Identification of Damping

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Outline of the Presentation

- Introduction
- Models of damping
- Complex frequencies and modes
- Viscous and Non-viscous damping identification
- Generalized proportional damping
- Identification of Generalized proportional damping
- Simulation and Experimental Results
- Conclusions

Introduction

There are two ways to understand the dynamics of complex structures:

- The first is the experimental approach. A carefully conducted experiment can provide crucial information regarding the system dynamics. However, the experimental process is time consuming, expensive and it may be not be possible to dynamically test a complex structure under desired loading conditions.
- The alternative is to ‘replace’ the actual structure by a *mathematical model* and perform numerical experiments in a computer.

Quality of a model

The quality of a model depends on:

- *Fidelity to (experimental) data:* The results obtained from a numerical or mathematical model undergoing a given excitation force should be close to the results obtained from the vibration testing of the same structure undergoing the same excitation.
- *Robustness with respect to (random) errors:* Errors in estimating the system parameters, boundary conditions and dynamic loads are unavoidable in practice. The output of the model should not be very sensitive to such errors.

Quality of a model

- *Predictive capability* In general it is not possible to experimentally validate a model over the entire domain of its scope of application. The model should predict the response well beyond its validation domain.

Viscously damped systems

- Equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \qquad (1)$$

- Proportional damping (Rayleigh 1877)

$$\mathbf{C} = \alpha_1 \mathbf{M} + \alpha_2 \mathbf{K}$$

  - Classical normal modes
  - Simplifies analysis methods
  - Identification of damping becomes easier

Models of damping

- Non-proportional viscous damping
- Non-viscous damping models: fractional derivative model, GHM model, convolution integral model
- Non-linear damping models

In general, the use of these damping models will result in complex modes

Non-proportional damping

- Modes becomes complex if damping is non-proportional
- Approximate natural frequencies and modes:

$$\lambda_j \approx \pm\omega_j + \mathrm{i}C'_{jj}/2, \quad \mathbf{z}_j \approx \mathbf{x}_j + \mathrm{i}\sum_{\substack{k=1\\k\neq j}}^{N} \frac{\omega_j C'_{kj}}{(\omega_j^2 - \omega_k^2)}\mathbf{x}_k.$$

ω_j: undamped natural frequencies;
$\mathbf{x}_k$: undamped modes;
$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X}$: modal damping matrix.

Non-viscous damping models

- **Fractional derivative model:**
  $\mathbf{F}_d = \sum_{j=1}^{l} \mathbf{g}_j D^{\nu_j}[\mathbf{q}(t)]$, where
  $$D^{\nu_j}[\mathbf{q}(t)] = \frac{\mathrm{d}^{\nu_j}\mathbf{q}(t)}{\mathrm{d}t^{\nu_j}} = \frac{1}{\Gamma(1-\nu_j)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathbf{q}(t)}{(t-\tau)^{\nu_j}}\mathrm{d}\tau$$
  Special case: $\nu_j = 1 :\Rrightarrow$ viscous damping
- **Convolution integral model:**
  $\mathbf{F}_d = \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau$
  $\boldsymbol{\mathcal{G}}(t)$ is a matrix of the damping kernel functions.
  Special case: $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\delta(t-\tau) :\Rrightarrow$ viscous damping

Non-viscously damped systems

Equation of motion:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{y}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{y}(t) = \mathbf{0} \qquad (2)$$

Approximate natural frequencies and modes:

$$\lambda_j \approx \pm\omega_j + \mathrm{i}G'_{jj}(\pm\omega_j)/2, \qquad \mathbf{z}_j \approx \mathbf{x}_j + \mathrm{i}\sum_{\substack{k=1\\k\neq j}}^{N} \frac{\omega_j G'_{kj}(\omega_j)}{(\omega_j^2 - \omega_k^2)}\mathbf{x}_k.$$

$\mathbf{G}(\omega)$: Fourier transform of $\boldsymbol{\mathcal{G}}(t)$; $\mathbf{G}'(\omega_j) = \mathbf{X}^T\mathbf{G}(\omega_j)\mathbf{X}$

Damping functions in the Laplace domain

Damping functions	Author, Year
$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot (1955, 1958)
$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta}$ $0 < \alpha < 1, \quad 0 < \beta < 1$	Bagley and Torvik (1983)
$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2} \right]$	Golla and Hughes (1985) and McTavish and Hughes (1993)
$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori (1990)
$G(s) = c \frac{1 - e^{-st_0}}{st_0}$	Adhikari (1998)
$G(s) = c \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari (1998)

Basic questions of interest

- From experimentally determined complex modes can one identify the *underlying damping mechanism*? Is it viscous or non-viscous? Can the correct model parameters be found experimentally?
- Is it possible to establish experimentally the *spatial distribution* of damping?

Basic questions of interest

- Is it possible that more than one damping model with corresponding correct sets of parameters may represent the system response equally well, so that the identified model becomes *non-unique*?
- Does the selection of damping model matter from an engineering point of view? Which aspects of behaviour are *wrongly predicted* by an incorrect damping model?

Viscous damping identification

If natural frequencies ($\mathbf{\Omega} \in \mathbb{R}^{n \times n}$), damping ratios ($\boldsymbol{\zeta} \in \mathbb{R}^{n \times n}$) and complex modes ($\mathbf{Z} \in \mathbb{R}^{m \times n}$) are known from measurments, then the damping matrix can be identified [a]:

$$\mathbf{U} = \Re\left(\mathbf{Z}\right), \quad \mathbf{V} = \Im\left(\mathbf{Z}\right)$$

$$\mathbf{B} = \mathbf{U}^{+}\mathbf{V}$$

$$\mathbf{C}' = \left[\mathbf{\Omega}^2\mathbf{B} - \mathbf{B}\mathbf{\Omega}^2\right]\mathbf{\Omega}^{-1} + 2\boldsymbol{\zeta}\mathbf{\Omega}$$

$$\mathbf{C} = {\mathbf{U}^{+}}^{T}\mathbf{C}'\mathbf{U}^{+}$$

[a] Adhikari and Woodhouse, J.of Sound & Vibration, 243[1] (2001) 43-61

Non-viscous damping identification

Damping model used for fitting: $\mathcal{G}(t) = \mu e^{-\mu t}\,\mathbf{C}$

$$\mu = \frac{\omega_1 \mathbf{v}_1^T \mathbf{M} \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{M} \mathbf{u}_1}$$

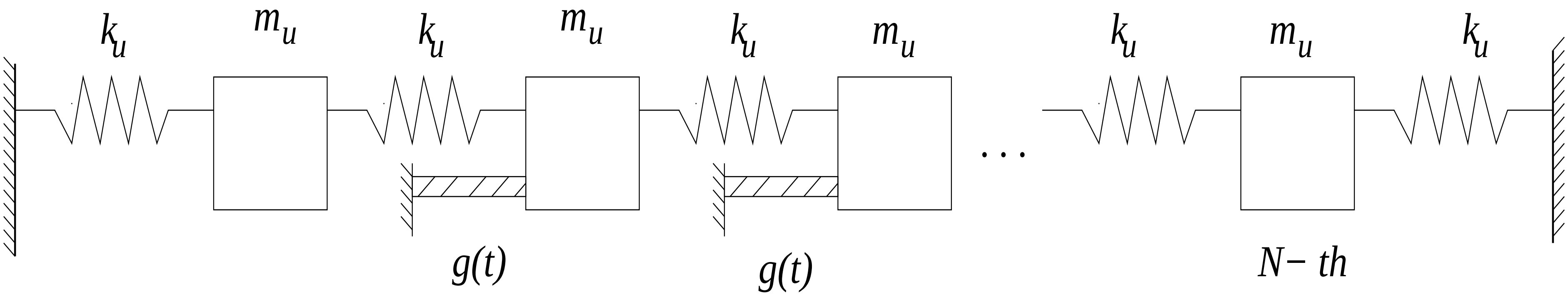
$$\mathbf{X} = \mathbf{U} - \frac{1}{\mu}\left[\mathbf{V}\mathbf{\Omega}\right]$$

$$\mathbf{B} = \mathbf{X}^{+}\mathbf{V}$$

$$\mathbf{C}' = \left[\mathbf{\Omega}^2\mathbf{B} - \mathbf{B}\mathbf{\Omega}^2\right]\mathbf{\Omega}^{-1} + 2\zeta\mathbf{\Omega}$$

$$\mathbf{C} = \mathbf{X}^{+^T}\mathbf{C}'\mathbf{U}^{+}$$

Simulation example



Linear array of N spring-mass oscillators, $N = 30$, $m_u = 1\,Kg$, $k_u = 4 \times \times 10^3 N/m$.

The kernel functions have the form

$$\boldsymbol{\mathcal{G}}(t) = \mathbf{C}\, g(t) \tag{3}$$

Here $g(t)$ is some damping function and $\mathbf{C}$ is a positive definite constant matrix.

Models of non-viscous damping

- MODEL 1 (exponential): $g^{(1)}(t) = \mu_1 e^{-\mu_1 t}$
- MODEL 2 (Gaussian): $g^{(2)}(t) = 2\sqrt{\frac{\mu_2}{\pi}} e^{-\mu_2 t^2}$

The damping models are normalized such that the damping functions have unit area when integrated to infinity, i.e.,

$$\int_0^\infty g^{(j)}(t)\,\mathrm{d}t = 1.$$

Characteristic time constant

For each damping function the *characteristic time constant* is defined via the first moment of $g(t)$ as

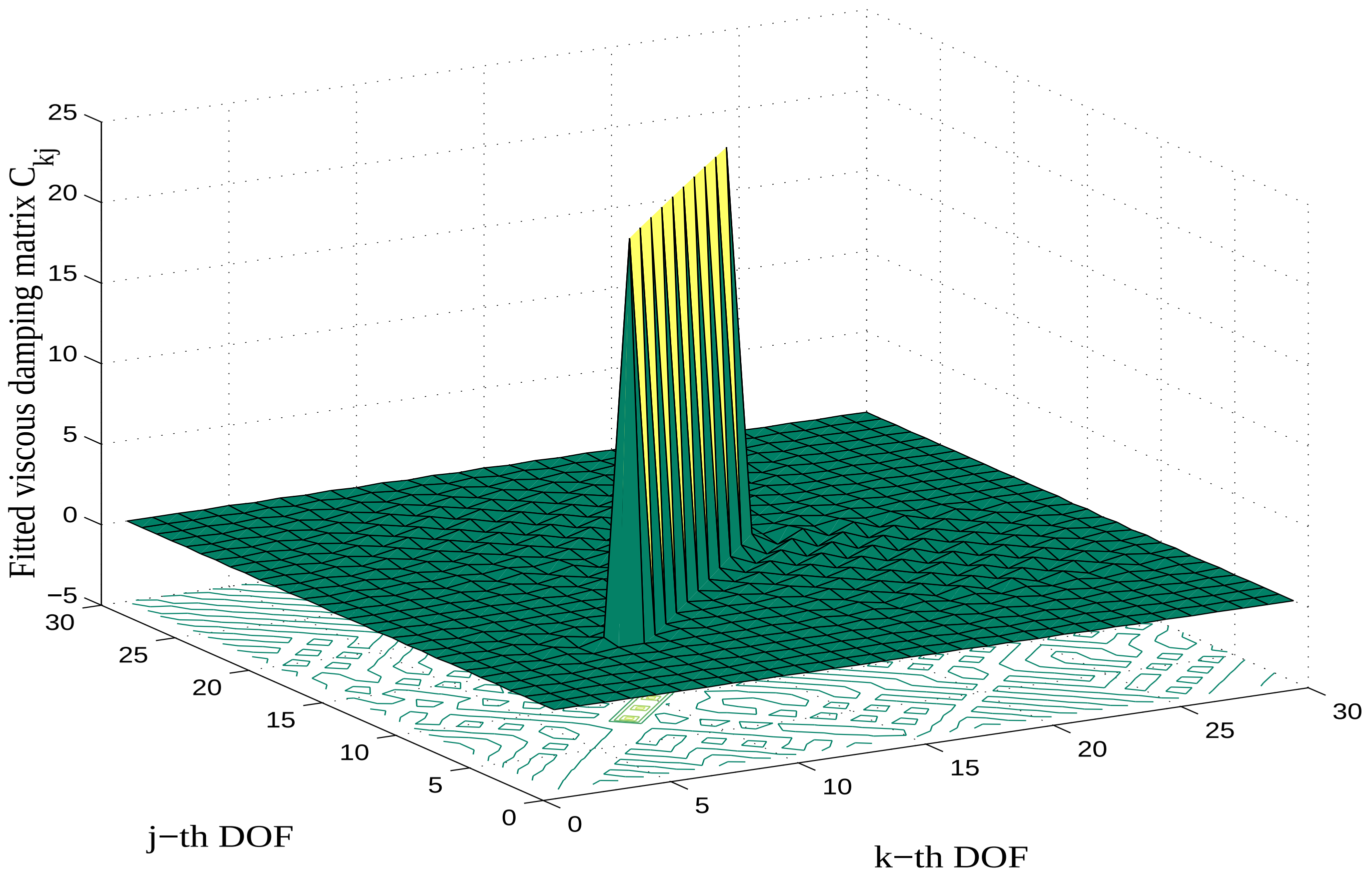
$$\theta = \int_0^\infty t\, g(t)\, \mathrm{d}t.$$

Express θ as: $\theta = \gamma\, T_{min}$.
γ is the *non-dimensional characteristic time constant* and T_{min} is the minimum time period. We expect:

$$\gamma \ll 1: \quad \text{near viscous}$$

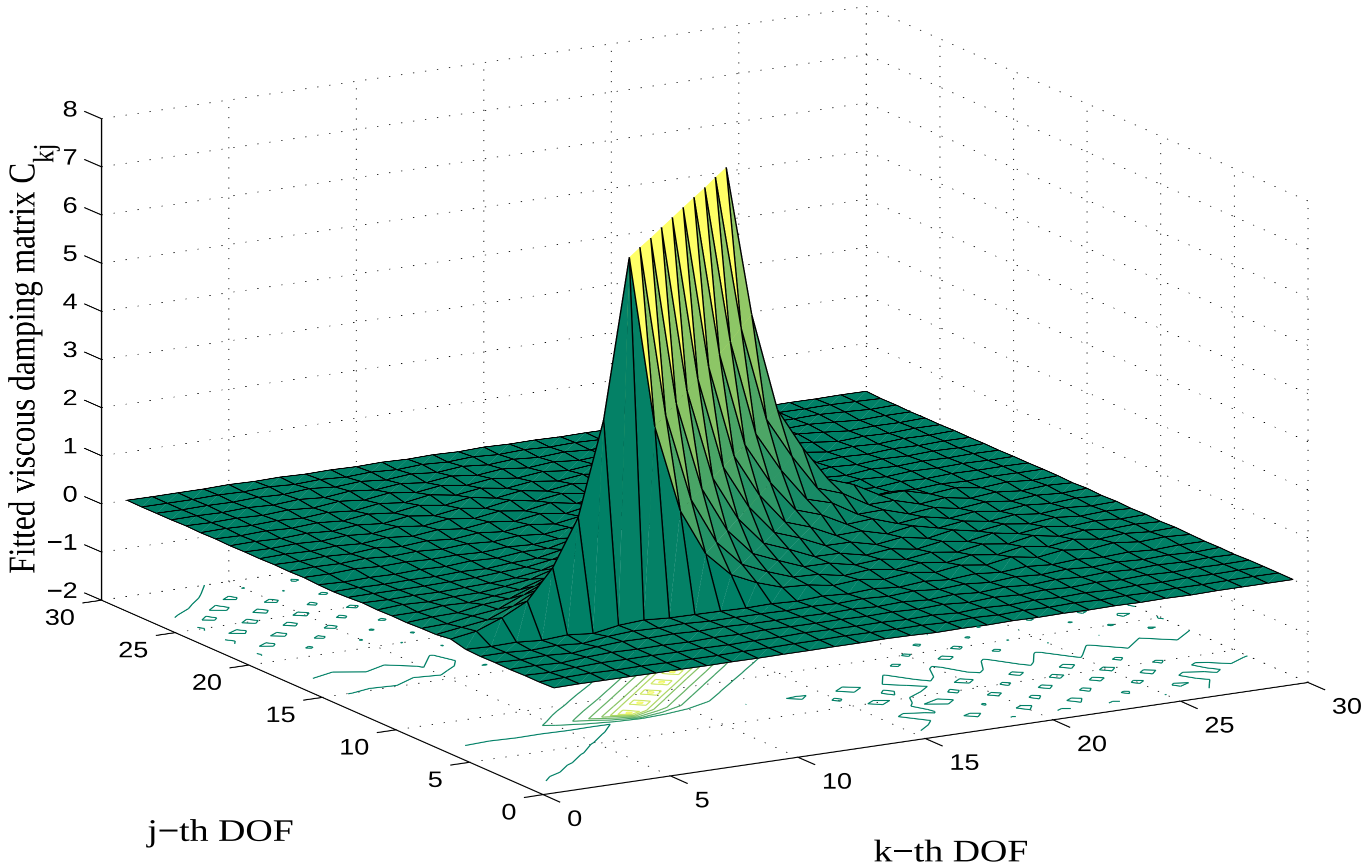
$$\gamma \to \mathcal{O}(1): \quad \text{strongly non-viscous}$$

Viscous damping identification



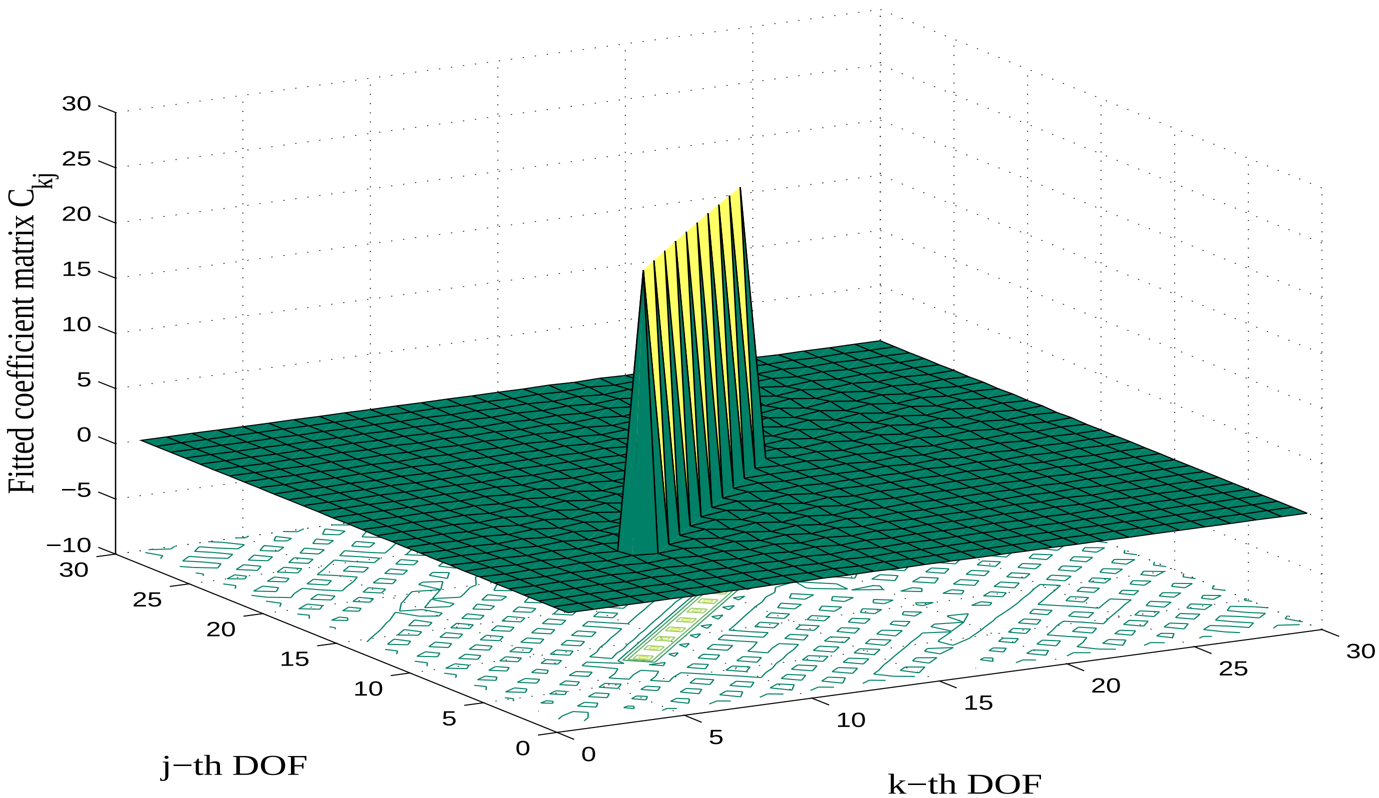
Fitted viscous damping matrix: $\gamma = 0.02$, damping model 2

Viscous damping identification



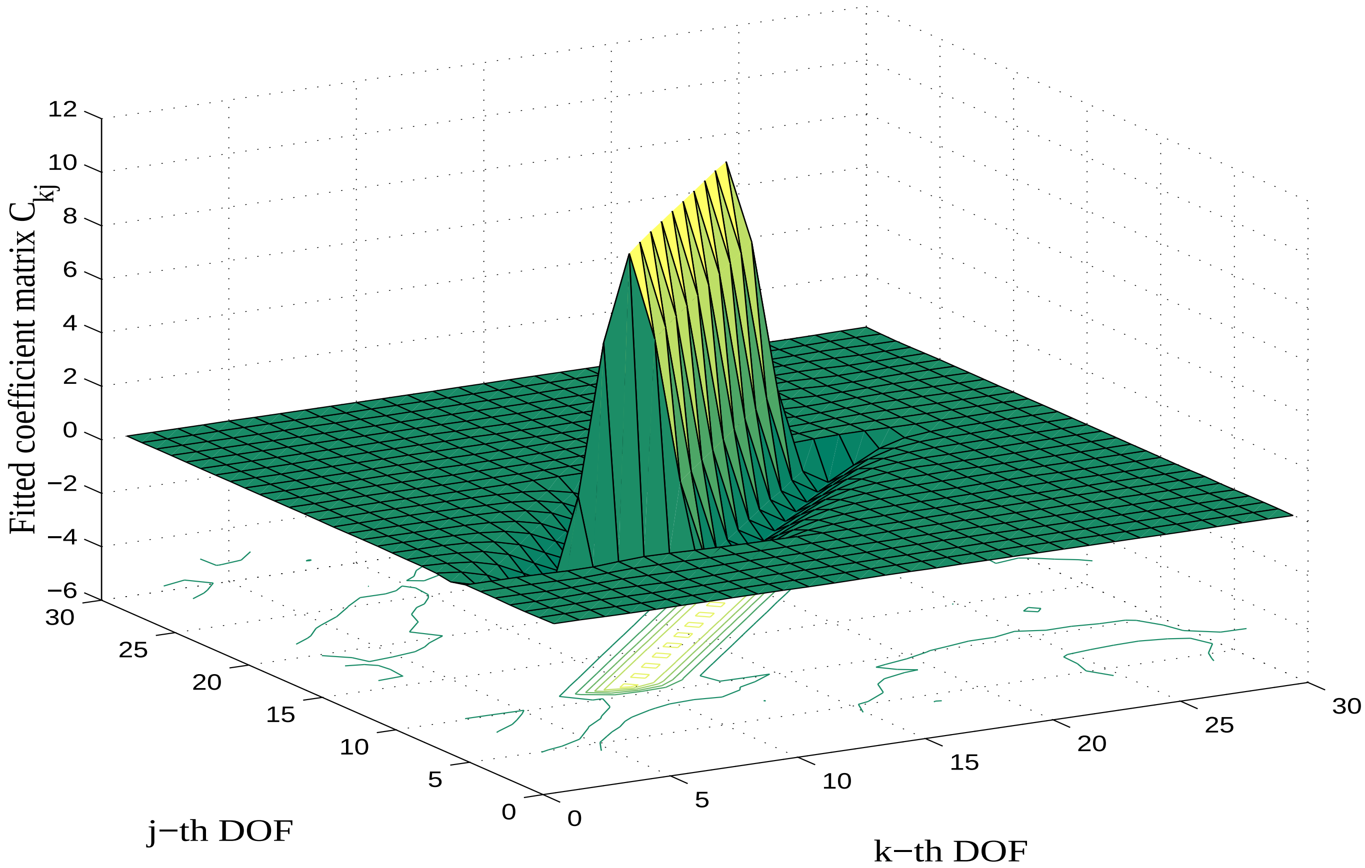
Fitted viscous damping matrix: $\gamma = 0.5$, damping model 1

Non-viscous Damping Identification



Fitted coefficient matrix: $\gamma = 0.5$, damping model 1; $\gamma_{\text{fit}} = 0.49$

Non-viscous damping identification

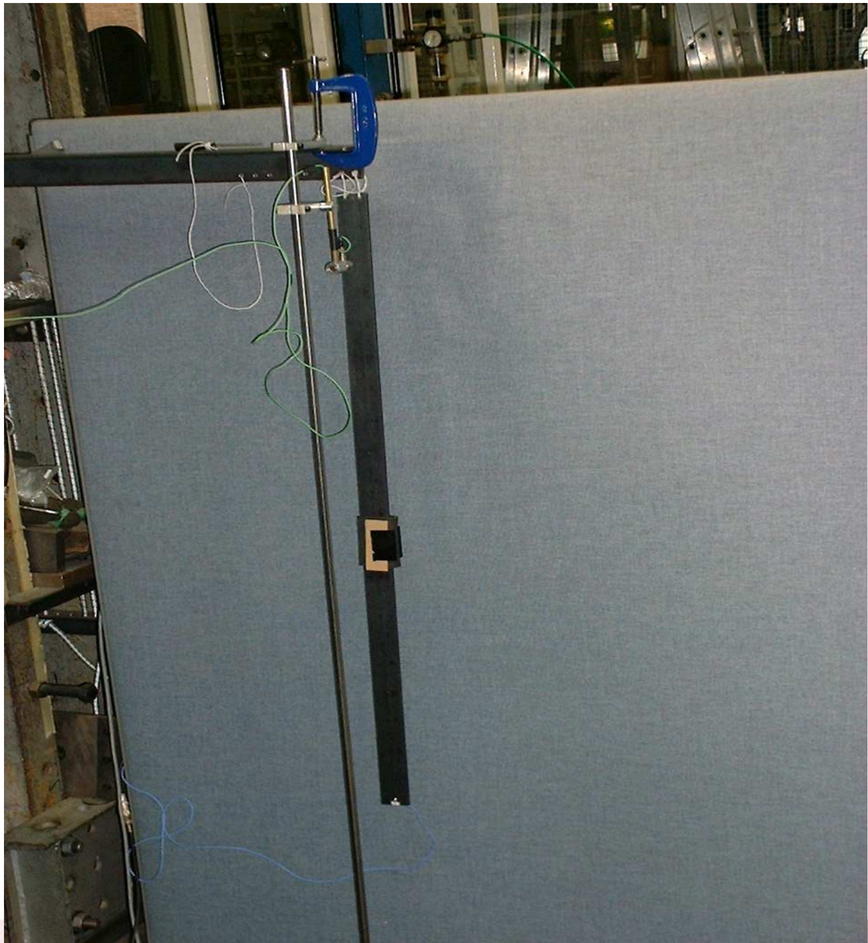


Fitted coefficient matrix: $\gamma = 0.5$, damping model 2; $\gamma_{\text{fit}} = 0.63$

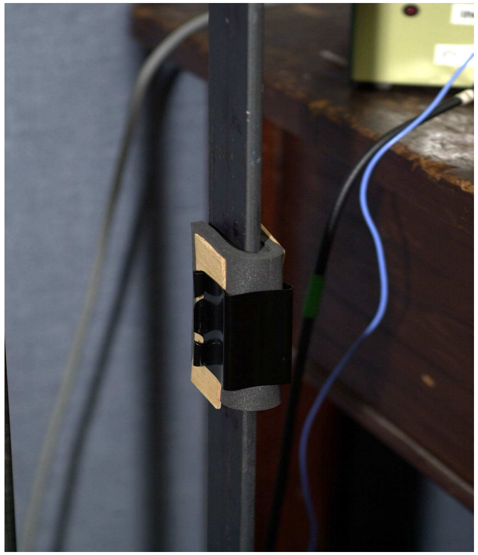
Experimental setup

Damped free-free beam, $L = 1$m,

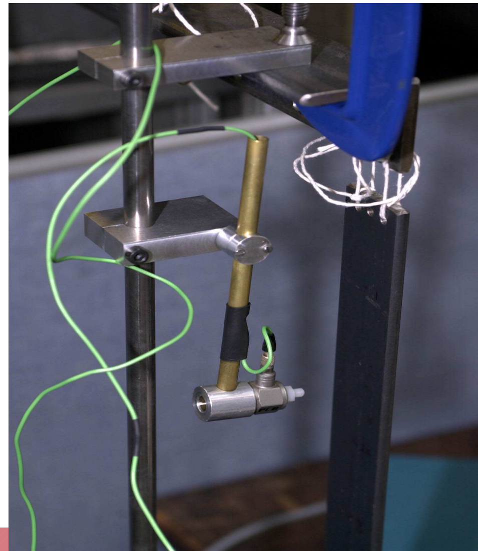
width = 39.0 mm, thickness = 5.93 mm



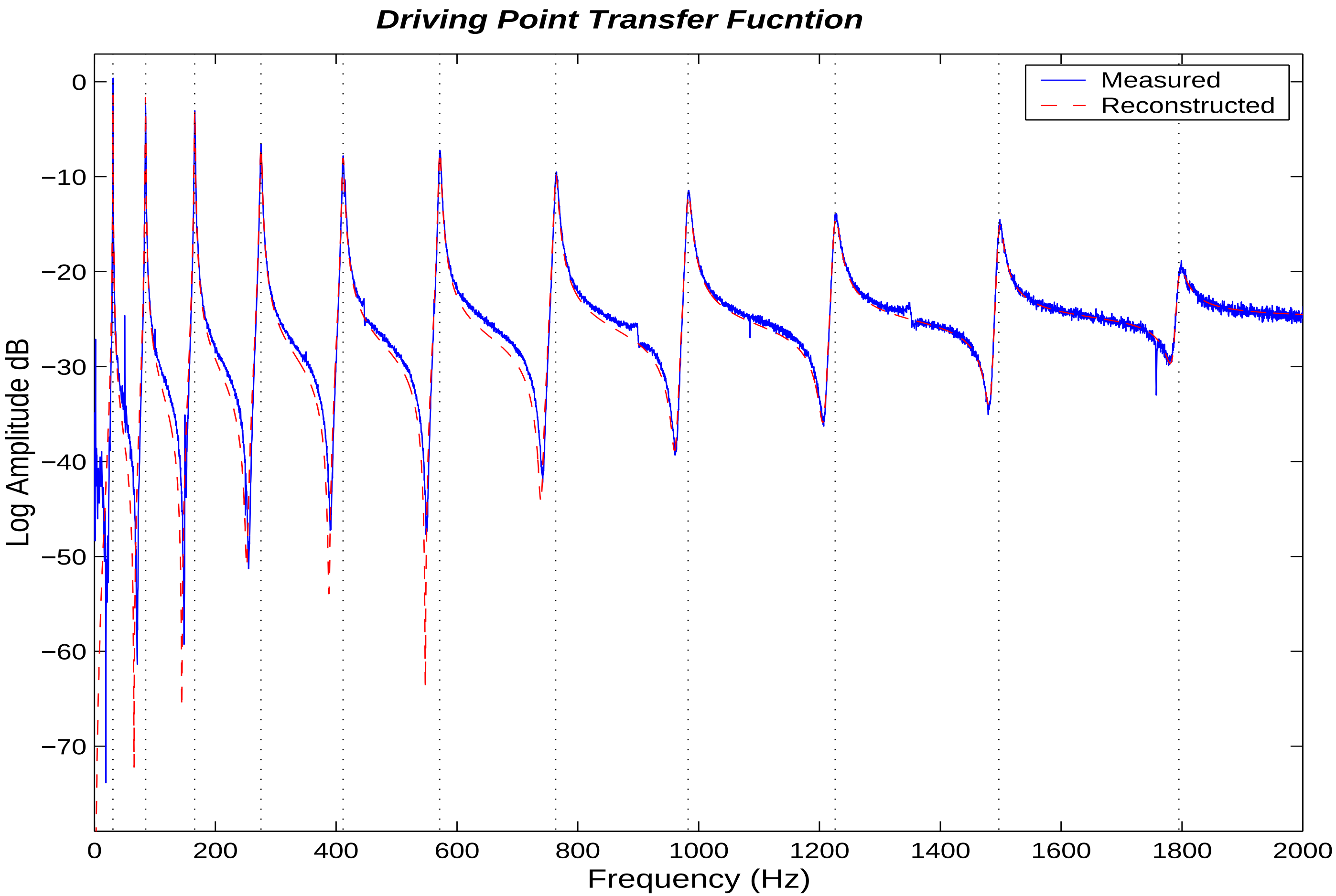
Clamped damping mechanism



Instrumented hammer for impulse input

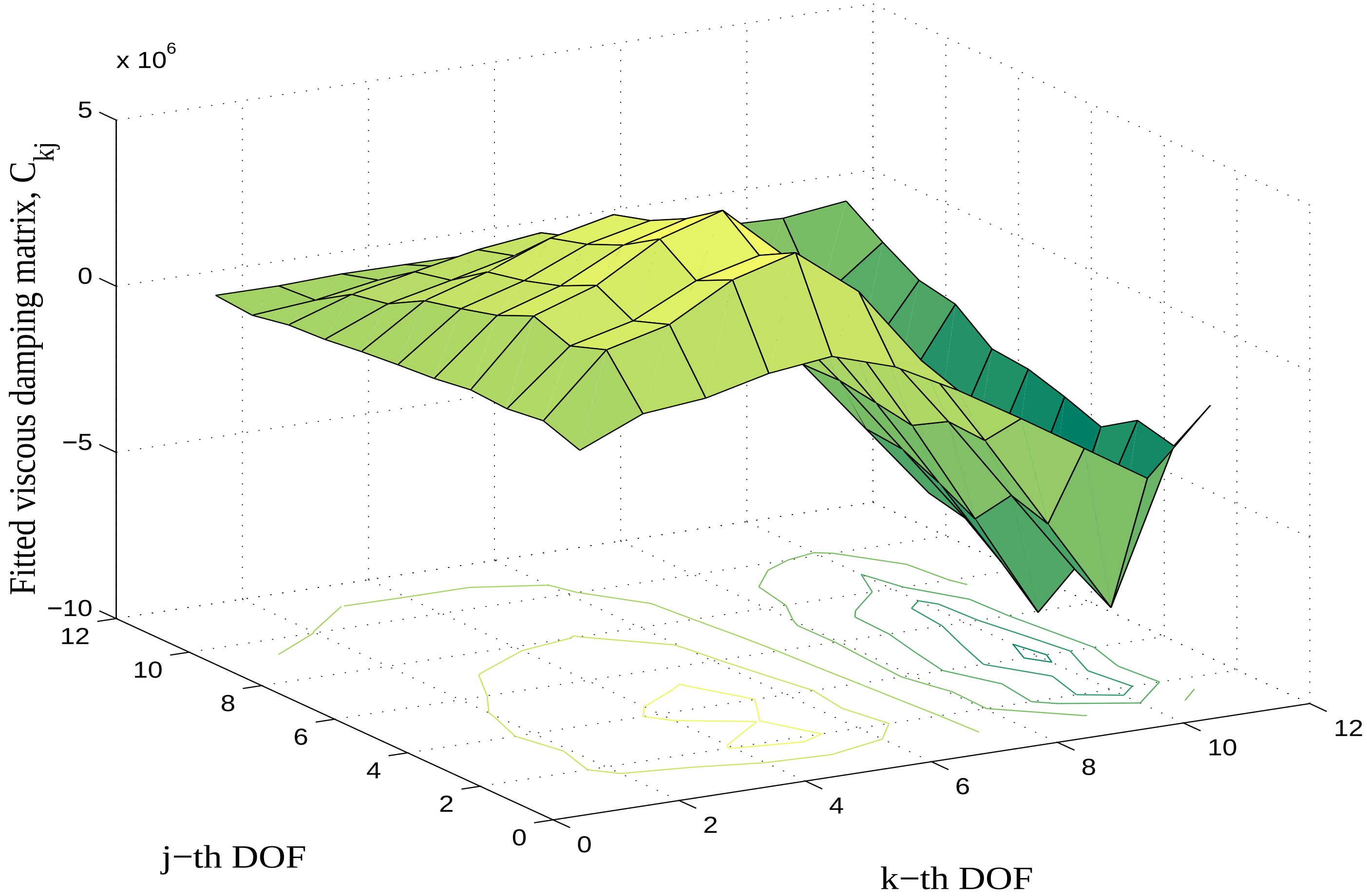


Measured transfer functions



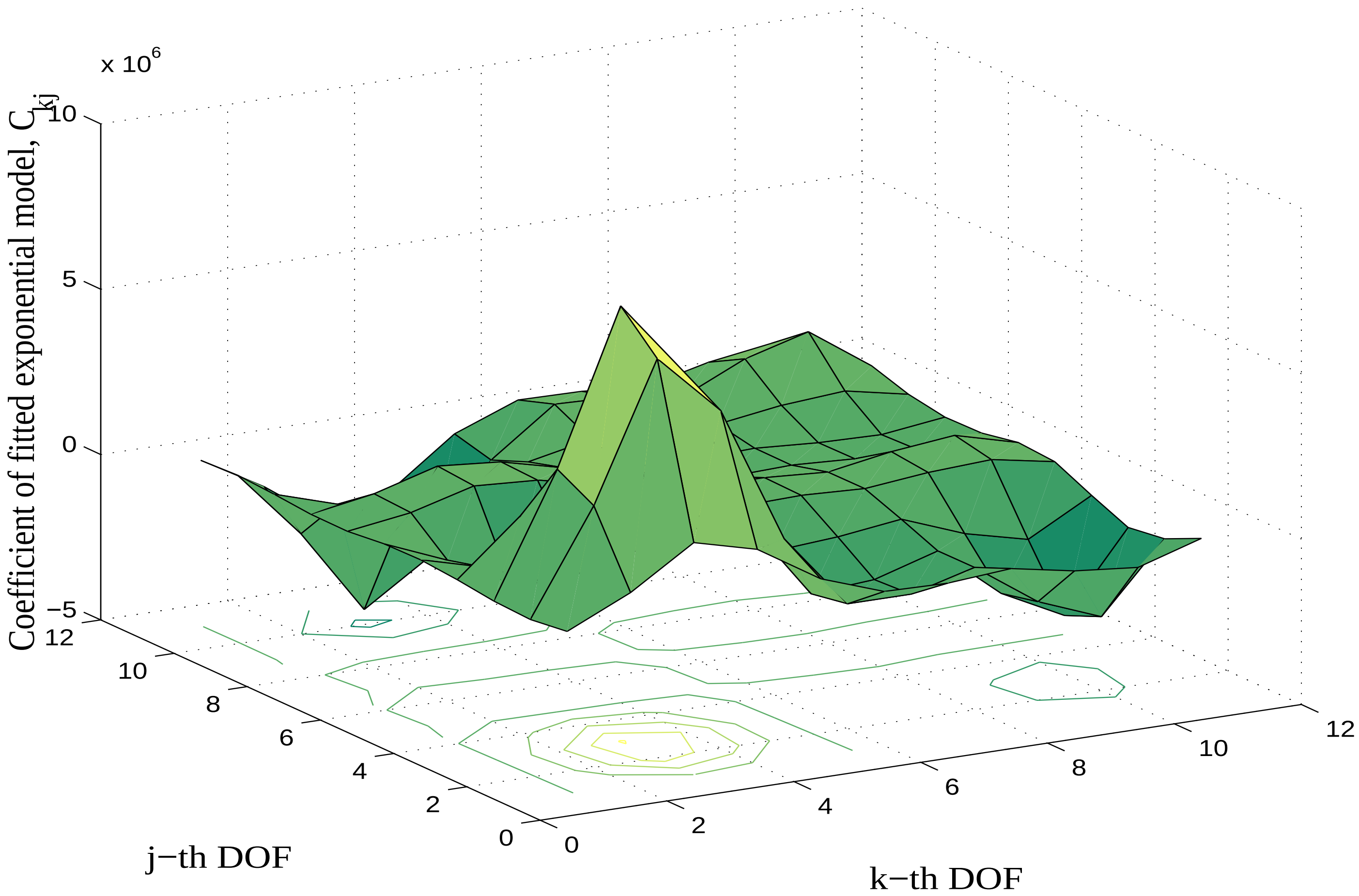
Measured and fitted transfer function of the beam

Viscous damping fitting



Fitted viscous damping matrix (damping between 4-5 nodes)

Non-viscous damping fitting



Fitted coefficient matrix (damping between 4-5 nodes); $\gamma_{\text{fit}} = 1.31$

Summary so far

- A method is proposed to identify a non-proportional non-viscous damping model in vibrating systems from complex modes and natural frequencies.
- If the fitted damping model is wrong, the procedure yields a non-physical result by fitting a non-symmetric coefficient matrix. That is, the procedure gives an indication that a wrong model is selected for fitting.

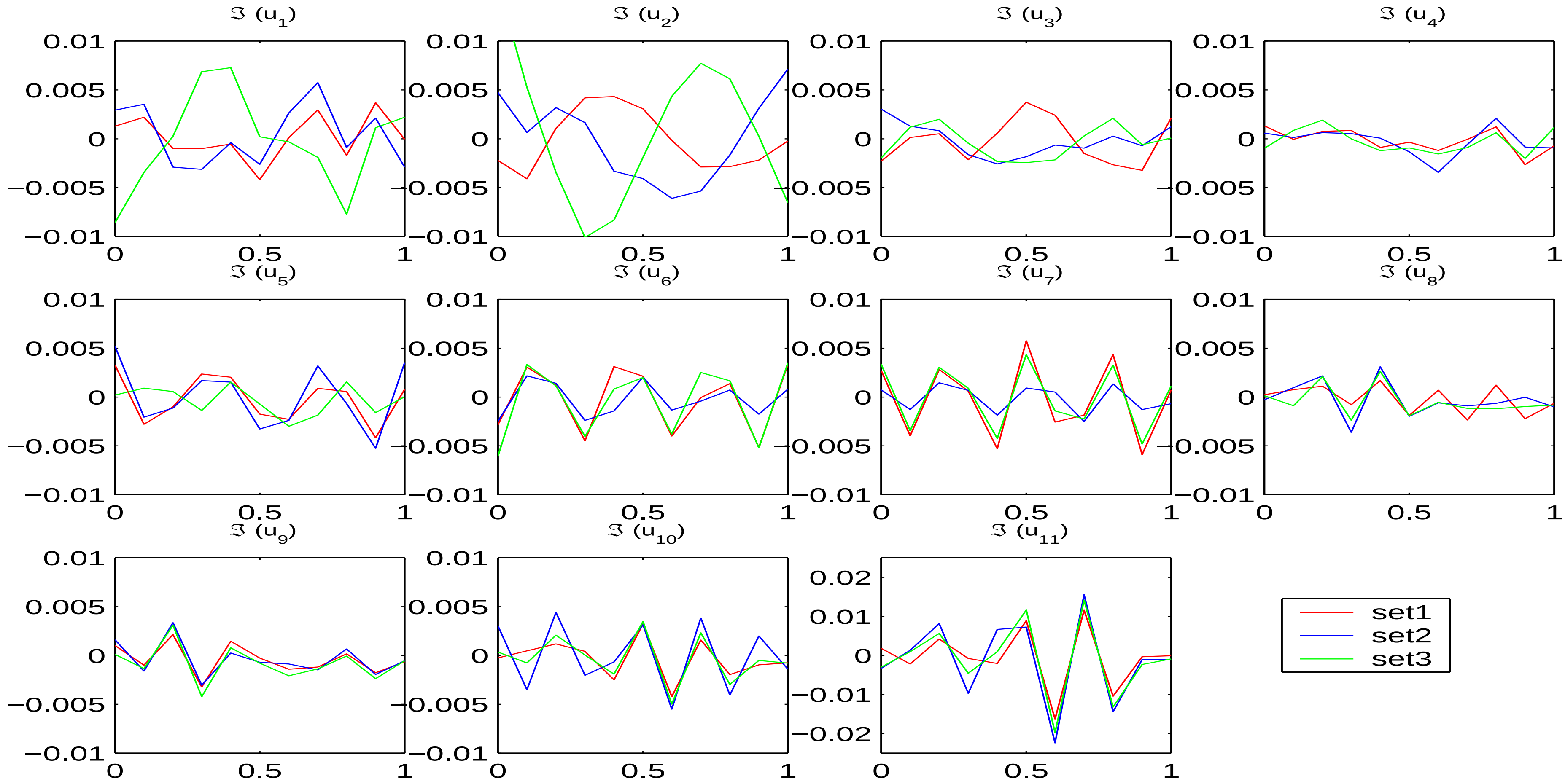
Difficulties with complex modes

- the expected ‘shapes’ of complex modes are not clear
- (complex) scaling of complex modes can change their geometric appearances
- the imaginary parts of the complex modes are usually very small compared to the real parts – makes it difficult to reliably extract complex modes

Difficulties with complex modes

- the phases of complex modes are highly sensitive to experimental errors, ambient conditions and measurement noise and often not repeatable in a satisfactory manner

Difficulties with complex modes

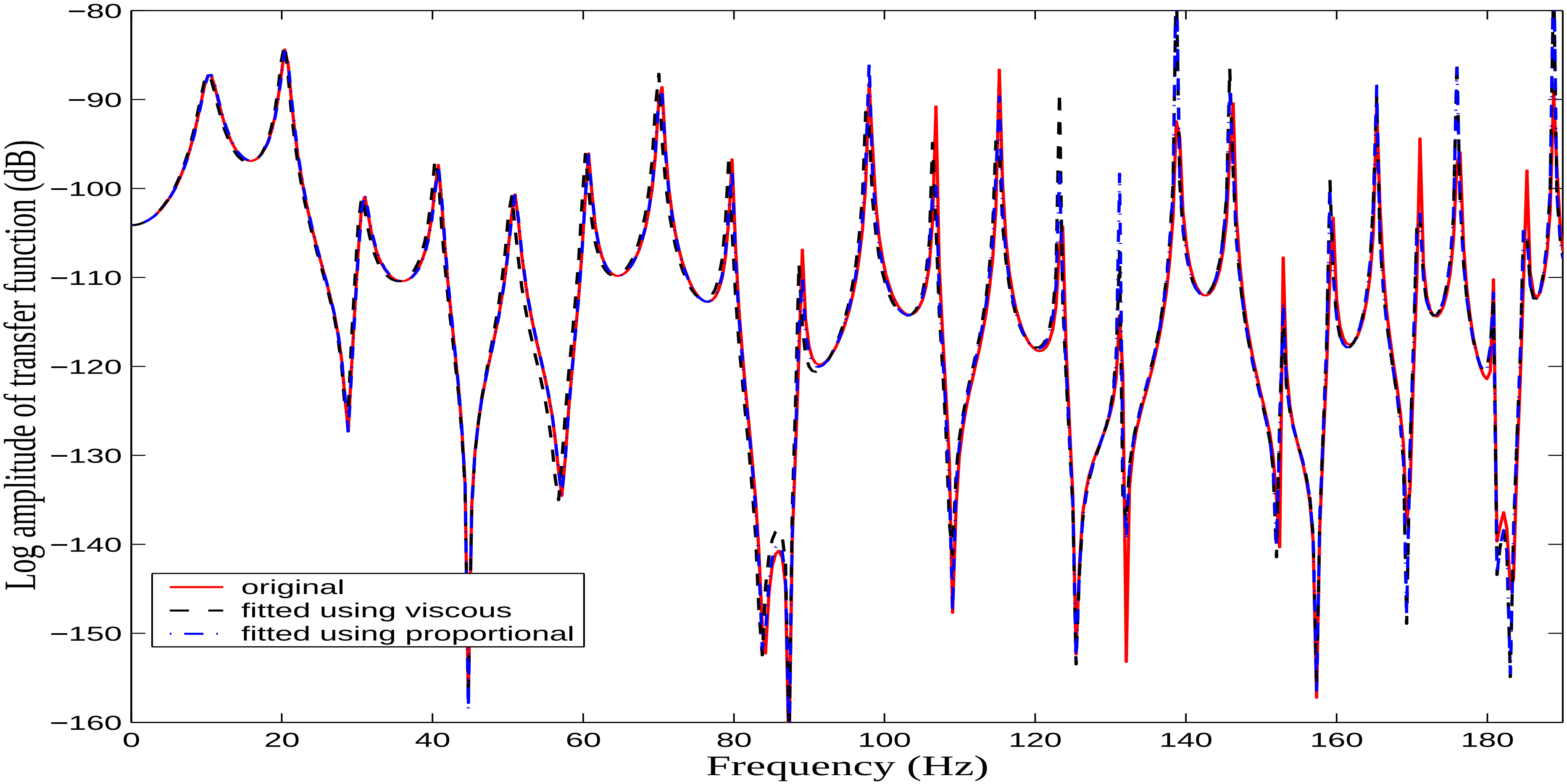


Imaginary parts of the identified complex modes

Proportional damping

- Avoids most of the problems associated with complex modes
- Can accurately reproduce transfer functions for systems with light damping

Transfer function



Some observations

- It is possible that more than one damping model with corresponding correct sets of parameters may represent the system response equally well.
- Different damping models can be fitted with the identified poles and residues of the transfer functions so that they are approximated accurately by all models.
- As a consequence proportional viscous damping can be used as a valid model.

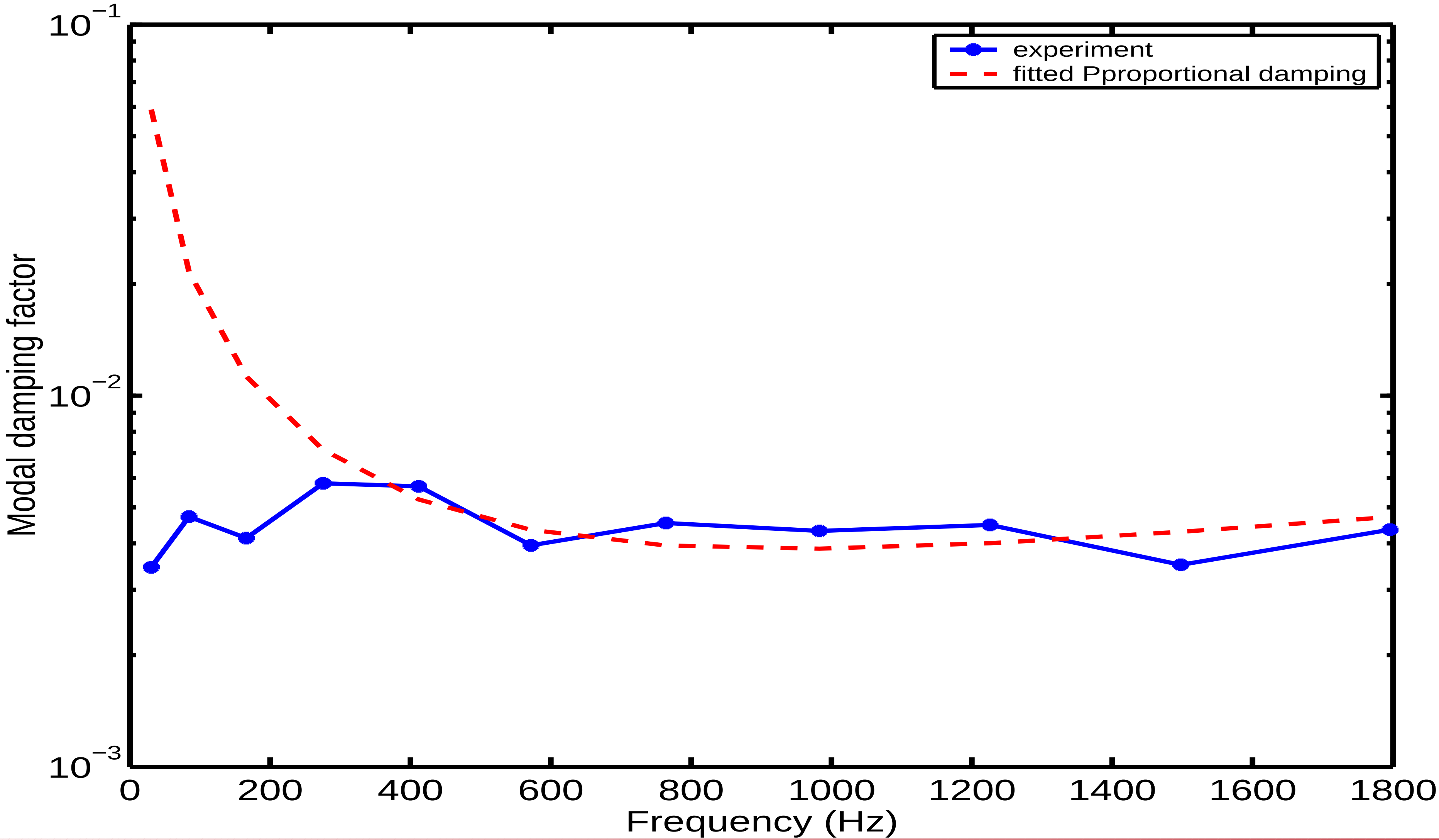
Limitations of proportional damping

- The modal damping factors:

$$\zeta_j = \frac{1}{2}\left(\frac{\alpha_1}{\omega_j} + \alpha_2 \omega_j\right)$$

- Not all forms of variation can be captured

Damping factors



Conditions for proportional damping

Theorem 1 *A viscously damped linear system can possess classical normal modes if and only if at least one of the following conditions is satisfied: (a)* $\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K}$*, (b)* $\mathbf{M}\mathbf{K}^{-1}\mathbf{C} = \mathbf{C}\mathbf{K}^{-1}\mathbf{M}$*, (c)* $\mathbf{M}\mathbf{C}^{-1}\mathbf{K} = \mathbf{K}\mathbf{C}^{-1}\mathbf{M}$*.*

This can be easily proved by following Caughey and O'Kelly's (1965) approach and interchanging $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ successively.

Caughey series

- Caughey series:

$$\mathbf{C} = \mathbf{M} \sum_{j=0}^{N-1} \alpha_j \left(\mathbf{M}^{-1}\mathbf{K}\right)^j$$

- The modal damping factors:

$$\zeta_j = \frac{1}{2}\left(\frac{\alpha_1}{\omega_j} + \alpha_2\omega_j + \alpha_3\omega_j^3 + \cdots\right)$$

- More general than Rayleigh's version of proportional damping

Generalized proportional damping

- Premultiply condition (a) of the theorem by $\mathbf{M}^{-1}$:

$$\left(\mathbf{M}^{-1}\mathbf{K}\right)\left(\mathbf{M}^{-1}\mathbf{C}\right)=\left(\mathbf{M}^{-1}\mathbf{C}\right)\left(\mathbf{M}^{-1}\mathbf{K}\right)$$

- Since $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{M}^{-1}\mathbf{C}$ are commutative matrices

$$\mathbf{M}^{-1}\mathbf{C}=f_1(\mathbf{M}^{-1}\mathbf{K})$$

- Therefore, we can express the damping matrix as

$$\mathbf{C}=\mathbf{M}f_1(\mathbf{M}^{-1}\mathbf{K})$$

Generalized proportional damping

- Premultiply condition (b) of the theorem by $\mathbf{K}^{-1}$:

$$\left(\mathbf{K}^{-1}\mathbf{M}\right)\left(\mathbf{K}^{-1}\mathbf{C}\right)=\left(\mathbf{K}^{-1}\mathbf{C}\right)\left(\mathbf{K}^{-1}\mathbf{M}\right)$$

- Since $\mathbf{K}^{-1}\mathbf{M}$ and $\mathbf{K}^{-1}\mathbf{C}$ are commutative matrices

$$\mathbf{K}^{-1}\mathbf{C}=f_2(\mathbf{K}^{-1}\mathbf{M})$$

- Therefore, we can express the damping matrix as

$$\mathbf{C}=\mathbf{K}f_1(\mathbf{K}^{-1}\mathbf{M})$$

Generalized proportional damping

- Combining the previous two cases

$$\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$$

- Similarly, postmultiplying condition (a) of Theorem 1 by $\mathbf{M}^{-1}$ and (b) by $\mathbf{K}^{-1}$ we have

$$\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$$

- Special case: $\beta_i(\bullet) = \alpha_i \mathbf{I} \to$ Rayleigh damping.

Generalized proportional damping

Theorem 2 *A viscously damped positive definite linear system possesses classical normal modes if and only if* $\mathbf{C}$ *can be represented by*

(a) $\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$*, or*

(b) $\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$

for any $\beta_i(\bullet), i = 1, \cdots, 4$*.*

Example 1

Equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}} + \left[\mathbf{M}e^{-\left(\mathbf{M}^{-1}\mathbf{K}\right)^2/2}\sinh(\mathbf{K}^{-1}\mathbf{M}\ln(\mathbf{M}^{-1}\mathbf{K})^{2/3}) + \mathbf{K}\cos^2(\mathbf{K}^{-1}\mathbf{M})\sqrt[4]{\mathbf{K}^{-1}\mathbf{M}}\,\tan^{-1}\frac{\sqrt{\mathbf{M}^{-1}\mathbf{K}}}{\pi}\right]\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

It can be shown that the system has real modes and

$$2\xi_j\omega_j = e^{-\omega_j^4/2}\sinh\left(\frac{1}{\omega_j^2}\ln\frac{4}{3}\omega_j\right) + \omega_j^2\cos^2\left(\frac{1}{\omega_j^2}\right)\frac{1}{\sqrt{\omega_j}}\tan^{-1}\frac{\omega_j}{\pi}.$$

Damping identification method

To simplify the identification procedure, express the damping matrix by

$$\mathbf{C} = \mathbf{M} f\left(\mathbf{M}^{-1}\mathbf{K}\right)$$

Using this simplified expression, the modal damping factors can be obtained as

$$2\zeta_j\omega_j = f\left(\omega_j^2\right)$$

$$\text{or} \quad \zeta_j = \frac{1}{2\omega_j} f\left(\omega_j^2\right) = \widehat{f}(\omega_j) \quad \text{(say)}$$

Damping identification method

- The function $\widehat{f}(\bullet)$ can be obtained by fitting a continuous function representing the variation of the measured modal damping factors with respect to the frequency
- With the fitted function $\widehat{f}(\bullet)$, the damping matrix can be identified as

$$2\zeta_j\omega_j = 2\omega_j\widehat{f}(\omega_j)$$

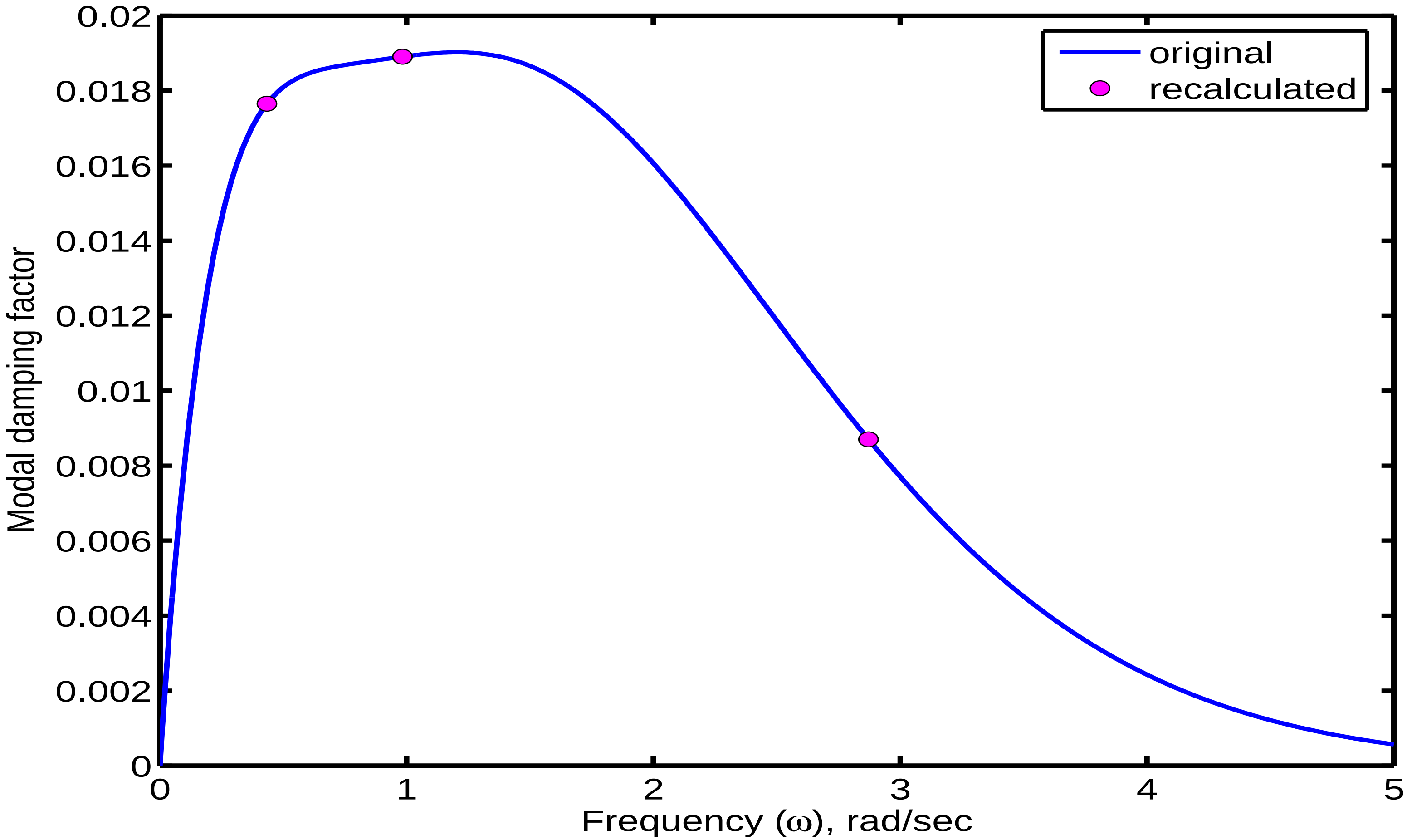
$$\text{or} \quad \widehat{\mathbf{C}} = 2\mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\,\widehat{f}\left(\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)$$

Example 2

Consider a 3DOF system with mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix}$$

Example 2



Damping factors

Example 2

Here this (continuous) curve was simulated using the equation

$$\widehat{f}(\omega) = \frac{1}{15}\left(e^{-2.0\omega} - e^{-3.5\omega}\right)\left(1 + 1.25\sin\frac{\omega}{7\pi}\right)\left(1 + 0.75\omega^3\right)$$

From the above equation, the modal damping factors in terms of the discrete natural frequencies, can be obtained by

$$2\xi_j\omega_j = \frac{2\omega_j}{15}\left(e^{-2.0\omega_j} - e^{-3.5\omega_j}\right)\left(1 + 1.25\sin\frac{\omega_j}{7\pi}\right)\left(1 + 0.75\omega_j^3\right).$$

Example 2

To obtain the damping matrix, consider the preceding equation as a function of ω_j^2 and replace ω_j^2 by $\mathbf{M}^{-1}\mathbf{K}$ and any constant terms by that constant times $\mathbf{I}$. Therefore:

$$\mathbf{C} = \mathbf{M}\,\frac{2}{15}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\left[e^{-2.0\sqrt{\mathbf{M}^{-1}\mathbf{K}}} - e^{-3.5\sqrt{\mathbf{M}^{-1}\mathbf{K}}}\right] \times \left[\mathbf{I} + 1.25\sin\left(\frac{1}{7\pi}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)\right]\left[\mathbf{I} + 0.75(\mathbf{M}^{-1}\mathbf{K})^{3/2}\right]$$

Steps to follow

1. Measure a suitable transfer function $H_{ij}(\omega)$
2. Obtain the undamped natural frequencies ω_j and modal damping factors ζ_j
3. Fit a function $\zeta = \widehat{f}(\omega)$ which represents the variation of ζ_j with respect to ω_j for the range of frequency considered in the study
4. Calculate the matrix $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$
5. Obtain the damping matrix using $\widehat{\mathbf{C}} = 2\,\mathbf{M}\,\mathbf{T}\,\widehat{f}\,(\mathbf{T})$

Summary(1)

- Rayleigh s proportional damping is generalized
- The generalized proportional damping expresses the damping matrix in terms of any non-linear function involving specially arranged mass and stiffness matrices so that the system still posses classical normal modes
- This enables one to model practically any type of variations in the modal damping factors with respect to the frequency

Summary(2)

- Once a scalar function is fitted to model such variations, the damping matrix can be identified very easily using the proposed method
- The method is very simple and requires the measurement of damping factors and natural frequencies only (that is, the measurements of the mode shapes are not necessary)
- The proposed method is applicable to any linear structures as long as one have validated mass and stiffness matrix models which can predict the natural frequencies accurately and modes are not significantly complex

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