Probabilistic Structural Analysis Using Matrix Variate Distributions

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Stochastic structural dynamics

- The equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The objective is to quantify uncertainties in the response vector $\mathbf{x}$.

Current Methods

Three different approaches are currently available

- Low frequency : Stochastic Finite Element Method (SFEM) - considers parametric uncertainties in details
- High frequency : Statistical Energy Analysis (SEA) - do not consider parametric uncertainties in details
- Mid-frequency : Hybrid method - ‘combination’ of the above two

Random Matrix Method (RMM)

- The objective: To have a simple unified method which will work across the frequency range.
- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Outline of the presentation

In what follows next, I will discuss:

- Introduction to Matrix variate distributions
- ‘Maximal uncertain’ distribution
- Distributions under inverse moment constraints
- Optimal Wishart distribution
- Response statistics using Wishart distribution
- Numerical examples
- Open problems & discussions

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (1)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Gaussian orthogonal ensembles

A random matrix $\mathbf{H} \in \mathbb{R}_{n,n}$ belongs to the Gaussian orthogonal ensemble (GOE) provided its pdf of is given by

$$p_{\mathbf{H}}(\mathbf{H}) = \exp\left(-\theta_2 \mathrm{Trace}\left(\mathbf{H}^2\right) + \theta_1 \mathrm{Trace}\left(\mathbf{H}\right) + \theta_0\right)$$

where θ_2 is real and positive and θ_1 and θ_0 are real.

This is a good model for high-frequency vibration problems.

Wishart matrix

An $n \times n$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n \left(\frac{1}{2} p \right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{S} \right\} \tag{2}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Matrix variate Gamma distribution

An $n \times n$ random symmetric positive definite matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) |\mathbf{\Psi}|^{-a}\right\}^{-1} |\mathbf{W}|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\}; \quad \Re(a) > (n-1)/2 \quad (3)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \quad (4)$$

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{5}$$

and $$\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} \tag{6}$$

MEnt Distribution - 1

The Lagrangian to be maximised:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} \, d\mathbf{G} + \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - 1\right) + \operatorname{Trace}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - \overline{\mathbf{G}}\right]\right) \tag{7}$$

$\lambda_0 \in \mathbb{R}$ and $\mathbf{\Lambda}_1 \in \mathbb{R}_{n,n}$ are the unknown Lagrange multiplies to be determined.

MEnt Distribution - 2

Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

or $$(\lambda_0 - 1) + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \mathbf{G}\right) - \left(1 + \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\}\right) = 0$$

or $$-\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \mathbf{G}\right)$$

or $$p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\} \mathrm{etr}\left\{-\mathbf{\Lambda}_1 \mathbf{G}\right\} \tag{8}$$

MEnt Distribution - 3

Substituting into the constraint equations results

$$p_{\mathbf{G}}(\mathbf{G}) = r^{nr} \{\Gamma_n(r)\}^{-1} \left|\overline{\mathbf{G}}\right|^{-r} \operatorname{etr}\left\{-r\overline{\mathbf{G}}^{-1}\mathbf{G}\right\} \quad (9)$$

where $r = \frac{1}{2}(n+1)$. Comparing, it can be observed that $\mathbf{G}$ has the Wishart distribution with parameters $p = n+1$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/(n+1)$.

Theorem 1. *If only the mean of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ is available, say $\overline{\mathbf{G}}$, then the matrix has a Wishart distribution with parameters $(n+1)$ and $\overline{\mathbf{G}}/(n+1)$, that is $\mathbf{G} \sim W_n\left(n+1, \overline{\mathbf{G}}/(n+1)\right)$.*

Further constraints

- Suppose the inverse moments (say up to order ν) of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}}=\left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^{T}\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln|\mathbf{G}|^{-\nu}\right]<\infty$$

MEnt Distribution - Again!

The new Lagrangian becomes:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} \, d\mathbf{G} + \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - 1\right) - \nu \int_{\mathbf{G}>0} \ln\left|\mathbf{G}\right| p_{\mathbf{G}} \, d\mathbf{G} + \mathrm{Trace}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0} \mathbf{G} \, p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - \overline{\mathbf{G}}\right]\right) \quad (10)$$

Note: ν cannot be obtained uniquely!

MEnt Distribution - 2A

Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

or $-\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \mathbf{G}\right) - \ln\left|\mathbf{G}\right|^{\nu}$

or $p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\} \left|\mathbf{G}\right|^{\nu} \mathrm{etr}\left\{-\mathbf{\Lambda}_1 \mathbf{G}\right\}$

We use the matrix variate Laplace transform:

$$\int_{\mathbf{\Lambda}>0} \mathrm{etr}\left\{-\mathbf{\Lambda}\mathbf{Z}\right\} \left|\mathbf{\Lambda}\right|^{a-(p+1)/2} d\mathbf{\Lambda} = \Gamma_p(a) \left|\mathbf{Z}\right|^{-a}$$

MEnt Distribution - 3A

Substituting into the constraint equations we have:

Theorem 2. *If ν-th order inverse moment of a system matrix $\mathbf{G}$ exists and only the mean is available, say $\overline{\mathbf{G}}$, then the matrix has a Wishart distribution with parameters $(2\nu + n + 1)$ and $\overline{\mathbf{G}}/(2\nu + n + 1)$, that is*
$$\mathbf{G} \sim W_n\left(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1)\right).$$

Note that $\nu = 0$ gives us the 'maximal uncertain distribution' derived before.

Properties of the Distribution

- Covariance tensor of G:

$$\operatorname{cov}\left(G_{ij}, G_{kl}\right) = \frac{1}{2\nu + n + 1}\left(\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}\right) \tag{11}$$

- Normalized standard deviation matrix $\mathrm{E}\left[(\mathbf{G} - \overline{\mathbf{G}})^2\right]\overline{\mathbf{G}}^{-2}$:

$$\sigma^2_{\mathbf{G}} = \frac{1}{2\nu + n + 1}\left[\mathbf{I}_n + \overline{\mathbf{G}}^{-1}\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\right] \tag{12}$$

- $\nu \uparrow \Rightarrow \sigma^2_{\mathbf{G}} \downarrow$.

Distribution of the inverse - 1

- If $\mathbf{G}$ is $W_n(p, \mathbf{\Sigma})$ then $\mathbf{V} = \mathbf{G}^{-1}$ has the inverted Wishart distribution:

$$P_{\mathbf{V}}(\mathbf{V}) = \frac{2^{m-n-1}n/2 \left|\mathbf{\Psi}\right|^{m-n-1} /2}{\Gamma_n[(m-n-1)/2] \left|\mathbf{V}\right|^{m/2}} \mathrm{etr}\left\{-\frac{1}{2}\mathbf{V}^{-1}\mathbf{\Psi}\right\}$$

where $m = n + p + 1$ and $\mathbf{\Psi} = \mathbf{\Sigma}^{-1}$ (recall that $p = 2\nu + n + 1$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/p$)

Distribution of the inverse - 2

- Mean: $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \dfrac{p\overline{\mathbf{G}}^{-1}}{p-n-1}$
- Normalized standard deviation matrix $\mathrm{E}\left[\left(\mathbf{G}^{-1} - \overline{\mathbf{G}}^{-1}\right)^2\right]\overline{\mathbf{G}}^2$:

$$\sigma^2_{\mathbf{G}^{-1}} = \frac{(p-n-1)}{(p-n)(p-n-3)}\left[\mathbf{I}_n + \overline{\mathbf{G}}\mathrm{Trace}\left(\overline{\mathbf{G}}^{-1}\right)\right] \tag{13}$$

Distribution of the inverse - 3

- Suppose $n = 101$ & $\nu = 2$. So $p = 2\nu + n + 1 = 106$ and $p - n - 1 = 4$. Therefore, $\mathrm{E}\left[\mathbf{G}\right] = \overline{\mathbf{G}}$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \frac{106}{4}\overline{\mathbf{G}}^{-1} = 26.5\overline{\mathbf{G}}^{-1}$!!!!!!!!!!
- Of course there is no reason why $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \overline{\mathbf{G}}^{-1}$. But from a practical point of view do we expect them to be so far apart?
- One way to reduce the gap is to increase p. But this implies reduction of variance.

Some questions

- What do we really need: $\mathrm{E}\left[\mathbf{G}\right] = \overline{\mathbf{G}}$ or $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \overline{\mathbf{G}}^{-1}$ or any other powers.
- $\overline{\mathbf{G}}$ is just one ‘observation’ - not an ensemble mean.
- What happens if we know the covariance tensor of $\mathbf{G}$ (e.g., using Stochastic Finite element Method)?
- What if the zeros in $\mathbf{G}$ are not preserved?

Optimal Wishart Distribution - 1

- Suppose $\mathbf{G} \sim W_n(m, \mathbf{\Sigma})$ and $\mathbf{A} \in \mathbb{R}_n^+$ is the deterministic value of a system matrix.
- My argument: The distribution of $\mathbf{G}$ must be such that $\mathrm{E}[\mathbf{G}]$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right]$ should be closest to $\mathbf{A}$ and $\mathbf{A}^{-1}$ respectively.
- Therefore, define (and subsequently minimize) 'normalized errors':
$\boldsymbol{\varepsilon}_1 = [\mathbf{A} - \mathrm{E}[\mathbf{G}]] \mathbf{A}^{-1} \in \mathbb{R}_n$
$\boldsymbol{\varepsilon}_2 = \left[\mathbf{A}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right] \mathbf{A} \in \mathbb{R}_n$

Optimal Wishart Distribution - 2

- Obtain m and $\mathbf{\Sigma}$ such that $\chi^2 = \|\boldsymbol{\varepsilon}_1\|_{\mathrm{F}}^2 + \|\boldsymbol{\varepsilon}_2\|_{\mathrm{F}}^2 = \mathrm{Trace}\left(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1^T\right) + \mathrm{Trace}\left(\boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_2^T\right)$ is minimized.
- Suppose $m = n + 1 + \theta$ and $\mathbf{\Sigma} = \mathbf{AX}$. Using these we have $\boldsymbol{\varepsilon}_1 = [\mathbf{I}_n - (n+1+\theta)\mathbf{X}]$ and $\boldsymbol{\varepsilon}_2 = \left[\mathbf{I}_n - \{\theta\mathbf{X}\}^{-1}\right]$
- Since $\mathbf{\Sigma} = \mathbf{\Sigma}^T$ and $\mathbf{A} = \mathbf{A}^T$, we have $\mathbf{AX} = \mathbf{X}^T\mathbf{A}$ or $\mathbf{X}^T = \mathbf{AXA}^{-1}$
- Set $\frac{\partial \chi^2}{\partial \theta} = 0$ and $\frac{\partial \chi^2}{\partial \mathbf{X}} = \mathbf{O}$. Total n^2+1 unknowns and n^2+1 equations - can be solved.

Optimal Wishart Distribution - 3

We obtain

$$\theta^4 \mathrm{Trace}\left(\mathbf{XAXA}^{-1}\right) + \theta^3 \left\{(n+1)\mathrm{Trace}\left(\mathbf{XAXA}^{-1}\right) - \mathrm{Trace}\left(\mathbf{X}\right)\right\} + \theta \mathrm{Trace}\left(\mathbf{X}^{-1}\right) - \mathrm{Trace}\left(\mathbf{X}^{-1}\mathbf{AX}^{-1}\mathbf{A}^{-1}\right) = 0 \quad (14)$$

$$\mathbf{AXA}^{-1} + \mathbf{A}^{-1}\mathbf{XA} = \frac{2\mathbf{I}_n}{n+1+\theta} - \boldsymbol{\xi}(\theta, \mathbf{X}) \quad (15)$$

$$\boldsymbol{\xi}(\theta, \mathbf{X}) = (n+1+\theta)^{-2}[2\theta^{-1}\mathbf{X}^{-2} + \theta^{-2}\mathbf{X}^{-1}[\mathbf{AX}^{-1}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{X}^{-1}\mathbf{A}]\mathbf{X}^{-1}] \quad (16)$$

Optimal Wishart Distribution - 4

Use iteration to solve the coupled nonlinear scalar-matrix equations

1. Start with $\theta = 2$, $\mathbf{X} = \mathbf{I}_n/(n+1+\theta)$
2. Solve θ from the fourth order equation (14)
3. Obtain $\mathbf{X}_{new}$ from (by taking $\mathrm{vec}\,(\bullet)$ of Eq. (15))

$$[\mathbf{A}^{-1} \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{A}^{-1}]\mathrm{vec}\,(\mathbf{X}_{new}) = \frac{2\mathbf{I}_n}{n+1+\theta}\mathrm{vec}\,(\mathbf{I}_n) - \mathrm{vec}\,(\boldsymbol{\xi}(\theta, \mathbf{X}))$$

4. If $\|\mathbf{X}_{new} - \mathbf{X}\|_{\mathrm{F}} <$ ‘small number’ then stop. Otherwise, set $\mathbf{X} = \mathbf{X}_{new}$ and go back to step 2.

Response statistics - 1

- The equation of motion is $\mathbf{D}\mathbf{x} = \mathbf{p}$, $\mathbf{D}$ is in general $n \times n$ complex random matrix.
- The response is given by

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{p}$$

- Consider static problems so that all matrices/vectors are real.

Response statistics - 2

- We may want statistics of few elements or some linear combinations of the elements in $\mathbf{x}$. So the quantify of interest is

$$\mathbf{y} = \mathbf{R}\mathbf{x} = \mathbf{R}\mathbf{D}^{-1}\mathbf{p} \tag{17}$$

  Here $\mathbf{R}$ is in general $r \times n$ rectangular matrix. For the special case when $\mathbf{R} = \mathbf{I}_n$, we have $\mathbf{y} = \mathbf{x}$.

- Eq. (17) arises in SFEM. There are many papers on its solution. Mainly perturbation methods are used.

Response statistics - 3

Suppose $\mathbf{D} = \mathbf{D}_0 + \mathbf{\Delta D}$, where $\mathbf{D}_0$ is the deterministic part and $\mathbf{\Delta D}$ is the (small) random part. It can be shown that

$$\mathbf{D}^{-1} = \mathbf{D}_0 - \mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1} + \mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1} + \cdots$$

From, this

$$\mathbf{y} = \mathbf{y}_0 - \mathbf{R}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{x}_0 + \mathbf{R}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{x}_0 + \cdots \quad (18)$$

where $\mathbf{x}_0 = \mathbf{D}_0^{-1}\mathbf{p}$ and $\mathbf{y}_0 = \mathbf{R}\mathbf{x}_0$.

Response statistics - 4

The statistics of $\mathbf{y}$ can be calculated from Eq. (18). However,

- The calculation is difficult if $\mathbf{\Delta D}$ is non-Gaussian.
- Even if $\mathbf{\Delta D}$ is Gaussian, inclusion of higher-order terms results very messy calculations (I have not seen any published work for more than second-order)
- For these reasons, the response statistics will be inaccurate for large randomness.

Response statistics - 5

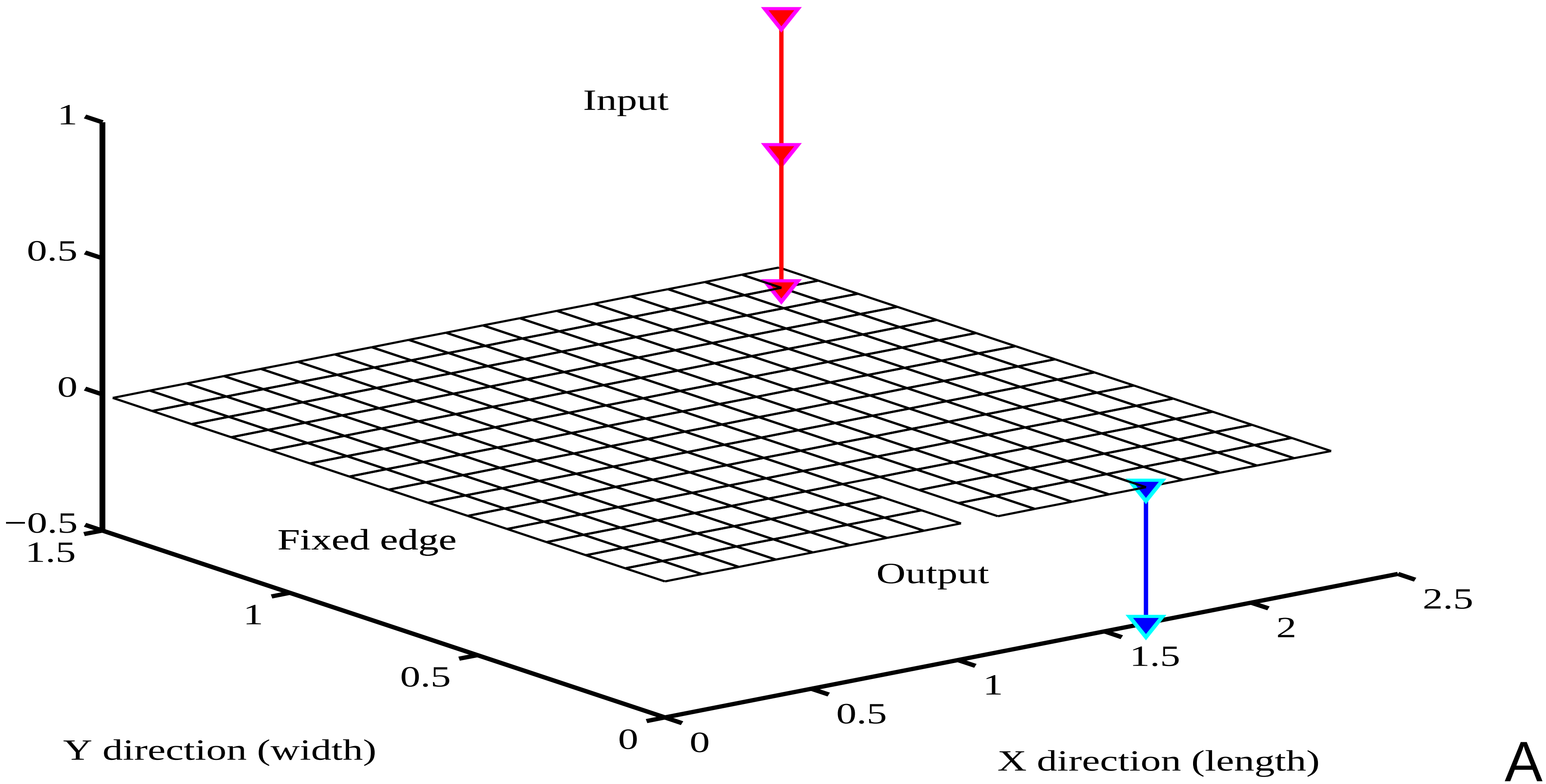
I will propose an exact method using RMM.
Suppose $\mathbf{D} \sim W_n(m, \mathbf{\Sigma})$.

$$\mathrm{E}[\mathbf{y}] = \mathrm{E}\left[\mathbf{R}\mathbf{D}^{-1}\mathbf{p}\right] = \mathbf{R}\,\mathrm{E}\left[\mathbf{D}^{-1}\right]\mathbf{p} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{p}/\theta \quad (19)$$

The complete covariance matrix of $\mathbf{y}$

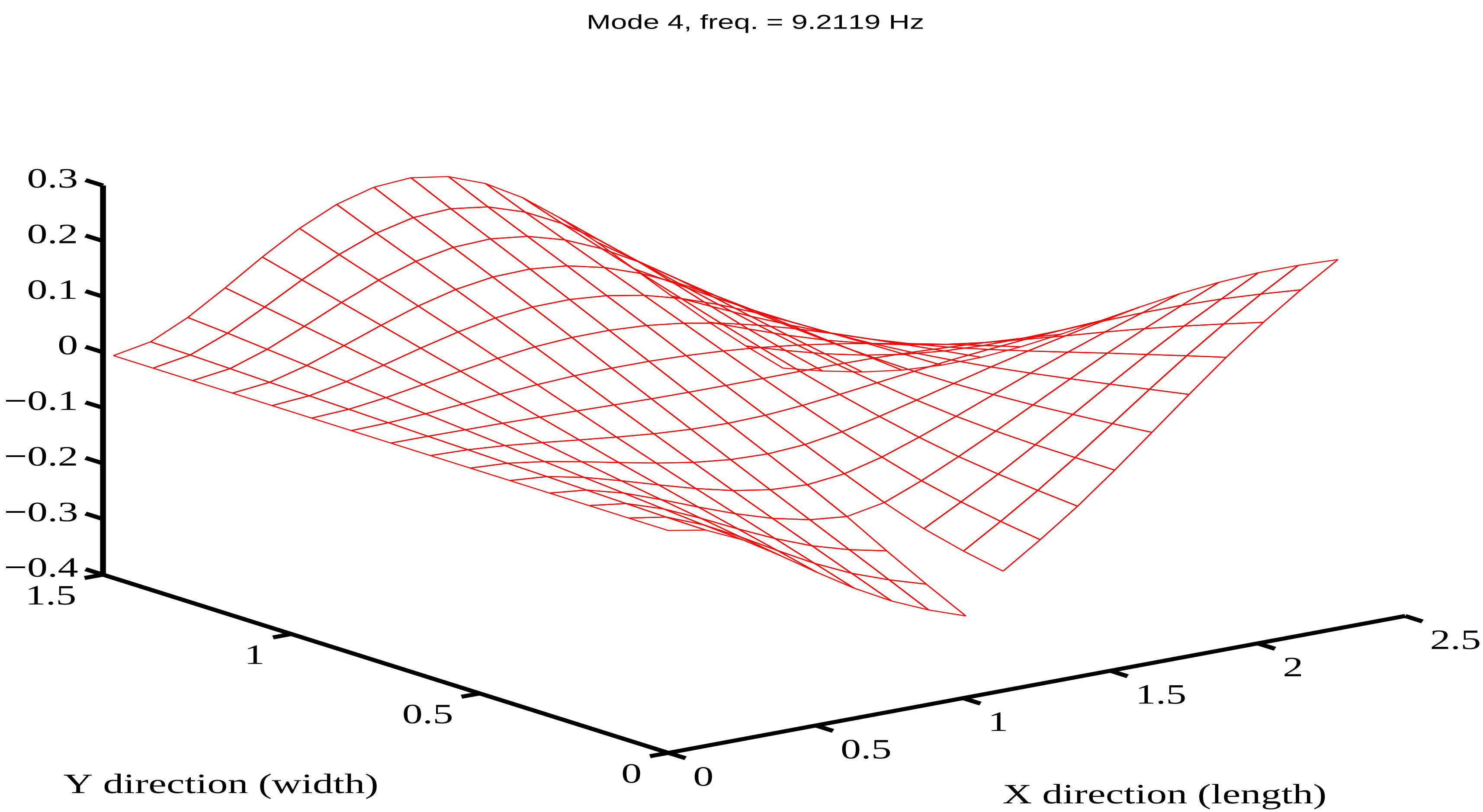
$$\begin{aligned}
&\mathrm{E}\left[(\mathbf{y} - \mathrm{E}[\mathbf{y}])(\mathbf{y} - \mathrm{E}[\mathbf{y}])^T\right] \\
&= \mathbf{R}\,\mathrm{E}\left[\mathbf{D}^{-1}\mathbf{p}\mathbf{p}^T\mathbf{D}^{-1}\right]\mathbf{R}^T - \mathrm{E}[\mathbf{y}]\,(\mathrm{E}[\mathbf{y}])^T \\
&= \frac{\mathrm{Trace}\left(\mathbf{\Sigma}^{-1}\mathbf{p}\mathbf{p}^T\right)\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{R}^T}{\theta(\theta+1)(\theta-2)} + \frac{(\theta+2)\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{p}\mathbf{p}^T\mathbf{\Sigma}^{-1}\mathbf{R}^T}{\theta^2(\theta+1)(\theta-2)}
\end{aligned} \quad (20)$$

Example: A cantilever Plate



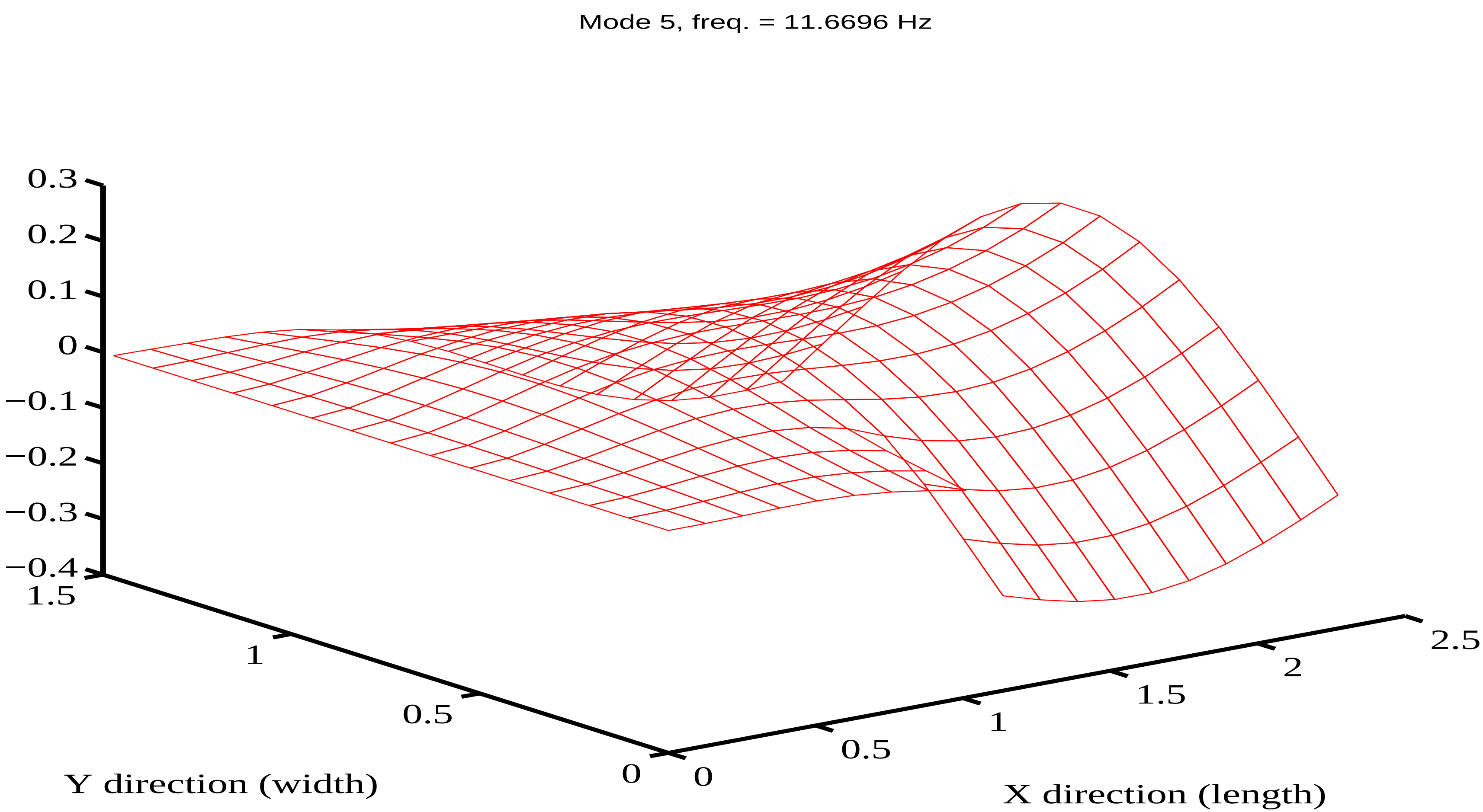
Cantilever plate with a slot: $\mu = 0.3$, $\rho = 8000$ kg/m^3, $t = 5$mm, $L_x = 2.27$m, $L_y = 1.47$m

Plate Mode 4



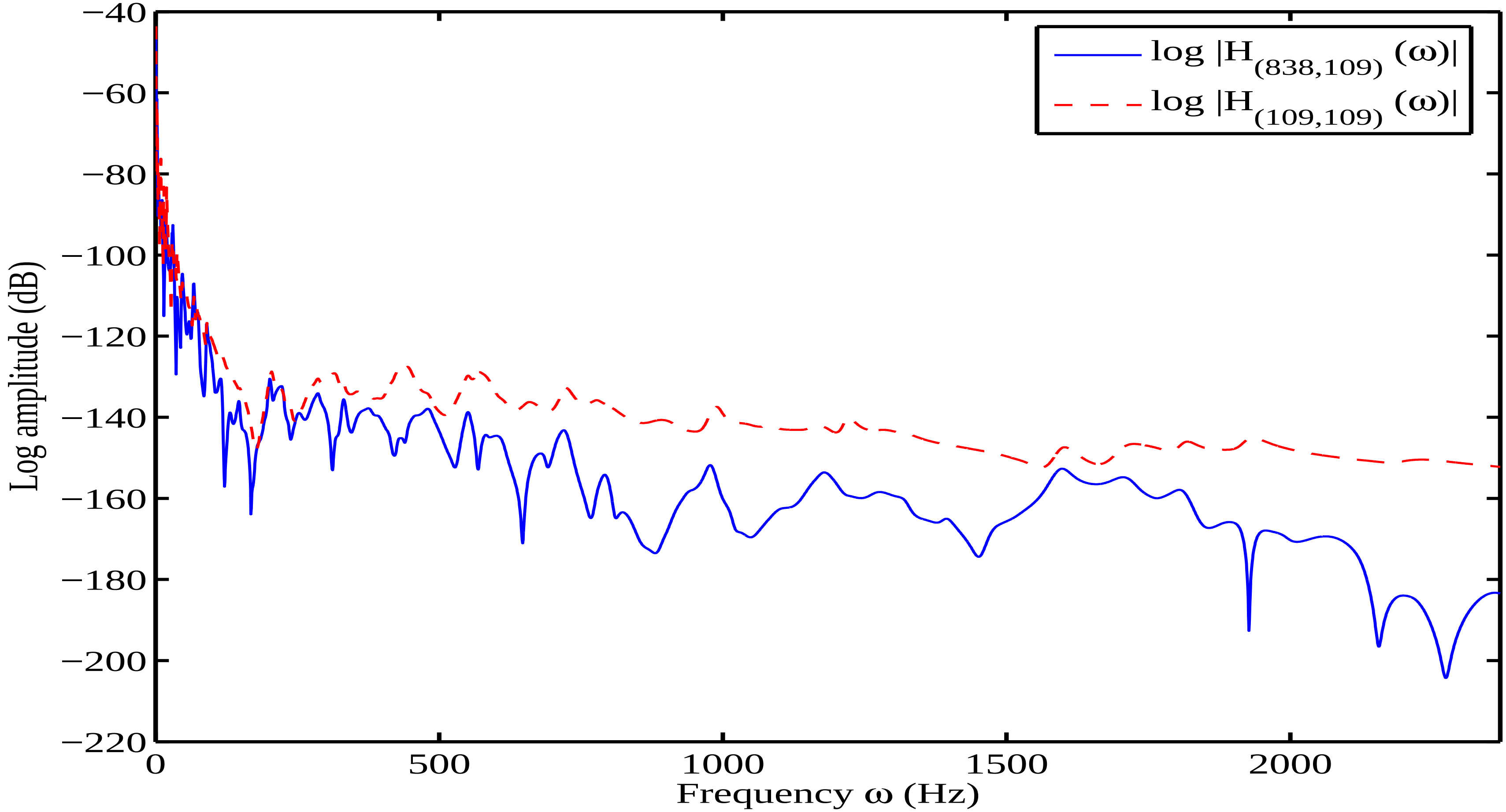
Fourth Mode shape

Plate Mode 5



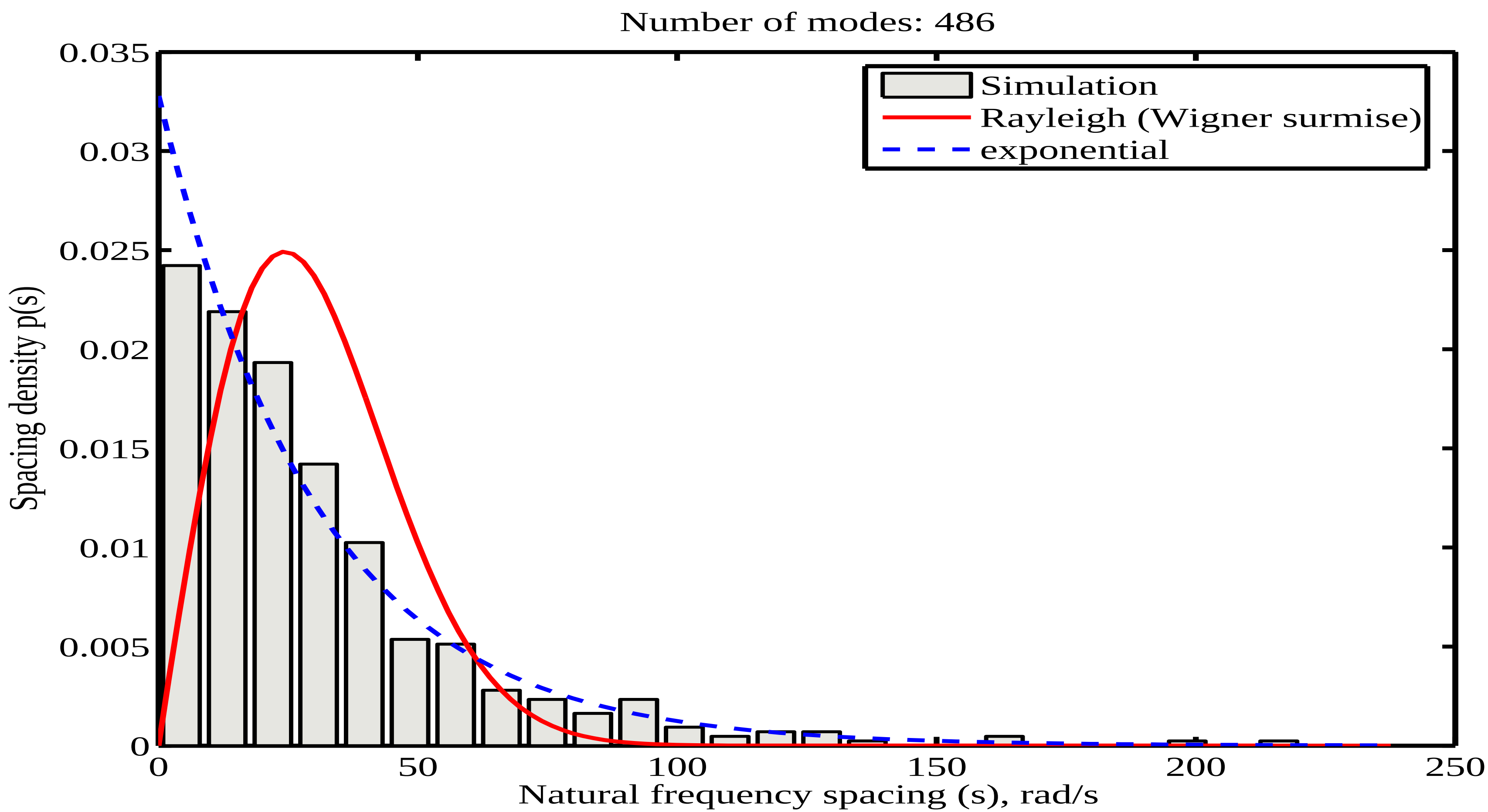
Fifth Mode shape

Deterministic FRF



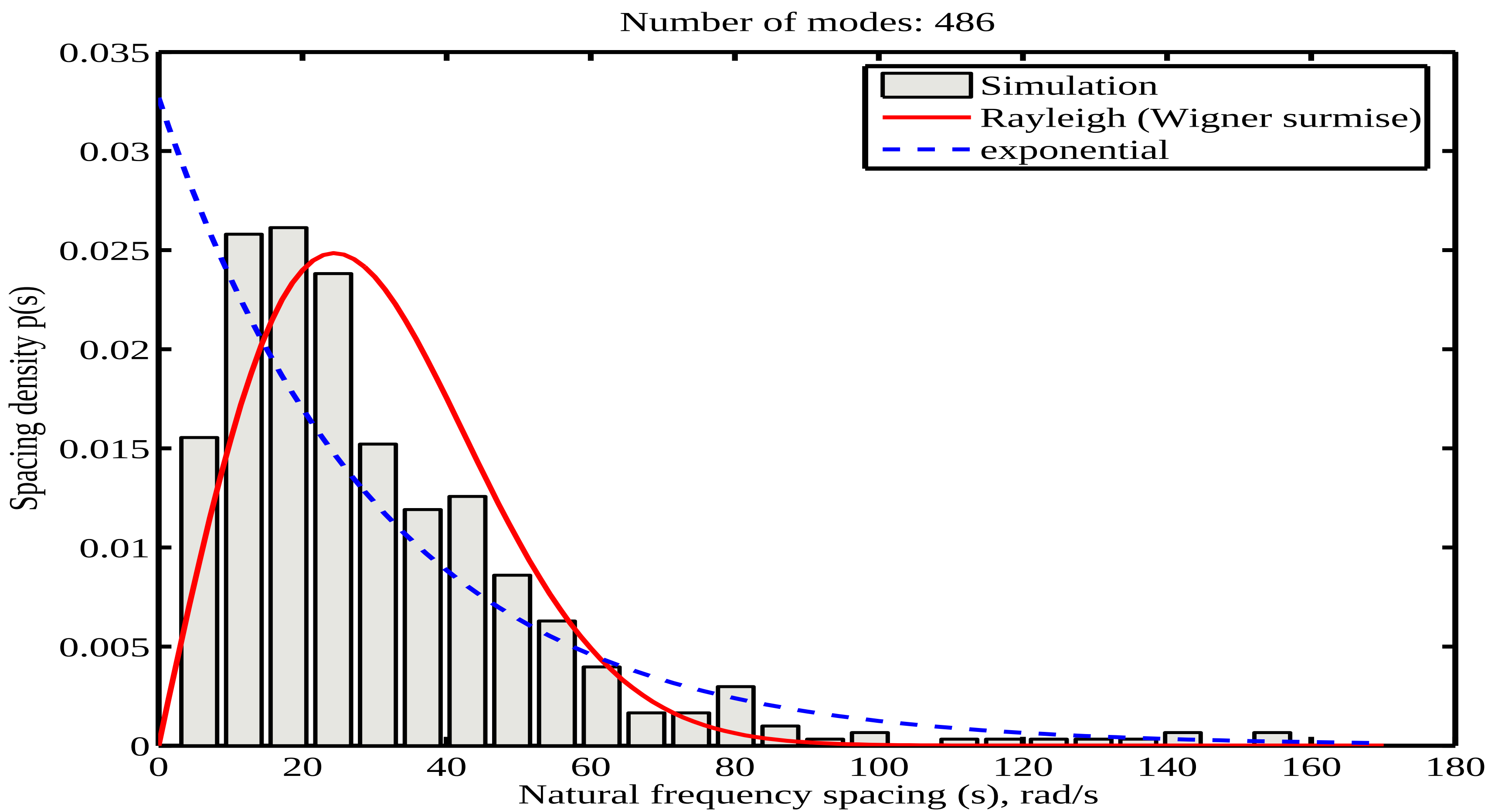
FRF of the deterministic plate

Frequency Spacing



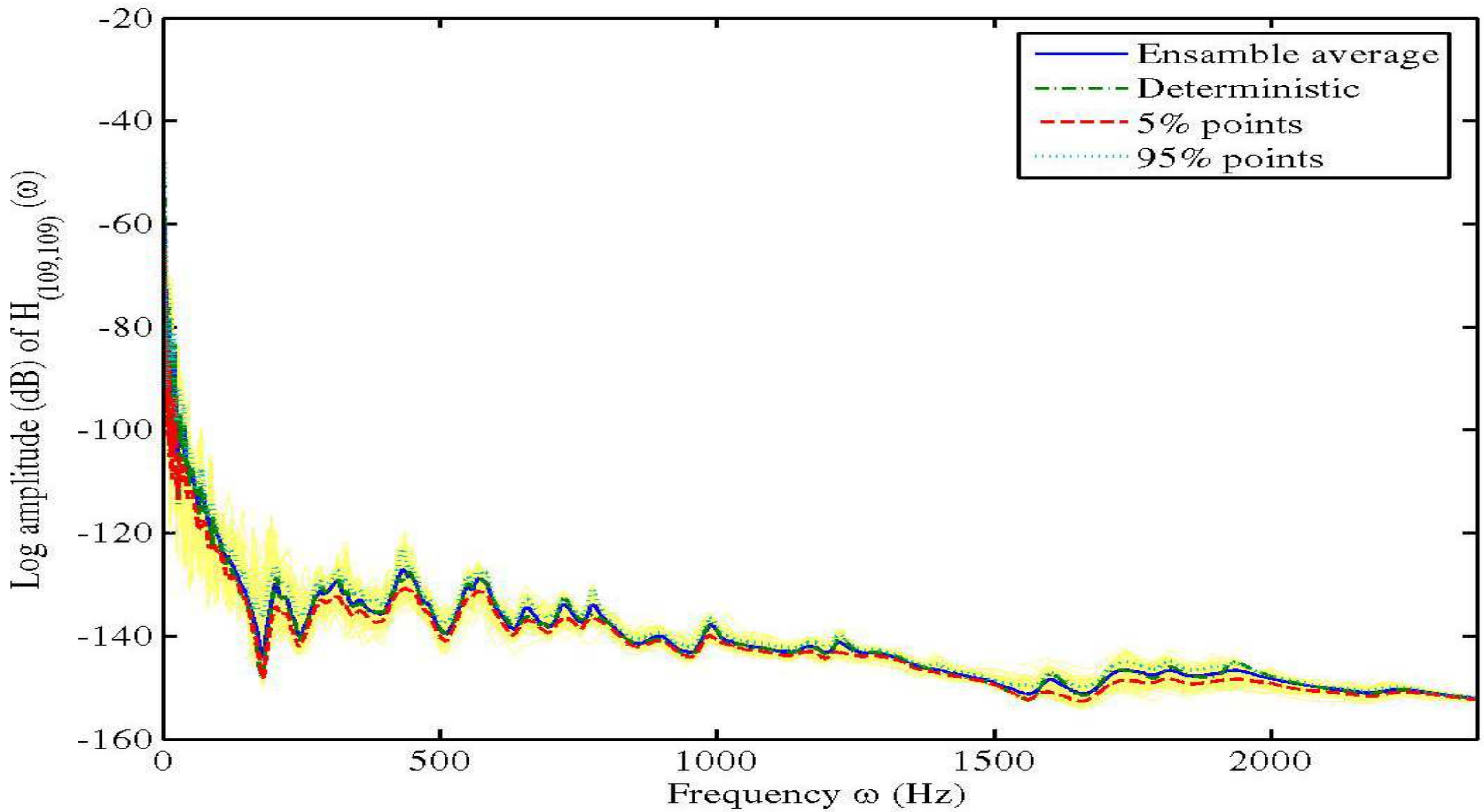
Natural frequency spacing distribution (without slot)

Frequency Spacing



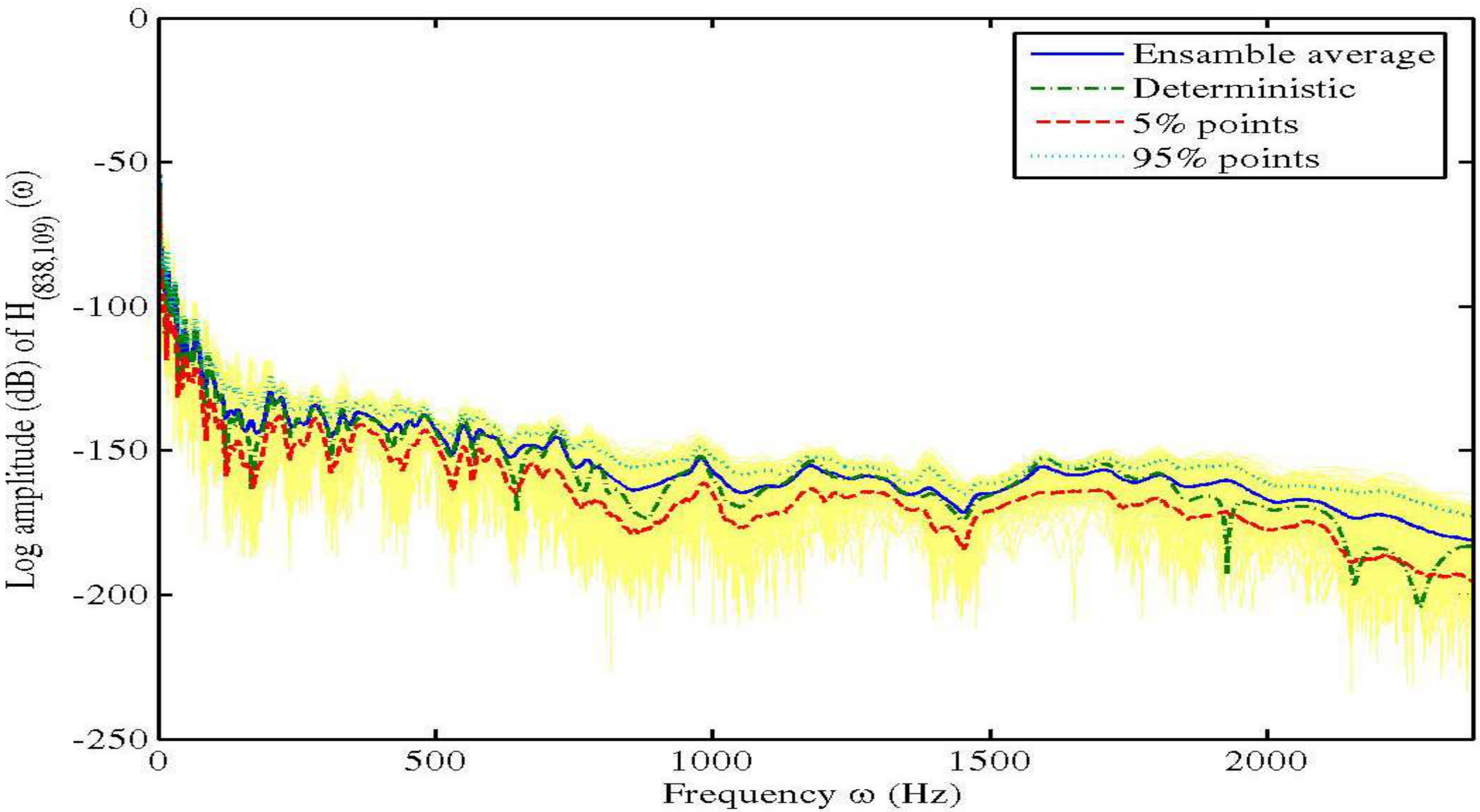
Natural frequency spacing distribution (with slot)

Random FRF - 1



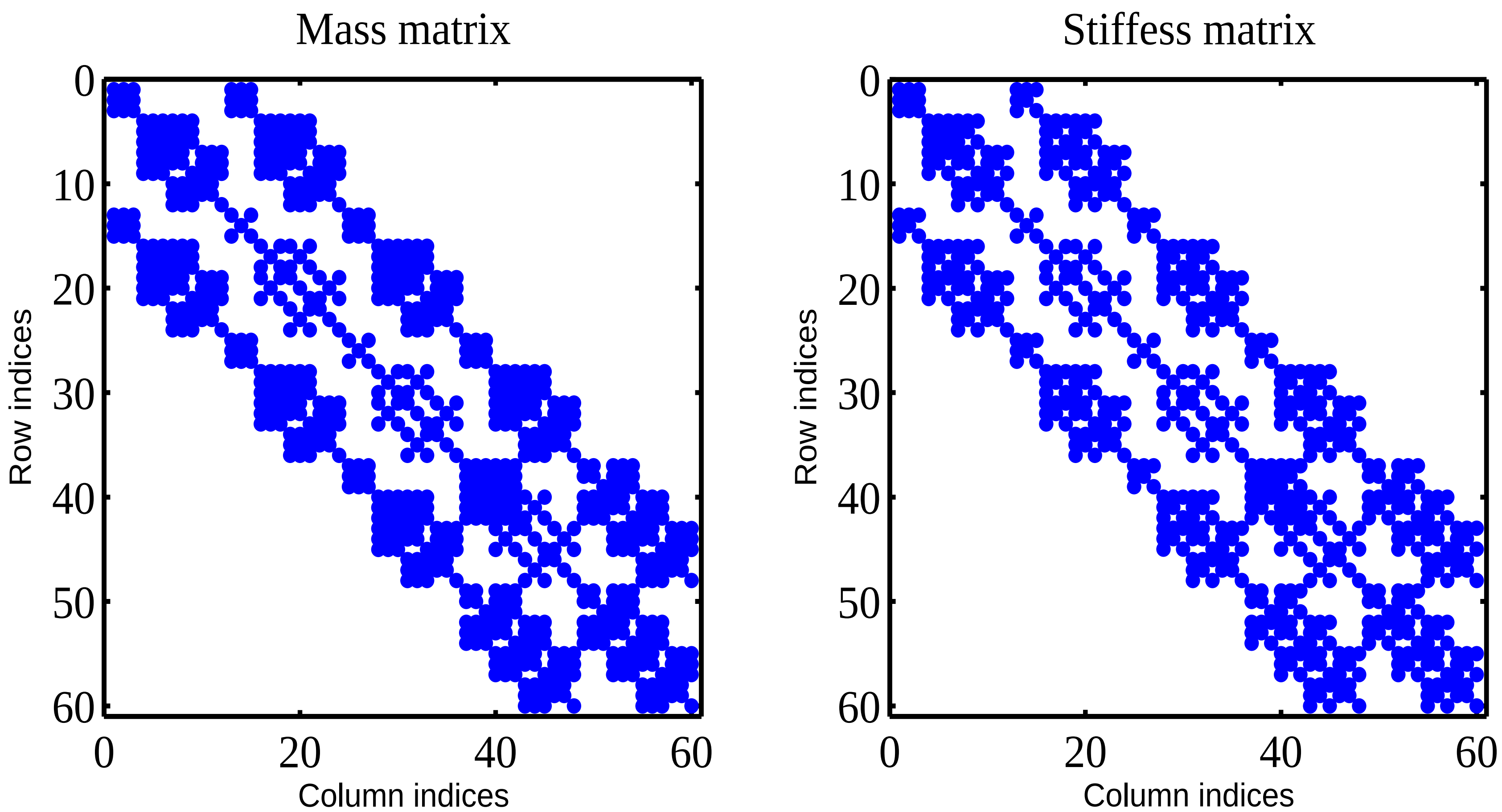
Driviing point FRF using optimal Wishart distribution

Random FRF - 2



A cross FRF using optimal Wishart distribution

Structure of the Matrices



Nonzero elements of the system matrices

Summary & conclusions

- Wishart matrices can used as the distribution for the system matrices in structural dynamics.
- The parameters of the distribution can be obtained by solving an optimisation problem

Next steps

- Numerical works (validation against??)
- Distribution of the dynamic stiffness matrix (complex Wishart matrix?)
- Inversion of the dynamic stiffness matrix (FRFs)
- Distribution of $\mathbf{Y}(\omega) = \left[\mathbf{R}\mathbf{D}(\omega)^{-1}\mathbf{P}\right]$ where $\mathbf{P} \in \mathbb{C}_{n,r}$ and $\mathbf{R} \in \mathbb{R}_{p,n}$
- Eigenvalues, eigenvector statistics and calculation of dynamic response.
- Cumulative distribution function of the response

Open problems & discussions

- Structure preserving random matrices (low-mid frequency?)
- Non-central Wishart matrices (preservation of covariance structure - parametric uncertainty models?)
- Solution of SFEM using RMT (connection with polynomial chaos expansions)?
- Eigenvalue problem, Wigner surmise
- Analytical expression of the pdf of dynamic response
- Energy statistics - SEA