The Nature of Epistemic Uncertainty in Linear Dynamical Systems

S. Adhikari* and A. Sarkar†

*Department of Aerospace Engineering, University of Bristol, Bristol, U.K.
Email: S.Adhikari@bristol.ac.uk

†Department of Civil and Environmental Engineering, Carleton University, Ottawa, Canada. Email: abhijit_sarkar@carleton.ca

Outline of the presentation

- The origin of epistemic uncertainty in linear dynamical systems
- Modeling of epistemic uncertainty
- Random matrix models
- Wishart random matrices
- Uncertainty propagation using random matrix theory
- Numerical example
- Conclusions

Acknowledgments

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Epistemic uncertainty

Uncertainties can be broadly divided into two categories:

- The first type is due to the inherent variability in the system parameters. This is often referred to as aleatoric uncertainty or parametric uncertainty. If enough samples are present, it is possible to characterize the variability using well established statistical methods.
- The second type of uncertainty is mainly due to the lack of knowledge regarding a system, referred to as epistemic uncertainty or non-parametric uncertainty. This generally arise in the modelling of complex systems. Due to its very nature, it is comparatively difficult to quantify.

The origins of epistemic uncertainty

- In the modelling of complex systems, such as the marine (e.g. ships and submarines) and aerospace systems (e.g. helicopters, aircrafts and space shuttles), modeling uncertainty arises naturally due to the lack of complete knowledge of the system.
- We assume that modeling uncertainty can be represented by random subsystems attached to the 'master system'.
- For typical marine and aerospace structures, the cargo, piping, fuel, control cables, electronics and avionic systems, hydraulics and bulkheads constitute such subsystems.

The origins of epistemic uncertainty

The ‘lack of knowledge’ may arise due to, but not restricted to:

- the lack of knowledge regarding the presence of such subsystems at the first place,
- the lack of knowledge regarding their spatial locations with respect to the primary structure,
- imprecise and incomplete information about their constitutive and geometric properties.
- unknown coupling characteristics.

Modeling of epistemic uncertainty

- For many complex dynamical systems, the main structural parts (often known as the primary or master structure) can often be modeled *deterministically* using the conventional finite element method.
- On the other hand, the substructures (often known as the secondary systems) attached to the primary structure may not be practically accessible for conventional finite element modeling due to the lack of knowledge of such subsystems.
- Here we use random oscillators to model this lack of knowledge arising in the context of linear dynamical systems.

Modeling of epistemic uncertainty

- Randomly distributed sprung-masses can be used to simulate the effect of *uncertain* secondary systems whose spatial attachment locations and dynamic characteristics are not available *a priori*.
- In contrast to the case of data uncertainty (traditionally modeled in the framework of stochastic finite element method), the model uncertainty arising from the sprung-mass oscillators gives rise to new variety of dynamical system for each sample. This can be observed from the variation in the sparsity structure of the mass, stiffness and damping matrices of the total system from sample to sample.

Variation in the sparsity pattern

- In the frequency domain equation of motion can be expressed as

$$\mathbf{A}(\omega)\bar{\mathbf{q}}(\omega) = \bar{\mathbf{f}}(\omega) \tag{1}$$

  where $\mathbf{A}(\omega) = -\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}$ is known as the dynamic stiffness matrix

- Suppose $\bar{\mathbf{q}}_m$ denotes the degrees-of-freedom of baseline system and $\bar{\mathbf{q}}_u$ denotes the degrees-of-freedom of secondary systems.

Variation in the sparsity pattern

- Eq. (1) can be partitioned as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{mu} \\ \mathbf{A}_{um} & \mathbf{A}_{uu} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{q}}_m \\ \bar{\mathbf{q}}_u \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{Bmatrix}. \tag{2}$$

- In reality, one only knows $\bar{\mathbf{q}}_m$ and $\mathbf{A}_{mm}$ using the conventional finite element method.
- In most cases, no information regarding $\bar{\mathbf{q}}_u$ is available. The uncertainty associated with these 'unknown' DOFs include their dimension, nature and locations. As a result $\mathbf{A}_{uu}$ and the coupling matrix $\mathbf{A}_{um}$ are also unknown.

Variation in the sparsity pattern

- Eliminating $\bar{\mathbf{q}}_u$ from Eq. (2) by condensation, one has

$$\left[\mathbf{A}_{mm} - \mathbf{A}_{mu}\mathbf{A}_{uu}^{-1}\mathbf{A}_{um}\right]\bar{\mathbf{q}}_m = \bar{\mathbf{f}} \tag{3}$$

$$\text{or} \quad \left[\mathbf{A}_{mm} + \mathbf{\Delta A}\right]\bar{\mathbf{q}}_m = \bar{\mathbf{f}} \tag{4}$$

where $\mathbf{\Delta A} = -\mathbf{A}_{mu}\mathbf{A}_{uu}^{-1}\mathbf{A}_{um} \in \mathbb{R}^{n \times n}$.

- This equation shows that whatever may be the nature of uncertainty associated with the DOFs arising from the secondary systems, they randomly perturb the condensed 'baseline' matrix $\mathbf{A}_{mm}$ by $\mathbf{\Delta A}$. Moreover, from Eq. (4) it is clear that sparsity structures associated with deterministic matrix $\mathbf{A}_{mm}$ and $\mathbf{A}_{mm} + \mathbf{\Delta A}$ are different.

Modeling of epistemic uncertainty

- Change in the sample-wise sparsity pattern cannot be modeled by data uncertainty alone. In the case of data uncertainty, the actual configuration of dynamical system remains unchanged, just its local parameters change from sample to sample.
- Our conjecture: Episematic or non-parametric uncertainty leads to the variation in the sparsity structure of the system matrices.

Modeling of epistemic uncertainty

- Question 1: What global probabilistic model can be used which will result in the difference in the sparsity structure of the system matrices?
- Question 2: How much information regarding uncertainty we need to model epistemic uncertainty?
- We investigate the feasibility of using random matrix theory to address these issues.

Structural dynamics

- The objective: To have a general method to model epistemic uncertainty in discrete linear dynamical systems.
- The equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.

Random Matrix Method (RMM)

- The methodology :
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \left|\mathbf{\Sigma}\right|^{-p/2} \left|\mathbf{\Psi}\right|^{-n/2} \operatorname{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T \right\} \tag{5}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n\left(\frac{1}{2}p\right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S} \right\} \tag{6}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \left|\mathbf{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\}; \quad \Re(a) > \frac{1}{2}(n- \tag{7}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \tag{8}$$

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$

Matrix Factorization Approach (MFA)

- Suppose $\mathbf{G}$ denotes any one of the system matrices. Because $\mathbf{G}$ is a symmetric and positive-definite random matrix, it can be always factorized as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^T \tag{9}$$

where $\mathbf{X} \in \mathbb{R}^{n \times p}, p \geq n$ is in general a rectangular matrix.

- The simplest case is when the mean of $\mathbf{X}$ is $\mathbf{O} \in \mathbb{R}^{n \times p}, p \geq n$ and the covariance tensor of $\mathbf{X}$ is given by $\mathbf{\Sigma} \otimes \mathbf{I}_p \in \mathbb{R}^{np \times np}$ where $\mathbf{\Sigma} \in \mathbb{R}_n^+$.
- $\mathbf{X}$ is a Gaussian random matrix with mean $\mathbf{O} \in \mathbb{R}^{n \times p}, p \geq n$ and covariance $\mathbf{\Sigma} \otimes \mathbf{I}_p \in \mathbb{R}^{np \times np}$.

Wishart Pdf

After some algebra it can be shown that $\mathbf{G}$ is a $W_n(p, \boldsymbol{\Sigma})$ Wishart random matrix, whose pdf is given given by

$$p_{\mathbf{G}}\left(\mathbf{G}\right) = \left\{ 2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)\left|\boldsymbol{\Sigma}\right|^{\frac{1}{2}p}\right\}^{-1} \left|\mathbf{G}\right|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{G}\right\} \tag{10}$$

Parameter Estimation of Wishart Distribution

- The distribution of $\mathbf{G}$ must be such that $\mathrm{E}\left[\mathbf{G}\right]$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right]$ should be closest to $\overline{\mathbf{G}}$ and $\overline{\mathbf{G}}^{-1}$ respectively.
- Since $\mathbf{G} \sim W_n\left(p, \boldsymbol{\Sigma}\right)$, there are two unknown parameters in this distribution, namely, p and $\boldsymbol{\Sigma}$. This implies that there are in total $1 + n(n+1)/2$ number of unknowns.
- We define and subsequently minimize 'normalized errors':

$$\varepsilon_1 = \left\|\overline{\mathbf{G}} - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}\right\|_{\mathrm{F}}$$
$$\varepsilon_2 = \left\|\overline{\mathbf{G}}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}^{-1}\right\|_{\mathrm{F}}$$

MFA Distribution

Solving the optimization problem we have:

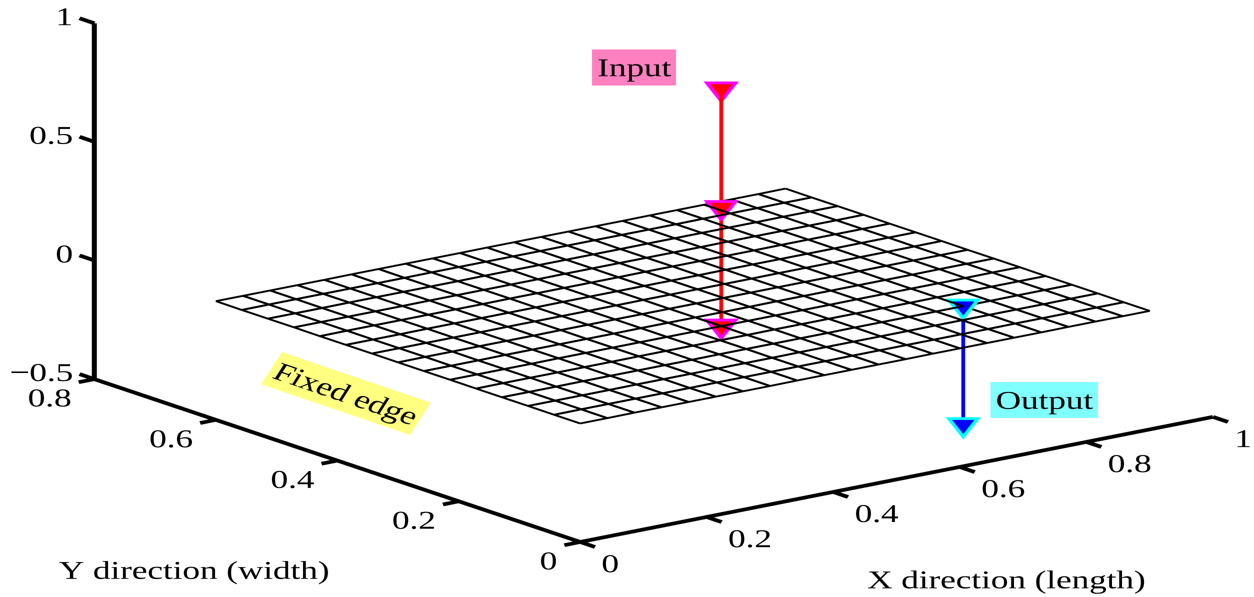
Theorem 1. *If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the distribution of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/\sqrt{2\nu(2\nu + n + 1)}$, that is*

$$\mathbf{G} \sim W_n\left(2\nu + n + 1, \overline{\mathbf{G}}/\sqrt{2\nu(2\nu + n + 1)}\right).$$

Simulation Algorithm: Dynamical Systems

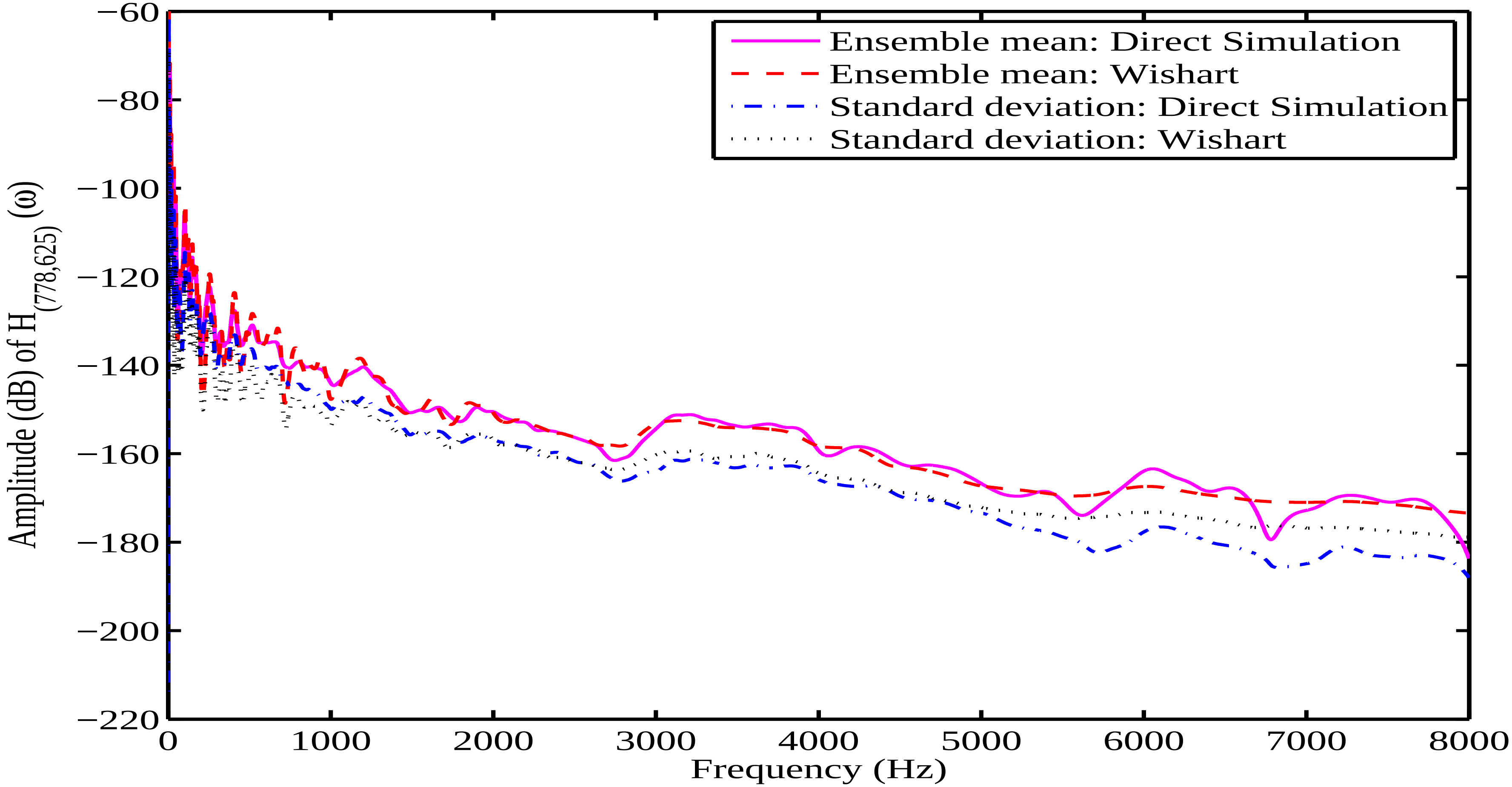
- Obtain $\theta = \frac{1}{\delta_{\mathbf{G}}^2} \left\{ 1 + \frac{\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)} \right\} - (n+1)$
- If $\theta < 4$, then select $\theta = 4$.
- Calculate $\alpha = \sqrt{\theta(n+1+\theta)}$
- Generate samples of $\mathbf{G} \sim W_n\left(n+1+\theta, \overline{\mathbf{G}}/\alpha\right)$ (MATLAB® command `wishrnd` can be used to generate the samples)
- Repeat the above steps for all system matrices and solve for every samples

Example 1: A cantilever Plate



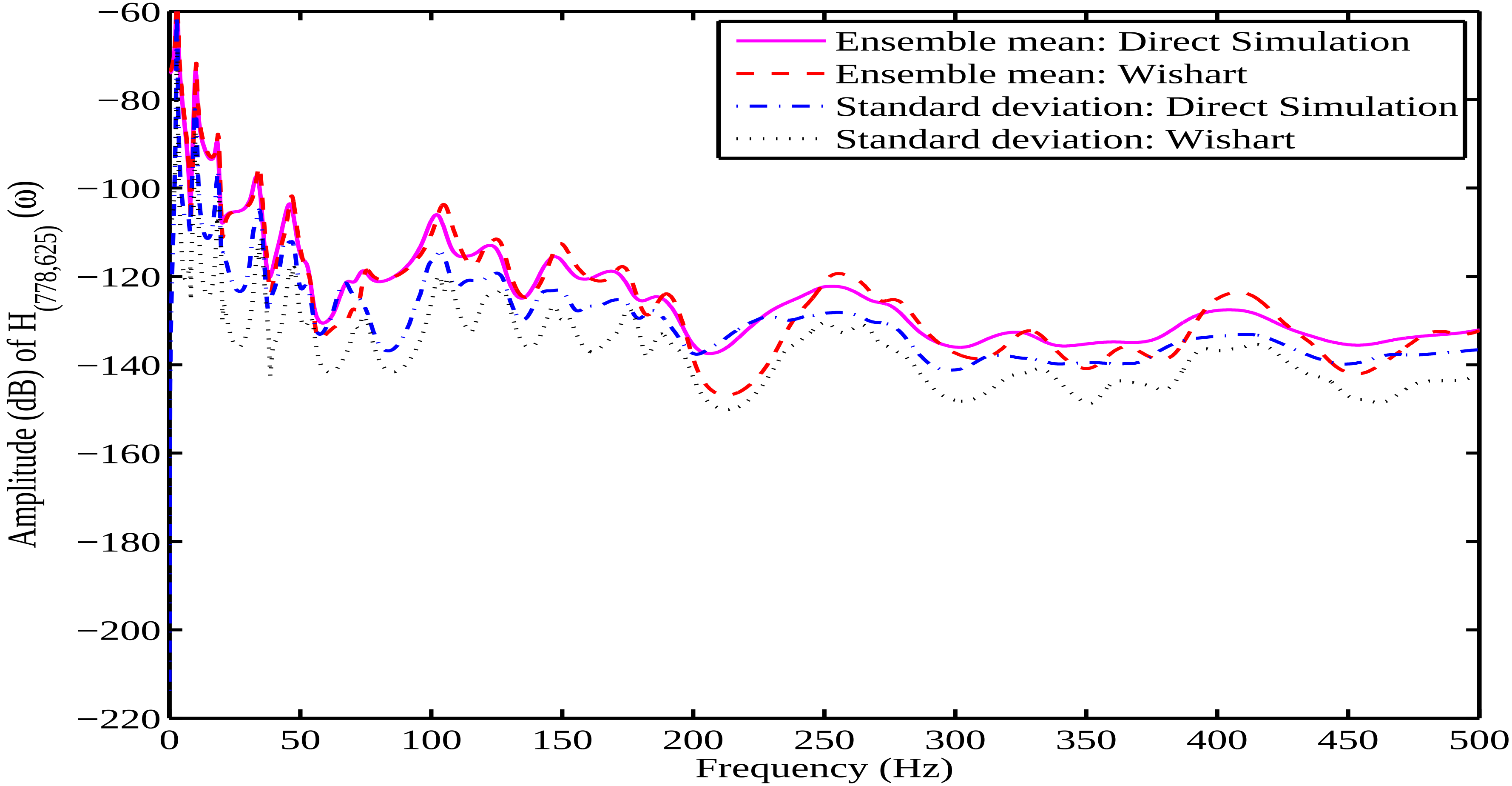
The Finite Element (FE) model of a steel cantilever plate: 21×14 elements, 330 nodes, 945 degrees-of-freedom, $\bar{E} = 200 \times 10^9$N/m^2, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860$kg/m^3, $\bar{t} = 3.0$mm, $L_x = 0.9$m, $L_y = 0.6$m, 2% modal damping for all modes. Ten randomly placed sprung-mass oscillators having natural frequencies uniformly distributed between $(1 - 2)$ KHz are considered.

Comparison of Amplitude



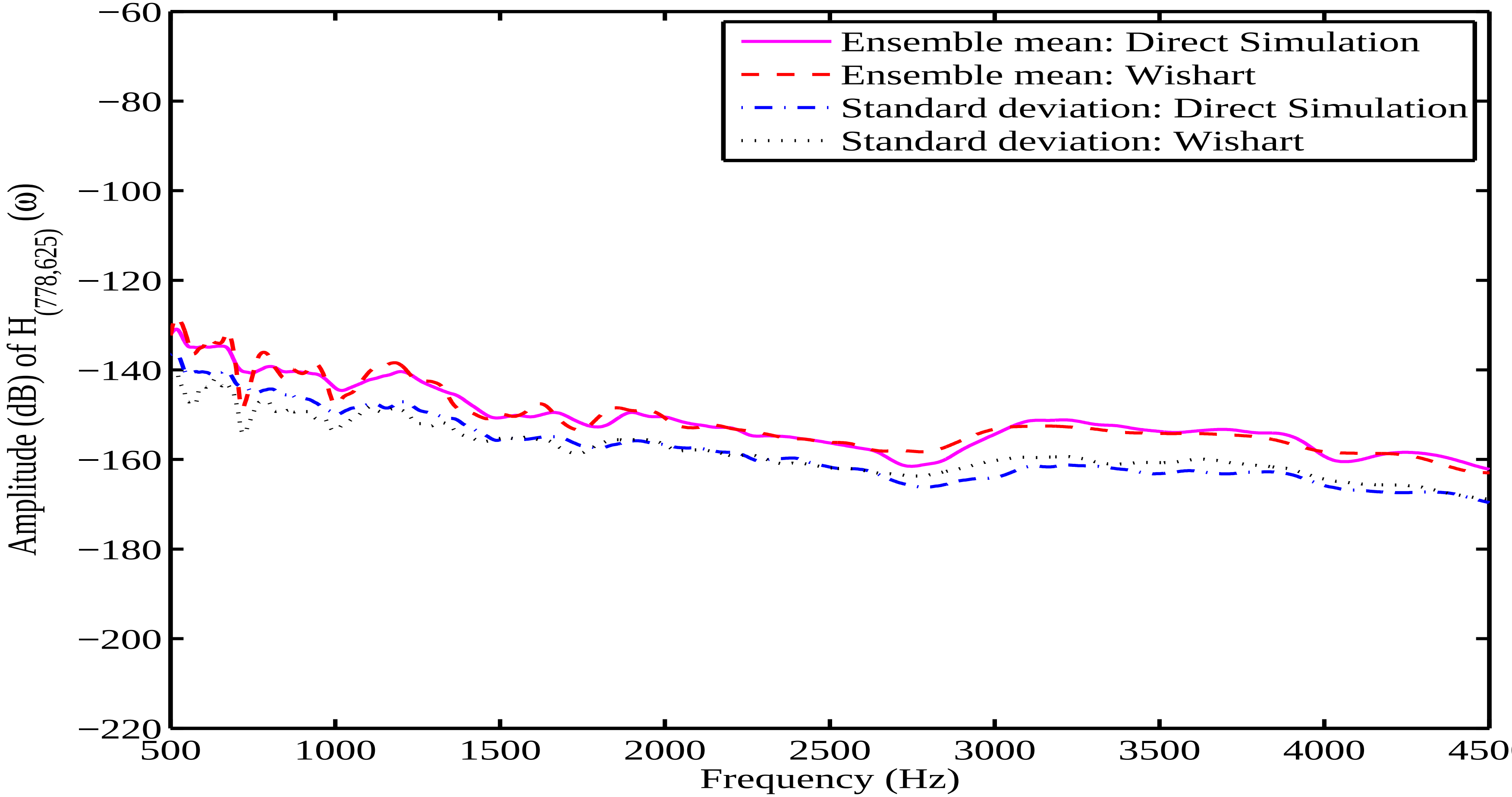
Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 945$ and $\delta_K = 0.1699$.

Comparison of Amplitude: Low Freq



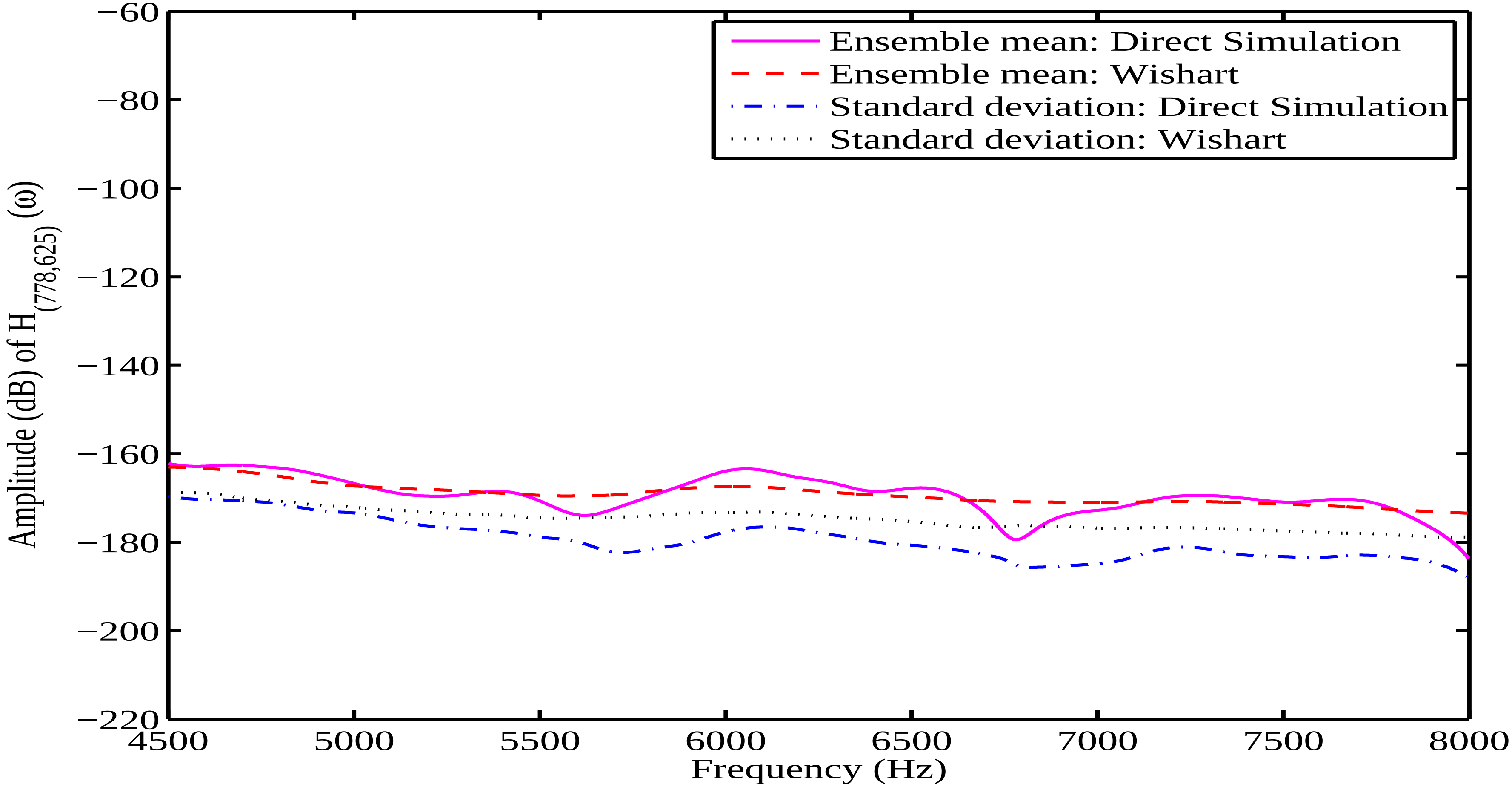
Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 945$ and $\delta_K = 0.1699$.

Comparison of Amplitude: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 945$ and $\delta_K = 0.1699$.

Comparison of Amplitude: High Freq



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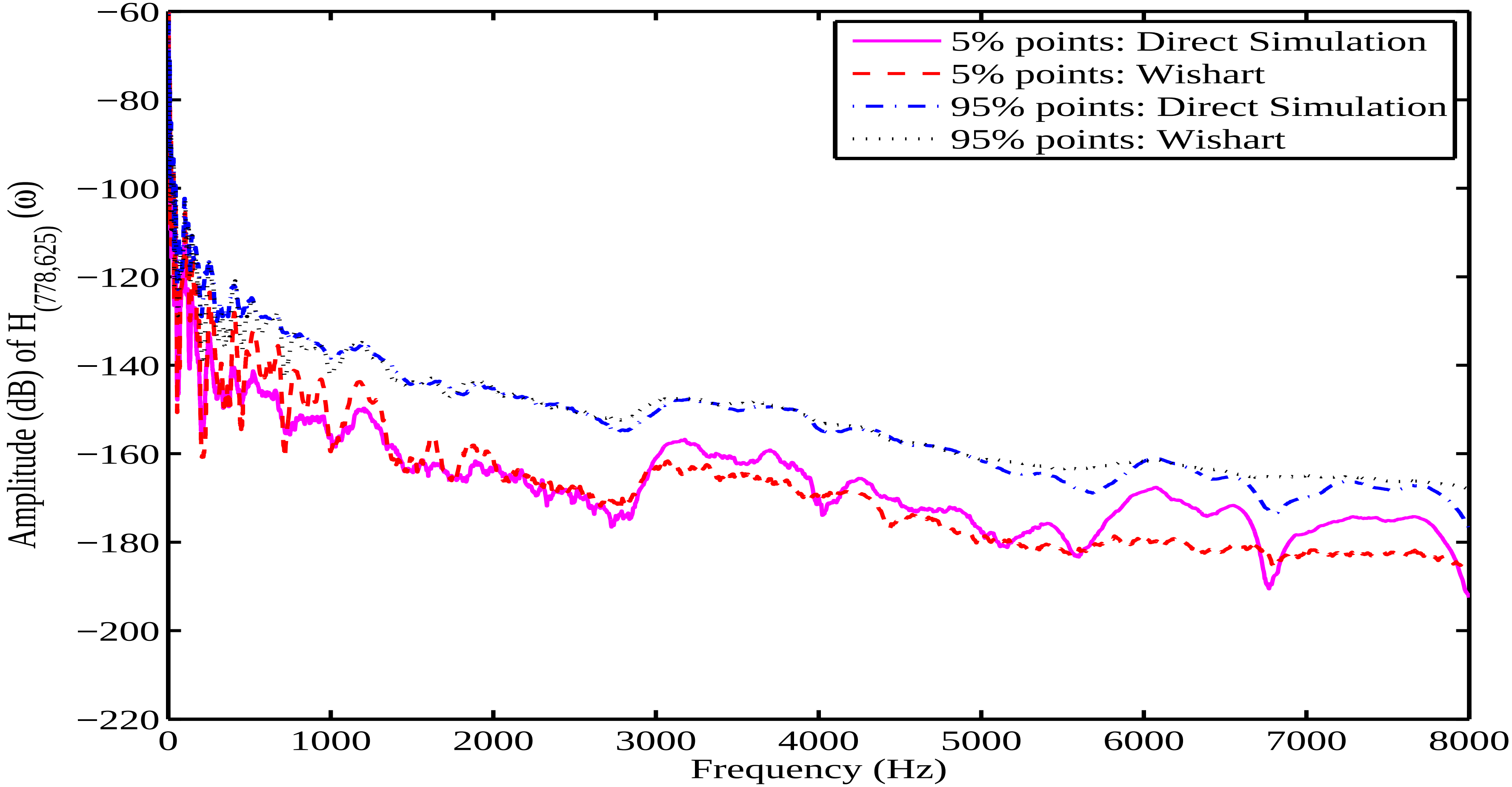
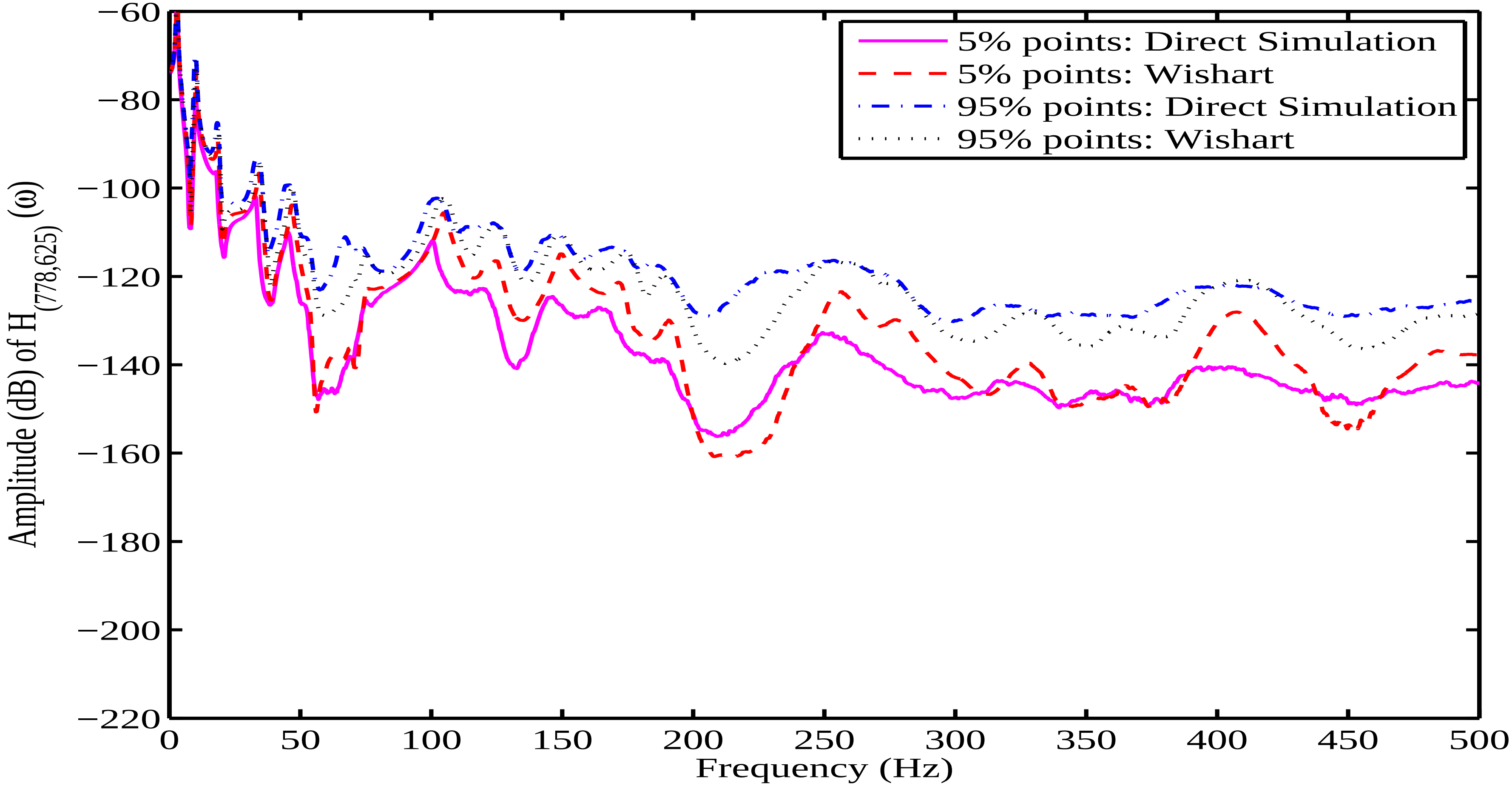
Comparison of Amplitude



Comparison of the 5% and 95% probability points of the amplitude of the cross-FRF.

Comparison of Amplitude: Low Freq



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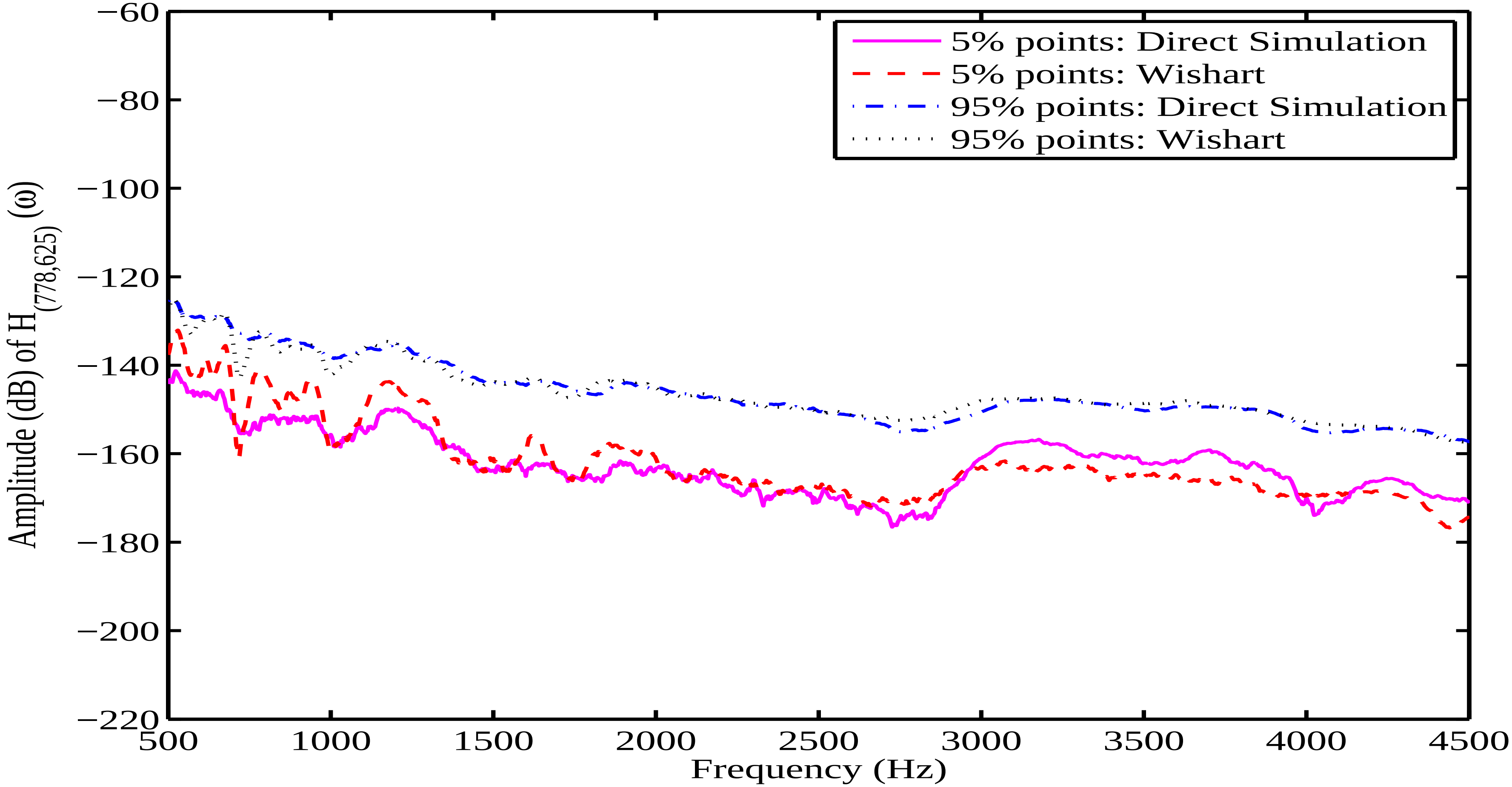

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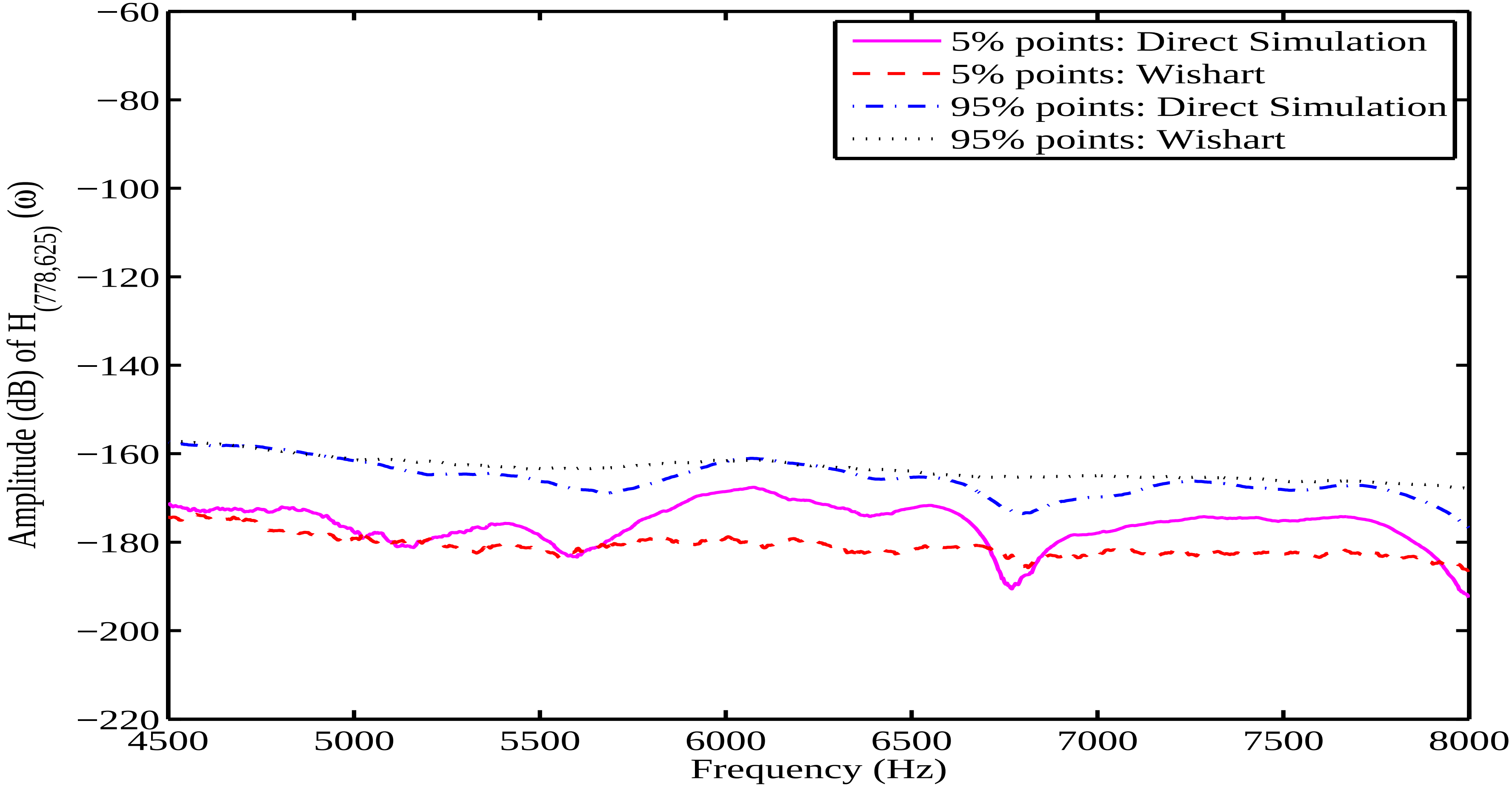
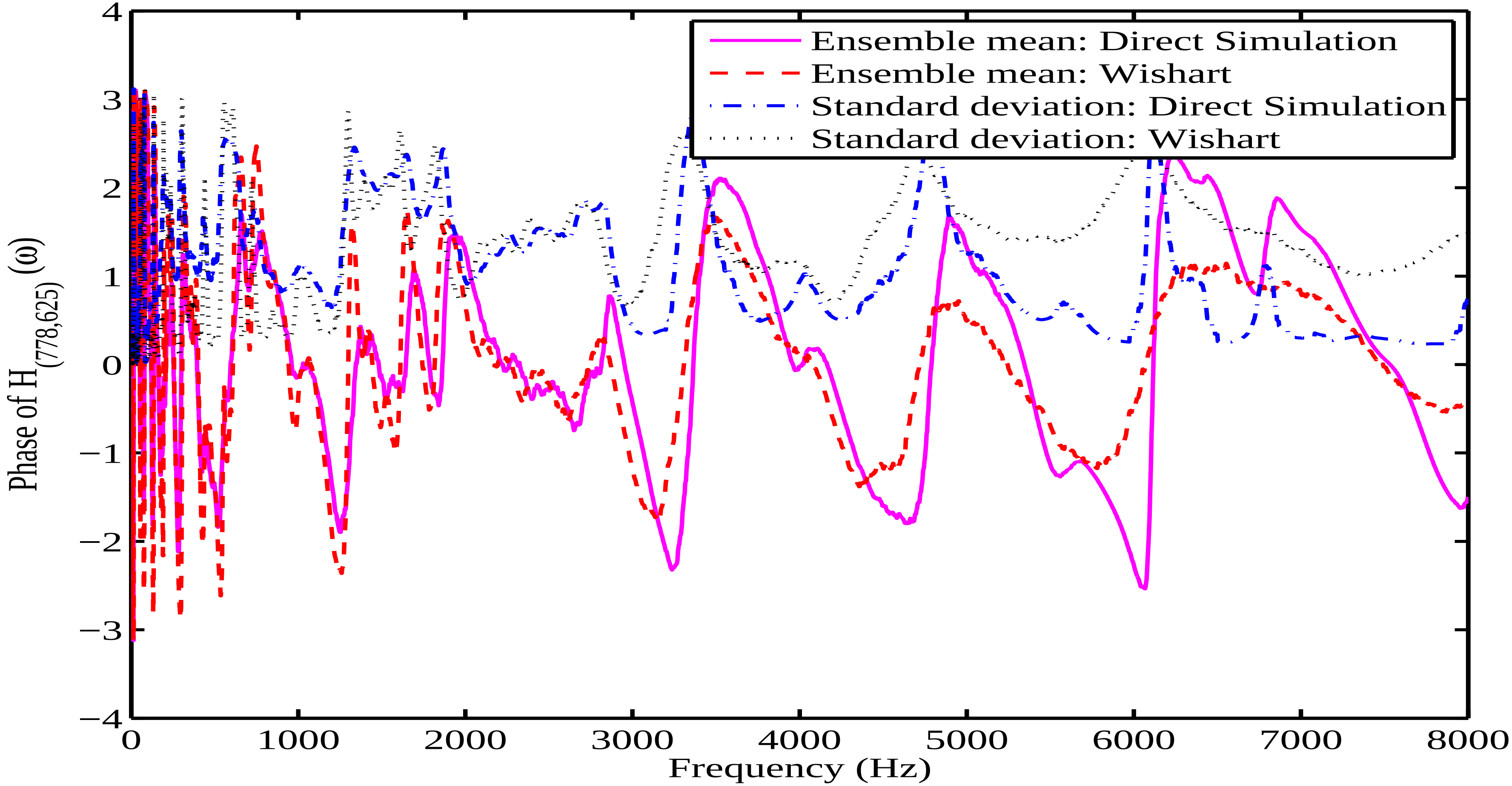
Comparison of Amplitude: High Freq



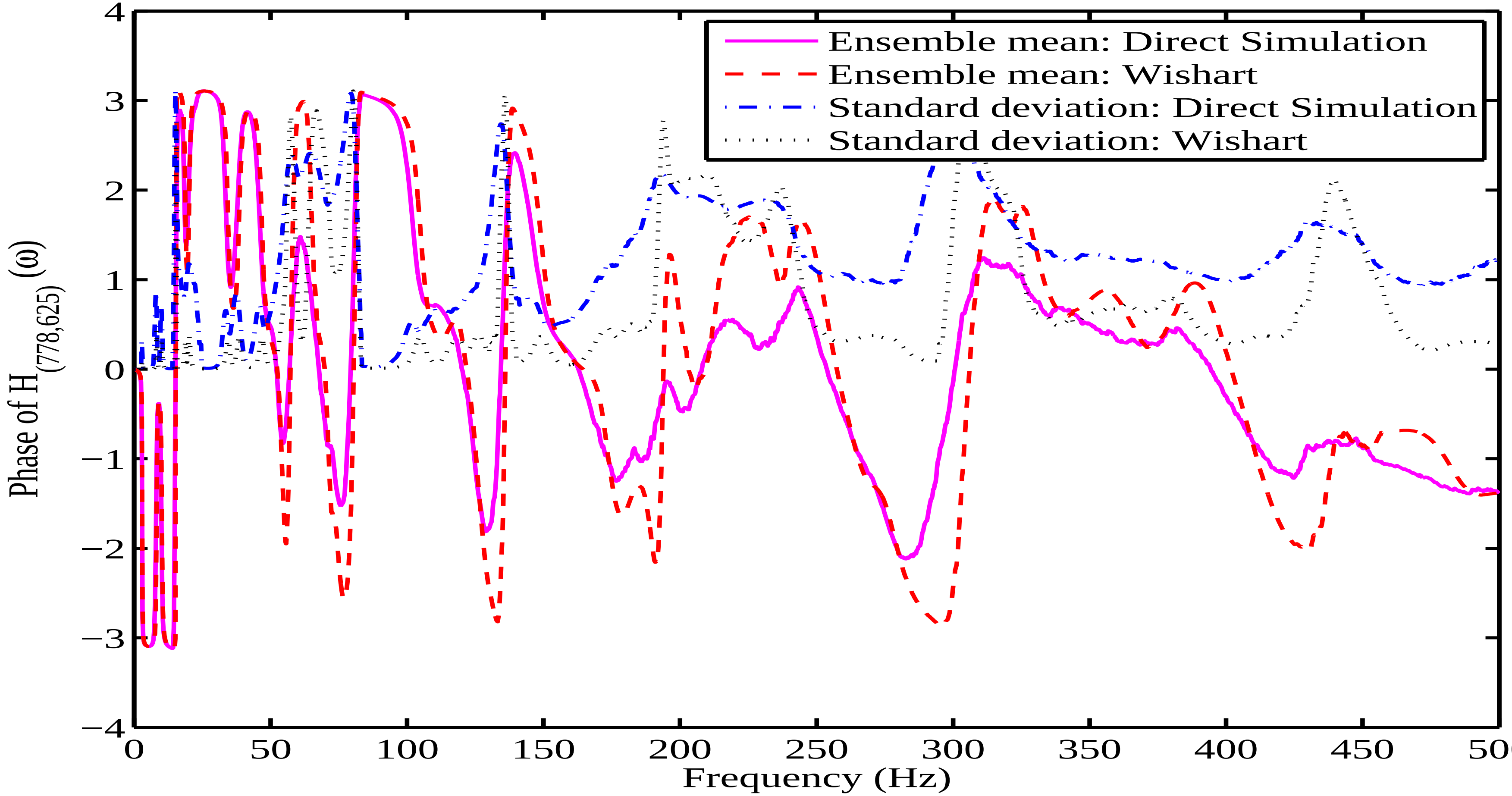
Comparison of the 5% and 95% probability points of the amplitude of the cross-FRF.

Comparison of phase



Comparison of the mean and standard deviation of the phase of the cross-FRF, $n = 945$ and $\delta_K = 0.1699$.

Comparison of phase: Low Freq



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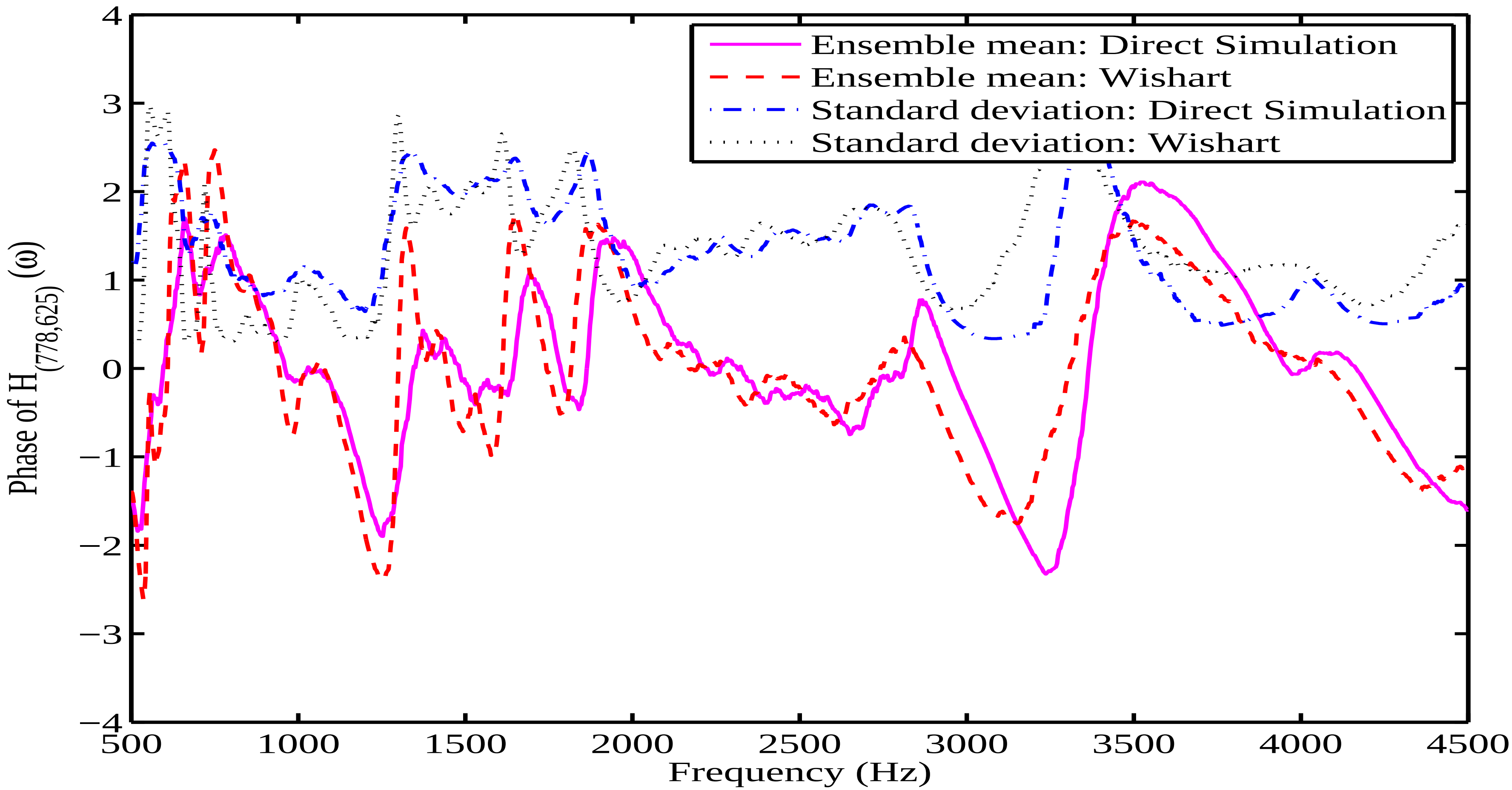
Comparison of phase: Mid Freq



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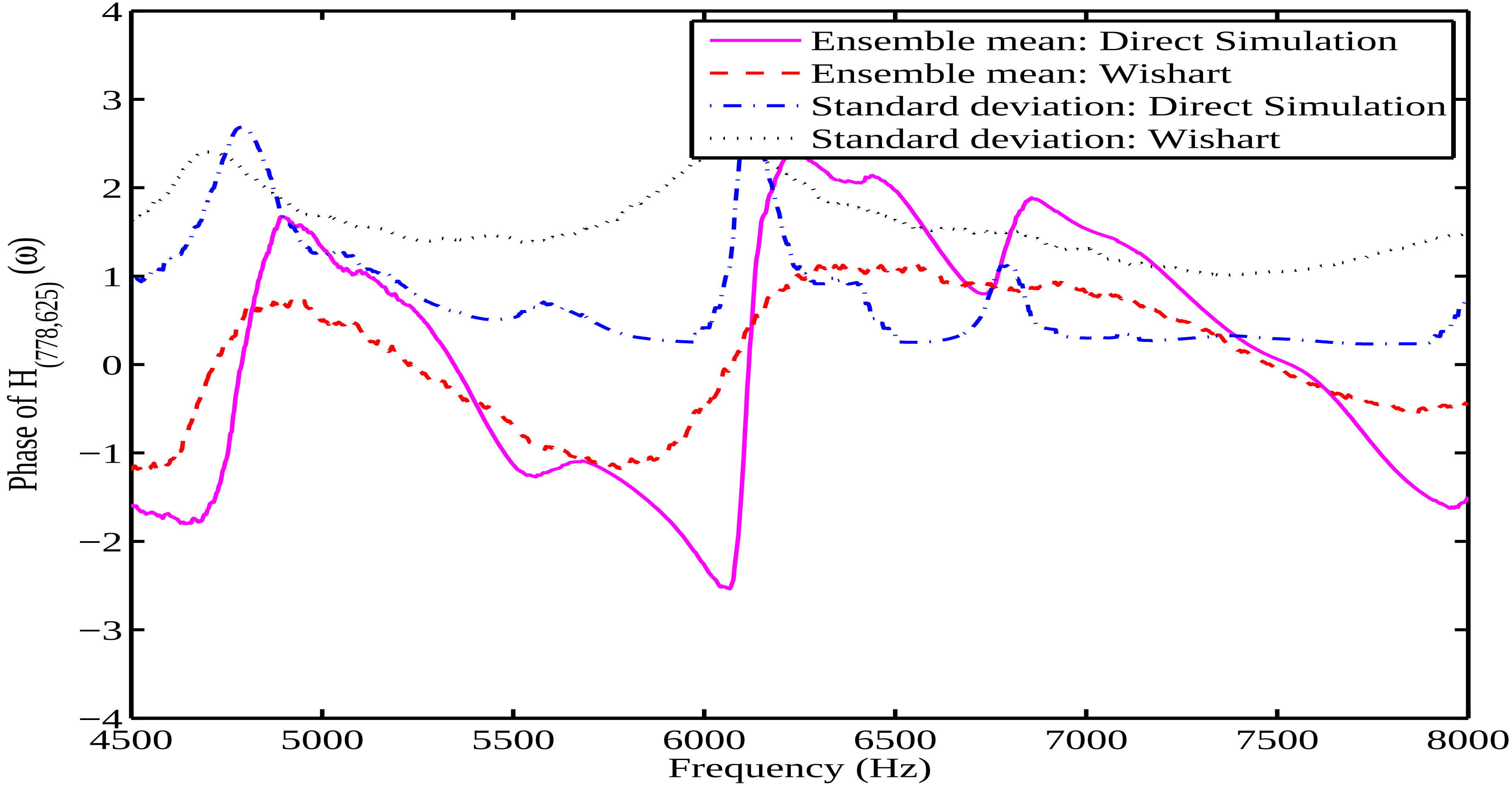
Comparison of phase: High Freq



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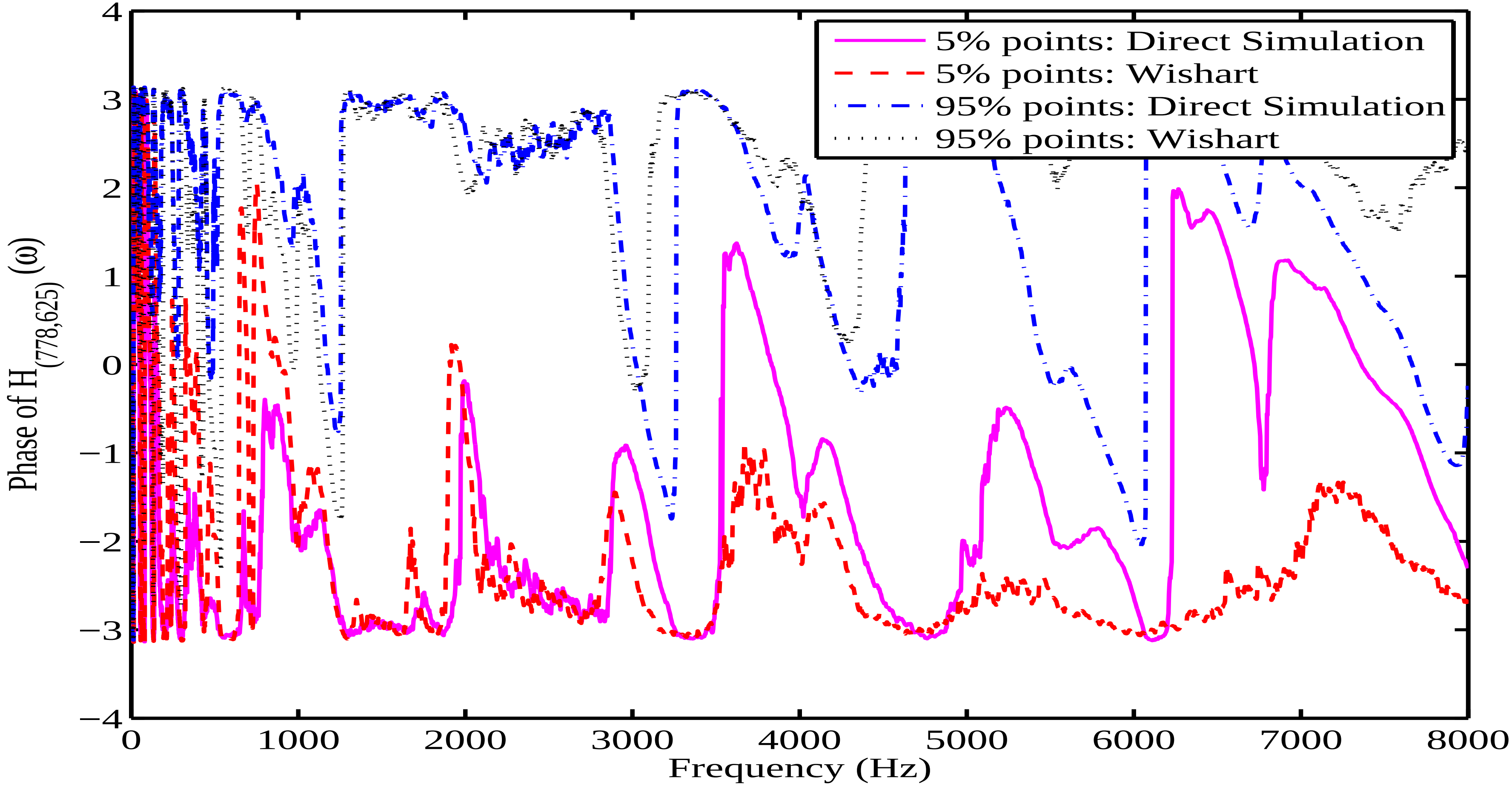
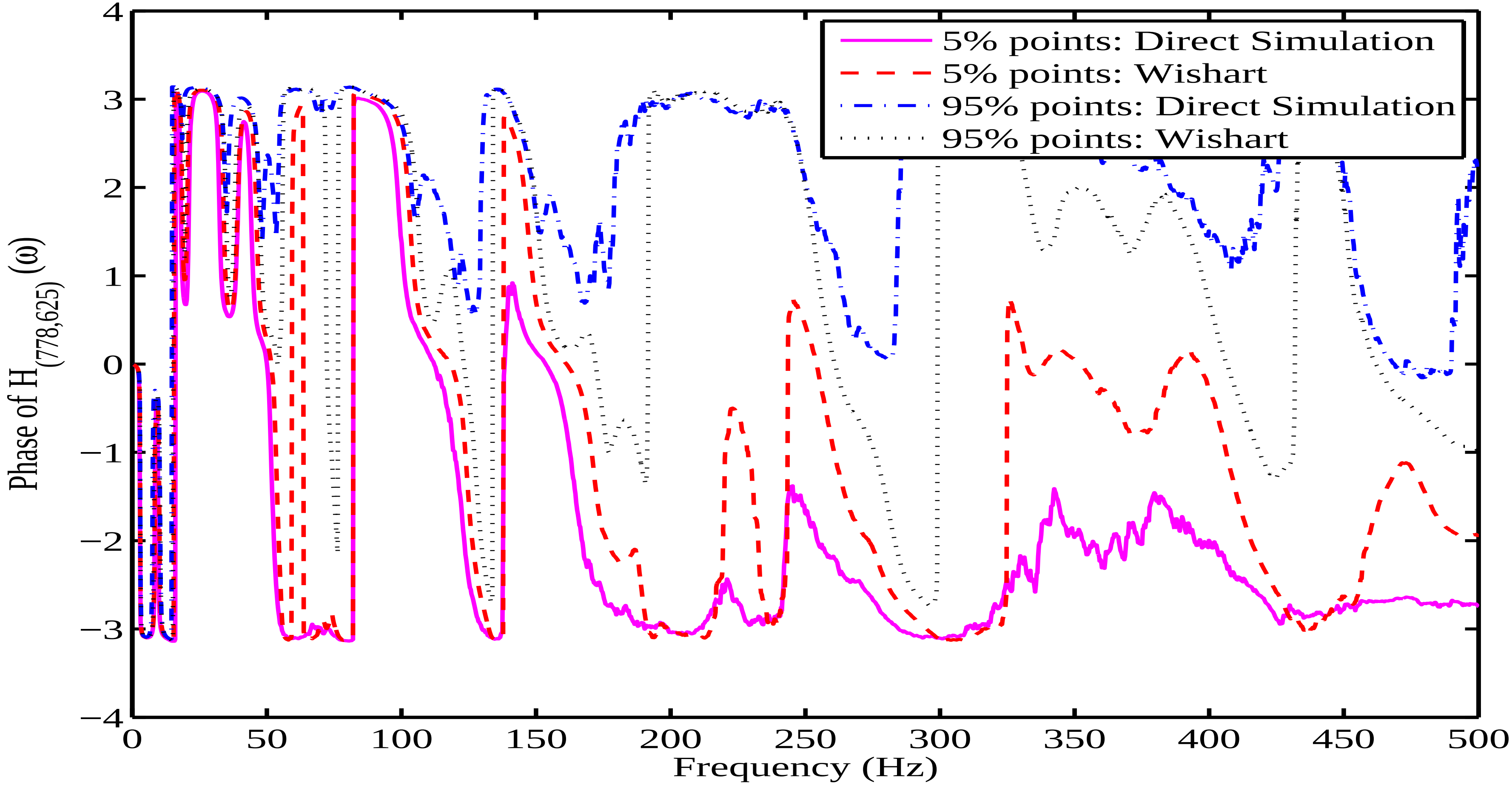
Comparison of phase



Comparison of the 5% and 95% probability points of the phase of the cross-FRF.

Comparison of phase: Low Freq



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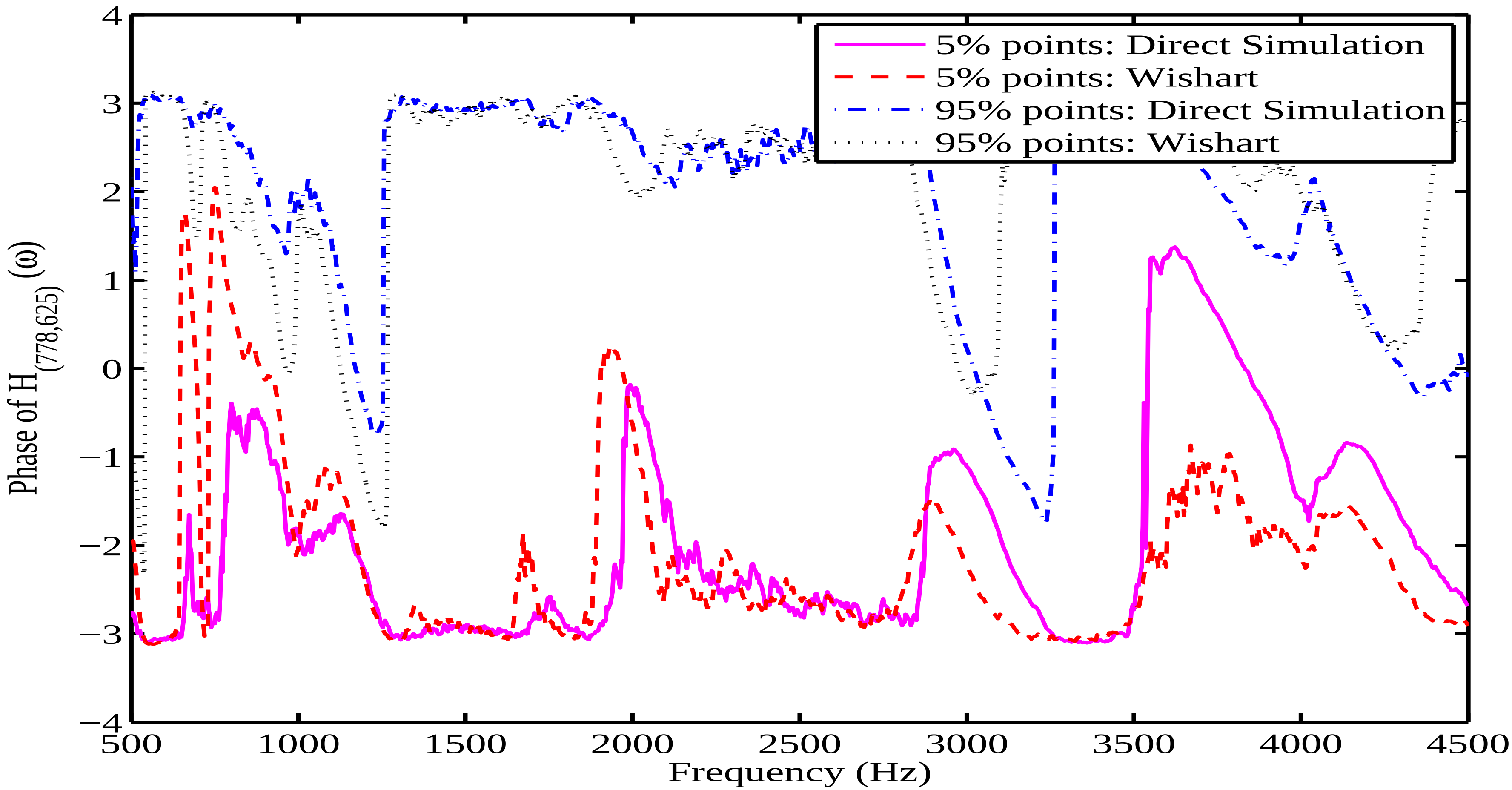
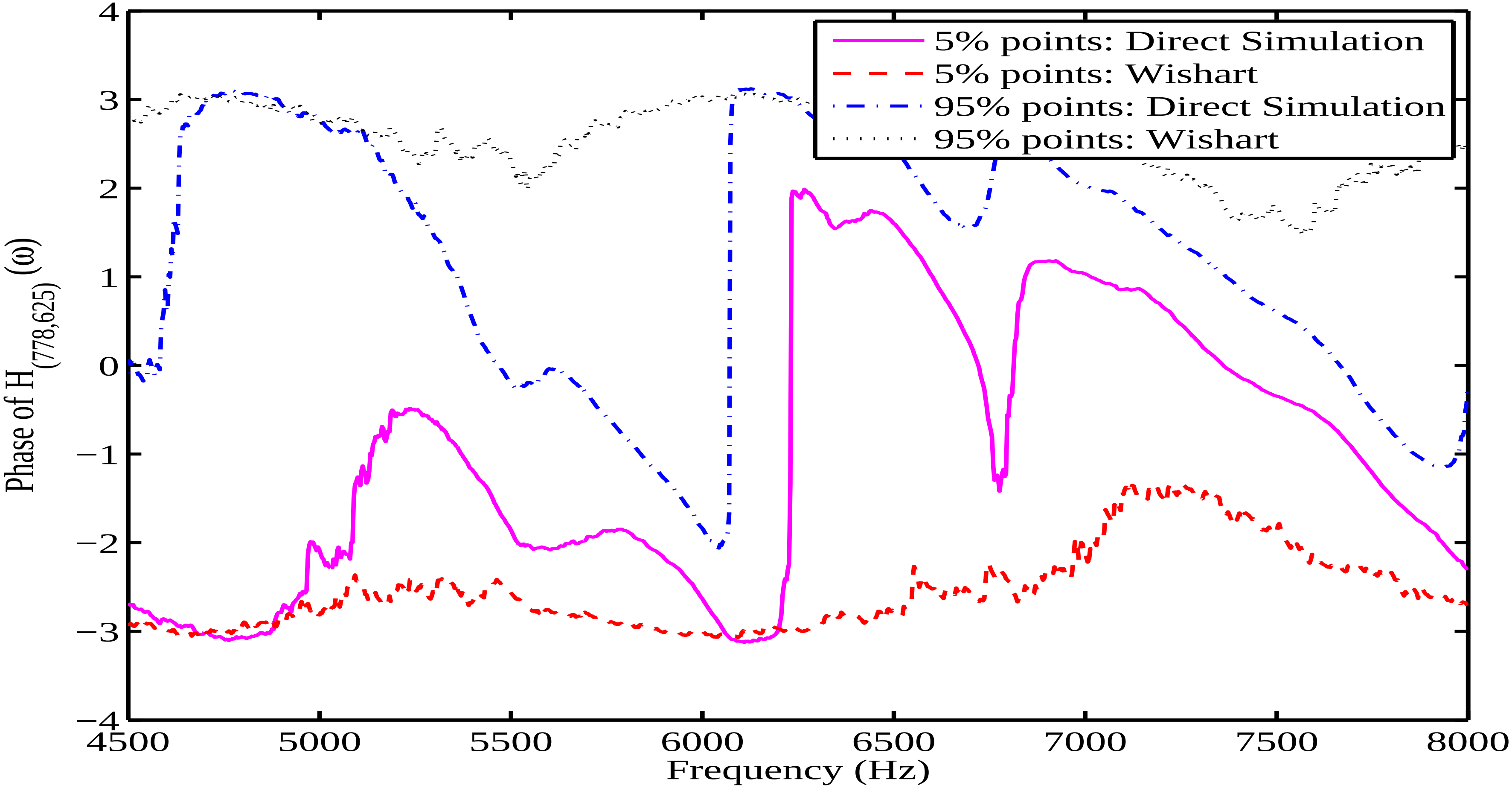


Comparison of the 5% and 95% probability points of the phase of the cross-FRF.

Comparison of phase: High Freq



Comparison of the 5% and 95% probability points of the phase of the cross-FRF.

Conclusions

- We attempted to represent model uncertainty in linear dynamical systems using Random Matrix Theory (RMT). The study explored practical means to represent an ensemble of dynamical systems derived from model perturbation of a baseline system.
- We have shown that episematic or non-parametric uncertainty leads to the sample-wise variation in the sparsity structure of the system matrices.
- Using a Matrix Factorization Approach (MFA), it was shown that Wishart matrices may be used as the model for the random system matrices in structural dynamics.

Conclusions

- Only the mean matrix and the normalized standard deviation is required to model the system.
- As an illustration, we considered model uncertainty in a vibrating plate due to disorderly attached sprung-mass oscillators having random natural frequencies.
- The encouraging agreements (in the mid and high frequency region) between the results obtained the from Wishart matrix model and direct Monte Carlo simulation suggest that it may be a practical method to represent the statistical dispersion observed in the response of dynamical systems arising from model uncertainty.