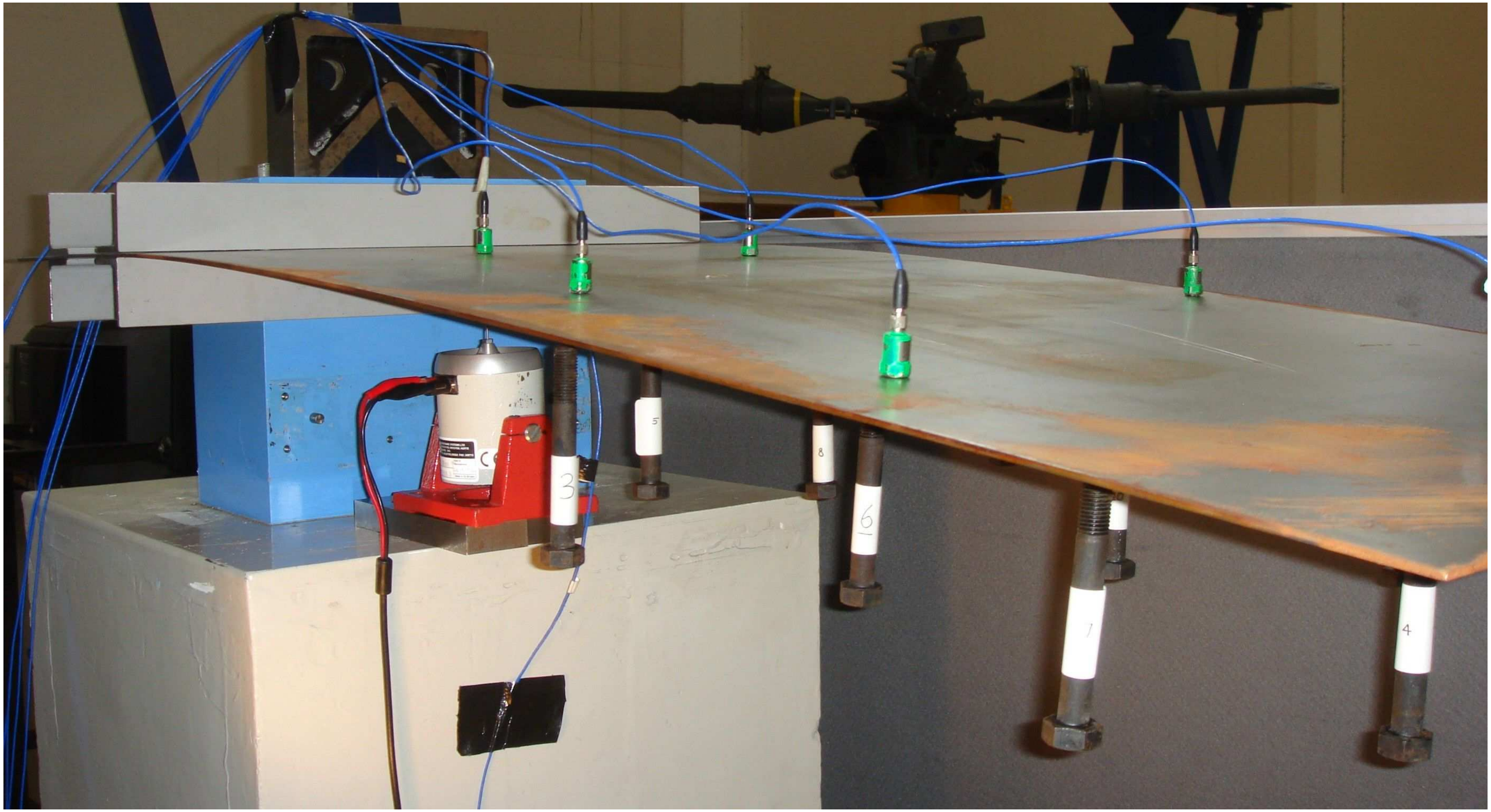
1. Criteria 3: $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \mathbf{G}_0{}^{-1}$ and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0/\theta \tag{5}$$

2. Criteria 4: The mean of the eigenvalues of the distribution is same as the ‘measured’ eigenvalues of the mean matrix and the (normalized) standard deviation is same as the measured standard deviation:

$$\mathrm{E}\left[\mathbf{M}^{-1}\right] = \mathbf{M}_0{}^{-1}, \, \mathrm{E}\left[\mathbf{K}\right] = \mathbf{K}_0, \, \sigma_M = \widetilde{\sigma}_M \quad \text{and} \quad \sigma_K = \widetilde{\sigma}_K. \tag{6}$$

A cantilever plate: front view



The test rig for the cantilever plate; front view.

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A cantilever plate: side view

The test rig for the cantilever plate; side view.

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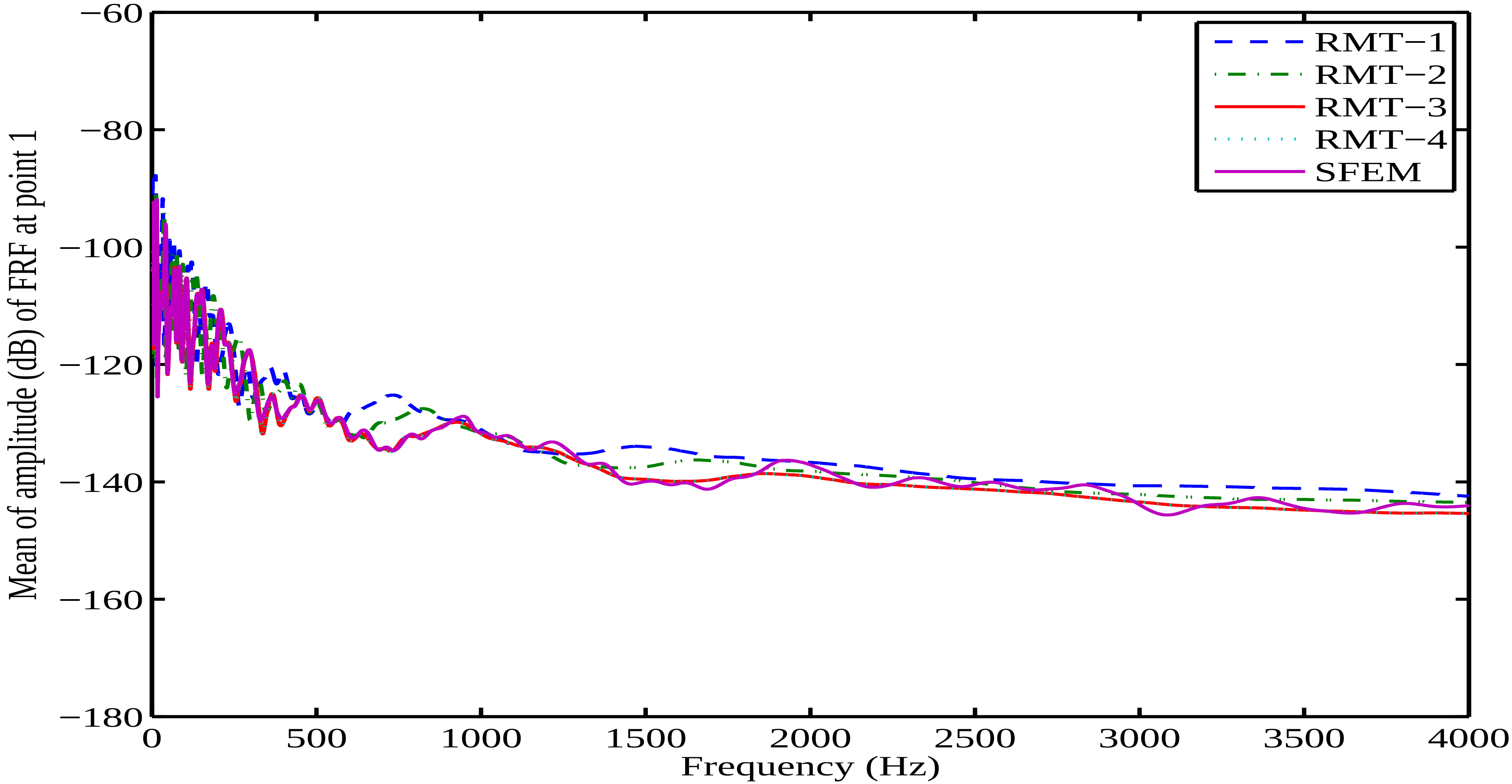
Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/∼adhikaris/uq/.

Mean of cross-FRF



Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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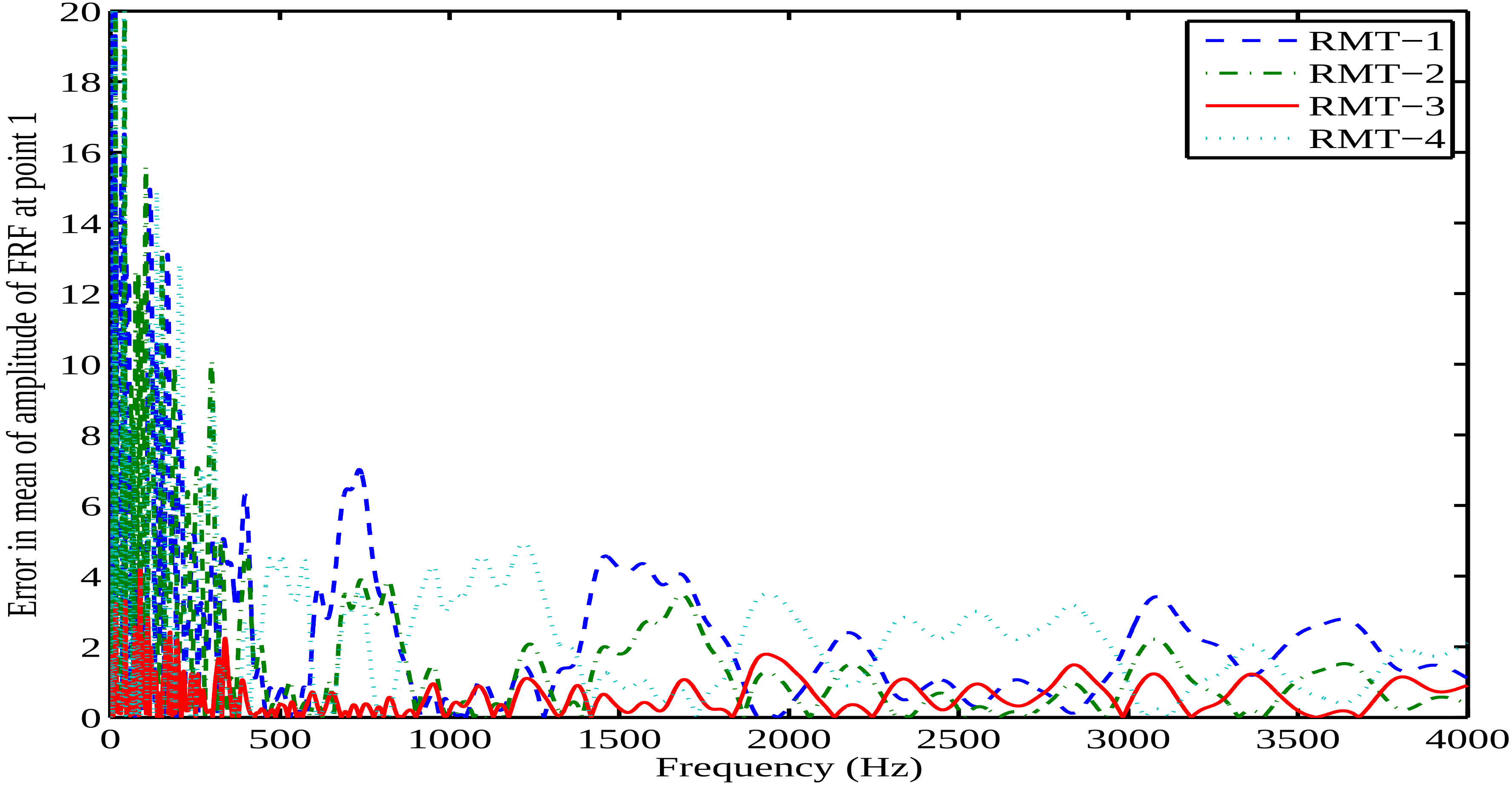
Error in the mean of cross-FRF



Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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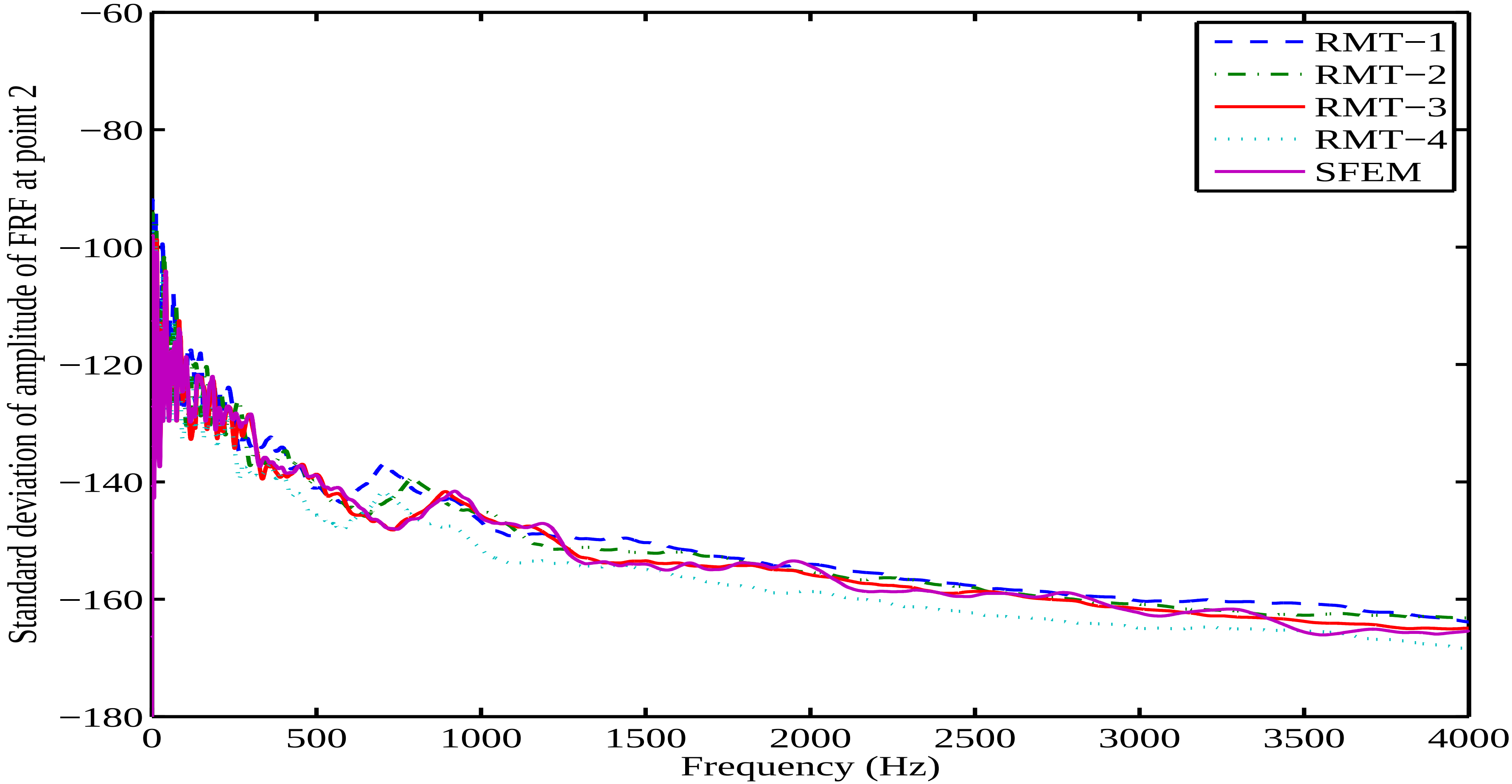
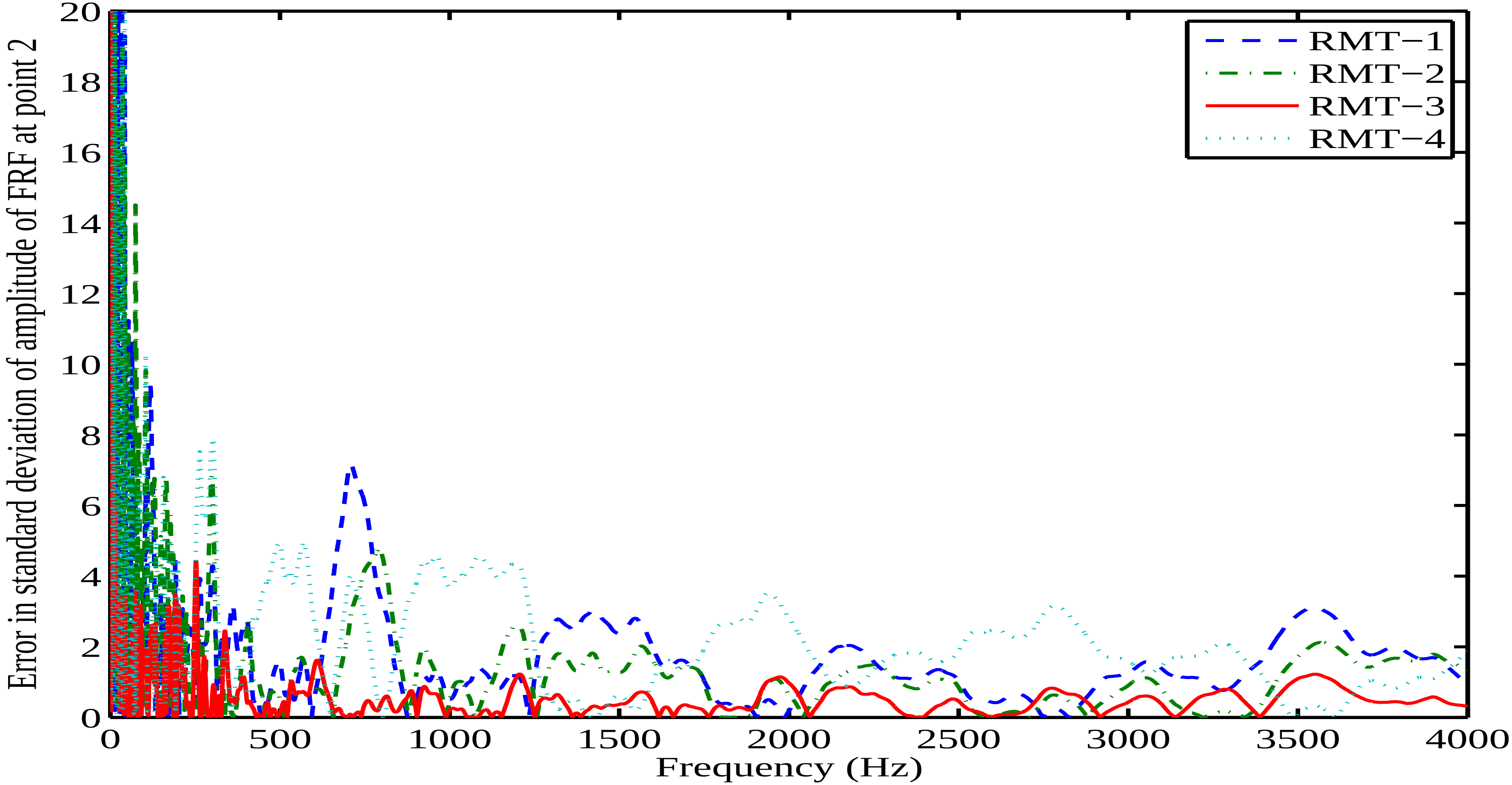
Standard deviation of driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the standard deviation of driving-point-FRF



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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Main observations

- Error in the low frequency region is higher than that in the higher frequencies
- In the high frequency region all methods are similar
- Overall, parameter selection 3 performs best; especially in the low frequency region.

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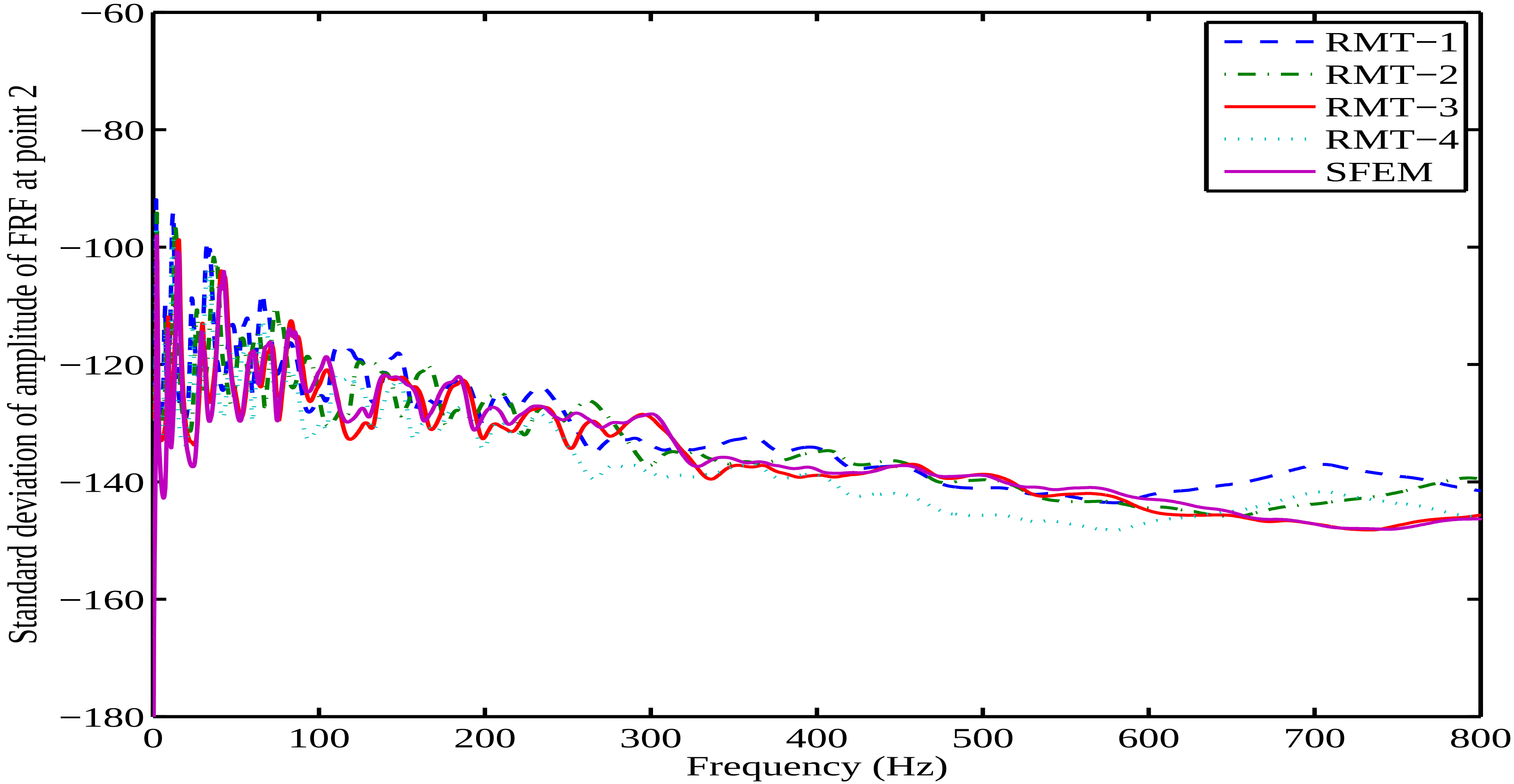
Standard deviation: low frequency



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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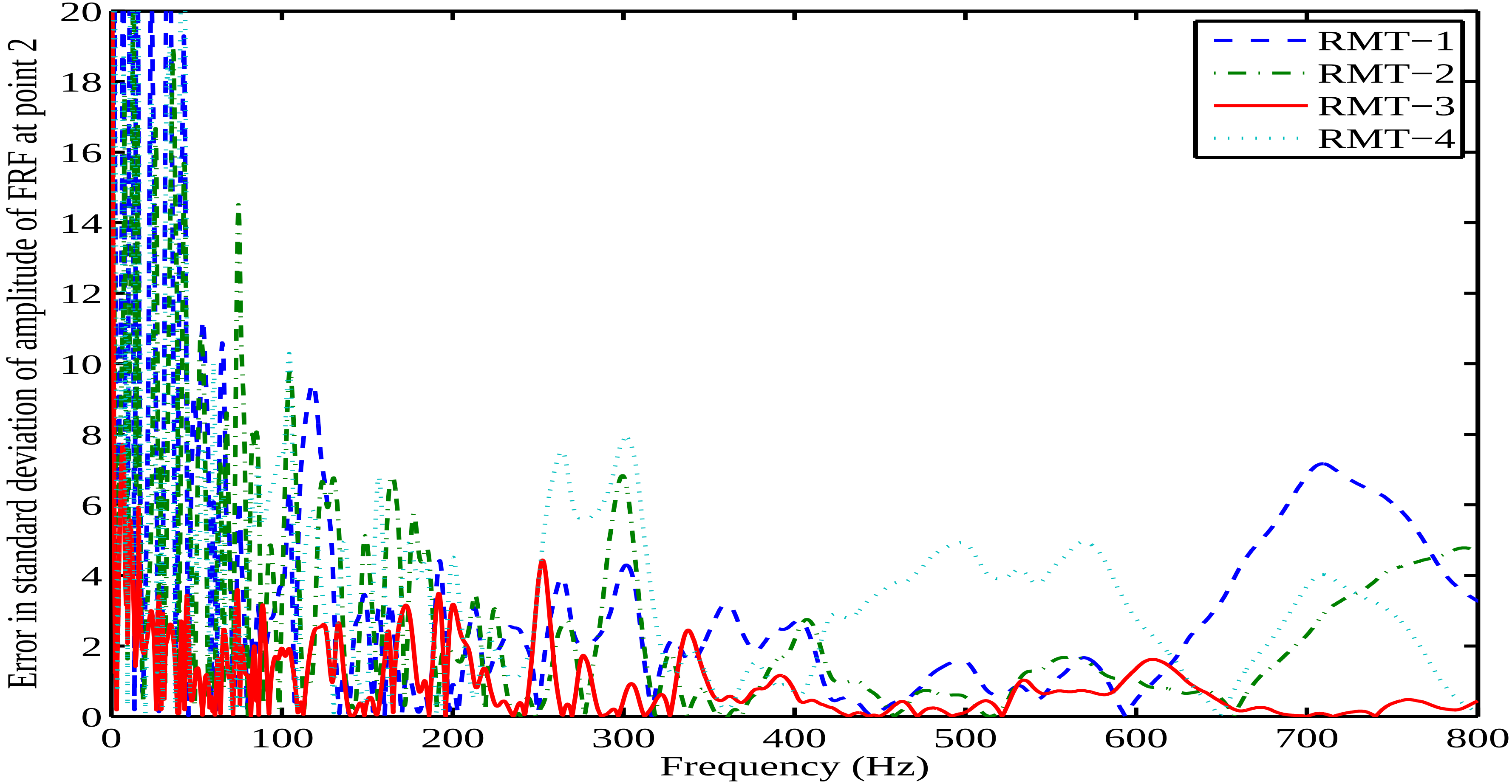
Error in the standard deviation: low frequency



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Dynamic response: analytical approach

- The dynamic response of the system can be expressed in the frequency domain as

$$\mathbf{q}(\omega) = \mathbf{D}^{-1}(\omega)\mathbf{f}(\omega) \tag{7}$$

where the dynamic stiffness matrix is defined as

$$\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}. \tag{8}$$

This is a complex symmetric random matrix.

- The calculation of the response statistics requires the calculation of statistical moments of the inverse of this matrix.

Main assumptions

1. Damping matrix is 'small' compared to the mass and stiffness matrices.
2. The damping matrix is deterministic.
3. The mass and stiffness matrices are statistically independent Wishart matrices.
4. The input force is deterministic.

(no assumptions related to proportional damping, small randomness or Gaussianity).

Response moments - 1

- The first-order moment of the absolute of the response:

$$\bar{\mathbf{q}} = \mathrm{E}\left[|\mathbf{q}|\right] = \mathrm{E}\left[|\mathbf{D}|^{-1}\right]\bar{\mathbf{f}} \tag{9}$$

where $\bar{\mathbf{f}} = |\mathbf{f}|$.

- The second-order moment of the absolute of the response:

$$\begin{aligned} \mathrm{cov}_{|\mathbf{q}|} &= \mathrm{E}\left[(|\mathbf{q}| - \mathrm{E}\left[|\mathbf{q}|\right])(|\mathbf{q}| - \mathrm{E}\left[|\mathbf{q}|\right])^T\right] = \mathrm{E}\left[|\mathbf{q}|\,|\mathbf{q}|^T\right] - \bar{\mathbf{q}}\bar{\mathbf{q}}^T \\ &= \mathrm{E}\left[|\mathbf{D}|^{-1}\,\bar{\mathbf{f}}\bar{\mathbf{f}}^T\,|\mathbf{D}|^{-1}\right] - \bar{\mathbf{q}}\bar{\mathbf{q}}^T. \end{aligned} \tag{10}$$

Response moments - 2

The dynamic response statistics is obtained in two steps:

- A Wishart distribution is fitted to $|\mathbf{D}(\omega)| = \left\{[-\omega^2\mathbf{M} + \mathbf{K}]^2 + \omega^2\mathbf{C}^2\right\}^{1/2}$, which is symmetric and non-negative definite random matrix. Note that $\mathbf{D}(\omega)$ cannot be a Wishart matrix unless the system is undamped.
- Once the parameters of the Wishart distribution corresponding to $|\mathbf{D}|$ is identified, the inverse moments are obtained exactly in closed-from using the inverted Wishart distribution.

Response moments - 3

After some algebra we have the mean

$$\bar{\mathbf{q}} = \frac{p_D(\omega)}{\theta_D(\omega)} \mathbf{q}_0(\omega) \tag{11}$$

Here $\mathbf{q}_0(\omega)$ is the absolute value of the response for the baseline or 'mean' system

$$\mathbf{q}_0(\omega) = |\mathbf{D}_0(\omega)|^{-1} |\mathbf{f}(\omega)| \tag{12}$$

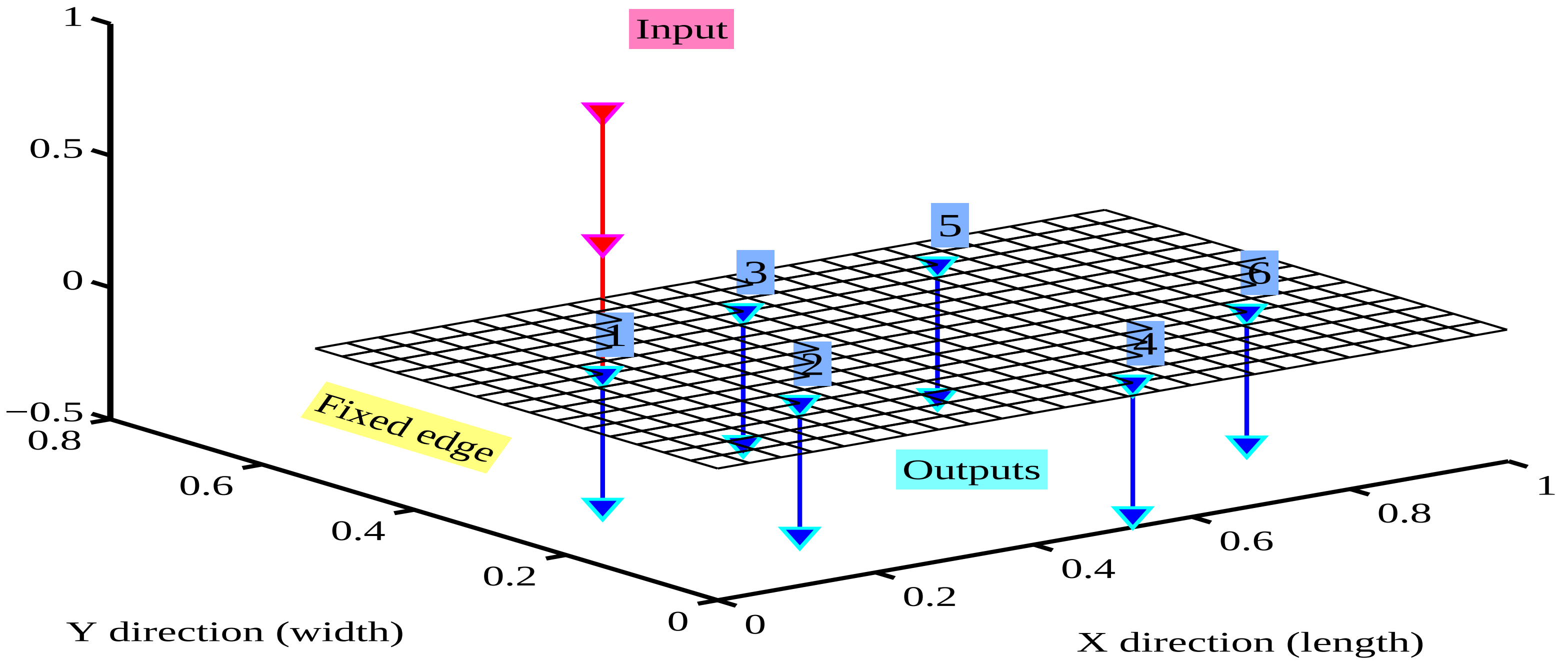
with $|\mathbf{D}_0(\omega)| = \left|-\omega^2 \mathbf{M}_0 + i\omega\mathbf{C} + \mathbf{K}_0\right|$
$\theta_D(\omega) = p_D(\omega) - n - 1$, $p_D(\omega) = \mathrm{Trace}\left(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}\right) / \mathrm{Trace}\left(\boldsymbol{\mathcal{A}}^2\right)$
where

$$\boldsymbol{\mathcal{A}} = \omega^4 p_M \left({\mathbf{M}_0}^2 + \mathbf{M}_0 \mathrm{Trace}\left(\mathbf{M}_0\right)\right) / \theta_M + p_K \left({\mathbf{K}_0}^2 + \mathbf{K}_0 \mathrm{Trace}\left(\mathbf{K}_0\right)\right) / \theta_K$$
$$\boldsymbol{\mathcal{B}} = |\mathbf{D}_0(\omega)|^2 + |\mathbf{D}\mathbf{D}_0| \, \mathrm{Trace}\left(|\mathbf{D}_0(\omega)|\right).$$

The covariance of the absolute of the response can be obtained as

$$\mathrm{cov}_{|\mathbf{q}|}(\omega) = \frac{(\theta_D(\omega) + n + 1)\mathrm{Trace}\left(\mathbf{q}_0(\omega)\bar{\mathbf{f}}(\omega)^T\right) \boldsymbol{\Sigma}_D^{-1}(\omega) + (\theta_D(\omega) + 2)\mathbf{q}_0(\omega)\mathbf{q}_0^T(\omega)}{(\theta_D(\omega) + 1)(\theta_D(\omega) - 2)}. \tag{13}$$

Finite element & Wishart matrix model



Baseline Model: 25×15 elements, 416 nodes, 1200 degrees-of-freedom. Input node number: 481, Output node numbers: 481, 877, 268, 1135, 211 and 844, 0.7% modal damping is assumed for all modes..

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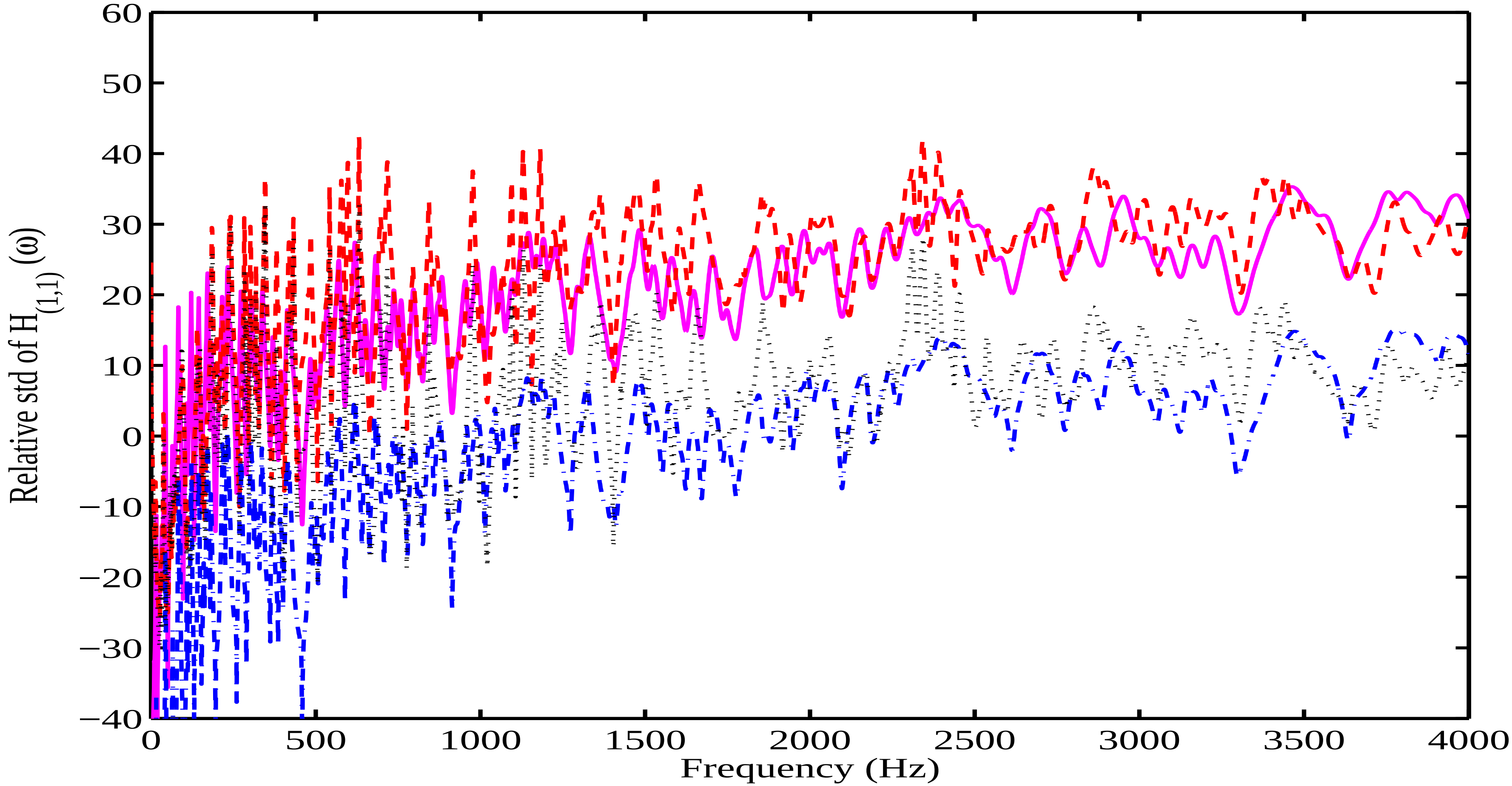
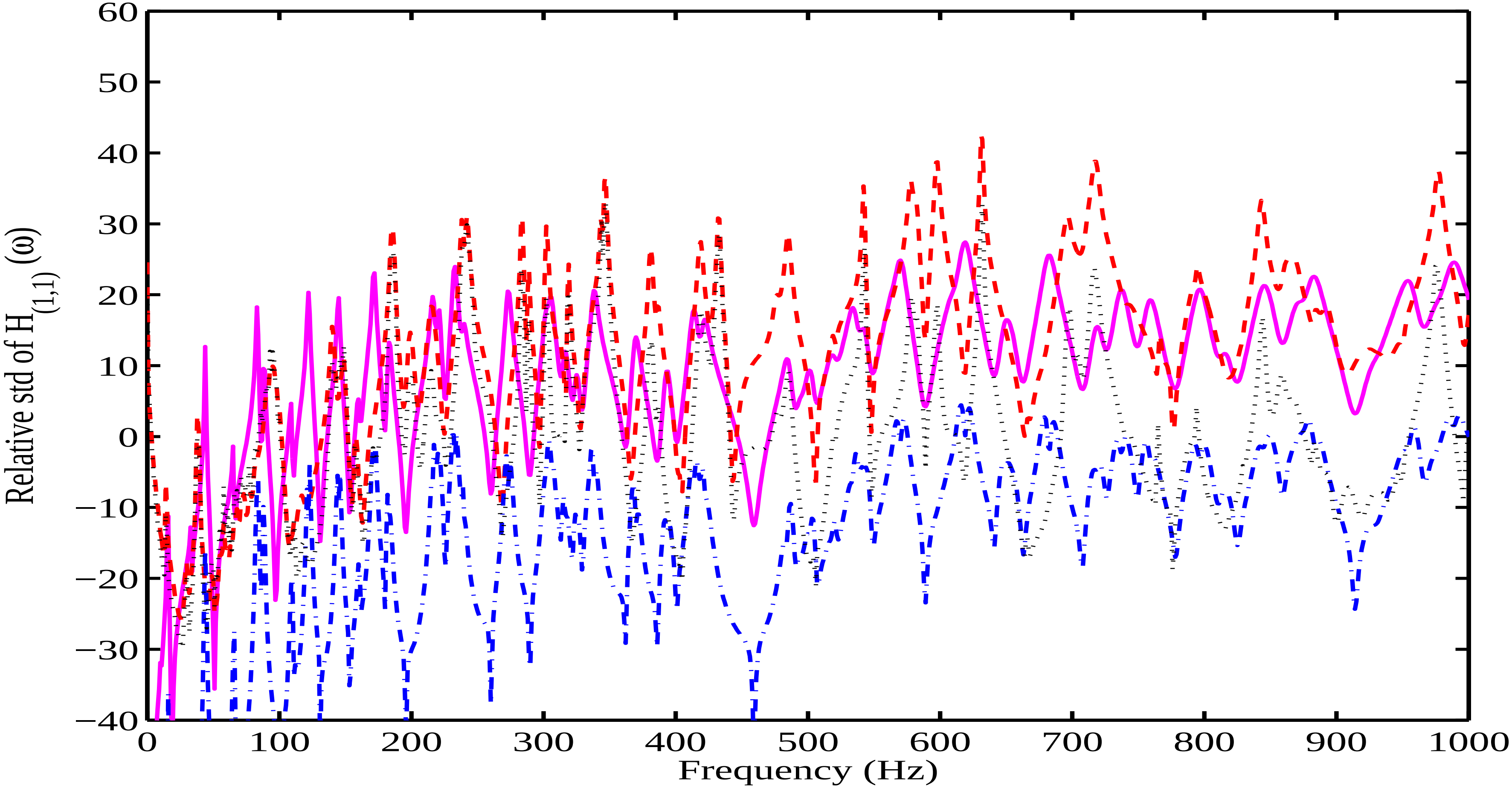
Comparison of driving-point-FRF



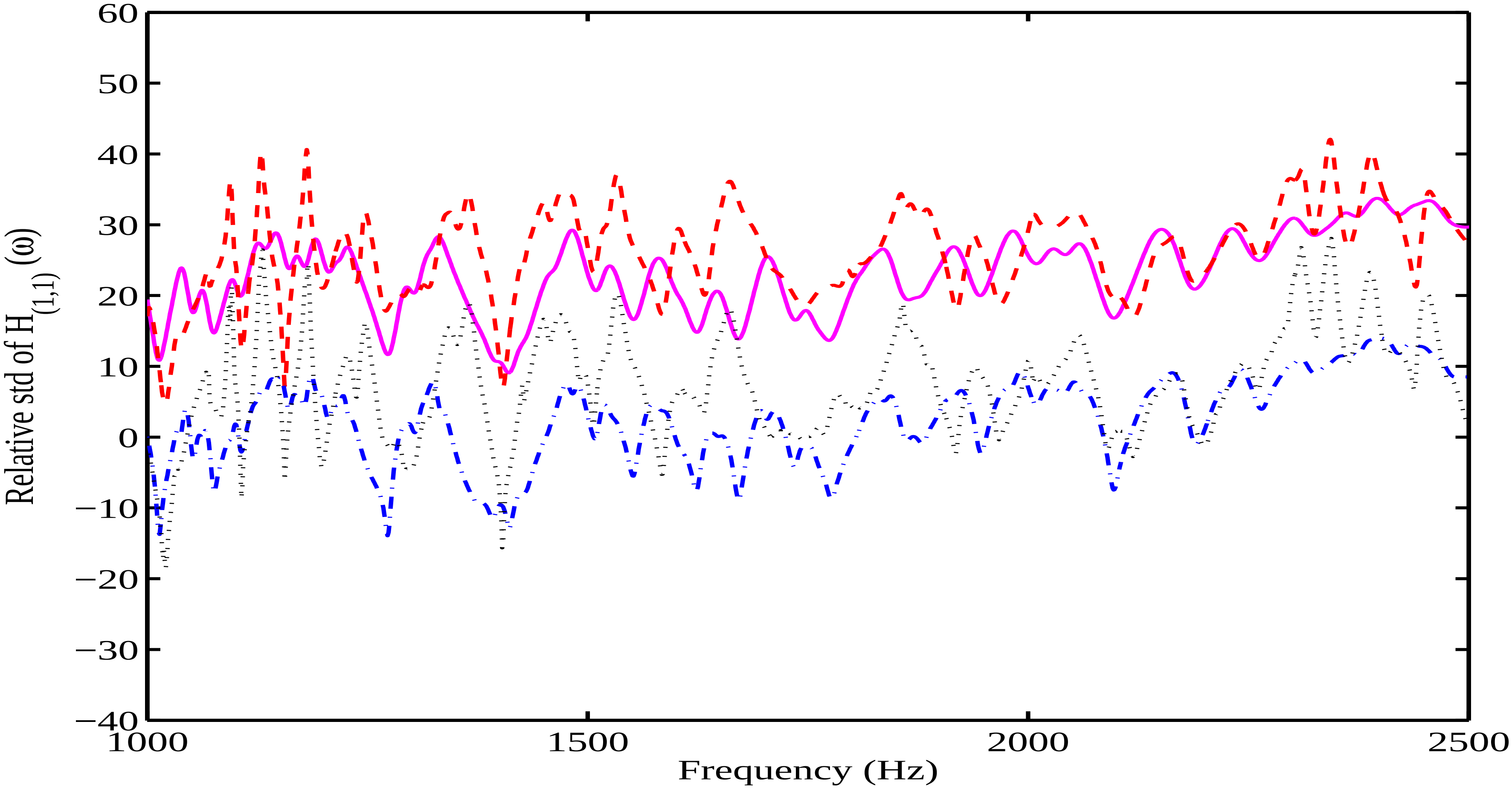
Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: Low Freq



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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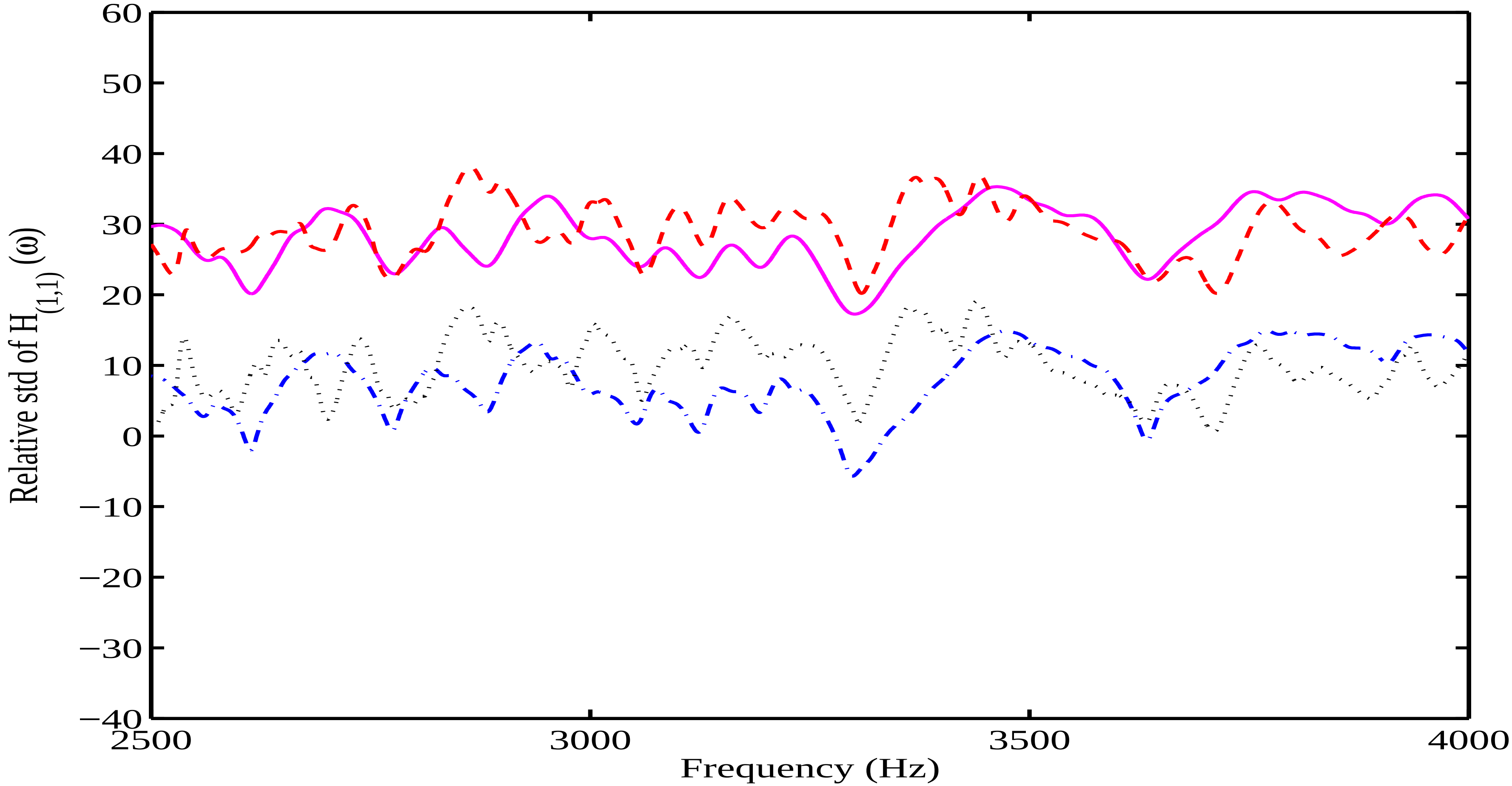
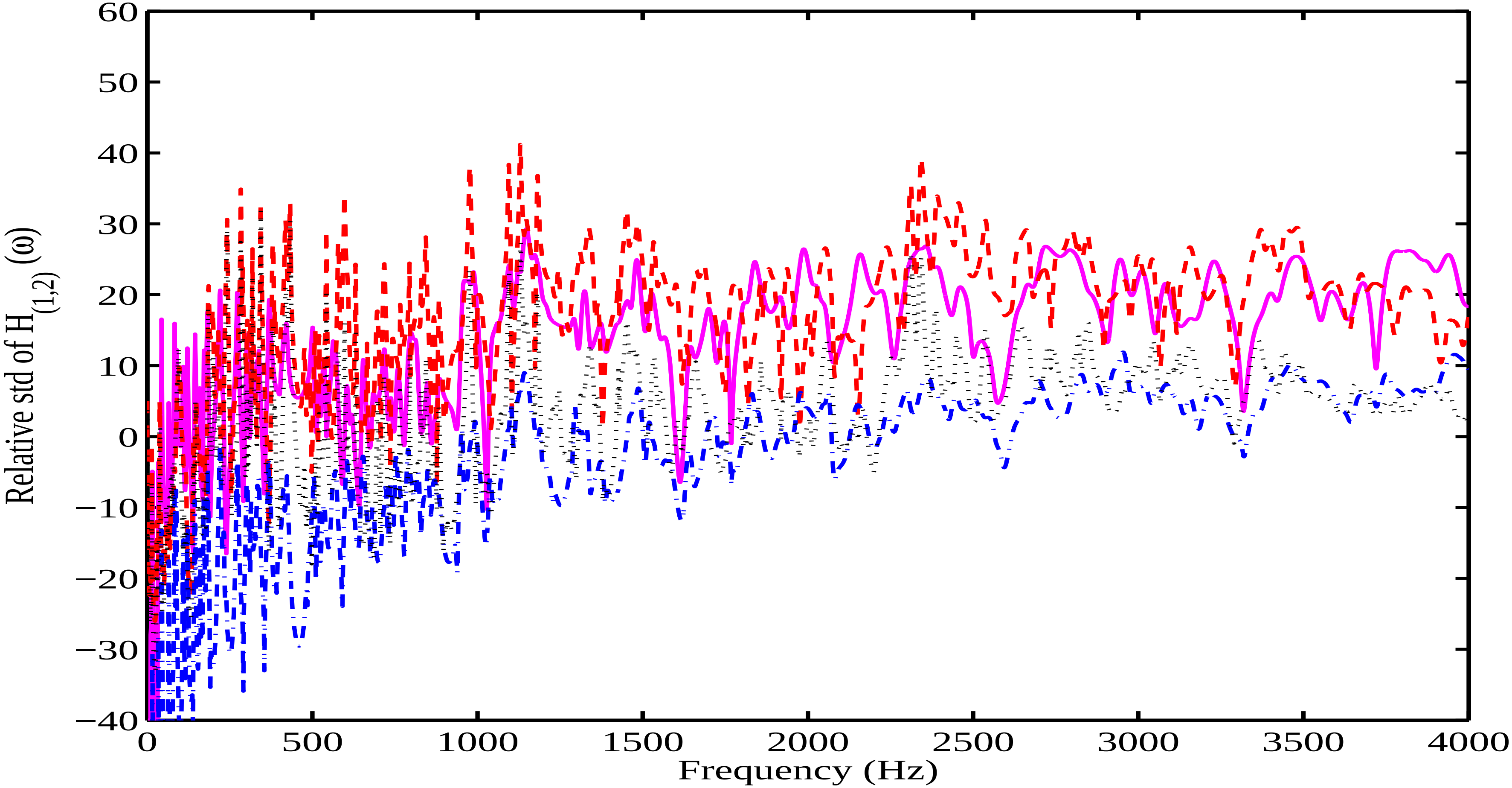
Comparison of driving-point-FRF: High Freq



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of cross-FRF



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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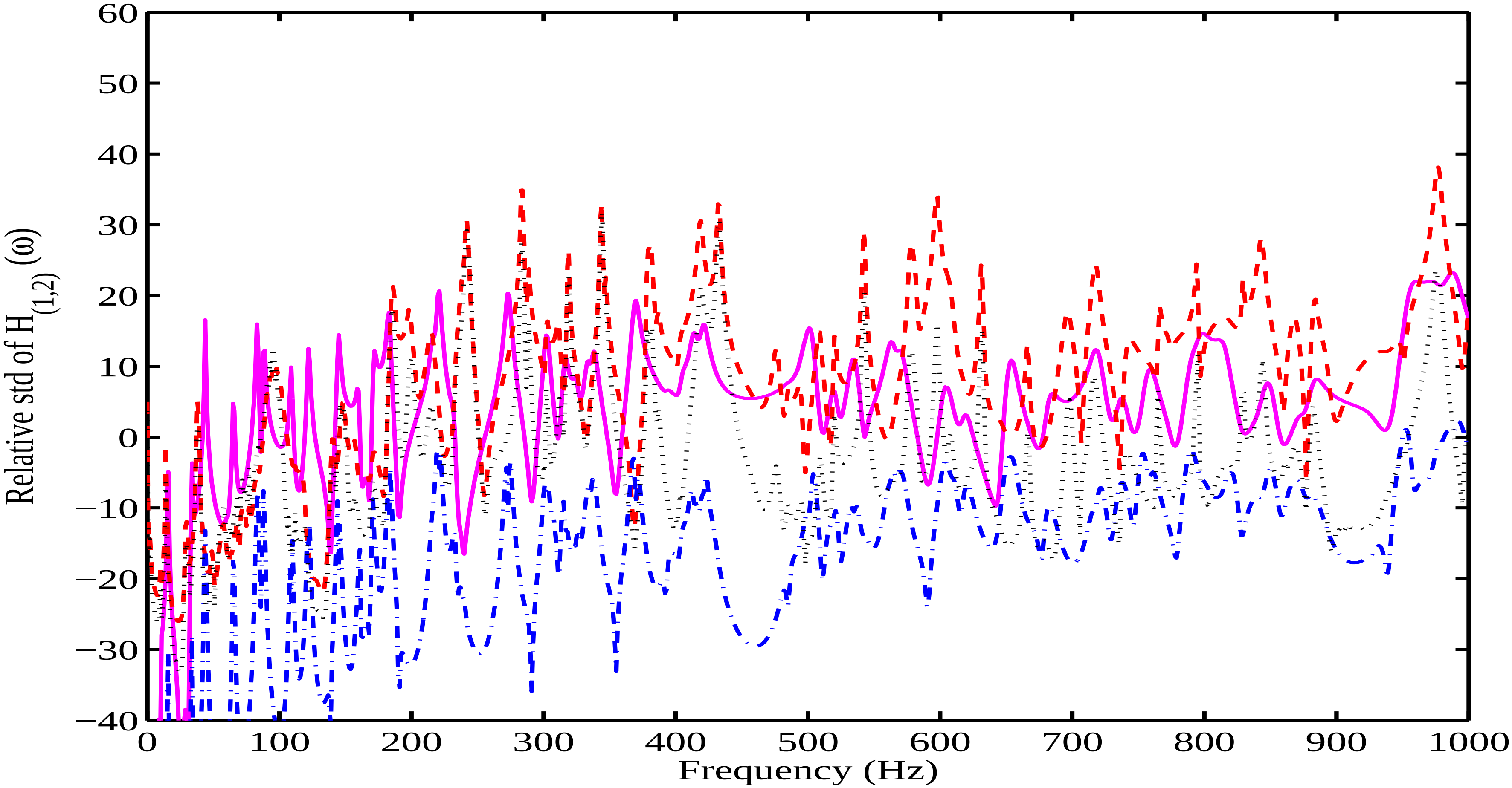
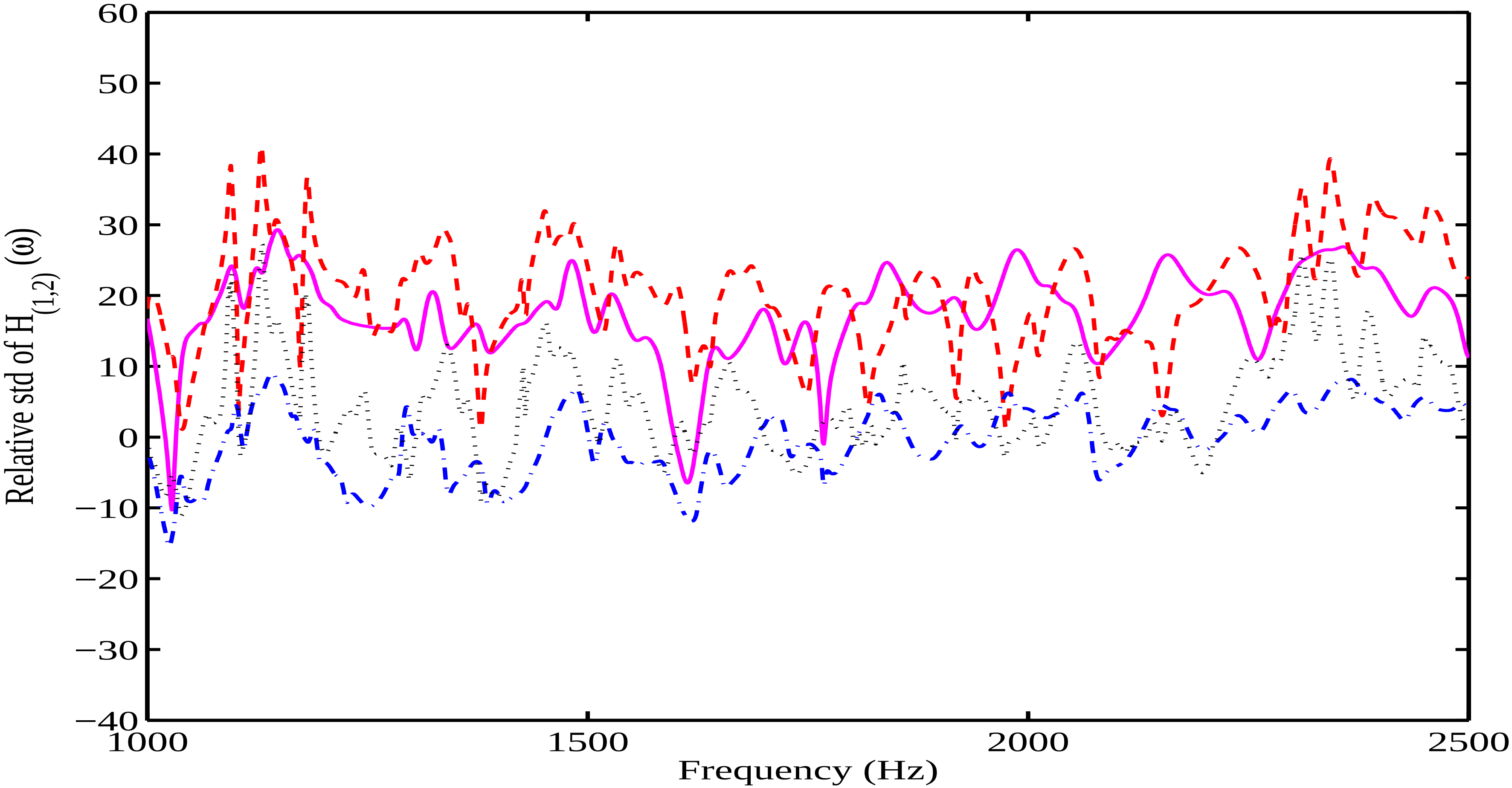
Comparison of cross-FRF: Low Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of cross-FRF: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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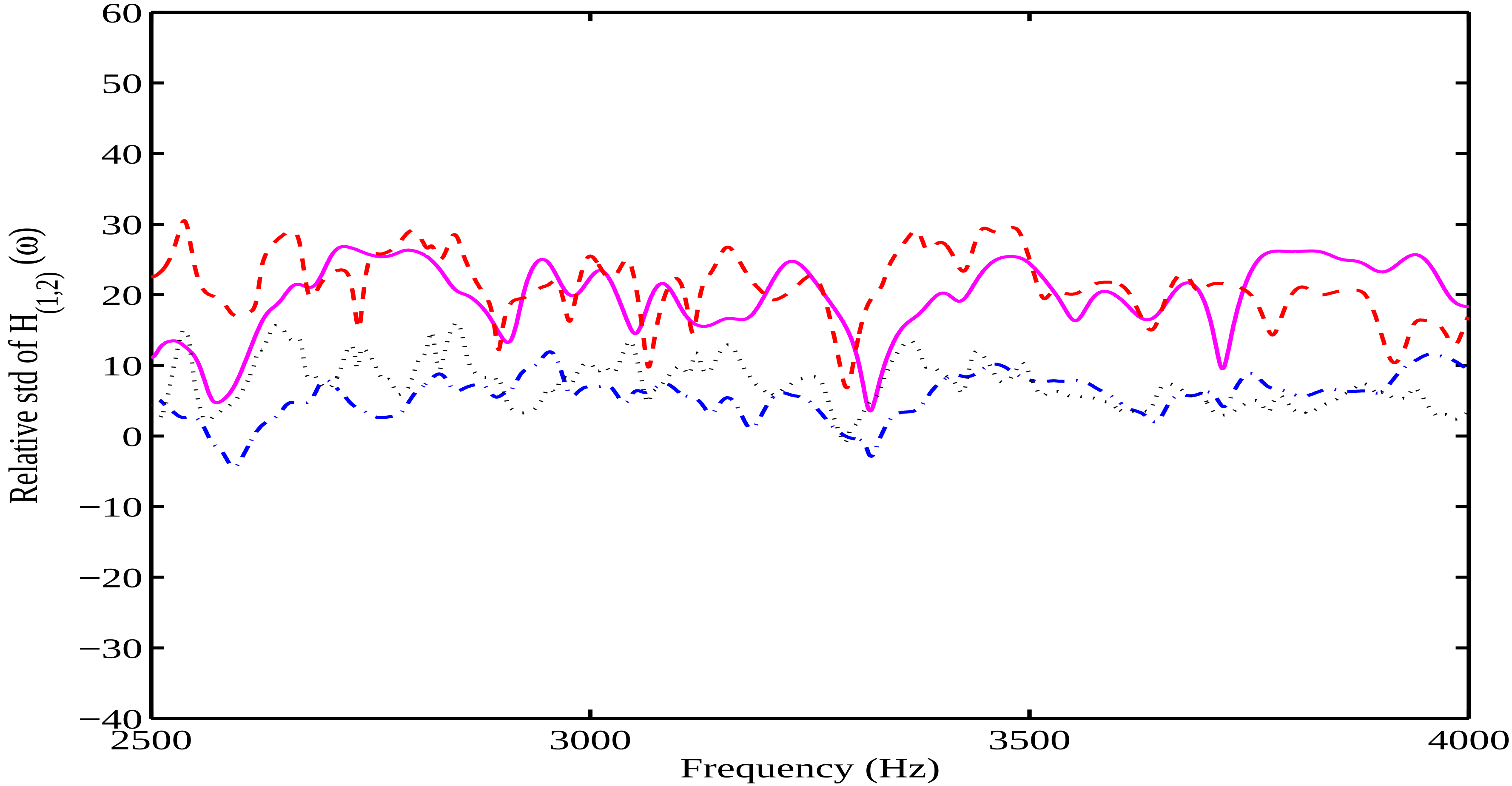
Comparison of cross-FRF: High Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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Random matrix approach: Future works

- Refine random matrix inversion approach:
  - relax some of the simplifying approximations employed in the current work
  - explore different random matrix parameter fitting options
- Random eigenvalue based computational method:
  - utilize eigensolution density function of Wishart matrices in response calculation
  - simple analytical expressions
- Non-central Wishart matrices:
  - better approximation of the covariance of the system matrices

Parametric uncertainty

- Complex engineering dynamical systems with parametric uncertainty are often investigated running computer codes (e.g, with Monte Carlo Simulation), also known as simulators (O'Hagan, 2006).
- A simulator is a function $\eta(\cdot)$ that, given an input **x**, it produces an output **y**.
- Sophisticated simulators can have a high cost of execution, measured in terms of:
  - CPU time employed
  - Floating point operations performed
  - Computer capability required

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Emulator - 1

- A possible solution is to build an emulator of the expensive simulator.
- An emulator is a statistical approximation to the simulator, i.e., it provides a probability distribution for $\eta(\cdot)$.
- Emulators have already been implemented in a number of fields, which include:
  - Environmental science (Challenor et al., 2006)
  - Climate modeling (Rougier, 2007)
  - Medical science (Haylock and O'Hagan, 1996)

Emulator - 2

- An emulator is built by first choosing n design points in the input domain of the simulator and obtaining the training set $\{\eta(\mathbf{x}_1), \ldots, \eta(\mathbf{x}_n)\}$.
- After that initial choice is made, an emulator should:
  - Reproduce the known output at any design point.
  - At any untried input, provide a distribution whose mean value constitutes a plausible interpolation of the training data. The probability distribution around this mean value should also express the uncertainty about how the emulator might interpolate.

Emulator: simple example - 1

- To illustrate what do the above criteria mean, an emulator was constructed to approximate the simple simulator $\mathbf{y} = cos(\mathbf{x})$.
- In the following figures, the solid line is the true output of the simulator. The circles represent the training runs, and the dots are the mean of the distribution provided by the emulator, which is the approximation.
- Note how the approximation improves when more design points are chosen.

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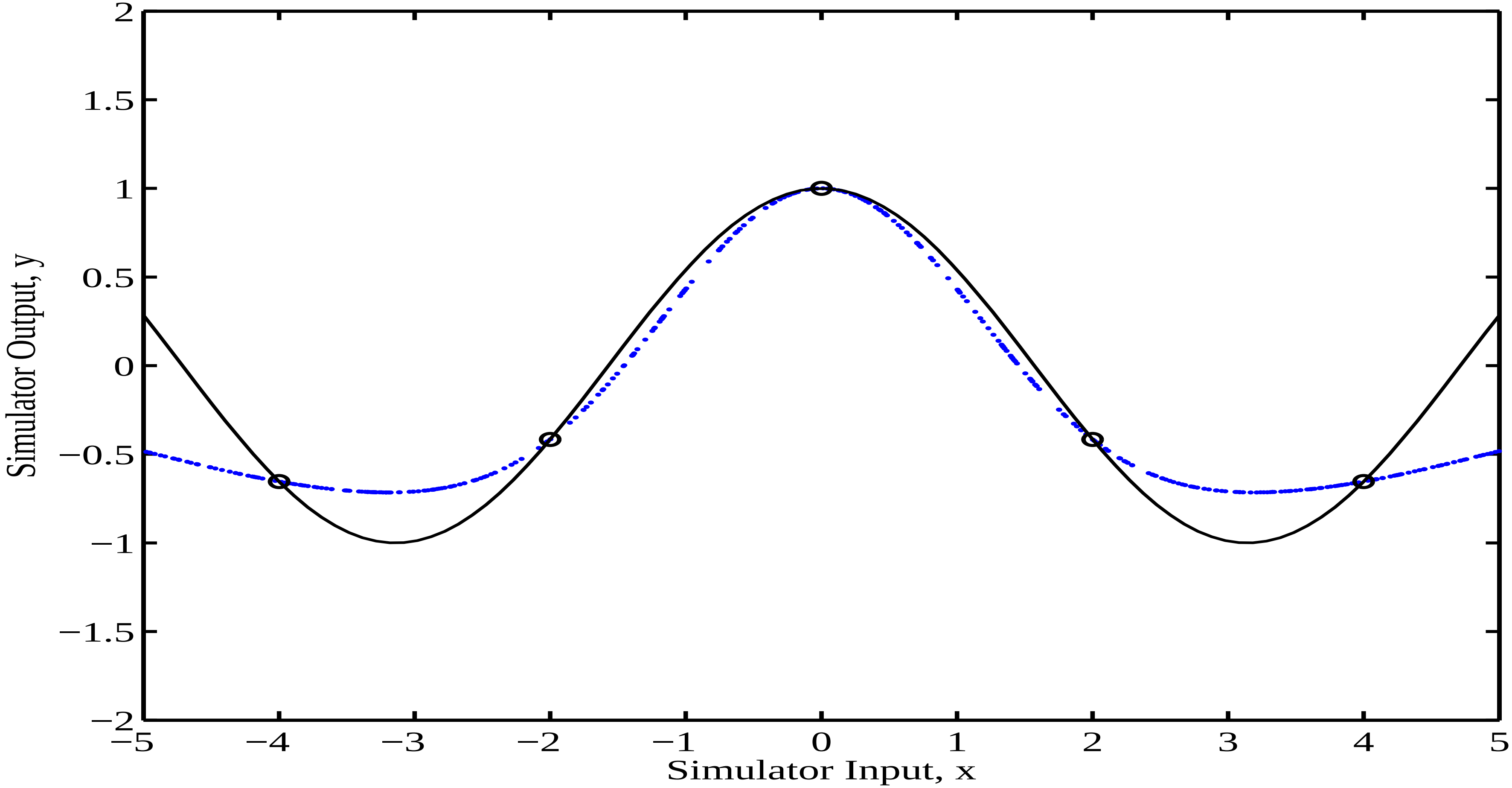
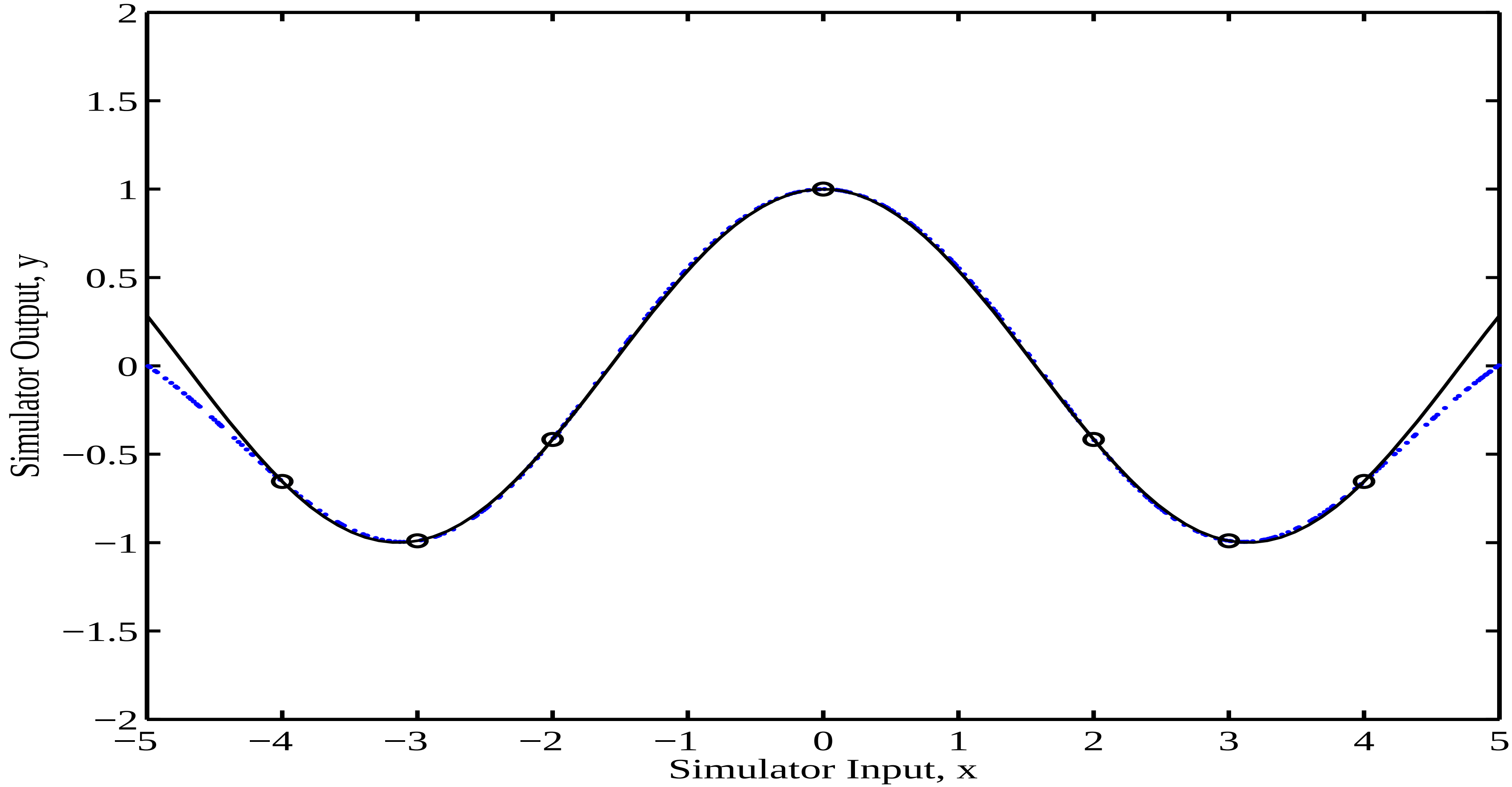
Emulator: simple example - 2



Approximation using 5 design points.

Emulator: simple example - 3



Approximation using 7 design points.

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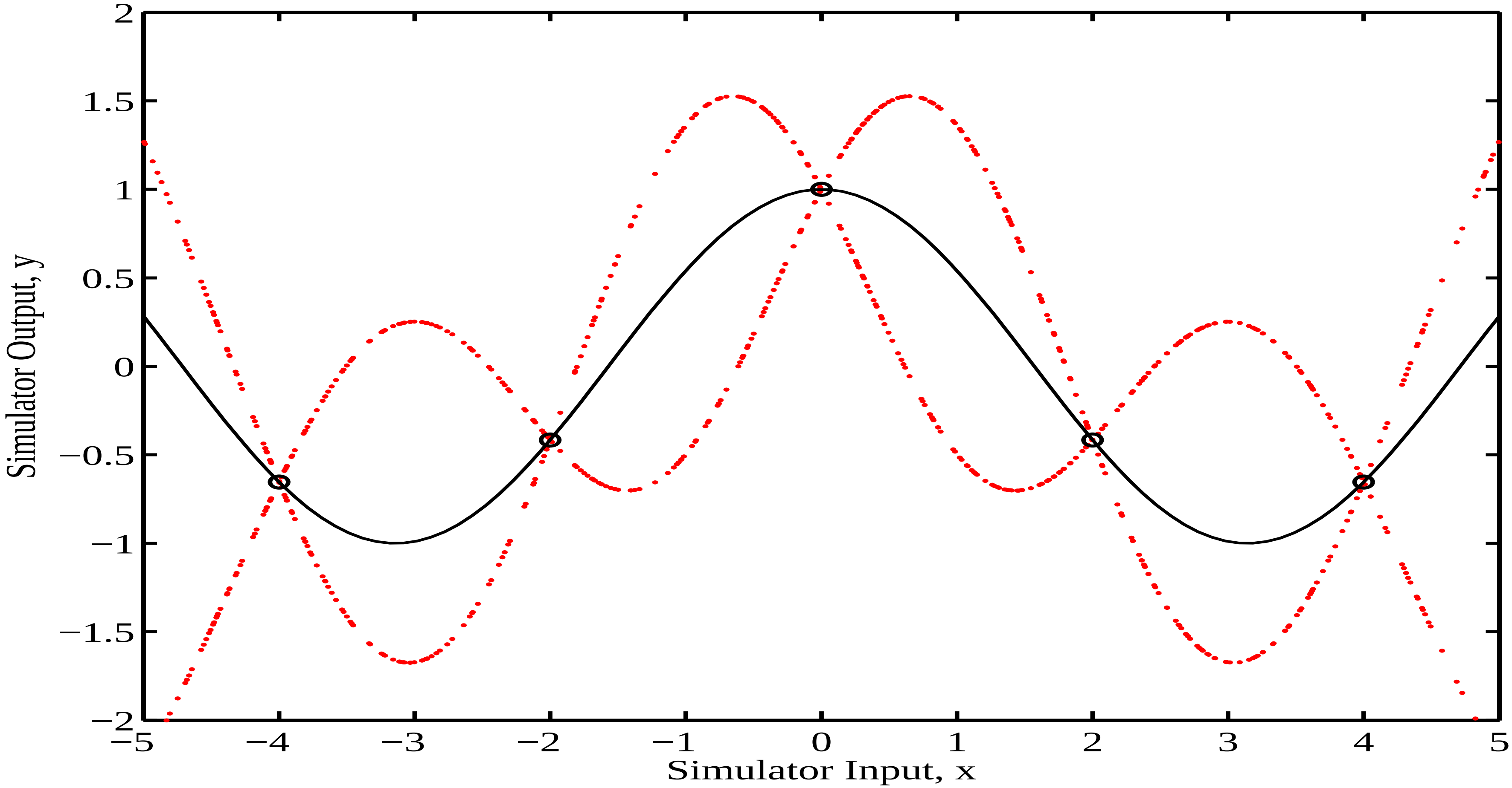
Emulator: simple example - 4

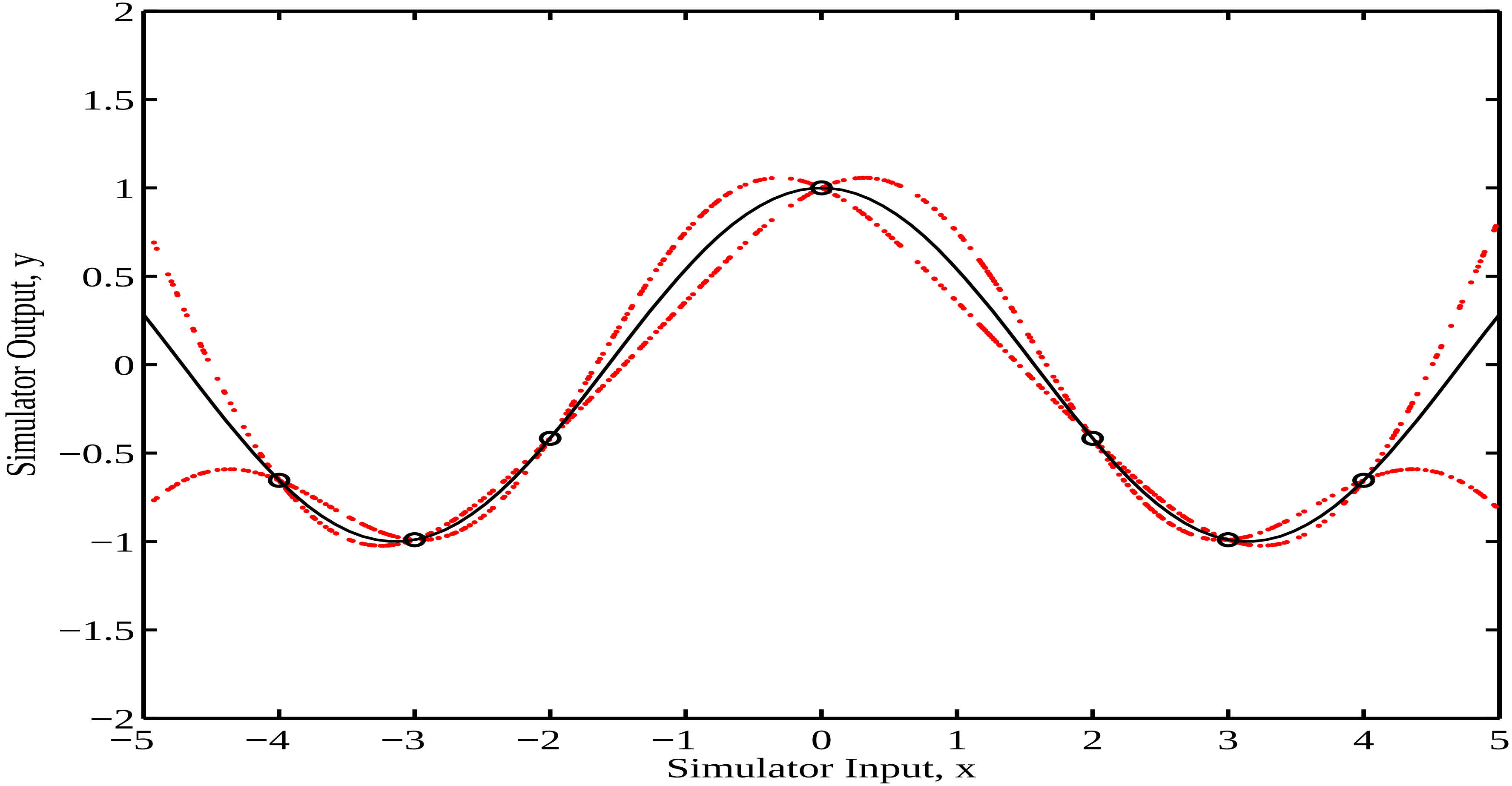
- In the same way, the following figures show upper and lower probability bounds of two standard deviations for the mean of the emulator. The solid line is the true output of the simulator. The circles represent the training runs, and the dots are the bounds.
- Note how the uncertainty about the approximation is reduces as more design points are chosen.

Emulator: simple example - 5



Uncertainty using 5 design points.

Emulator: simple example - 6



Uncertainty using 7 design points.

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Emulator: theory - 1

- From the perspective of Bayesian Statistics, $\eta(\cdot)$ is a random variable in the sense that it is unknown until the simulator is run.
- Assume that $\eta(\cdot)$ deviates from the mean of its distribution in the following way

$$\eta(\mathbf{x}) = \sum_{j=1}^{n} \beta_j h_j(\mathbf{x}) + Z(\mathbf{x}) \quad (14)$$

where for all j, $h_j(\mathbf{x})$ is a known function and β_j is an unknown coefficient.

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Emulator: theory - 2

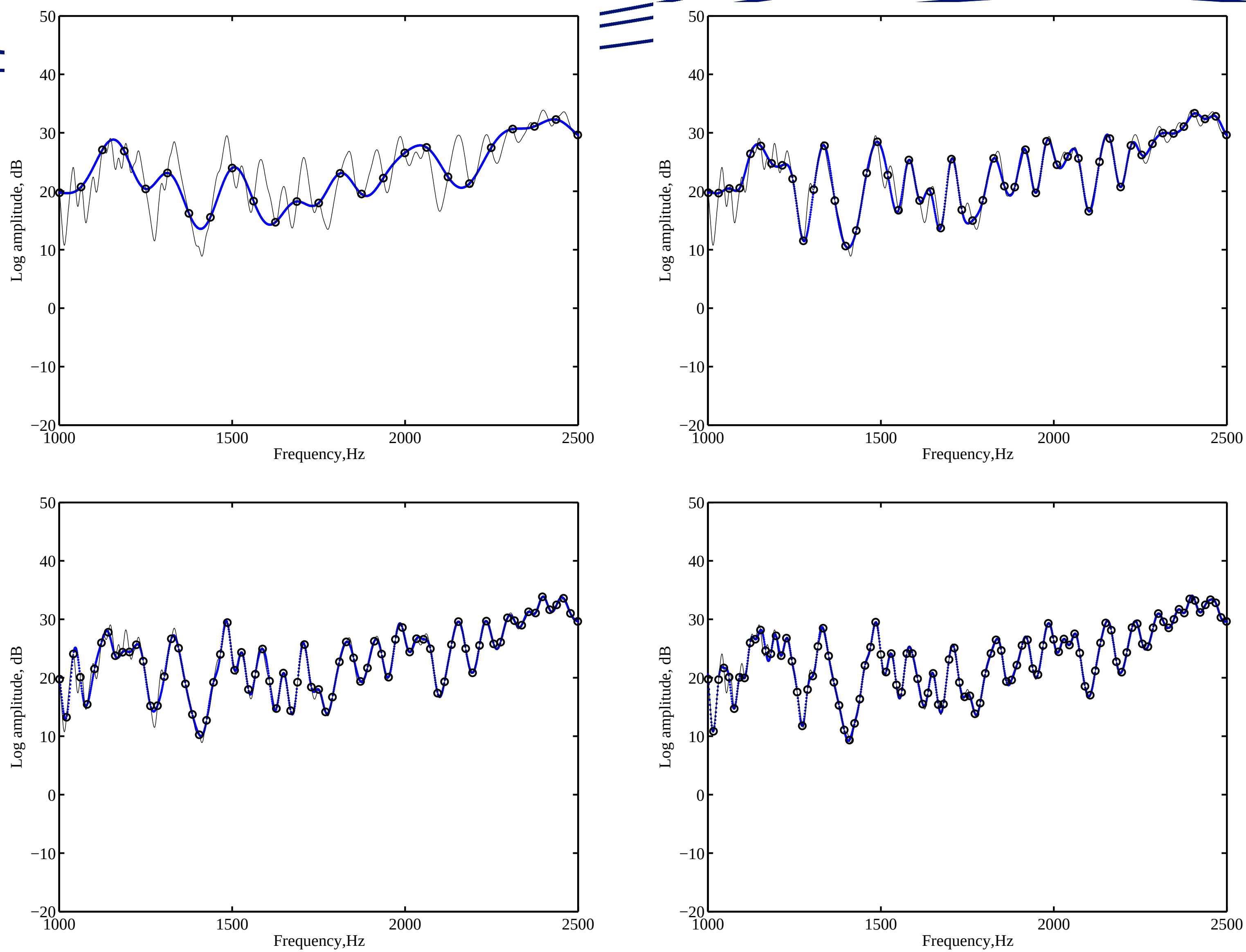
- The function $Z(\cdot)$ in Eq.(14) is assumed to be a Gaussian stochastic process (GP) with mean zero and covariance given by

$$Cov(\eta(\mathbf{x}), \eta(\mathbf{x}')) = \sigma^2 e^{-(\mathbf{x}-\mathbf{x}')^T \mathbf{B}(\mathbf{x}-\mathbf{x}')} \tag{15}$$

  where **B** is a positive definite diagonal matrix that contains smoothness parameters.

- If the mean of $\eta(\cdot)$ is of the form $m(\cdot) = \mathbf{h}(\cdot)^T \boldsymbol{\beta}$ then $\eta(\cdot)$ has a GP distribution with mean $m(\cdot)$ and covariance given by Eq.(2).

Application: experimentally measured FRF of a plate



Emulation with 25, 50, 75 and 100 design points, mid-freq range.

Stochastic Finite Element (SFE) problems

- A random field $\mathcal{H}(\mathbf{x},\theta)$ can be discretized using the Karhunen-Loeve expansion (KLE) as

$$\mathcal{H}(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(\mathbf{x}) \tag{16}$$

- Using this, the system equation can be represented as

$$[\mathbf{K}_0 + \sum_{i=0}^{M} \mathbf{K}_i\xi(\theta)]\mathbf{u} = \mathbf{f} \tag{17}$$

where each $\mathbf{K}_i$ is a deterministic matrix.

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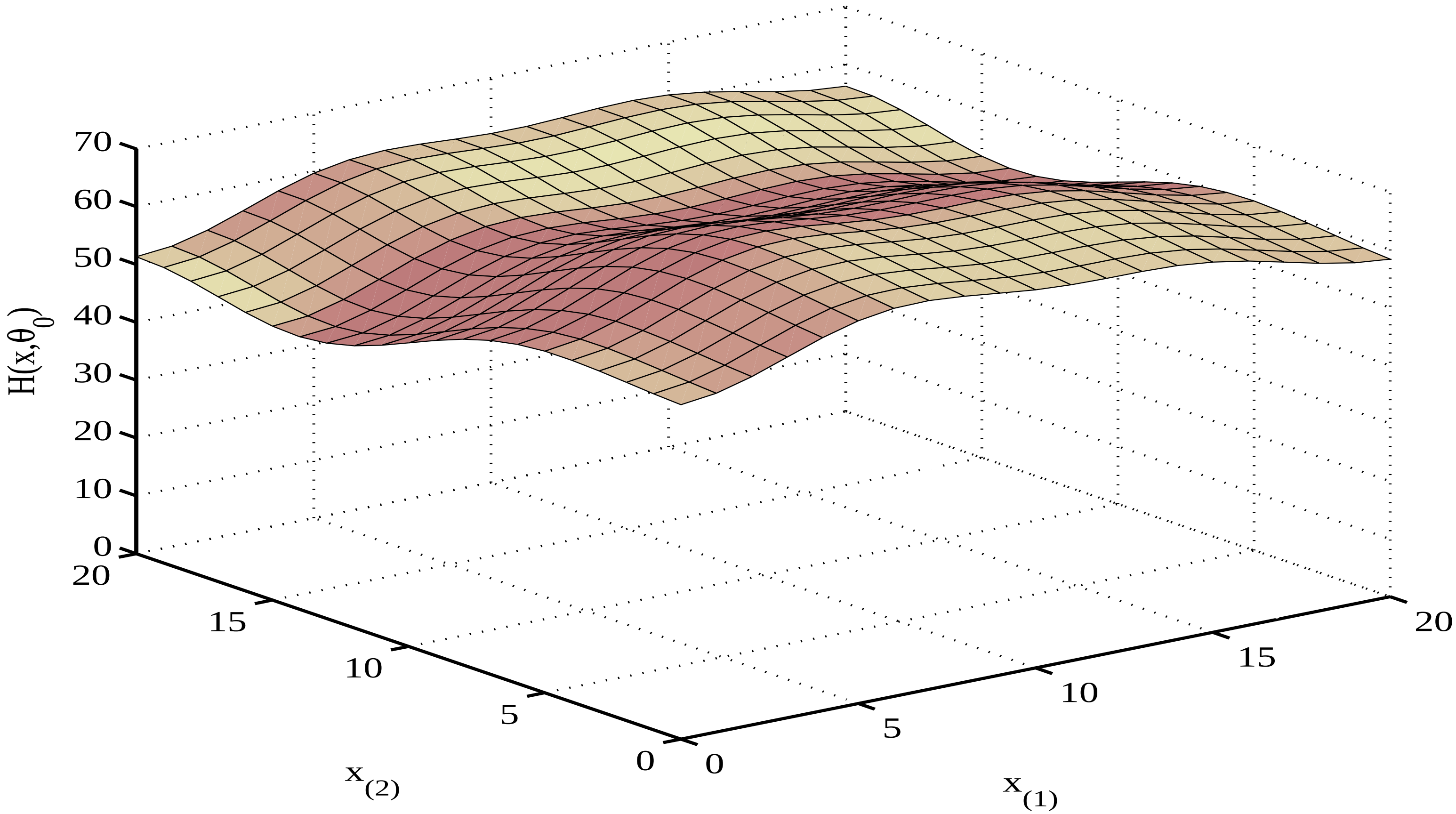
Simulation of random field - 1



Realization of the Gaussian homogeneous random field $\mathcal{H}(\mathbf{x}, \theta)$; autocorrelation $\rho(\mathbf{x}, \mathbf{x}') = e^{-\alpha_1 |\mathbf{x}_{(1)} - \mathbf{x}'_{(1)}| - \alpha_2 |\mathbf{x}_{(2)} - \mathbf{x}'_{(2)}|}$; mean $\mu = 50$; variance $\sigma^2 = 0.09$.

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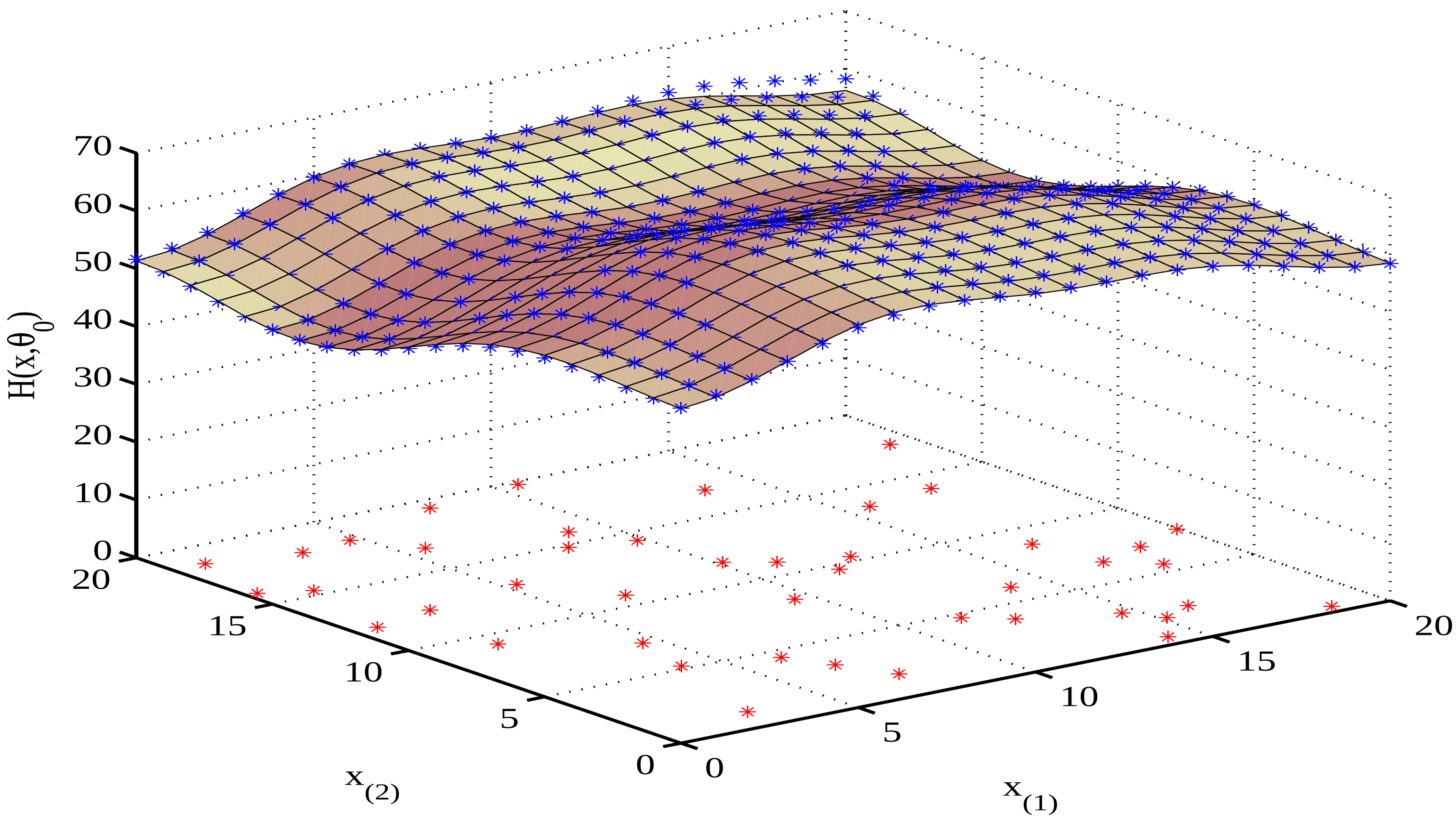
Simulation of random field - 2



Emulation of the values of $\mathcal{H}(\mathbf{x}, \theta)$ at the nodal points. The initial design is shown lying on the lower plane.

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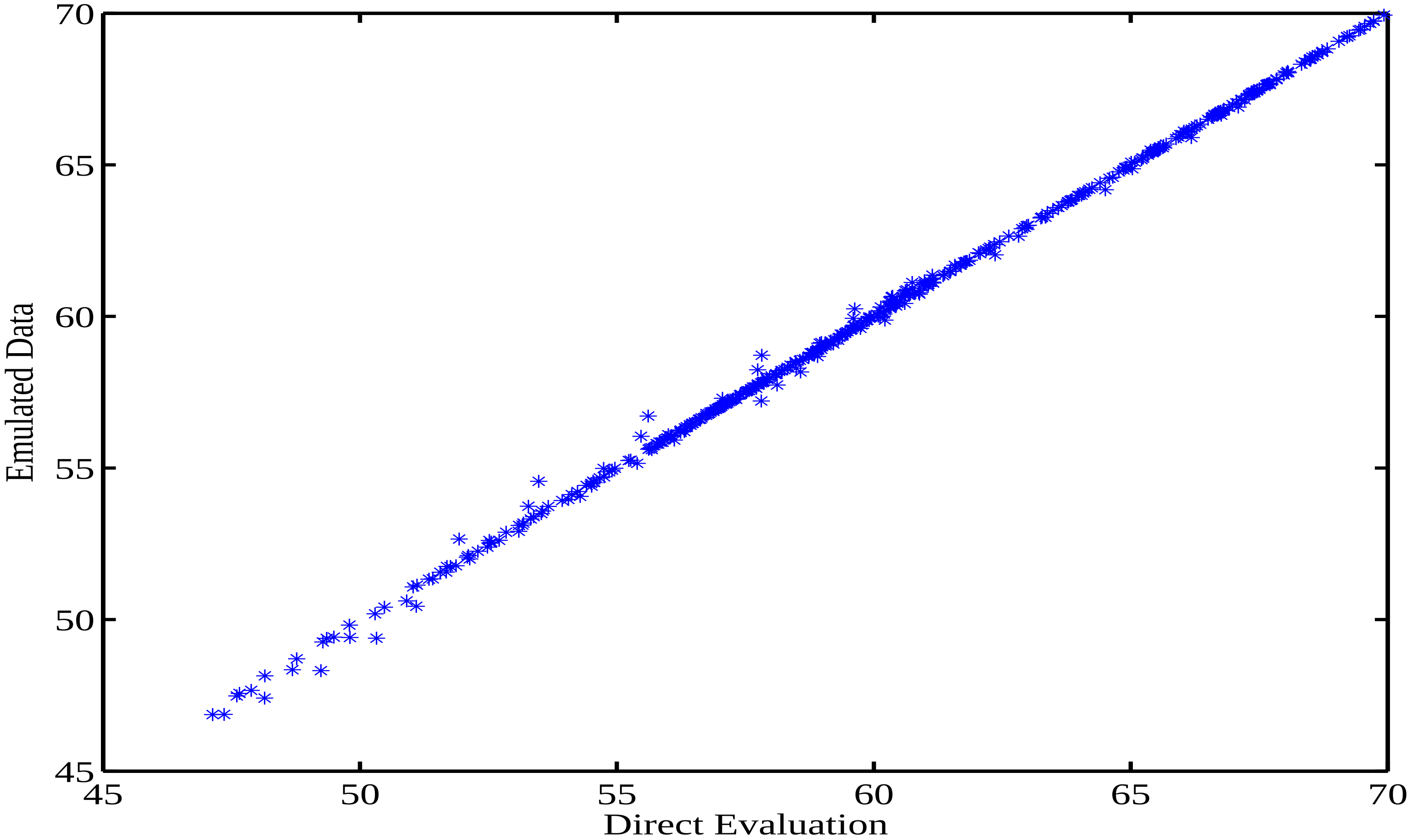
Simulation of random field - 3



Correlation of the emulated values at the nodal points

Computational effort

No. Nodes	Time (secs.) Direct	Time (secs.) Emulator
121	9.56	0.07
256	19.92	0.24
441	34.43	0.75
961	76.23	6.05
1681	131.29	17.76
2601	273.18	59.66

Number of nodes vs. CPU time employed for a typical sample of the random field

Emulator: Future works

- Parametric eigenvalue problem:
  - Express the eigenvalues of interest by emulator (probabilistic response surface)
  - Exploit explicit parametric sensitivity expressions
- Representation of stochastic response field:
  - Monte Carlo simulation using emulator
  - polynomial chaos representation by emulator
- Domain decomposition and substructure problem (Guyan reduction type approach)

Conclusions - 1

- When uncertainties in the system parameters (parametric uncertainty) and modelling (nonparametric uncertainty) are considered, the discretized equation of motion of linear dynamical systems is characterized by random mass, stiffness and damping matrices.
- Two different approaches are discussed:
  - Wishart random matrix method: $\longrightarrow$ non-parametric uncertainty problem
  - Gaussian emulator method: $\longrightarrow$ parametric uncertainty problem

Conclusions - 2

- Approximate closed-form expressions of the mean and covariance of the amplitude of the dynamic response in the frequency domain is derived. These expressions are simple post-processing of the results corresponding to the baseline system. Selected experimental and numerical results were shown.
- Samples of random field has been emulated using Gaussian emulators.

Future direction

- Model calibration/updating: taking model and measurement uncertainties into account
- Model validation: development of physically appealing and mathematically correct generalized norms
- Predictive capability assessment: how good are our model when no data is available to validate?
- Hybrid parametric-nonparametric uncertainty quantification: data assimilation and uncertainty propagation