Rayleigh's Classical Damping Revisited

S. Adhikari and A. Srikantha Phani

Department of Aerospace Engineering, University of Bristol, Bristol, U.K.

Email: S.Adhikari@bristol.ac.uk

URL: http://www.aer.bris.ac.uk/contact/academic/adhikari/home.html

Bristol Aerospace

Outline of the presentation

- Introduction
- Background of proportionally damped systems
- Generalized proportional damping
- Damping identification method
- Examples
- Summary and conclusions

Introduction

- Equation of motion of viscously damped systems:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{f}(t)$$

- Proportional damping (Rayleigh 1877)

$$\mathbf{C} = \alpha_1\mathbf{M} + \alpha_2\mathbf{K}$$

  - Classical normal modes
  - Simplifies analysis methods
  - Identification of damping becomes easier

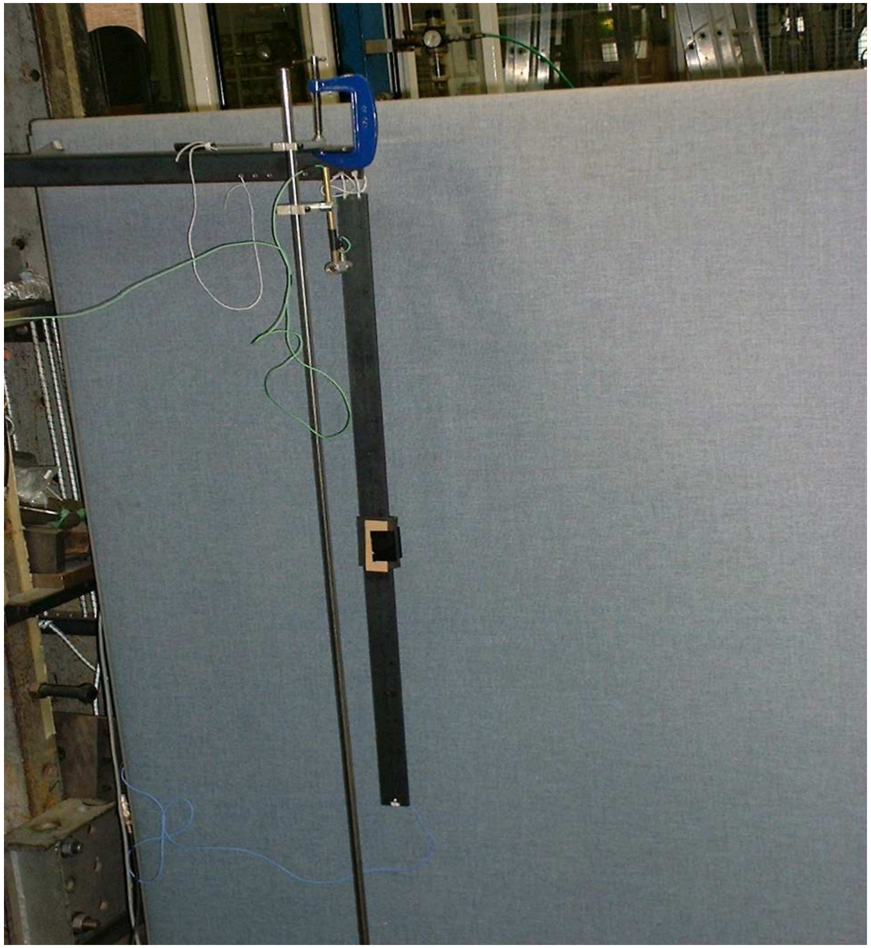
Limitations of proportional damping

- The modal damping factors:

$$\zeta_j = \frac{1}{2}\left(\frac{\alpha_1}{\omega_j} + \alpha_2\omega_j\right)$$

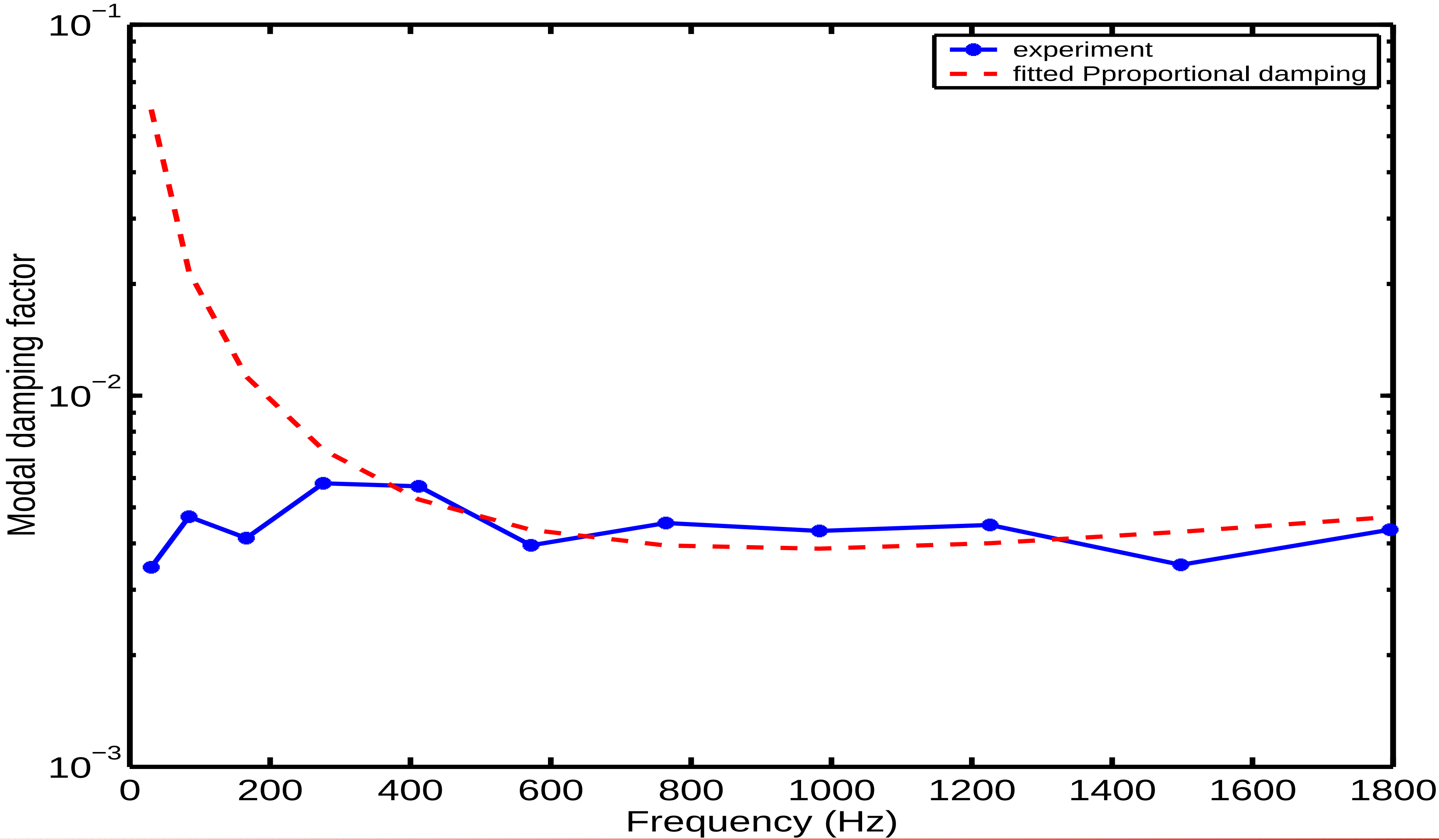
- Not all forms of variation can be captured

Damped Beam Example



Damped free-free beam:
$L = 1$m, width = 39.0 mm
thickness = 5.93 mm

Damping factors



Our Objective

- Can we improve the Classical Damping proposed by Lord Rayleigh in 1877 so that we can take account of the frequency variation of the damping factors?

Conditions for proportional damping

Theorem 1 *A viscously damped linear system can possess classical normal modes if and only if at least one of the following conditions is satisfied:*
(a) $\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K}$, *(b)* $\mathbf{M}\mathbf{K}^{-1}\mathbf{C} = \mathbf{C}\mathbf{K}^{-1}\mathbf{M}$, *(c)* $\mathbf{M}\mathbf{C}^{-1}\mathbf{K} = \mathbf{K}\mathbf{C}^{-1}\mathbf{M}$.

This can be easily proved by following Caughey and O'Kelly's (1965) approach and interchanging $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ successively.

Caughey series

- Caughey series:

$$\mathbf{C} = \mathbf{M} \sum_{j=0}^{N-1} \alpha_j \left(\mathbf{M}^{-1}\mathbf{K}\right)^j$$

- The modal damping factors:

$$\zeta_j = \frac{1}{2}\left(\frac{\alpha_1}{\omega_j} + \alpha_2\omega_j + \alpha_3\omega_j^3 + \cdots\right)$$

- More general than Rayleigh's version of proportional damping

Generalized proportional damping

- Premultiply condition (a) of the theorem by $\mathbf{M}^{-1}$:

$$\left(\mathbf{M}^{-1}\mathbf{K}\right)\left(\mathbf{M}^{-1}\mathbf{C}\right)=\left(\mathbf{M}^{-1}\mathbf{C}\right)\left(\mathbf{M}^{-1}\mathbf{K}\right)$$

- Since $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{M}^{-1}\mathbf{C}$ are commutative matrices

$$\mathbf{M}^{-1}\mathbf{C}=f_1(\mathbf{M}^{-1}\mathbf{K})$$

- Therefore, we can express the damping matrix as

$$\mathbf{C}=\mathbf{M}f_1(\mathbf{M}^{-1}\mathbf{K})$$

Generalized proportional damping

- Premultiply condition (b) of the theorem by $\mathbf{K}^{-1}$:

$$\left(\mathbf{K}^{-1}\mathbf{M}\right)\left(\mathbf{K}^{-1}\mathbf{C}\right) = \left(\mathbf{K}^{-1}\mathbf{C}\right)\left(\mathbf{K}^{-1}\mathbf{M}\right)$$

- Since $\mathbf{K}^{-1}\mathbf{M}$ and $\mathbf{K}^{-1}\mathbf{C}$ are commutative matrices

$$\mathbf{K}^{-1}\mathbf{C} = f_2(\mathbf{K}^{-1}\mathbf{M})$$

- Therefore, we can express the damping matrix as

$$\mathbf{C} = \mathbf{K} f_1(\mathbf{K}^{-1}\mathbf{M})$$

Generalized proportional damping

- Combining the previous two cases

$$\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$$

- Similarly, postmultiplying condition (a) of Theorem 1 by $\mathbf{M}^{-1}$ and (b) by $\mathbf{K}^{-1}$ we have

$$\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$$

- Special case: $\beta_i(\bullet) = \alpha_i \mathbf{I} \rightarrow$ Rayleigh damping.

Generalized proportional damping

Theorem 2 *A viscously damped positive definite linear system possesses classical normal modes if and only if* $\mathbf{C}$ *can be represented by*

(a) $\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$*, or*

(b) $\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$

for any $\beta_i(\bullet), i = 1, \cdots, 4$*.*

Example 1

Equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}} + \left[\mathbf{M}e^{-\left(\mathbf{M}^{-1}\mathbf{K}\right)^2/2}\sinh(\mathbf{K}^{-1}\mathbf{M}\ln(\mathbf{M}^{-1}\mathbf{K})^{2/3}) + \mathbf{K}\cos^2(\mathbf{K}^{-1}\mathbf{M})\sqrt[4]{\mathbf{K}^{-1}\mathbf{M}}\tan^{-1}\frac{\sqrt{\mathbf{M}^{-1}\mathbf{K}}}{\pi}\right]\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

It can be shown that the system has real modes and

$$2\xi_j\omega_j = e^{-\omega_j^4/2}\sinh\left(\frac{1}{\omega_j^2}\ln\frac{4}{3}\omega_j\right) + \omega_j^2\cos^2\left(\frac{1}{\omega_j^2}\right)\frac{1}{\sqrt{\omega_j}}\tan^{-1}\frac{\omega_j}{\pi}.$$

Damping identification method

To simplify the identification procedure, express the damping matrix by

$$\mathbf{C} = \mathbf{M} f \left(\mathbf{M}^{-1} \mathbf{K} \right)$$

Using this simplified expression, the modal damping factors can be obtained as

$$2\zeta_j \omega_j = f\left(\omega_j^2\right)$$

$$\text{or} \quad \zeta_j = \frac{1}{2\omega_j} f\left(\omega_j^2\right) = \widehat{f}(\omega_j) \quad \text{(say)}$$

Damping identification method

- The function $\widehat{f}(\bullet)$ can be obtained by fitting a continuous function representing the variation of the measured modal damping factors with respect to the frequency
- With the fitted function $\widehat{f}(\bullet)$, the damping matrix can be identified as

$$2\zeta_j\omega_j = 2\omega_j\widehat{f}(\omega_j)$$

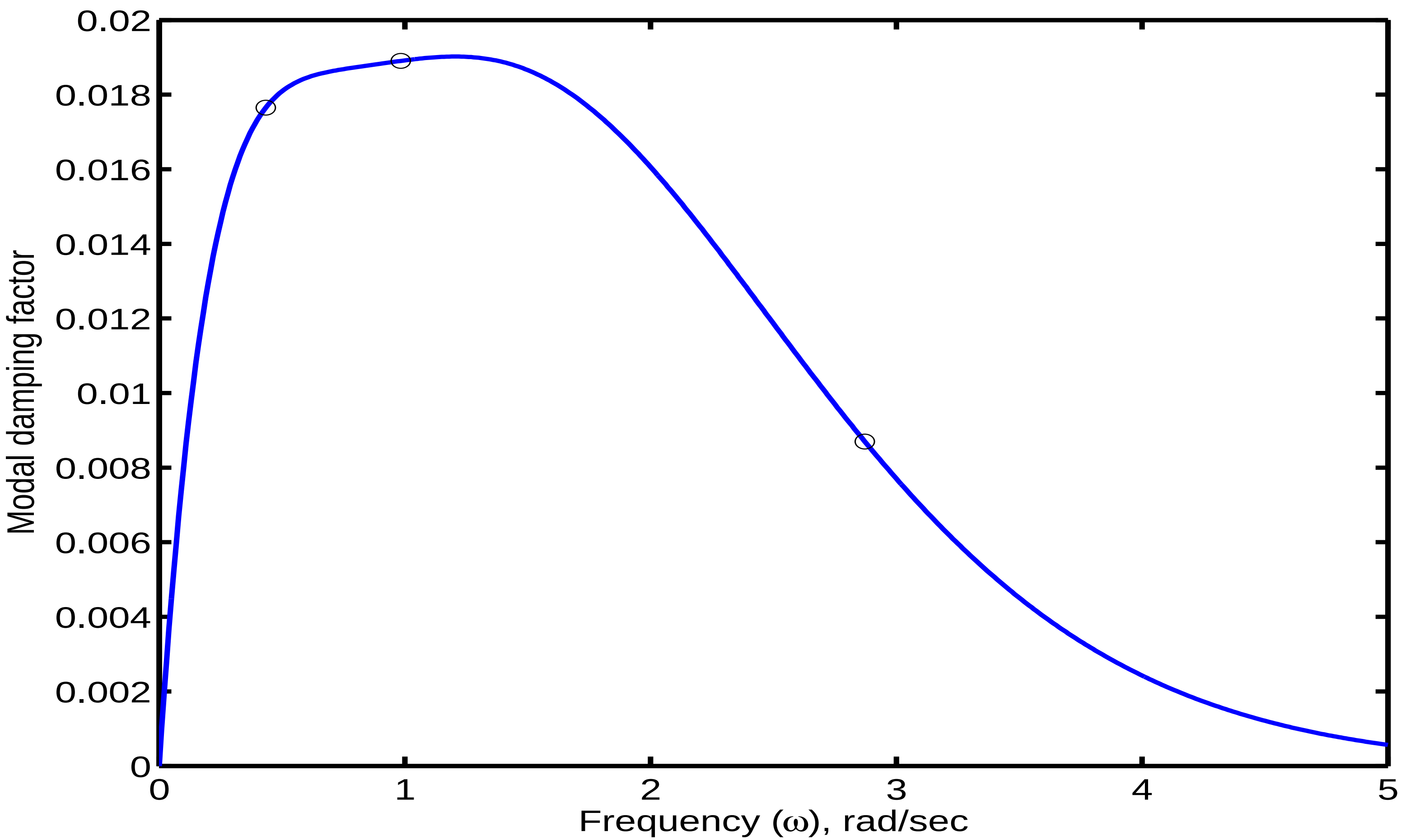
$$\text{or} \quad \widehat{\mathbf{C}} = 2\mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\,\widehat{f}\left(\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)$$

Example 2

Consider a 3DOF system with mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix}$$

Example 2



Damping factors

Example 2

Here this (continuous) curve was simulated using the equation

$$\widehat{f}(\omega) = \frac{1}{15}\left(e^{-2.0\omega} - e^{-3.5\omega}\right)\left(1 + 1.25\sin\frac{\omega}{7\pi}\right)\left(1 + 0.75\omega^3\right)$$

From the above equation, the modal damping factors in terms of the discrete natural frequencies, can be obtained by

$$2\xi_j\omega_j = \frac{2\omega_j}{15}\left(e^{-2.0\omega_j} - e^{-3.5\omega_j}\right)\left(1 + 1.25\sin\frac{\omega_j}{7\pi}\right)\left(1 + 0.75\omega_j^3\right).$$

Example 2

To obtain the damping matrix, consider the preceding equation as a function of ω_j^2 and replace ω_j^2 by $\mathbf{M}^{-1}\mathbf{K}$ and any constant terms by that constant times $\mathbf{I}$. Therefore:

$$\mathbf{C} = \mathbf{M}\,\frac{2}{15}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\left[e^{-2.0\sqrt{\mathbf{M}^{-1}\mathbf{K}}} - e^{-3.5\sqrt{\mathbf{M}^{-1}\mathbf{K}}}\right] \times \left[\mathbf{I} + 1.25\sin\left(\frac{1}{7\pi}\sqrt{\mathbf{M}^{-1}\mathbf{K}}\right)\right]\left[\mathbf{I} + 0.75(\mathbf{M}^{-1}\mathbf{K})^{3/2}\right]$$

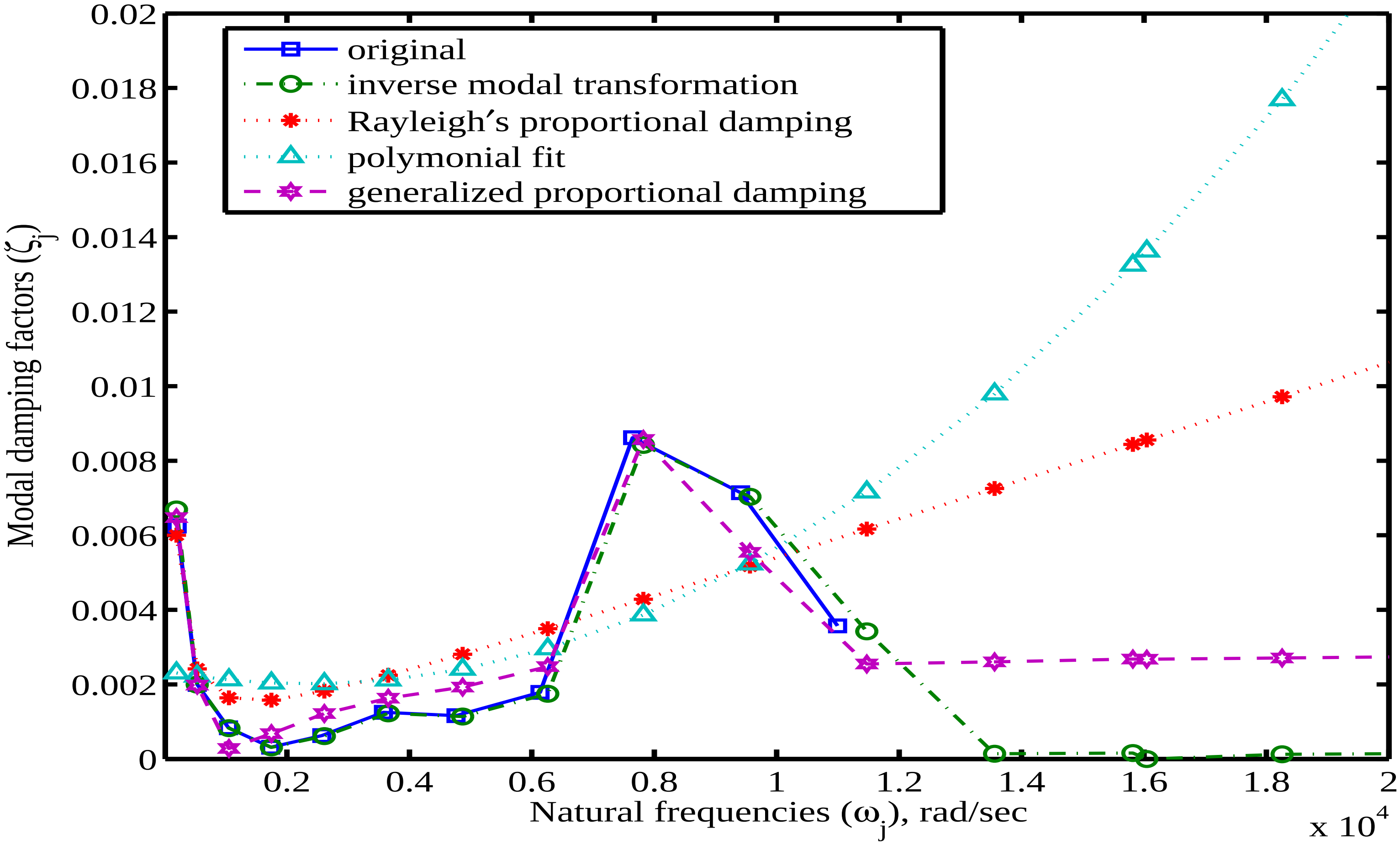
Experimental Example 1

Natural frequencies, Hz (experimental)	Damping factors (in % of critical damping)	Natural frequencies, Hz (from FE)
33.00	0.6250	30.81 (-6.64 %)
85.00	0.2000	85.24 (0.29 %)
166.00	0.0833	167.61 (0.97 %)
276.00	0.0313	277.73 (0.63 %)
409.00	0.0625	415.67 (1.63 %)
569.00	0.1250	581.42 (2.18 %)
758.00	0.1163	774.94 (2.24 %)
976.00	0.1786	996.20 (2.07 %)
1217.00	0.8621	1245.15 (2.31 %)
1498.00	0.7143	1521.77 (1.59 %)
1750.00	0.3571	1826.06 (4.35 %)

Measured data for the beam example

$\widehat{\mathbf{C}}_d = 2\mathbf{M}\mathbf{T}\left[p_1\mathbf{I} + p_2\mathbf{T} + p_3\mathbf{T}^2\right] = 2p_2\mathbf{K} + 2(p_1\mathbf{M} + p_3\mathbf{K})\sqrt{\mathbf{M}^{-1}\mathbf{K}}.$

Experimental Example 1



Fitted and measured damping factors

Summary

1. Measure a suitable transfer function $H_{ij}(\omega)$
2. Obtain the undamped natural frequencies ω_j and modal damping factors ζ_j
3. Fit a function $\zeta = \widehat{f}(\omega)$ which represents the variation of ζ_j with respect to ω_j for the range of frequency considered in the study
4. Calculate the matrix $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$
5. Obtain the damping matrix using $\widehat{\mathbf{C}} = 2\,\mathbf{M}\,\mathbf{T}\,\widehat{f}\,(\mathbf{T})$

Conclusions(1)

- Rayleigh s proportional damping is generalized.
- The generalized proportional damping expresses the damping matrix in terms of any non-linear function involving specially arranged mass and stiffness matrices so that the system still posses classical normal modes.
- This enables one to model practically any type of variations in the modal damping factors with respect to the frequency.

Conclusions(2)

- Once a scalar function is fitted to model such variations, the damping matrix can be identified very easily using the proposed method.
- The method is very simple and requires the measurement of damping factors and natural frequencies only (that is, the measurements of the mode shapes are not necessary).
- The proposed method is applicable to any linear structures as long as one have validated mass and stiffness matrix models which can predict the natural frequencies accurately and modes are not significantly complex.