Stochastic Methods in Structural Dynamics Part 1

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About me

Education:

- PhD (Engineering), 2001, University of Cambridge (Trinity College), Cambridge, UK.
- MSc (Structural Engineering), 1997, Indian Institute of Science, Bangalore, India.
- B. Eng, (Civil Engineering), 1995, Calcutta University, India.

Work:

- 04/2007-Present: Professor of Aerospace Engineering, Swansea University (Civil and Computational Engineering Research Centre).
- 01/2003-03/2007: Lecturer in dynamics: Department of Aerospace Engineering, University of Bristol.
- 11/2000-12/2002: Research Associate, Cambridge University Engineering Department (Junior Research Fellow, Fitzwilliam College, Cambridge) .

Overview of the course

The course is dived into eight topics:

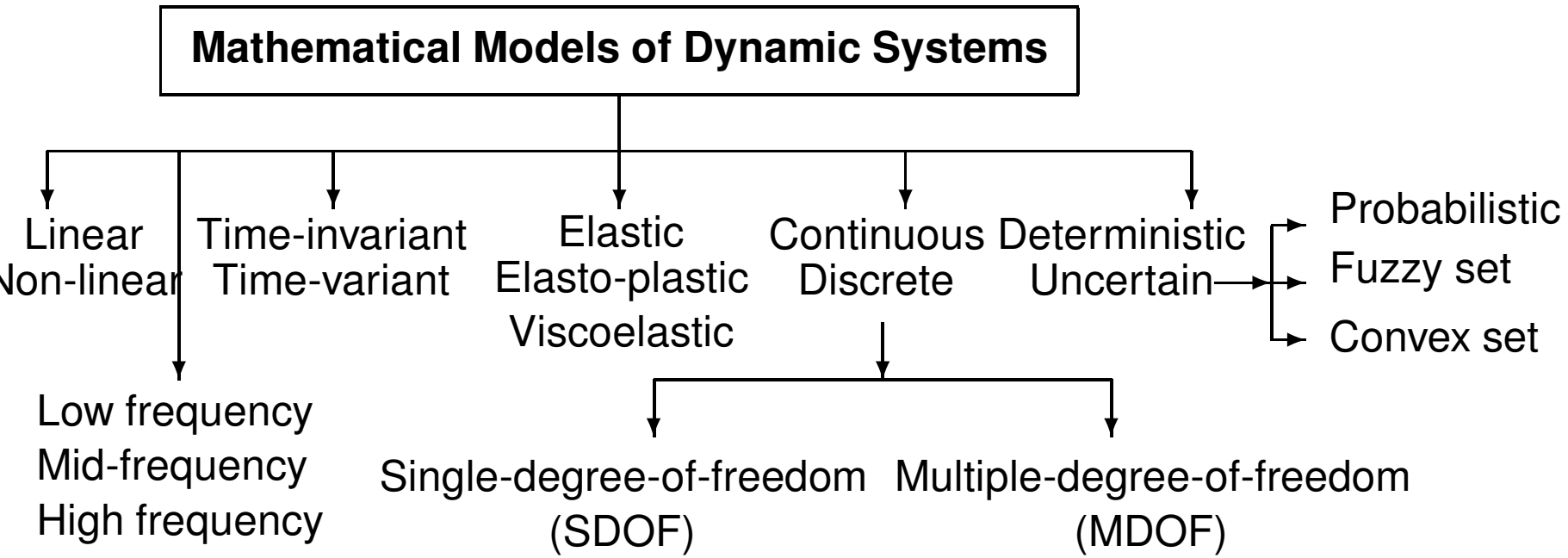
- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

Outline of this talk

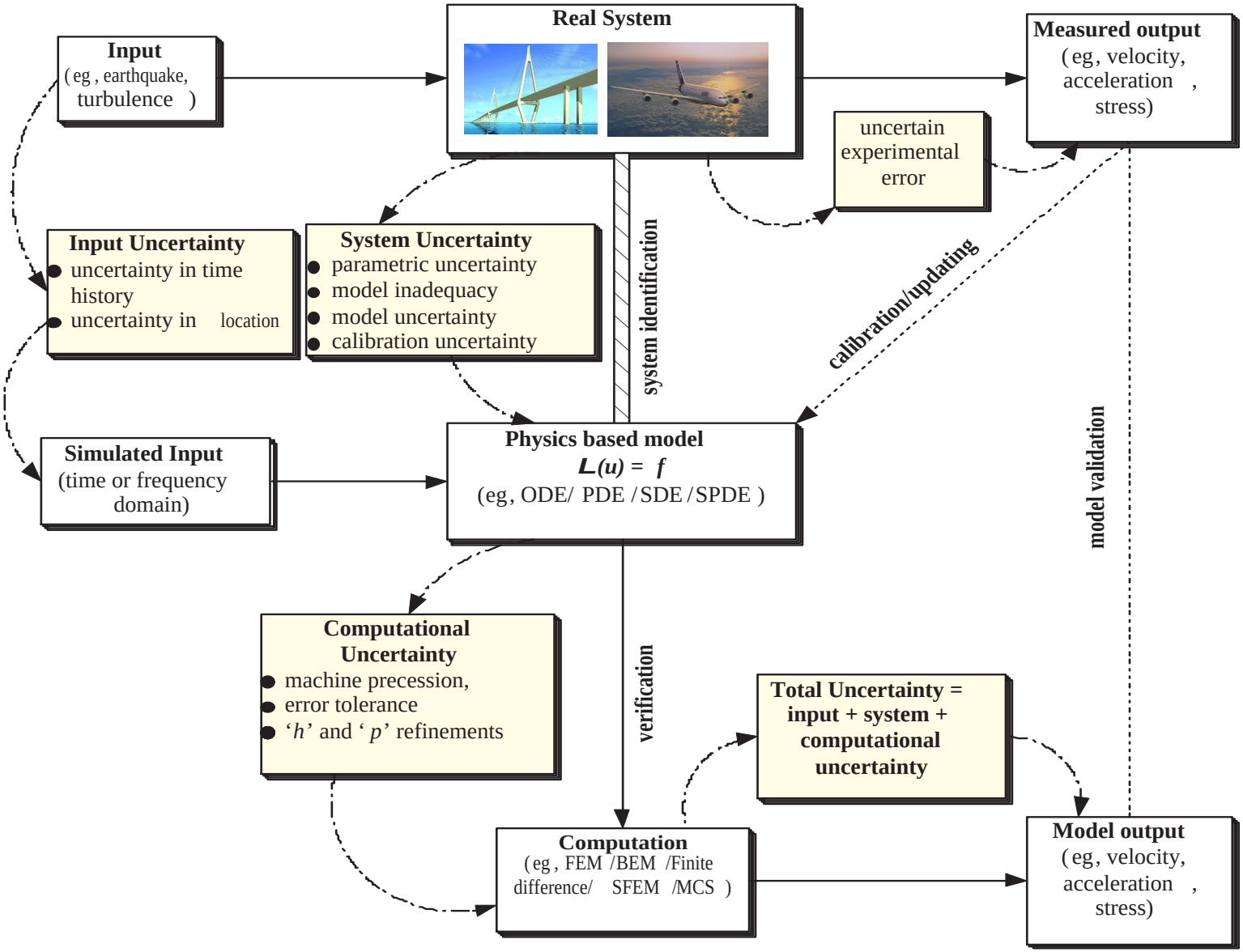
Few general questions

- How does system uncertainty impact the dynamic response? Does it matter?
- What is the underlying physics?
- How can we model uncertainty in dynamic systems? Do we ‘know’ the uncertainties?
- How can we efficiently quantify uncertainty in the dynamic response for large multi degrees of freedom systems?
- What about using ‘black box’ type response surface methods?
- Can we use modal analysis for stochastic systems? Does stochastic systems has natural frequencies and mode shapes?

Mathematical models for dynamic systems



A general overview of computational mechanics



Ensembles of structural dynamical systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems). On the other hand, some models are complex!

Complex structural dynamical systems

Complex aerospace system can have millions of degrees of freedom and there can be ‘errors’ and/or ‘lack of knowledge’ in its numerical (Finite Element) model

Model quality

The quality of a model of a dynamic system depends on the following three factors:

- *Fidelity to (experimental) data:*
  The results obtained from a numerical or mathematical model undergoing a given excitation force should be close to the results obtained from the vibration testing of the same structure undergoing the same excitation.
- *Robustness with respect to (random) errors:*
  Errors in estimating the system parameters, boundary conditions and dynamic loads are unavoidable in practice. The output of the model should not be very sensitive to such errors.
- *Predictive capability*
  In general it is not possible to experimentally validate a model over the entire domain of its scope of application. The model should predict the response well beyond its validation domain.

Sources of uncertainty

Different sources of uncertainties in the modeling and simulation of dynamic systems may be attributed, but not limited, to the following factors:

- Mathematical models: equations (linear, non-linear), geometry, damping model (viscous, non-viscous, fractional derivative), boundary conditions/initial conditions, input forces;
- Model parameters: Young's modulus, mass density, Poisson's ratio, damping model parameters (damping coefficient, relaxation modulus, fractional derivative order)
- Numerical algorithms: weak formulations, discretisation of displacement fields (in finite element method), discretisation of stochastic fields (in stochastic finite element method), approximate solution algorithms, truncation and roundoff errors, tolerances in the optimization and iterative methods, artificial intelligent (AI) method (choice of neural networks)
- Measurements: noise, resolution (number of sensors and actuators), experimental hardware, excitation method (nature of shakers and hammers), excitation and measurement point, data processing (amplification, number of data points, FFT), calibration

Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM/Finite difference
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating/calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM/SEA/RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating/calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random/deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random/deterministic)	Known (random)	Known from experiment and model (random)	*Model validation*	Validation methods
Known (random/deterministic)	Known (random)	Known from different computations (random)	*Model verification*	verification methods

Equation of motion

- The equations of motion of an undamped non-gyroscopic system with N degrees of freedom can be given by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

  where $\mathbf{M} \in \mathbb{R}^n$ is the mass matrix, $\mathbf{K} \in \mathbb{R}^n$ is the stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector.
- Equation (1) represents a set of coupled second-order ordinary-differential equations. The solution of this equation also requires the knowledge of the initial conditions in terms of displacements and velocities of all the coordinates. The *initial conditions* can be specified as

$$\mathbf{q}(0) = \mathbf{q_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_\mathbf{0} \in \mathbb{R}^N. \tag{2}$$

Modal analysis

- The natural frequencies (ω_j) and the mode shapes ($\mathbf{x}_j$) are intrinsic characteristic of a system and can be obtained by solving the associated matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j, \quad \forall\, j = 1, \cdots, N. \tag{3}$$

- The eigensolutions satisfy the orthogonality condition

$$\mathbf{x}_l^T \mathbf{M}\mathbf{x}_j = \delta_{lj} \tag{4}$$

$$\text{and} \quad \mathbf{x}_l^T \mathbf{K}\mathbf{x}_j = \omega_j^2 \delta_{lj}, \quad \forall\, l, j = 1, \cdots, N \tag{5}$$

- Using the orthogonality relationships in (4) and (5), the equations of motion in the modal coordinates may be obtained as

$$\begin{aligned} \ddot{\mathbf{y}}(t) + \mathbf{\Omega}^2 \mathbf{y}(t) &= \tilde{\mathbf{f}}(t) \\ \text{or} \quad \ddot{y}_j(t) + \omega_j^2 y_j(t) &= \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N \end{aligned} \tag{6}$$

where $\tilde{\mathbf{f}}(t) = \mathbf{X}^T \mathbf{f}(t)$ is the forcing function in modal coordinates.

Dynamic response

- Taking the Laplace transform of (1) and considering the initial conditions in (2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{7}$$

$$\text{or} \quad \left[s^2\mathbf{M} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0} = \bar{\mathbf{p}}(s) \ \text{(say)}. \tag{8}$$

- Using the mode orthogonality the response in the frequency domain

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q_0}}{\omega_j^2 - \omega^2}\mathbf{x}_j. \tag{9}$$

This expression shows that the dynamic response of the system is a linear combination of the mode shapes.

Equation of motion

- The equations of motion can expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{10}$$

Theorem

Viscously damped system (10) possesses classical normal modes if and only if $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$.

- With proportional damping assumption, the damping matrix $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$. This implies that the damping matrix in the modal coordinate

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} \tag{11}$$

is a diagonal matrix. The damping ratios ζ_j are defined from the diagonal elements of the modal damping matrix as

$$C'_{jj} = 2\zeta_j\omega_j \quad \forall j = 1, \cdots, N. \tag{12}$$

Dynamic response

- The equations of motion in the modal coordinate can be decoupled as

$$\ddot{y}_j(t) + 2\zeta_j\omega_j\dot{y}_j(t) + \omega_j^2 y_j(t) = \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N. \tag{13}$$

- Taking the Laplace transform of (10) and considering the initial conditions in (2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{C}\bar{\mathbf{q}} - \mathbf{C}\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{14}$$

$$\text{or} \quad \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}. \tag{15}$$

- The transfer function matrix or the receptance matrix can be obtained as

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + 2i\omega\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T = \sum_{j=1}^{N}\frac{\mathbf{x}_j\mathbf{x}_j^T}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}. \tag{16}$$

Dynamic response

- The dynamic response in the frequency domain can be conveniently represented as

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T \bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{x}_j^T \mathbf{C}\mathbf{q}_\mathbf{0} + i\omega \mathbf{x}_j^T \mathbf{M}\mathbf{q}_\mathbf{0}}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2} \mathbf{x}_j. \tag{17}$$

  Therefore, like undamped systems, the dynamic response of proportionally damped system can also be expressed as a linear combination of the undamped mode shapes.

Dynamic response

- In the time-domain, taking the inverse Laplace transform we have

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t)\mathbf{x}_j \tag{18}$$

where the time dependent constants are given by

$$a_j(t) = \int_0^t \frac{1}{\omega_{d_j}} \mathbf{x}_j^T \mathbf{f}(\tau) e^{-\zeta_j \omega_j (t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau + e^{-\zeta_j \omega_j t} B_j \cos\left(\omega_{d_j} t + \theta_j\right) \tag{19}$$

where

$$B_j = \sqrt{\left(\mathbf{x}_j^T \mathbf{M}\mathbf{q_0}\right)^2 + \frac{1}{\omega_{d_j}^2}\left(\zeta_j \omega_j \mathbf{x}_j^T \mathbf{M}\mathbf{q_0} - \mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} - \mathbf{x}_j^T \mathbf{C}\mathbf{q_0}\right)^2} \tag{20}$$

$$\text{and} \quad \tan\theta_j = \frac{1}{\omega_{d_j}}\left(\zeta_j \omega_j - \frac{\mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{x}_j^T \mathbf{C}\mathbf{q_0}}{\mathbf{x}_j^T \mathbf{M}\mathbf{q_0}}\right) \tag{21}$$

Definition of a random variable

- A real random variable $Y(\theta)$, $\theta \in \Theta$ is a set of function defined on Θ such that for every real number y there exist a probability $P(\theta : Y(\omega) \leq y)$
- *Probability Distribution Function:* Consider the event $Y \leq y$. We define

$$F(y) = P(Y \leq y), y \in \mathbb{R}$$

  $F(y)$ is called Probability Distribution Function of Y. $F(y)$ is a monotonically increasing function y with $F(-\infty) = 0$ and $F(\infty) = 1$.
- *Probability Density Function:* The probability structure of a random variable can be described by the derivative of the probability distribution function $p(y)$, called the Probability Density Function. Thus

$$p(y) = \frac{\partial F(y)}{\partial y}$$

  This is normalised such that

$$\int_{-\infty}^{\infty} p(y)\mathrm{d}y = 1$$

Definition of a random field/process

- A random field $H(x, \theta)$ is defined as a set function of two arguments $\theta \in \Theta$ and $x \in X$, where Θ is the sample space of the family of random variables $H(x, \bullet)$ and X is the indexing set of parameter X.
- Since a random field $H(x, \theta)$ reduces to a set of random variables at fixed instances of $x = x_1, x_2, \cdots x_n, \cdots$, its probability structure may be defined by a hierarchy of joint probability density function

$$p(h_1, x_1), \quad p(h_1, x_1; h_2, x_2), \cdots, p(h_1, x_1; h_2, x_2; \cdots, h_n, x_n; \cdots) \tag{22}$$

- *Stationary random field:* A random field is said to be stationary if its probability structure is invariant under arbitrary translations of the indexing parameter. Thus $H(x, \theta)$ is stationary if for all $x_1, x_2, \cdots, x_n$ and an arbitrary constant τ if for all n

$$p(h_1, x_1; h_2, x_2; \cdots, h_n, x_n) = p(h_1, x_1 + \tau; h_2, x_2 + \tau; \cdots, h_n, x_n + \tau) \tag{23}$$

Moments of a random field

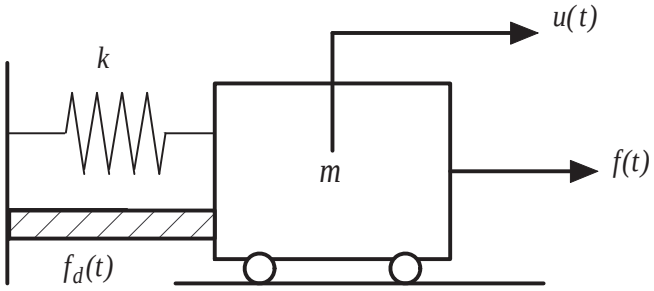
- The mean of a random field is given by

$$\mathrm{E}\left[H(x,\theta)\right] = \int H(x,\theta)p(h_1, x_1)\mathrm{d}h_1$$

- The autocorrelation is given by

$$C_{HH}(x_1, x_2) = \int H(x,\theta)p(h_1, x_1; h_2, x_2)\mathrm{d}h_1\,\mathrm{d}h_2$$

Stochastic SDOF systems



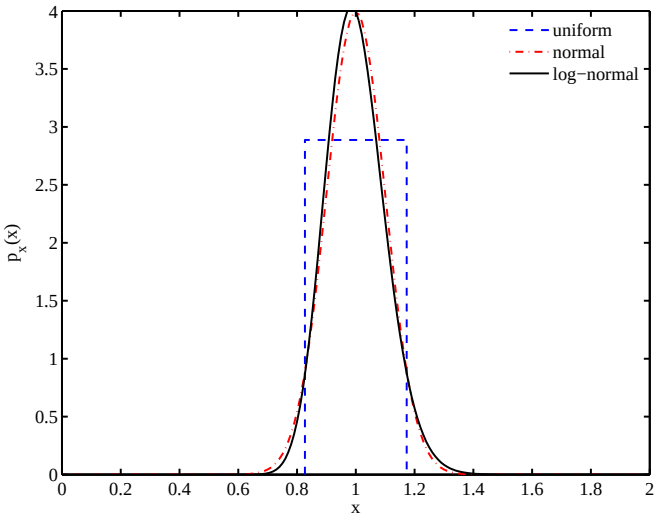
Consider a normalised single degrees of freedom system (SDOF):

$$\ddot{u}(t) + 2\zeta\omega_n \dot{u}(t) + \omega_n^2 u(t) = f(t)/m \tag{24}$$

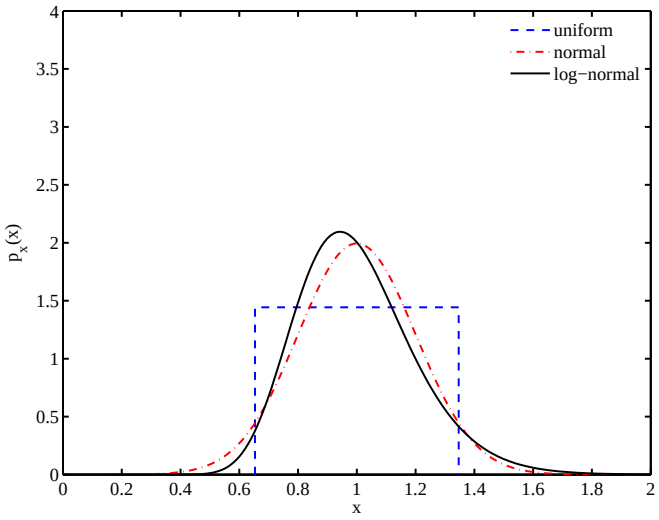
Here $\omega_n = \sqrt{k/m}$ is the natural frequency and $\xi = c/2\sqrt{km}$ is the damping ratio.

- We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.
- Stochastic perturbation can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency.

Frequency variability



(a) Pdf: $\sigma_a = 0.1$

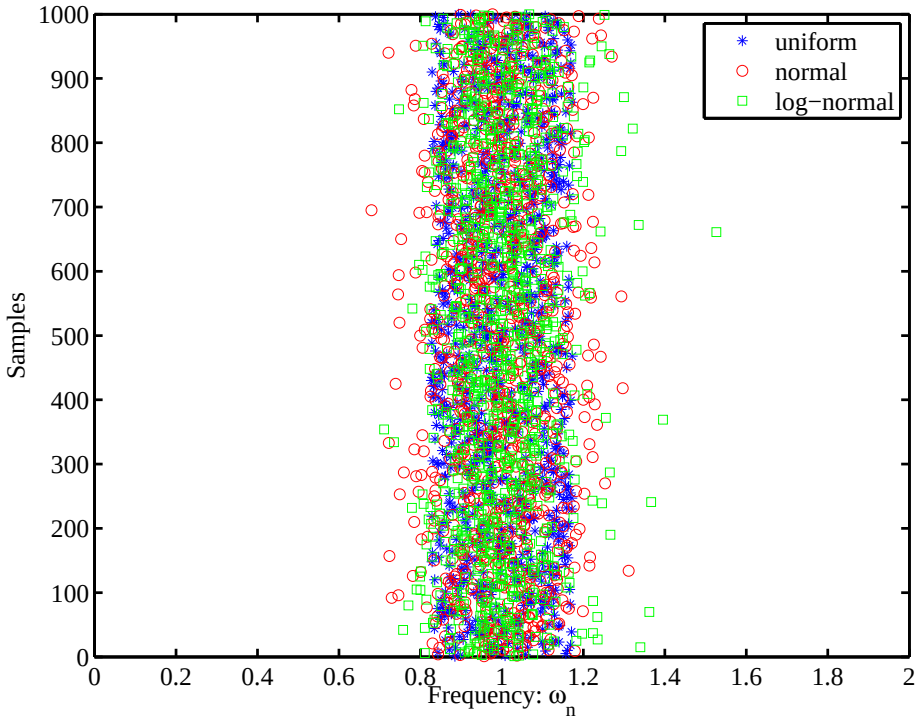


(b) Pdf: $\sigma_a = 0.2$

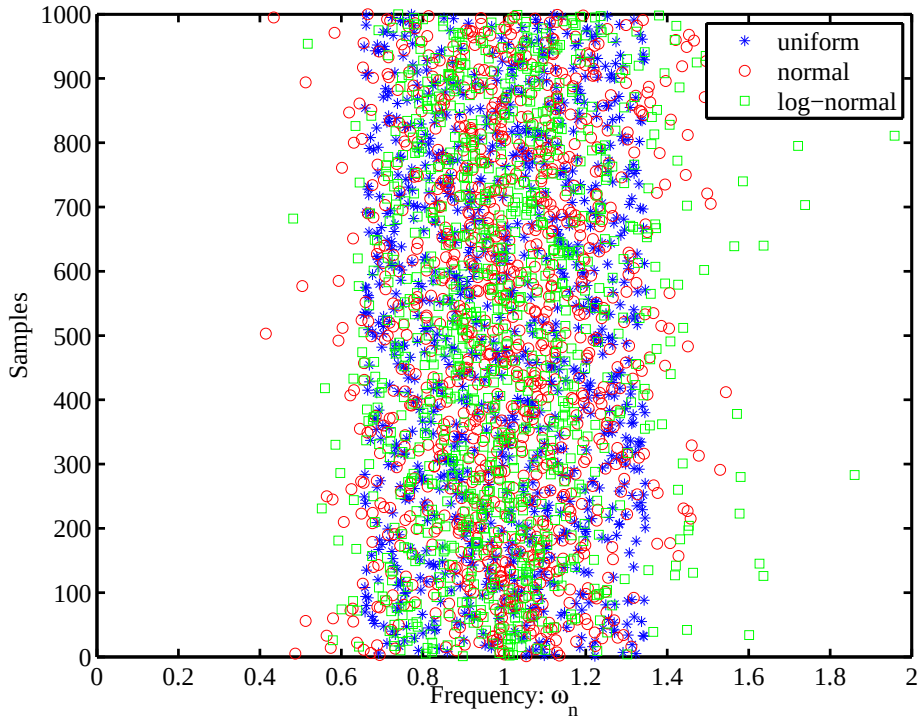
Figure : We assume that the mean of r is 1 and the standard deviation is σ_a.

- Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 r$, where ω_{n_0} is deterministic frequency and r is a random variable with a given probability distribution function.
- Several probability distribution function can be used.
- We use uniform, normal and lognormal distribution

Frequency samples



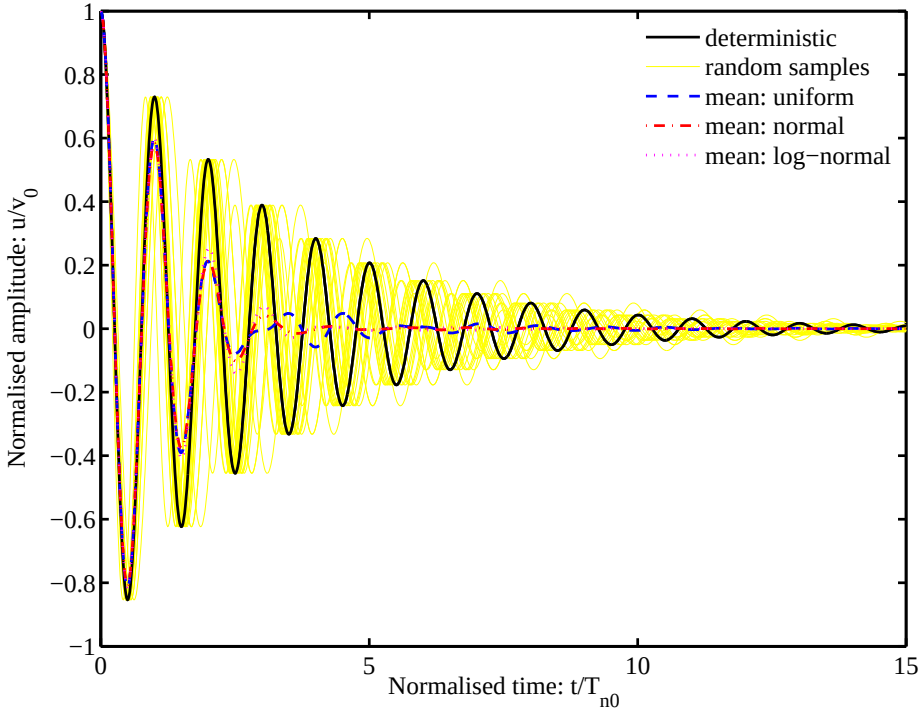
(a) Frequencies: $\sigma_a = 0.1$



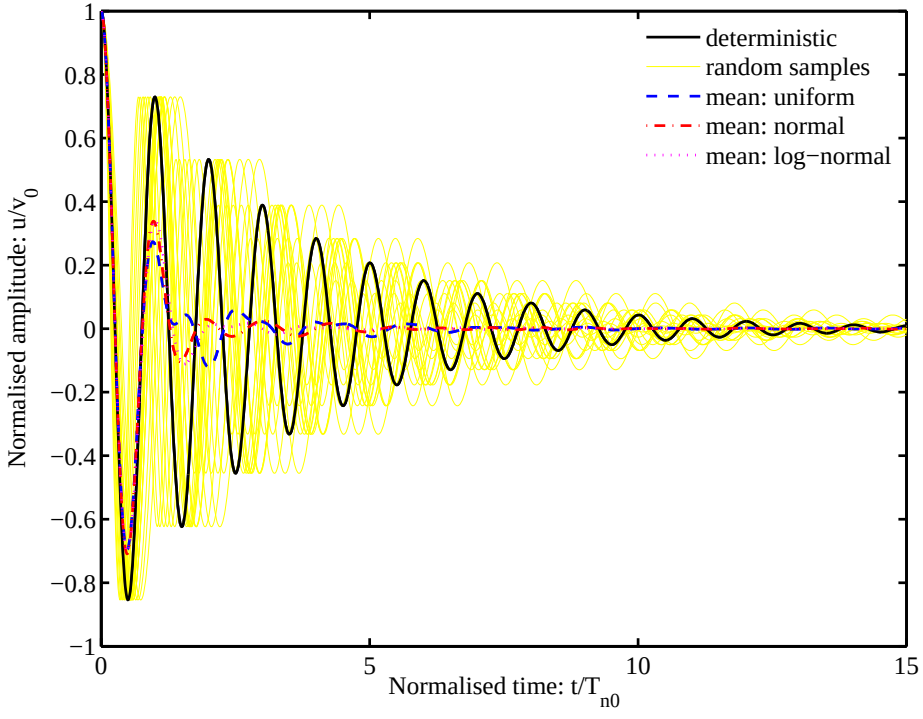
(b) Frequencies: $\sigma_a = 0.2$

Figure : 1000 sample realisations of the frequencies for the three distributions

Response in the time domain



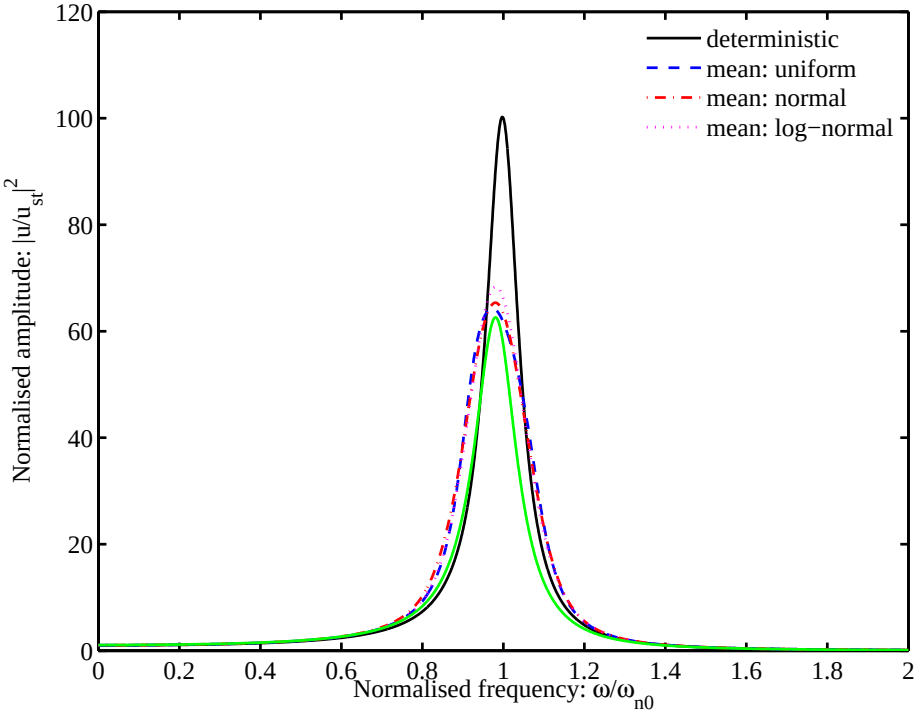
(a) Response: $\sigma_a = 0.1$



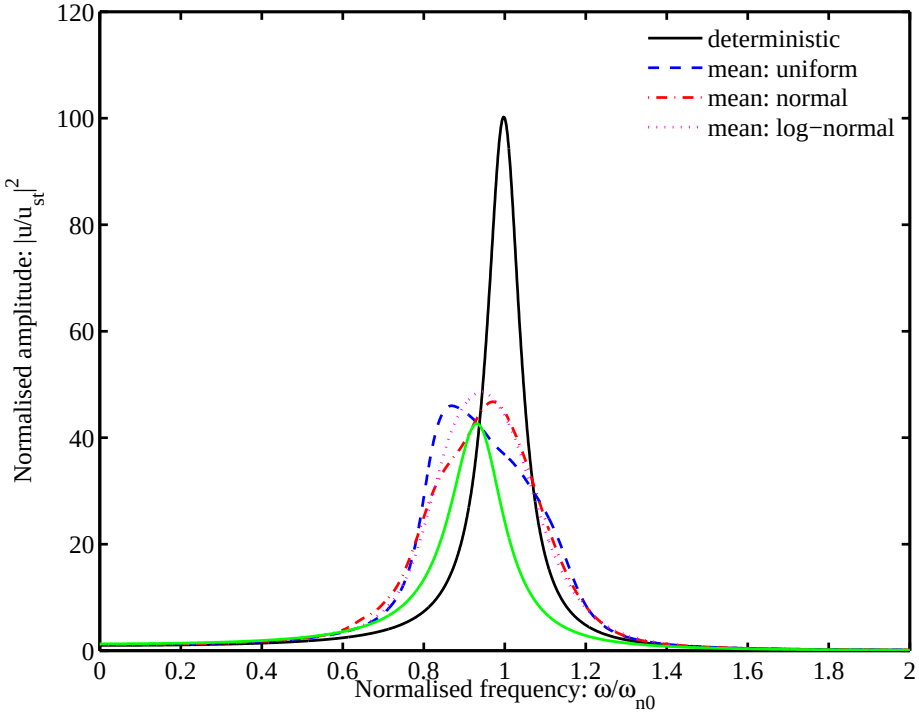
(b) Response: $\sigma_a = 0.2$

Figure : Response due to initial velocity v_0 with 5% damping

Frequency response function



(a) Response: $\sigma_a = 0.1$



(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function $|u/u_{st}|^2$, where $u_{st} = f/k$

Key observations

- The mean response is more damped compared to deterministic response.
- The higher the randomness, the higher the “effective damping”.
- The qualitative features are almost independent of the distribution the random natural frequency.
- We often use averaging to obtain more reliable experimental results - is it always true?

Assuming uniform random variable, we aim to explain some of these observations.

Equivalent damping

- Assume that the random natural frequencies are $\omega_n^2 = \omega_{n_0}^2(1+\epsilon x)$, where x has zero mean and unit standard deviation.
- The normalised harmonic response in the frequency domain

$$\frac{u(\mathrm{i}\omega)}{f/k} = \frac{k/m}{[-\omega^2 + \omega_{n_0}^2(1+\epsilon x)] + 2\mathrm{i}\xi\omega\omega_{n_0}\sqrt{1+\epsilon x}} \tag{25}$$

- Considering $\omega_{n_0} = \sqrt{k/m}$ and frequency ratio $r = \omega/\omega_{n_0}$ we have

$$\frac{u}{f/k} = \frac{1}{[(1+\epsilon x) - r^2] + 2\mathrm{i}\xi r\sqrt{1+\epsilon x}} \tag{26}$$

Equivalent damping

- The squared-amplitude of the normalised dynamic response at $\omega = \omega_{n_0}$ (that is $r = 1$) can be obtained as

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 = \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \tag{27}$$

- Since x is zero mean unit standard deviation uniform random variable, its pdf is given by $p_x(x) = 1/2\sqrt{3}, -\sqrt{3} \leq x \leq \sqrt{3}$
- The mean is therefore

$$\begin{aligned}
\mathrm{E}\left[\hat{U}\right] &= \int \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} p_x(x)\mathrm{d}x \\
&= \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} - \frac{\xi}{\sqrt{1-\xi^2}}\right) \\
&+ \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} + \frac{\xi}{\sqrt{1-\xi^2}}\right)
\end{aligned} \tag{28}$$

Equivalent damping

- Note that

$$\frac{1}{2}\left\{\tan^{-1}(a+\delta)+\tan^{-1}(a-\delta)\right\}=\tan^{-1}(a)+O(\delta^2) \tag{29}$$

- Neglecting terms of the order $O(\xi^2)$ we have

$$\mathrm{E}\left[\hat{U}\right]\approx\frac{1}{2\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}}\tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}}\right)=\frac{\tan^{-1}(\sqrt{3}\epsilon/2\xi)}{2\sqrt{3}\epsilon\xi} \tag{30}$$

Equivalent damping

- For small damping, the maximum deterministic amplitude at $\omega = \omega_{n_0}$ is $1/4\xi_e^2$ where ξ_e is the equivalent damping for the mean response
- Therefore, the equivalent damping for the mean response is given by

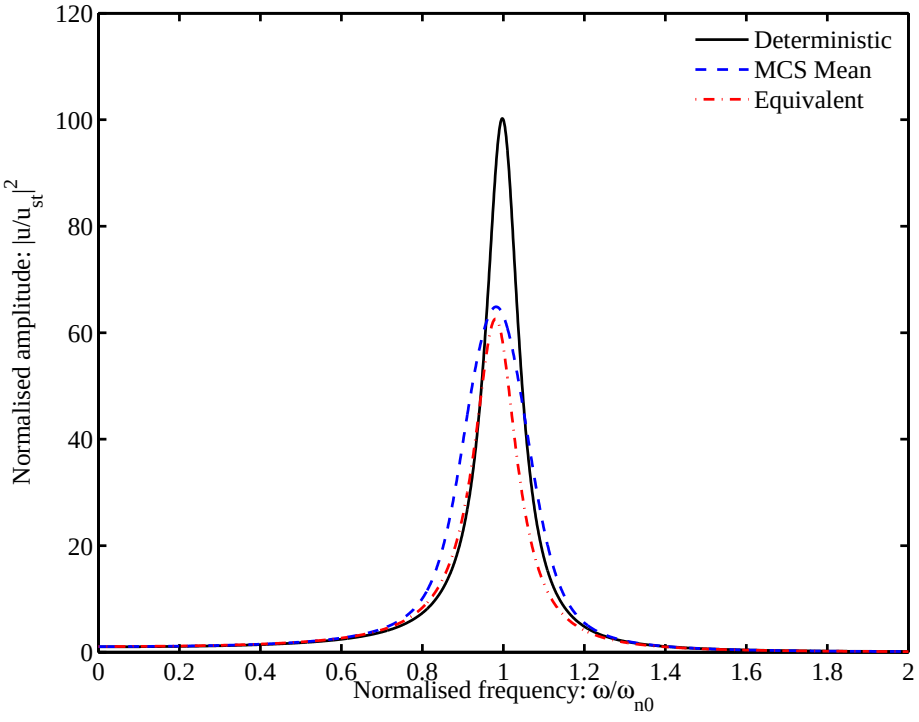
$$(2\xi_e)^2 = \frac{2\sqrt{3}\epsilon\xi}{\tan^{-1}(\sqrt{3}\epsilon/2\xi)} \tag{31}$$

- For small damping, taking the limit we can obtain

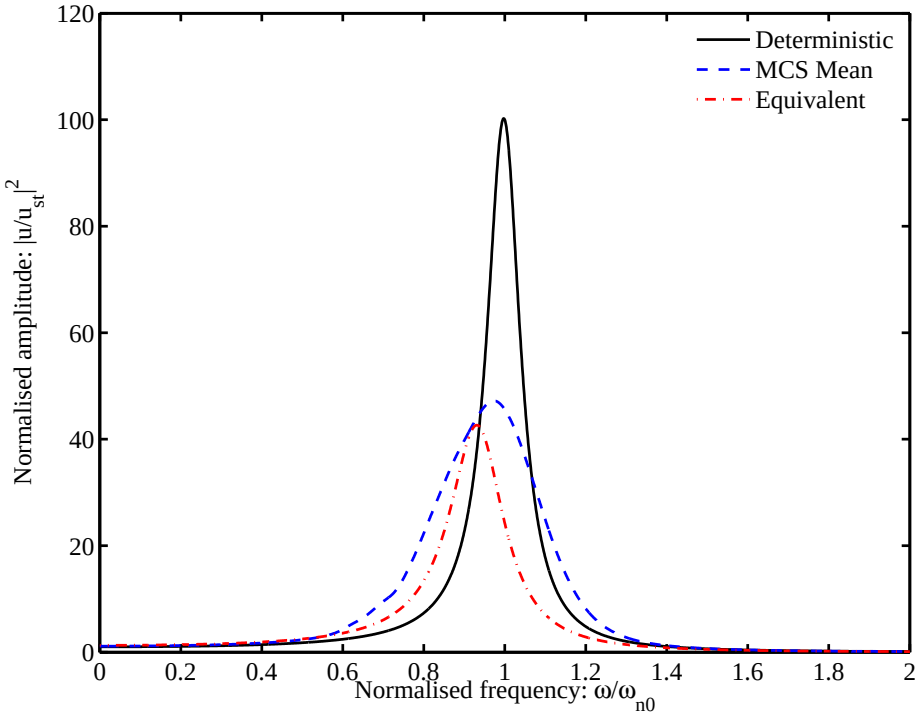
$$\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi} \tag{32}$$

- *The equivalent damping factor of the mean system is proportional to the square root of the damping factor of the underlying baseline system*

Equivalent frequency response function



(a) Response: $\sigma_a = 0.1$



(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function with equivalent damping ($\xi_e = 0.05$ in the ensembles). For the two cases $\xi_e = 0.0643$ and $\xi_e = 0.0819$ respectively.

Can we extend the ideas based on stochastic SDOF systems to stochastic MDOF systems?

Stochastic modal analysis

- Stochastic modal analysis to obtain the dynamic response needs further thoughts
- Consider the following 3DOF example:

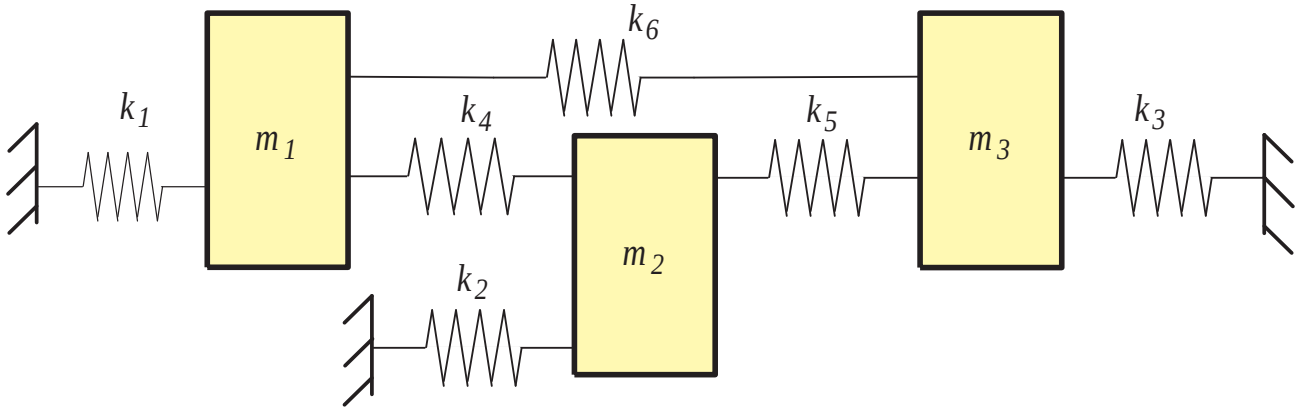
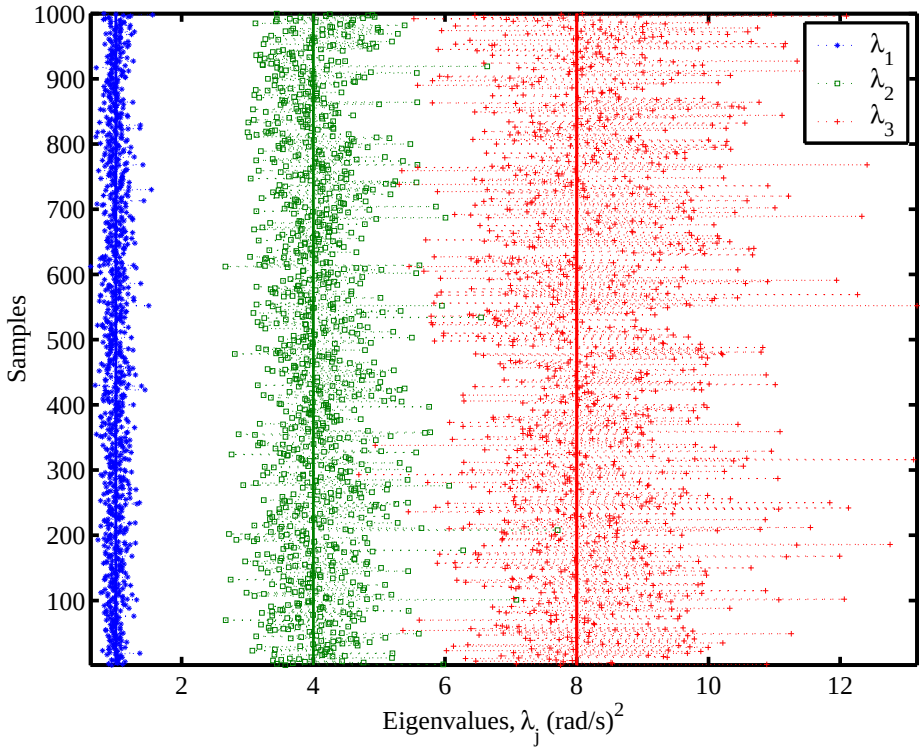
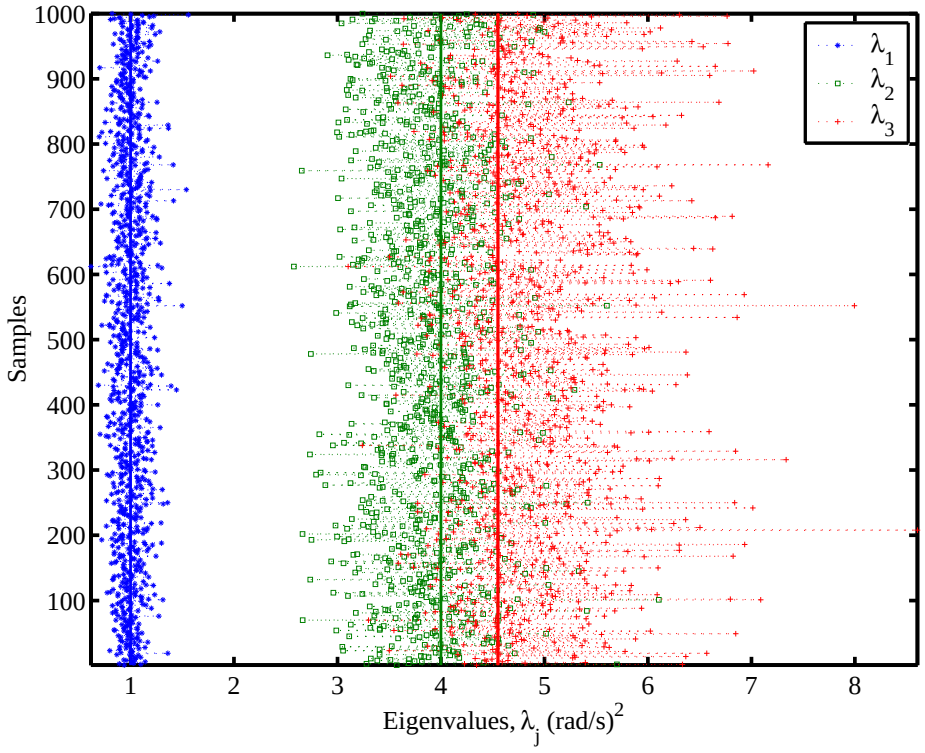


Figure : A 3DOF system with parametric uncertainty in m_i and k_i

Statistical overlap



(a) Eigenvalues are seperated



(b) Someeigenvalues are close

Figure : Scatter of the eigenvalues due to parametric uncertainties

We consider a stochastic partial differential equation (PDE)

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t,\theta)}{\partial t^2} + \mathcal{L}_\alpha \frac{\partial U(\mathbf{r},t,\theta)}{\partial t} + \mathcal{L}_\beta U(\mathbf{r},t,\theta) = p(\mathbf{r},t) \tag{33}$$

The stochastic operator $\mathcal{L}_\beta$ can be

- $\mathcal{L}_\beta \equiv \frac{\partial}{\partial x} AE(x,\theta)\frac{\partial}{\partial x}$ axial deformation of rods
- $\mathcal{L}_\beta \equiv \frac{\partial^2}{\partial x^2} EI(x,\theta)\frac{\partial^2}{\partial x^2}$ bending deformation of beams

$\mathcal{L}_\alpha$ denotes the stochastic damping, which is mostly proportional in nature. Here $\alpha, \beta : \mathbb{R}^d \times \Theta \to \mathbb{R}$ are stationary square integrable random fields, which can be viewed as a set of random variables indexed by $\mathbf{r} \in \mathbb{R}^d$. Based on the physical problem the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities (e.g., $AE(x,\theta)$, $EI(x,\theta)$).

Discretisation of random fields

- The random process $a(\mathbf{r}, \theta)$ can be expressed in a generalized Fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r}, \theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \tag{34}$$

- Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1, \mathbf{r}_2) \varphi_j(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 = \nu_j \varphi_j(\mathbf{r}_2), \quad \forall\, j = 1, 2, \cdots \tag{35}$$

Exponential autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{36}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève (KL) expansion in the interval $-a \leq x \leq a$ as

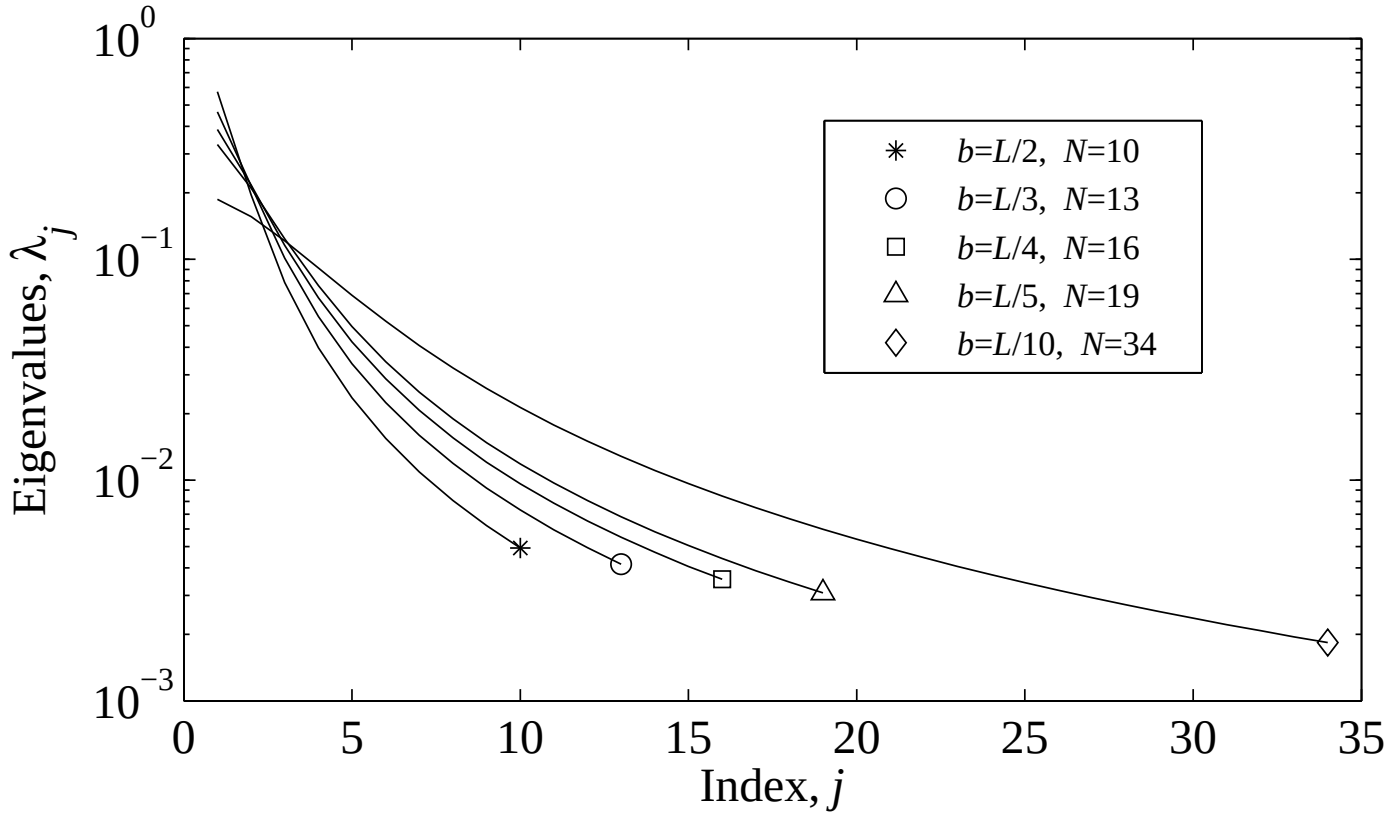
$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \tag{37}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions for odd j and even j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \qquad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{c}{\omega_j}, \tag{38}$$

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \qquad \varphi_j(x) = \frac{\sin(\omega_j x)}{\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{\omega_j}{-c}. \tag{39}$$

KL expansion



The eigenvalues of the Karhunen-Loève expansion for different correlation lengths, b, and the number of terms, N, required to capture 90% of the infinite series. An exponential correlation function with unit domain (i.e., $a = 1/2$) is assumed for the numerical calculations. The values of N are obtained such that $\lambda_N/\lambda_1 = 0.1$ for all correlation lengths. Only eigenvalues greater than λ_N are plotted.

Example: A beam with random properties

The equation of motion of an undamped Euler-Bernoulli beam of length L with random bending stiffness and mass distribution:

$$\frac{\partial^2}{\partial x^2}\left[EI(x,\theta)\frac{\partial^2 Y(x,t)}{\partial x^2}\right] + \rho A(x,\theta)\frac{\partial^2 Y(x,t)}{\partial t^2} = p(x,t). \tag{40}$$

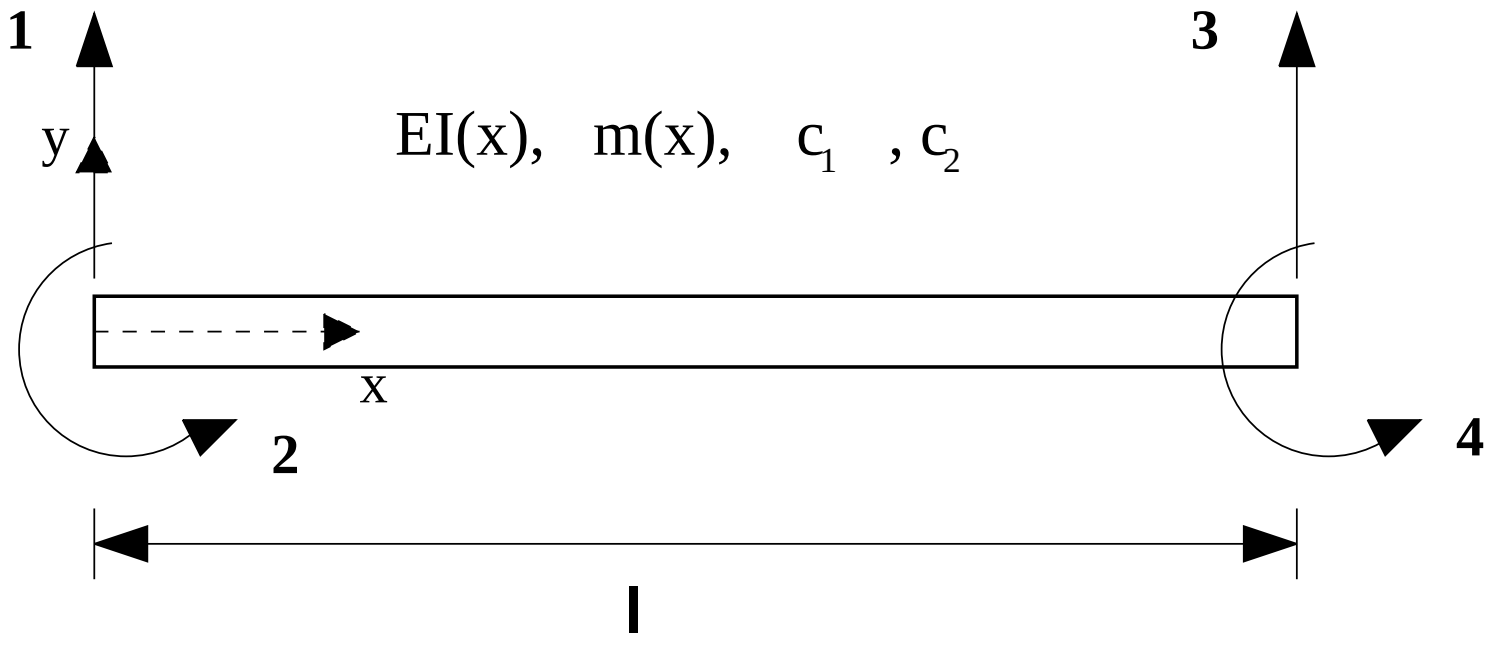
$Y(x,t)$: transverse flexural displacement, $EI(x)$: flexural rigidity, $\rho A(x)$: mass per unit length, and $p(x,t)$: applied forcing. Consider

$$EI(x,\theta) = EI_0\left(1+\epsilon_1 F_1(x,\theta)\right) \tag{41}$$

$$\text{and} \quad \rho A(x,\theta) = \rho A_0\left(1+\epsilon_2 F_2(x,\theta)\right) \tag{42}$$

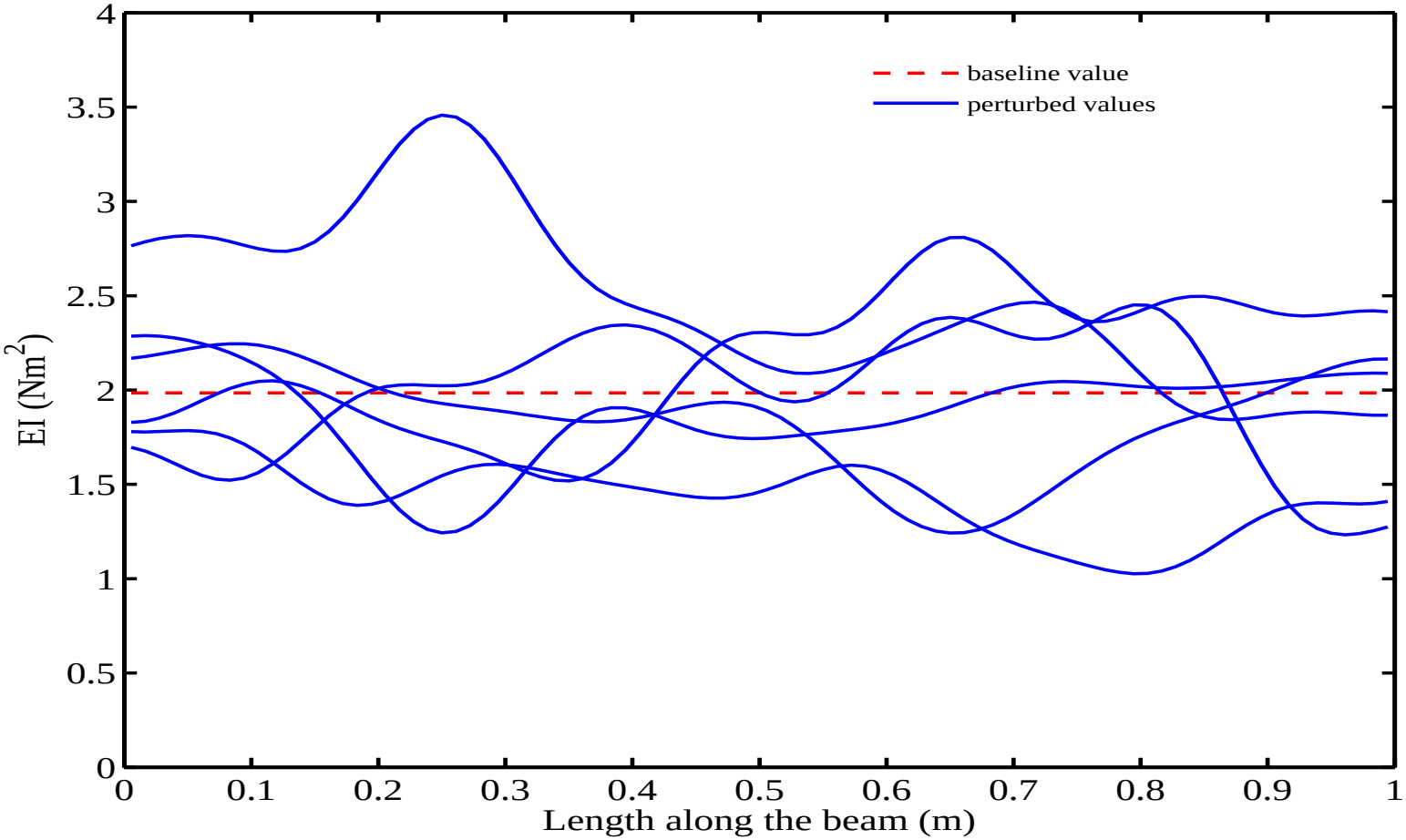
The subscript 0 indicates the mean values, $0 < \epsilon_i << 1$ (i=1,2) are deterministic constants and the random fields $F_i(x,\theta)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$.

Random beam element



Random beam element in the local coordinate.

Realisations of the random field



Some random realizations of the bending rigidity EI of the beam for correlation length $b = L/3$ and strength parameter $\epsilon_1 = 0.2$ (mean 2.0×10^5). Thirteen terms have been used in the KL expansion.

Example: A beam with random properties

We express the shape functions for the finite element analysis of Euler-Bernoulli beams as

$$\mathbf{N}(x) = \mathbf{\Gamma}\,\mathbf{s}(x) \tag{43}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & \dfrac{-3}{\ell_e{}^2} & \dfrac{2}{\ell_e{}^3} \\ 0 & 1 & \dfrac{-2}{\ell_e{}^2} & \dfrac{1}{\ell_e{}^2} \\ 0 & 0 & \dfrac{3}{\ell_e{}^2} & \dfrac{-2}{\ell_e{}^3} \\ 0 & 0 & \dfrac{-1}{\ell_e{}^2} & \dfrac{1}{\ell_e{}^2} \end{bmatrix} \quad \text{and} \quad \mathbf{s}(x) = \left[1,\ x,\ x^2,\ x^3\ \right]^T. \tag{44}$$

The element stiffness matrix:

$$\mathbf{K}_e(\theta) = \int_0^{\ell_e} \mathbf{N}^{''}(x) EI(x,\theta) \mathbf{N}^{''T}(x) dx = \int_0^{\ell_e} EI_0\left(1 + \epsilon_1 F_1(x,\theta)\right) \mathbf{N}^{''}(x) \mathbf{N}^{''T}(x) dx. \tag{45}$$

Example: A beam with random properties

Expanding the random field $F_1(x, \theta)$ in KL expansion

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e0} + \mathbf{\Delta K}_e(\theta) \tag{46}$$

where the deterministic and random parts are

$$\mathbf{K}_{e0} = EI_0 \int_0^{\ell_e} \mathbf{N}''(x)\mathbf{N}''^T(x)\, dx \quad \text{and} \quad \mathbf{\Delta K}_e(\theta) = \epsilon_1 \sum_{j=1}^{N_\mathrm{K}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_{ej}. \tag{47}$$

The constant N_K is the number of terms retained in the Karhunen-Loève expansion and $\xi_{\mathrm{K}j}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_{ej}$ can be expressed as

$$\mathbf{K}_{ej} = EI_0 \int_0^{\ell_e} \varphi_{\mathrm{K}j}(x_e + x)\mathbf{N}''(x)\mathbf{N}''^T(x)\, dx \tag{48}$$

Example: A beam with random properties

The mass matrix can be obtained as

$$\mathbf{M}_e(\theta) = \mathbf{M}_{e_0} + \mathbf{\Delta M}_e(\theta) \tag{49}$$

The deterministic and random parts is given by

$$\mathbf{M}_{e_0} = \rho A_0 \int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\, dx \quad \text{and} \quad \mathbf{\Delta M}_e(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_{ej}. \tag{50}$$

The constant N_{M} is the number of terms retained in Karhunen-Loève expansion and the constant matrices $\mathbf{M}_{ej}$ can be expressed as

$$\mathbf{M}_{ej} = \rho A_0 \int_0^{\ell_e} \varphi_{\mathrm{M}j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\, dx. \tag{51}$$

Both $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ can be obtained in closed-form.

Example: A beam with random properties

These element matrices can be assembled to form the global random stiffness and mass matrices of the form

$$\mathbf{K}(\theta) = \mathbf{K}_0 + \mathbf{\Delta K}(\theta) \quad \text{and} \quad \mathbf{M}(\theta) = \mathbf{M}_0 + \mathbf{\Delta M}(\theta). \tag{52}$$

Here the deterministic parts $\mathbf{K}_0$ and $\mathbf{M}_0$ are the usual global stiffness and mass matrices obtained form the conventional finite element method. The random parts can be expressed as

$$\mathbf{\Delta K}(\theta) = \epsilon_1 \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_j \quad \text{and} \quad \mathbf{\Delta M}(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_j \tag{53}$$

The element matrices $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ can be assembled into the global matrices $\mathbf{K}_j$ and $\mathbf{M}_j$. The total number of random variables depend on the number of terms used for the truncation of the infinite series. This in turn depends on the respective correlation lengths of the underlying random fields.

Stochastic equation of motion

- The equation for motion for stochastic linear MDOF dynamic systems:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t) + \mathbf{K}(\theta)\mathbf{u}(\theta,t) = \mathbf{f}(t) \tag{54}$$

- $\mathbf{M}(\theta) = \mathbf{M}_0 + \sum_{i=1}^{p} \mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{p} \nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta) \in \mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$). These can be obtained from discretising stochastic fields with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta) = \zeta_1\mathbf{M}(\theta) + \zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.

Frequency domain representation

- For the harmonic analysis of the structural system, taking the Fourier transform

$$\left[-\omega^2\mathbf{M}(\theta) + i\omega\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}(\omega) \tag{55}$$

  where $\widetilde{\mathbf{u}}(\omega,\theta)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}(\omega)$ is the amplitude of the harmonic force.
- For convenience we group the random variables associated with the mass and stiffness matrices as

$$\xi_i(\theta) = \mu_i(\theta) \quad \text{and} \quad \xi_{j+p_1}(\theta) = \nu_j(\theta) \quad \text{for} \quad i = 1, 2, \ldots, p_1 \quad \text{and} \quad j = 1, 2, \ldots, p_2$$

Frequency domain representation

- Using $M = p_1 + p_2$ which we have

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i(\omega) \right) \widetilde{\mathbf{u}}(\omega, \theta) = \widetilde{\mathbf{f}}(\omega) \tag{56}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n \times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble.

- For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1 \right] \mathbf{M}_0 + \left[i\omega\zeta_2 + 1 \right] \mathbf{K}_0, \tag{57}$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1 \right] \mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{58}$$

$$\text{and} \quad \mathbf{A}_{j+p_1}(\omega) = \left[i\omega\zeta_2 + 1 \right] \mathbf{K}_j \quad \text{for} \quad j = 1, 2, \ldots, p_2 \, .$$

Time domain representation

If the time steps are fixed to Δt, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{59}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)]\,\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{60}$$

$$\text{and,} \quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)) \tag{61}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t + \Delta t$ which consists of contributions of the system response at the previous time step.

Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0\left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2\dot{\mathbf{u}}_t(\theta) - a_3\ddot{\mathbf{u}}_t(\theta) \tag{62}$$

$$\text{and,} \quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6\ddot{\mathbf{u}}_t(\theta) + a_7\ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{63}$$

where the integration constants $a_i,\ i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha\Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha\Delta t}; \qquad a_2 = \frac{1}{\alpha\Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{64}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1-\delta); \qquad a_7 = \delta\Delta t \tag{65}$$

Following this development, the linear structural system in (60) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{66}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{67}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{68}$$

$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,.$$

General mathematical representation

- Whether time-domain or frequency domain methods were used, in general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i \right) \mathbf{u}(\theta) = \mathbf{f}(\theta) \tag{69}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.
- Generic response surface based methods have been used in literature - for example the Polynomial Chaos Method

Stochastic Methods in Structural Dynamics Part 2

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Overview of the course

The course is dived into eight topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

Outline of this talk

Random number generator

- Quasi-random number generators calculates a sequence of numbers that appear to be random $x_i = g(x_{i-1}, \ldots, x_{i-k})$, and the sequence is repeated after applying g a given number of times, called the period.
- These random number generators are used to simulate uniformly distributed random variables. The uniform univariate distribution $U(0,1)$ has a probability density function given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

  and its mean and variance are respectively $\mathrm{E}[X] = 1/2$, $\mathrm{Var}[X] = 1/12$.
- Generally, samples of random variables with pdfs different from the uniform pdf are needed. A random variable X with continuous cumulative density function P_X can be related to a uniform random variable $U(0,1)$ through the inverse CDF method

$$X = P_X^{-1}(U) \tag{2}$$

Random number generator

- For the case of a Gaussian random variable $N(0,1)$, samples can be obtained from samples of two independent uniform random variables U and V

$$X = (-2\ln U)^{1/2}\cos(2\pi V), \qquad Y = (-2\ln U)^{1/2}\sin(2\pi V) \tag{3}$$

  so that X and Y are independent random variables with standard normal distribution.
- Once the samples of the random variables are obtained, they are introduced in the PDE studied and the deterministic systems are solved. If MCS with N samples is used to obtain an estimation of the pdf of a random variable u (e.g. a term of the response vector $\mathbf{u}$), estimations of the mean and standard deviation are given by

$$\mathrm{E}[u] = \int up(u)\mathrm{d}u \approx \frac{1}{N}\sum_{i=1}^{N} u_i \tag{4}$$

$$\sigma = \int (u-\mathrm{E}[u])^2 p(u)\mathrm{d}u \approx \sqrt{\frac{1}{N}\sum_{i=1}^{N}(u_i-\mathrm{E}[u])^2} \tag{5}$$

Perturbation based methods

- One of the first methods used to study uncertainty propagation is the perturbation method where terms are expanded with their Taylor series expansion around the mean value of the random parameters α_i, $i = 1, \ldots, M$
- Taylor series expansions of stiffness **K**, response **u** and load vector **f** are truncated after the second order terms and introduced into $\mathbf{Ku} = \mathbf{f}$:

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{K}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \tag{6}$$

$$\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{N} \mathbf{u}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{u}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \tag{7}$$

$$\mathbf{f} = \mathbf{f}_0 + \sum_{i=1}^{N} \mathbf{f}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \tag{8}$$

Perturbation based methods

- The coefficients multiplying polynomials of the same order can be identified

$$\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}_0 \tag{9}$$

$$\mathbf{u}_i^I = \mathbf{K}_0^{-1}(\mathbf{f}_i^I - \mathbf{K}_i^I\mathbf{u}_0) \tag{10}$$

$$\mathbf{u}_{ij}^{II} = \mathbf{K}_0^{-1}(\mathbf{f}_{ij}^{II} - \mathbf{K}_i^I\mathbf{u}_j^I - \mathbf{K}_j^I\mathbf{u}_i^I - \mathbf{K}_{ij}^{II}\mathbf{u}_0) \tag{11}$$

  where terms with subindexes 0, i and ij are respectively the matrix or vector evaluated at $\alpha = 0$, its first derivative (e.g. $\mathbf{K}_i^I = \frac{\partial \mathbf{K}}{\partial \alpha_i}\Big|_{\alpha=0}$) and its second derivative (e.g. $\mathbf{K}_{ij}^{II} = \frac{\partial^2 \mathbf{K}}{\partial \alpha_i \partial \alpha_j}\Big|_{\alpha=0}$)

- The statistics of $\mathbf{u}$ are derived from the second order Taylor expansion of $\mathbf{u}$ and the statistics of α

$$\mathrm{E}[\mathbf{u}] \approx \mathbf{u}_0 + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\mathbf{u}_{ij}^{II}\mathrm{Cov}[\alpha_i, \alpha_j] \tag{12}$$

$$\mathrm{Cov}[\mathbf{u}, \mathbf{u}] \approx \sum_{i=1}^{M}\sum_{j=1}^{M}\mathbf{u}_i^I.(\mathbf{u}_j^I)^T\mathrm{Cov}[\alpha_i, \alpha_j] \tag{13}$$

Polynomial Chaos expansion

- If a function $f(\zeta)$ is a function of infinite number of variables $\{\zeta_{i_k}\}$ and square integrable, it can be expanded in Homogeneous Chaos as

$$
\begin{aligned}
f(\zeta) = \hat{y}_{i_0} h_0 + \sum_{i_1=1}^{\infty} \hat{y}_{i_1} \Gamma_1(\zeta_{i_1}) \\
+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \hat{y}_{i_1,i_2} \Gamma_2(\zeta_{i_1}, \zeta_{i_2}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \hat{y}_{i_1 i_2 i_3} \Gamma_3(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}) \qquad (14) \\
+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} \hat{y}_{i_1 i_2 i_3 i_4} \Gamma_4(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}, \zeta_{i_4}) + \ldots,
\end{aligned}
$$

  Here $\Gamma_p(\zeta_{i_1}, \zeta_{i_2}, \cdots \zeta_{i_m})$ is m-dimensional homogeneous chaos of order p.
- The polynomials are orthogonal with respect to the probability measure of the underlying random variables

Polynomial Chaos expansion

- For Gaussian random variables, Hermite polynomials are used.
- For Uniform random random variables, Legendre polynomials are used.
- Truncating Eq. (14) up to finite number of terms, we can concisely write

$$f(\zeta) = \sum_{j=0}^{P-1} y_j \Psi_j(\zeta) \tag{15}$$

  where the constant y_j and functions $\Psi_j(\bullet)$ are effectively constants $\hat{y}_k$ and functions $\Gamma_k(\bullet)$ for corresponding indices.
- Equation (15) can be viewed as the projection in the basis functions $\Psi_j(\zeta)$ with corresponding 'coordinates' y_j. The number of terms P in Eq. (15) depends on the number of variables m and maximum order of polynomials p as

$$P = \sum_{j=0}^{p} \frac{(m+j-1)!}{j!(m-1)!} = \binom{m+p}{p} \tag{16}$$

Polynomial Chaos expansion

j	p	Construction of Ψ_j	Ψ_j
0	$p=0$	L_0	1
1	$p=1$	$L_1(\zeta_1)$	ζ_1
2		$L_1(\zeta_2)$	ζ_2
3		$L_2(\zeta_1)$	$3/2\,{\zeta_1}^2 - 1/2$
4	$p=2$	$L_1(\zeta_1)L_1(\zeta_2)$	$\zeta_1\zeta_2$
5		$L_2(\zeta_2)$	$3/2\,{\zeta_2}^2 - 1/2$
6		$L_3(\zeta_1)$	$5/2\,{\zeta_1}^3 - 3/2\,\zeta_1$
7	$p=3$	$L_2(\zeta_1)L_1(\zeta_2)$	$\left(3/2\,{\zeta_1}^2 - 1/2\right)\zeta_2$
8		$L_1(\zeta_1)L_2(\zeta_2)$	$\zeta_1\left(3/2\,{\zeta_2}^2 - 1/2\right)$
9		$L_3(\zeta_2)$	$5/2\,{\zeta_2}^3 - 3/2\,\zeta_2$
10		$L_4(\zeta_1)$	$\frac{35}{8}\,{\zeta_1}^4 - \frac{15}{4}\,{\zeta_1}^2 + 3/8$
11		$L_3(\zeta_1)L_1(\zeta_2)$	$\left(5/2\,{\zeta_1}^3 - 3/2\,\zeta_1\right)\zeta_2$
12	$p=4$	$L_2(\zeta_1)L_2(\zeta_2)$	$\left(3/2\,{\zeta_1}^2 - 1/2\right)\left(3/2\,{\zeta_2}^2 - 1/2\right)$
13		$L_1(\zeta_1)L_3(\zeta_2)$	$\zeta_1\left(5/2\,{\zeta_2}^3 - 3/2\,\zeta_2\right)$
14		$L_4(\zeta_2)$	$\frac{35}{8}\,{\zeta_2}^4 - \frac{15}{4}\,{\zeta_2}^2 + 3/8$

Polynomial Chaos expansion

- A least-square error minimization approach can be used to obtain the constants y_j in Eq. (15). We define the inner product norm in $[-1,1]^m$ as

$$< \bullet, \bullet >= \frac{1}{V_m} \underbrace{\int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1}}_{m-\text{fold}} (\bullet)(\bullet) d\zeta_1 d\zeta_2 \cdots d\zeta_m \tag{17}$$

- Here the volume

$$V_m = 2^m \tag{18}$$

is used for normalization so that for two constants a and b we have $< a, b >= ab$. The error corresponding to Eq. (15) can be expressed as

$$\varepsilon = f(\zeta) - \sum_{j=0}^{P-1} y_j \Psi_j(\zeta) \tag{19}$$

- Using the inner product norm in (17), the norm of the error can be obtained as

$$\chi^2 = \langle \varepsilon, \varepsilon \rangle \tag{20}$$

Polynomial Chaos expansion

- Differentiating this with respect to y_k, it can be shown that (Galerkin approach) the optimal values of y_k can be obtained my making the basis functions orthogonal to the error, that is,

$$\varepsilon \perp \Psi_k \quad \text{or} \quad \langle \Psi_k, \varepsilon \rangle = 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{21}$$

- Substituting the expression of error from Eq. (19) into this equation we obtain

$$\sum_{j=0}^{P-1} y_j \left\langle \Psi_k(\zeta), \Psi_j(\zeta) \right\rangle = \langle \Psi_k(\zeta), f(\zeta) \rangle \tag{22}$$

  Using the orthogonality property of the basis function we have $\left\langle \Psi_k(\zeta), \Psi_j(\zeta) \right\rangle = c_k \delta_{jk}$.
- Therefore, the constants y_k can be obtained as

$$y_k = \frac{\langle \Psi_k(\zeta), f(\zeta) \rangle}{\langle \Psi_k(\zeta), \Psi_k(\zeta) \rangle}, \quad \forall\ k = 0, 2, \ldots, P-1 \tag{23}$$

  The integration appearing in the numerator and denominator can be obtained using any standard procedure for multidimensional integrals. In particular, the denominator can be calculated explicitly. The values of $\left\langle \Psi_j(\zeta), \Psi_j(\zeta) \right\rangle$ cab be obtained analytically.

Table : Values of $\langle \Psi_j(\zeta), \Psi_j(\zeta) \rangle$ for two variables ($m = 2$) with polynomial order 4 ($p = 4$).

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\langle \Psi_j(\zeta), \Psi_j(\zeta) \rangle$	1	1/3	1/3	1/5	1/9	1/5	1/7	1/15	1/15	1/7	1/9	1/21	1/25	1/21	1/9

- Substituting the values of y_k from (23) into the expansion (15) we have

$$\widehat{f}(\zeta) = \sum_{j=0}^{P-1} \left[\frac{\langle \Psi_j(\zeta), f(\zeta) \rangle}{\langle \Psi_j(\zeta), \Psi_j(\zeta) \rangle} \right] \Psi_j(\zeta) \tag{24}$$

- Here $\widehat{f}(\zeta)$ is an approximation to the original function $f(\zeta)$ for polynomial order upto p. The accuracy of this approximation can improve indefinitely by considering higher-order polynomials. If the evaluation of the original function $f(\zeta)$ is expensive, the surrogate model $\widehat{f}(\zeta)$ can be used instead of the original function.

Polynomial Chaos expansion: Example 1

- To illustrate the application of the Galerkin projection approach, we consider two problems involving bounded variables. We consider the function

$$\widehat{f}_1(\mathbf{x}) = \frac{89}{40} - \frac{\sqrt{2}}{1080}(x_1 + x_2 - 20)^3 + \frac{33}{140}(x_1 - x_2)\,;\; 4 \leq x_1, x_2 \leq 16 \quad (25)$$

- As the first step, we transform the variables in $[-1, 1]$:

$$x_1 = 6\zeta_1 + 10 \quad \text{and} \quad x_2 = 6\zeta_2 + 10 \quad (26)$$

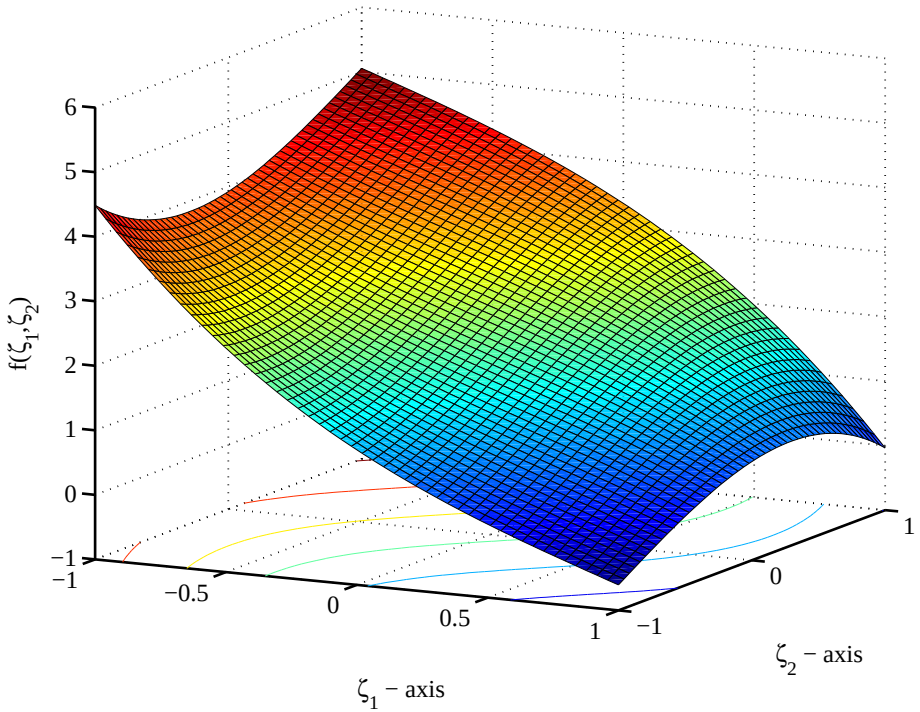
- Substituting these into Eq. (25) one obtains the function in the transformed variables as

$$f_1(\zeta) = \frac{89}{40} - \frac{1}{5}\sqrt{2}(\zeta_1 + \zeta_2)^3 + \frac{99}{70}\zeta_1 - \frac{99}{70}\zeta_2 \quad (27)$$

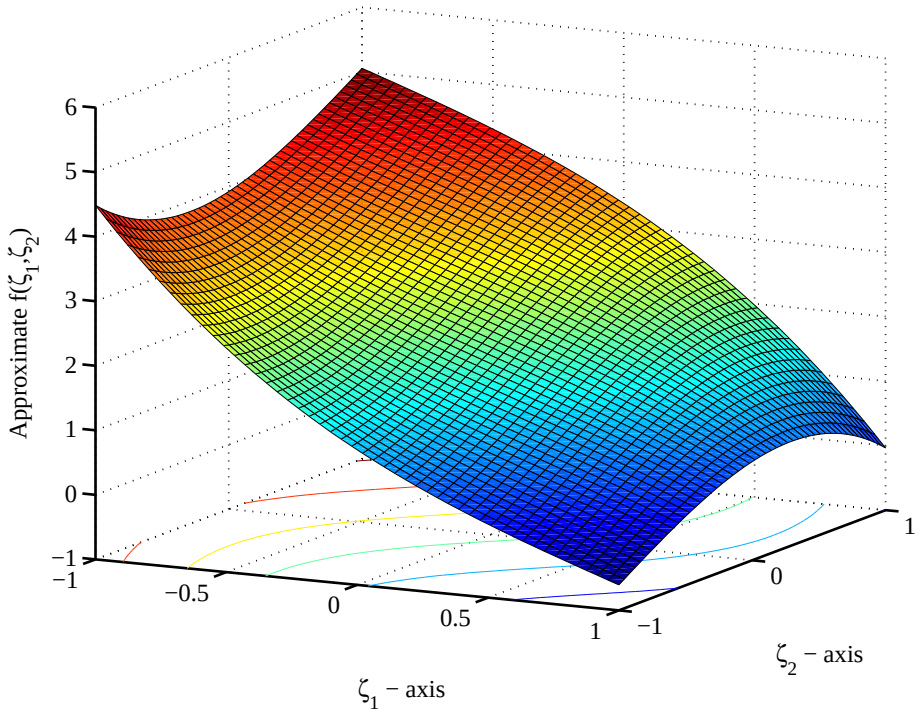
- Using Eq. (23) the nonzero values of y_j can be obtained as

$$\begin{aligned} &y_1 = \frac{89}{40}, y_2 = -\frac{8}{25}\sqrt{2} + \frac{99}{70}, y_3 = -\frac{8}{25}\sqrt{2} - \frac{99}{70}, y_7 = -\frac{2}{25}\sqrt{2}, \\ &y_8 = -2/5\sqrt{2}, y_9 = -2/5\sqrt{2} \quad \text{and} \quad y_{10} = \frac{2}{25}\sqrt{2} \end{aligned} \quad (28)$$

Polynomial Chaos expansion



(a) The exact function



(b) Fitted function using Legendre polynomials

Figure : The original function and the fitted function corresponding to Eq. (25).

Polynomial Chaos expansion: Example 2

- Consider the 'Camelback' function

$$f_1(\mathbf{x}) = (4 - 2.1x_1^2 + x_1^4/3)x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2; -3 \leq x_1 \leq 3; -2 \leq x_2 \leq 2 \tag{29}$$

- Transform the variables in $[-1, 1]$:

$$x_1 = 3\zeta_1 \quad \text{and} \quad x_2 = 2\zeta_2 \tag{30}$$

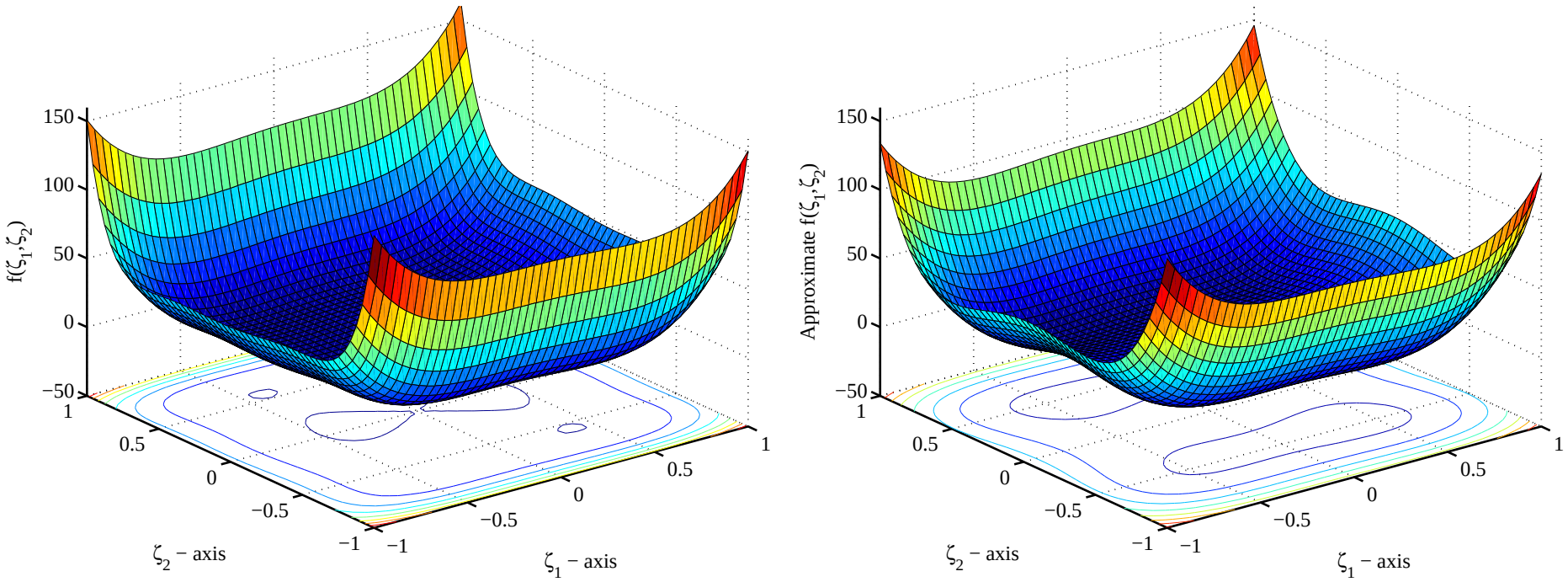
- Substituting these into Eq. (29) one obtains the function in the transformed variables as

$$f_1(\boldsymbol{\zeta}) = 9\left(4 - \frac{189}{10}{\zeta_1}^2 + 27{\zeta_1}^4\right){\zeta_1}^2 + 6\zeta_1\zeta_2 + 4\left(-4 + 16{\zeta_2}^2\right){\zeta_2}^2 \tag{31}$$

Using Eq. (23), carrying out the 2-dimensional integration analytically, the nonzero values of y_j can be obtained as

$$y_1 = \frac{21169}{1050}, y_4 = \frac{1488}{35}, y_5 = 6, y_6 = \frac{544}{21}, y_{11} = \frac{70956}{1925}, y_{15} = \frac{512}{35} \tag{32}$$

Polynomial Chaos expansion



(a) The exact function

(b) Fitted function using Legendre polynomials

Figure : The original function and the fitted function corresponding to Eq. (29).

Vector valued Polynomial Chaos

- The main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right)\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f} \tag{33}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.
- We project the solution vector $\mathbf{u}(\boldsymbol{\xi}) \in \mathbb{R}^n$ in the basis of orthogonal polynomials as

$$\mathbf{u}(\boldsymbol{\xi}) = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{34}$$

- The aim is to obtain the coefficient vectors $\mathbf{u}_j \in \mathbb{R}^n$ using a Galerkin type of error minimisation approach.

Vector valued Polynomial Chaos

- Substituting expansion of $\mathbf{u}(\xi)$ in the governing equation (33), the error vector can be obtained as

$$\varepsilon = \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i\right) \left(\sum_{j=1}^{P-1} \mathbf{u}_j \Psi_j(\xi)\right) - \mathbf{f} \in \mathbb{R}^n \tag{35}$$

  where $\xi_0 = 1$ is used to simplify the first summation expression.
- The expression (34) is viewed as a projection where $\Psi_j(\xi)$ are the orthogonal basis functions and $\mathbf{u}_j$ are the unknown 'coordinates' to be determined.
- We wish to obtain the vectors $\mathbf{u}_j$ using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon \perp \Psi_k(\xi) \quad \text{or} \quad < \Psi_k(\xi), \varepsilon >= 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{36}$$

  Imposing this condition and using the expression of ε from Eq. (35) one has

$$\left\langle \Psi_k(\xi), \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i\right) \left(\sum_{j=1}^{P-1} \mathbf{u}_j \Psi_j(\xi)\right) - \mathbf{f} \right\rangle = 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{37}$$

Vector valued Polynomial Chaos

- Interchanging the summation operations, this can be simplified to

$$\sum_{j=1}^{P-1}\sum_{i=0}^{M}\mathbf{A}_i\left\langle\xi_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})\right\rangle\mathbf{u}_j-\left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle=0\quad\forall\ k=0,2,\ldots,P-1 \tag{38}$$

- Introducing the notations

$$c_{ijk}=\left\langle\xi_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})\right\rangle\in\mathbb{R} \tag{39}$$

$$\text{and}\quad\mathbf{f}_k=\left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle\in\mathbb{R}^n \tag{40}$$

  one can express Eq. (38) as

$$\sum_{j=1}^{P-1}\sum_{i=0}^{M}c_{ijk}\mathbf{A}_i\mathbf{u}_j=\mathbf{f}_k\quad\forall\ k=0,2,\ldots,P-1 \tag{41}$$

- Since the forcing is assumed to be deterministic, $\mathbf{f}_k=\left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle=\left\langle\Psi_k(\boldsymbol{\xi})\right\rangle\mathbf{f}$. Using the definition of the orthogonal functions it can be easily shown that $\left\langle\Psi_1(\boldsymbol{\xi})\right\rangle=1$ and $\left\langle\Psi_k(\boldsymbol{\xi})\right\rangle=0$ for any other values of k.
- The constants c_{ijk} can be obtained in closed-form by performing the necessary integrals. In turns our that many of the c_{ijk} becomes 0.

Table : Values of c_{1jk} and c_{2jk} defined in Eq. (39) for two dimensional Legendre polynomial based homogeneous chaos basis up to $4th$ order

j	k	c_{1jk}	j	k	c_{2jk}
0	1	1/3	0	2	1/3
1	0	1/3	1	4	1/9
1	3	2/15	2	0	1/3
2	4	1/9	2	5	2/15
3	1	2/15	3	7	1/15
3	6	3/35	4	1	1/9
4	2	1/9	4	8	2/45
4	7	2/45	5	2	2/15
5	8	1/15	5	9	3/35
6	3	3/35	6	11	1/21
6	10	4/63	7	3	1/15
7	4	2/45	7	12	2/75
7	11	1/35	8	4	2/45
8	5	1/15	8	13	1/35
8	12	2/75	9	5	3/35
9	13	1/21	9	14	4/63

Vector valued Polynomial Chaos

- Once the values of c_{ijk} and $\mathbf{f}_k$ are obtained, further defining

$$\mathbf{A}_{jk} = \sum_{i=0}^{M} c_{ijk}\mathbf{A}_i \in \mathbb{R}^{n\times n} \tag{42}$$

one can rewrite Eq. (41) as

$$\sum_{j=1}^{P-1} \mathbf{A}_{jk}\mathbf{u}_j = \mathbf{f}_k, \quad \forall\ k = 0, 2, \ldots, P-1 \tag{43}$$

- For all values ok k, this equation can be expressed in a matrix form as

$$\begin{bmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots & \\ \mathbf{A}_{P-1,0} & \mathbf{A}_{P-1,1} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{44}$$

or in a compact notation

$$\mathcal{K}\mathcal{U} = \mathcal{F} \tag{45}$$

where $\mathcal{K} \in \mathbb{R}^{nP\times nP}$, $\mathcal{U}, \mathcal{F} \in \mathbb{R}^{nP}$. Once all $\mathbf{u}_j$ for $j = 0, 2, \ldots, P-1$ are obtained, the solution vector can be obtained from (34).

Vector valued Polynomial Chaos

- The main computational challenge posed by the method proposed here is the solution of the set of linear equations in (44), which of size nP. The value of the number of terms P depends on the number of random variables M and the order of the chaos expansion r as given by Eq. (16).
- Some values of P are shown for different number of random variables and order of chaos expansions.

M	2	3	5	10	20	50	100
1st order ($r = 1$)	3	4	6	11	21	51	101
2nd order ($r = 2$)	6	10	21	66	231	1326	5151
3rd order ($r = 3$)	10	20	56	286	1771	23426	176851
4th order ($r = 4$)	15	35	126	1001	10626	316251	4598126

- It can be seen that P increase significantly with the increase in M and r. The value of n depends on the finite element discretisation and can be large for complex problems. Therefore for practical problems nP can be very large.
- The solution of Eq. (44) can be a formidable task. The computational complexity of the matrix inversion problem scales in cubically with the dimension of the matrix in the worse case. Therefore, the computational time for solving Eq. (44) is in $\mathcal{O}(P^3 n^3)$.

Some observations of the PC solution

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- For an n-dimensional output vector, the number of terms in the projection can be more than n (depends on the number of random variables). This implies that many of the vectors $\mathbf{u}_k$ are linearly dependent.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

Possibilities of solution types

- As an example, consider the frequency domain response vector of the stochastic system $\mathbf{u}(\omega, \theta)$ governed by

$$\left[-\omega^2 \mathbf{M}(\boldsymbol{\xi}(\theta)) + i\omega \mathbf{C}(\boldsymbol{\xi}(\theta)) + \mathbf{K}(\boldsymbol{\xi}(\theta))\right] \mathbf{u}(\omega, \theta) = \mathbf{f}(\omega). \tag{46}$$

- Some possibilities are

$$\begin{aligned}
\mathbf{u}(\omega, \theta) &= \sum_{k=1}^{P_1} H_k(\boldsymbol{\xi}(\theta)) \mathbf{u}_k(\omega) \\
\text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\omega, \boldsymbol{\xi}(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_3} a_k(\omega) H_k(\boldsymbol{\xi}(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_4} a_k(\omega) H_k(\boldsymbol{\xi}(\theta)) \mathbf{U}_k(\boldsymbol{\xi}(\theta)) \quad \ldots \text{ etc.}
\end{aligned} \tag{47}$$

Deterministic classical modal analysis?

- For a deterministic system, the response vector $\mathbf{u}(\omega)$ can be expressed as

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} \Gamma_k(\omega)\mathbf{u}_k$$

$$\text{where} \quad \Gamma_k(\omega) = \frac{\phi_k^T \mathbf{f}}{-\omega^2 + 2i\zeta_k\omega_k\omega + \omega_k^2} \tag{48}$$

$$\mathbf{u}_k = \phi_k \quad \text{and} \quad P \leq n \ (\text{number of dominant modes})$$

- Can we extend this idea to stochastic systems?

Projection in the modal space

There exist a finite set of complex frequency dependent functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ *and a complete basis* $\boldsymbol{\phi}_k \in \mathbb{R}^n$ *for* $k = 1, 2, \ldots, n$ *such that the solution of the discretized stochastic finite element equation (46) can be expressed by the series*

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k(\omega, \boldsymbol{\xi}(\theta)) \boldsymbol{\phi}_k \tag{49}$$

Outline of the derivation: In the first step a complete basis is generated with the eigenvectors $\boldsymbol{\phi}_k \in \mathbb{R}^n$ of the generalized eigenvalue problem

$$\mathbf{K}_0 \boldsymbol{\phi}_k = \lambda_{0_k} \mathbf{M}_0 \boldsymbol{\phi}_k; \quad k = 1, 2, \ldots n \tag{50}$$

Projection in the modal space

- We define the matrix of eigenvalues and eigenvectors

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n\times n}; \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right] \in \mathbb{R}^{n\times n} \tag{51}$$

  Eigenvalues are ordered in the ascending order: $\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$.
- We use the orthogonality property of the modal matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi}^T \mathbf{K}_0 \boldsymbol{\Phi} = \boldsymbol{\lambda}_0, \quad \text{and} \quad \boldsymbol{\Phi}^T \mathbf{M}_0 \boldsymbol{\Phi} = \mathbf{I} \tag{52}$$

- Using these we have

$$\begin{aligned} \boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} &= \boldsymbol{\Phi}^T \left([-\omega^2 + i\omega\zeta_1]\mathbf{M}_0 + [i\omega\zeta_2 + 1]\mathbf{K}_0 \right) \boldsymbol{\Phi} \\ &= \left(-\omega^2 + i\omega\zeta_1\right)\mathbf{I} + \left(i\omega\zeta_2 + 1\right)\boldsymbol{\lambda}_0 \end{aligned} \tag{53}$$

  This gives $\boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} = \boldsymbol{\Lambda}_0$ and $\mathbf{A}_0 = \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}_0 \boldsymbol{\Phi}^{-1}$, where $\boldsymbol{\Lambda}_0 = \left(-\omega^2 + i\omega\zeta_1\right)\mathbf{I} + \left(i\omega\zeta_2 + 1\right)\boldsymbol{\lambda}_0$ and $\mathbf{I}$ is the identity matrix.

Projection in the modal space

- Hence, $\mathbf{\Lambda}_0$ can also be written as

$$\mathbf{\Lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{C}^{n \times n} \tag{54}$$

where $\lambda_{0_j} = \left(-\omega^2 + i\omega\zeta_1\right) + \left(i\omega\zeta_2 + 1\right)\lambda_j$ and λ_j is as defined in Eqn. (51). We also introduce the transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{\Phi}^T \mathbf{A}_i \mathbf{\Phi} \in \mathbb{C}^{n \times n}; i = 0, 1, 2, \ldots, M. \tag{55}$$

Note that $\widetilde{\mathbf{A}}_0 = \mathbf{\Lambda}_0$ is a diagonal matrix and

$$\mathbf{A}_i = \mathbf{\Phi}^{-T} \widetilde{\mathbf{A}}_i \mathbf{\Phi}^{-1} \in \mathbb{C}^{n \times n}; i = 1, 2, \ldots, M. \tag{56}$$

Projection in the modal space

Suppose the solution of Eq. (46) is given by

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1} \mathbf{f}(\omega) \tag{57}$$

Using Eqs. (51)–(56) and the mass and stiffness orthogonality of $\mathbf{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{\Phi}^{-T}\mathbf{\Lambda}_0(\omega)\mathbf{\Phi}^{-1} + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Phi}^{-T}\widetilde{\mathbf{A}}_i(\omega)\mathbf{\Phi}^{-1}\right]^{-1} \mathbf{f}(\omega)$$

$$\Rightarrow \quad \hat{\mathbf{u}}(\omega,\theta) = \mathbf{\Phi}\underbrace{\left[\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1}}_{\mathbf{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)}\mathbf{\Phi}^{-T}\mathbf{f}(\omega) \tag{58}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\}^T .$$

Projection in the modal space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{59}$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \operatorname{diag}\left[\widetilde{\mathbf{A}}\right] = \operatorname{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n \times n} \tag{60}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix.

$$\mathbf{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\underbrace{\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{\Lambda}_i(\omega)}_{\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right)} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta) \mathbf{\Delta}_i(\omega)}_{\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)} \right]^{-1} \tag{61}$$

where $\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{R}^{n \times n}$ is a diagonal matrix and $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix.

Projection in the modal space

We rewrite Eq. (61) as

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\boldsymbol{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n + \boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)\right]\right]^{-1} \tag{62}$$

The above expression can be represented using a Neumann type of matrix series as

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{s=0}^{\infty}(-1)^s \left[\boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)\right]^s \boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right) \tag{63}$$

Projection in the modal space

Taking an arbitrary r-th element of $\hat{\mathbf{u}}(\omega,\theta)$, Eq. (58) can be rearranged to have

$$\hat{u}_r(\omega,\theta) = \sum_{k=1}^{n} \Phi_{rk} \left(\sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\boldsymbol{\phi}_j^T \mathbf{f}(\omega) \right) \right) \tag{64}$$

Defining

$$\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\boldsymbol{\phi}_j^T \mathbf{f}(\omega) \right) \tag{65}$$

and collecting all the elements in Eq. (64) for $r = 1, 2, \ldots, n$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{n} \Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) \boldsymbol{\phi}_k \tag{66}$$

Spectral functions

Definition

The functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots n$ are the frequency-adaptive spectral functions as they are expressed in terms of the spectral properties of the coefficient matrices at each frequency of the governing discretized equation.

- Each of the spectral functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ contain infinite number of terms and they are highly nonlinear functions of the random variables $\xi_i(\theta)$.
- For computational purposes, it is necessary to truncate the series after certain number of terms.
- Different order of spectral functions can be obtained by using truncation in the expression of $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$

First-order and second order spectral functions

Definition

The different order of spectral functions $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots, n$ *are obtained by retaining as many terms in the series expansion in Eqn. (63).*

Retaining one and two terms in (63) we have

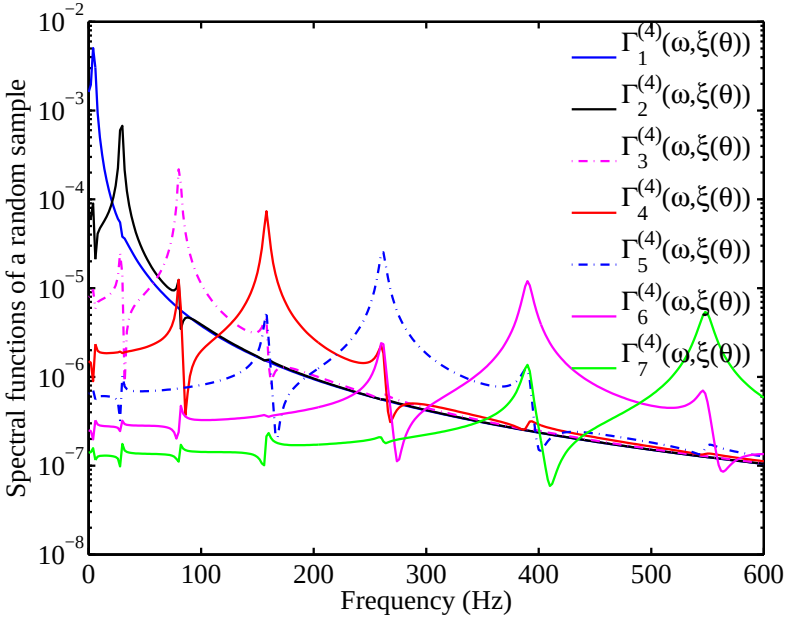
$$\boldsymbol{\Psi}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{67}$$

$$\boldsymbol{\Psi}^{(2)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) - \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))\, \boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))\, \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{68}$$

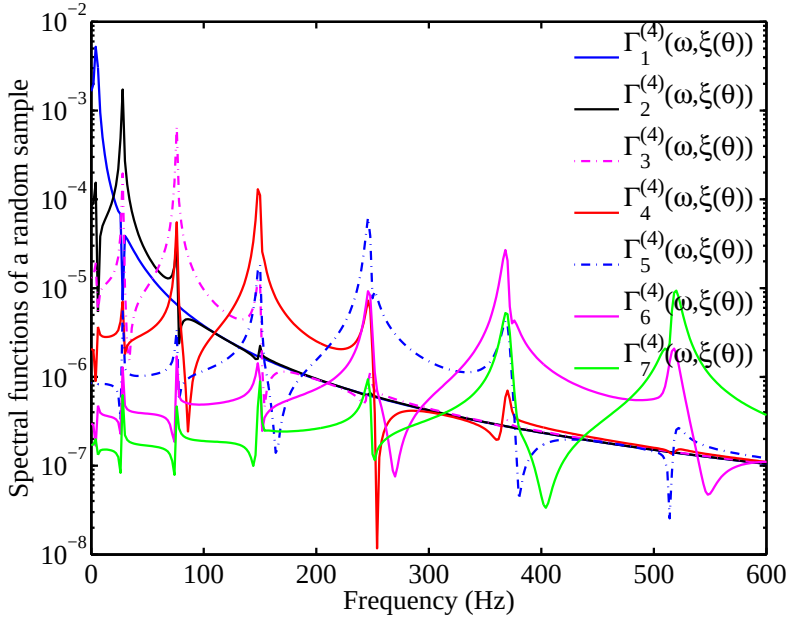
which are the first and second order spectral functions respectively.

- From these we find $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \sum_{j=1}^{n} \Psi_{kj}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) \left(\phi_j^T \mathbf{f}(\omega)\right)$ are non-Gaussian random variables even if $\xi_i(\theta)$ are Gaussian random variables.

Nature of the spectral functions



(a) Spectral functions for $\sigma_a = 0.1$.



(b) Spectral functions for $\sigma_a = 0.2$.

The amplitude of first seven spectral functions of order 4 for a particular random sample under applied force. The spectral functions are obtained for two different standard deviation levels of the underlying random field: $\sigma_a = \{0.10, 0.20\}$.

Summary of the basis functions (frequency-adaptive spectral functions)

The basis functions are:

1. not polynomials in $\xi_i(\theta)$ but ratio of polynomials.
2. independent of the nature of the random variables (i.e. applicable to Gaussian, non-Gaussian or even mixed random variables).
3. not general but specific to a problem as it utilizes the eigenvalues and eigenvectors of the system matrices.
4. such that truncation error depends on the off-diagonal terms of the matrix $\boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))$.
5. showing 'peaks' when ω is near to the system natural frequencies

Next we use these frequency-adaptive spectral functions as trial functions within a Galerkin error minimization scheme.

One can obtain constants $c_k \in \mathbb{C}$ such that the error in the following representation

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} c_k(\omega) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta)) \phi_k \tag{69}$$

can be minimised in the least-square sense. It can be shown that the vector $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^T$ satisfies the $n \times n$ complex algebraic equations $\mathbf{S}(\omega)\, \mathbf{c}(\omega) = \mathbf{b}(\omega)$ with

$$S_{jk} = \sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}; \quad \forall\, j, k = 1, 2, \ldots, n; \widetilde{A}_{i_{jk}} = \phi_j^T \mathbf{A}_i \phi_k, \tag{70}$$

$$D_{ijk} = \mathrm{E}\left[\xi_i(\theta) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\right], b_j = \mathrm{E}\left[\phi_j^T \mathbf{f}(\omega)\right]. \tag{71}$$

The Galerkin approach

- The error vector can be obtained as

$$\varepsilon(\omega,\theta) = \left(\sum_{i=0}^{M} \mathbf{A}_i(\omega)\xi_i(\theta)\right)\left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k\right) - \mathbf{f}(\omega) \in \mathbb{C}^{N\times N} \quad (72)$$

  The solution is viewed as a projection where $\boldsymbol{\phi}_k \in \mathbb{R}^n$ are the basis functions and c_k are the unknown constants to be determined. This is done for each frequency step.
- The coefficients c_k are evaluated using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon(\omega,\theta) \perp \boldsymbol{\phi}_j \Rrightarrow \left\langle \boldsymbol{\phi}_j, \varepsilon(\omega,\theta)\right\rangle = 0 \,\forall\, j = 1,2,\ldots,n \quad (73)$$

The Galerkin approach

- Imposing the orthogonality condition and using the expression of the error one has

$$\mathrm{E}\left[\phi_j^T\left(\sum_{i=0}^{M}\mathbf{A}_i\xi_i(\theta)\right)\left(\sum_{k=1}^{n}c_k\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\phi_k\right)-\phi_j^T\mathbf{f}\right]=0,\forall j \tag{74}$$

- Interchanging the $\mathrm{E}[\bullet]$ and summation operations, this can be simplified to

$$\sum_{k=1}^{n}\left(\sum_{i=0}^{M}\left(\phi_j^T\mathbf{A}_i\phi_k\right)\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)c_k\mathrm{E}\left[\phi_j^T\mathbf{f}\right] \tag{75}$$

$$\text{or}\quad \sum_{k=1}^{n}\left(\sum_{i=0}^{M}\widetilde{A}_{i_{jk}}D_{ijk}\right)c_k=b_j \tag{76}$$

Model Reduction by reduced number of basis

- Suppose the eigenvalues of $\mathbf{A}_0$ are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n} \tag{77}$$

- From the expression of the spectral functions observe that the eigenvalues ($\lambda_{0_k} = \omega_{0_k}^2$) appear in the denominator:

$$\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta) \Lambda_{i_k}(\omega)} \tag{78}$$

  where $\Lambda_{0_k}(\omega) = -\omega^2 + i\omega(\zeta_1 + \zeta_2 \omega_{0_k}^2) + \omega_{0_k}^2$

- The series can be truncated based on the magnitude of the eigenvalues relative to the frequency of excitation. Hence for the frequency domain analysis all the eigenvalues that cover almost twice the frequency range under consideration can be chosen.

- The mean vector can be obtained as

$$\bar{\mathbf{u}} = \mathrm{E}\left[\hat{\mathbf{u}}(\theta)\right] = \sum_{k=1}^{p} c_k \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right] \boldsymbol{\phi}_k \tag{79}$$

- The covariance of the solution vector can be expressed as

$$\boldsymbol{\Sigma}_u = \mathrm{E}\left[\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)^T\right] = \sum_{k=1}^{p}\sum_{j=1}^{p} c_k c_j \Sigma_{\Gamma_{kj}} \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T \tag{80}$$

where the elements of the covariance matrix of the spectral functions are given by

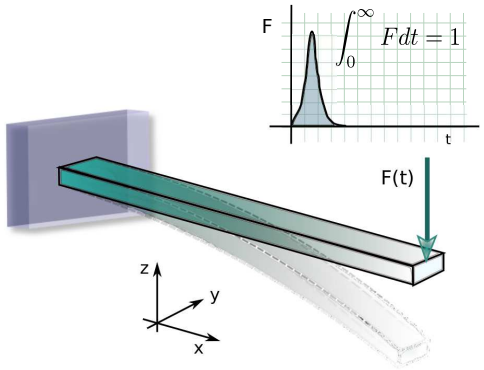
$$\Sigma_{\Gamma_{kj}} = \mathrm{E}\left[\left(\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)\left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\right]\right)\right] \tag{81}$$

Summary of the computational method

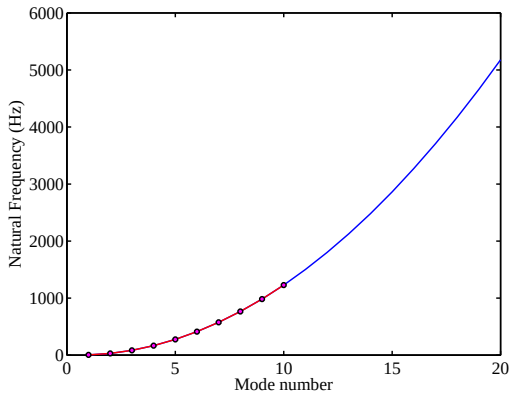
1. Solve the generalized eigenvalue problem associated with the mean mass and stiffness matrices to generate the orthonormal basis vectors: $\mathbf{K}_0\boldsymbol{\Phi} = \mathbf{M}_0\boldsymbol{\Phi}\boldsymbol{\lambda}_0$
2. Select a number of samples, say N_{samp}. Generate the samples of basic random variables $\xi_i(\theta), i = 1, 2, \ldots, M$.
3. Calculate the spectral basis functions (for example, first-order): $\Gamma_k(\omega, \boldsymbol{\xi}(\theta)) = \frac{\boldsymbol{\phi}_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)}$, for $k = 1, \cdots p$, $p < n$
4. Obtain the coefficient vector: $\mathbf{c}(\omega) = \mathbf{S}^{-1}(\omega)\mathbf{b}(\omega) \in \mathbb{R}^n$, where $\mathbf{b}(\omega) = \widetilde{\mathbf{f}(\omega)} \odot \overline{\boldsymbol{\Gamma}(\omega)}$, $\mathbf{S}(\omega) = \boldsymbol{\Lambda}_0(\omega) \odot \mathbf{D}_0(\omega) + \sum_{i=1}^{M} \widetilde{\mathbf{A}}_i(\omega) \odot \mathbf{D}_i(\omega)$ and $\mathbf{D}_i(\omega) = \mathrm{E}\left[\boldsymbol{\Gamma}(\omega, \theta)\xi_i(\theta)\boldsymbol{\Gamma}^T(\omega, \theta)\right], \forall\, i = 0, 1, 2, \ldots, M$
5. Obtain the samples of the response from the spectral series: $\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{p} c_k(\omega)\Gamma_k(\boldsymbol{\xi}(\omega, \theta))\boldsymbol{\phi}_k$

The Euler-Bernoulli beam example

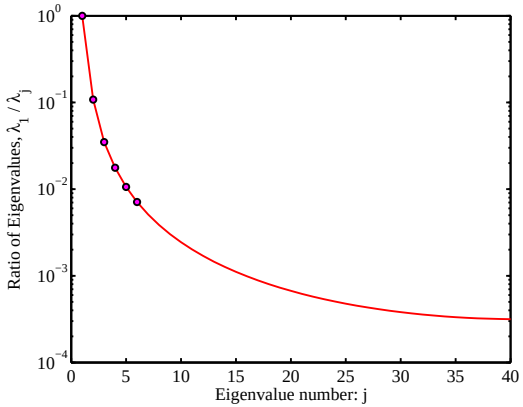
- An Euler-Bernoulli cantilever beam with stochastic bending modulus for a specified value of the correlation length and for different degrees of variability of the random field.



(c) Euler-Bernoulli beam



(d) Natural frequency distribution.



(e) Eigenvalue ratio of KL decomposition

- Length : 1.0 m, Cross-section : 39 $\times$ 5.93 mm^2, Young's Modulus: 2 $\times$ 10^{11} Pa.
- Load: Unit impulse at $t = 0$ on the free end of the beam.

Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{82}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The covariance kernel associated with this random field is

$$C_a(x_1, x_2) = \sigma_a^2 e^{-(|x_1 - x_2|)/\mu_a} \tag{83}$$

where μ_a is the correlation length and σ_a is the standard deviation.

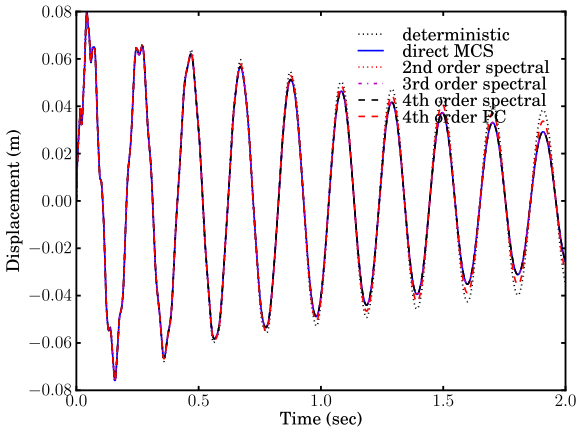
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

Problem details

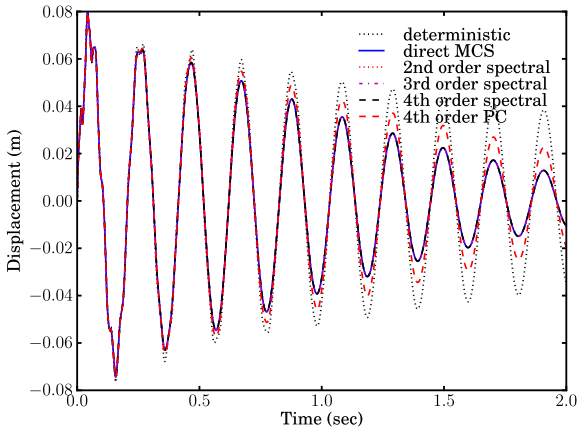
The random field is assumed to be Gaussian. The results are compared with the polynomial chaos expansion.

- The number of degrees of freedom of the system is n = 200.
- The K.L. expansion is truncated at a finite number of terms such that 90% variability is retained.
- direct MCS have been performed with 10,000 random samples and for three different values of standard deviation of the random field, $\sigma_a = 0.05, 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Time domain response of the free end of the beam is sought under the action of a unit impulse at $t = 0$
- Upto 4^{th} order spectral functions have been considered in the present problem. Comparison have been made with 4^{th} order Polynomial chaos results.

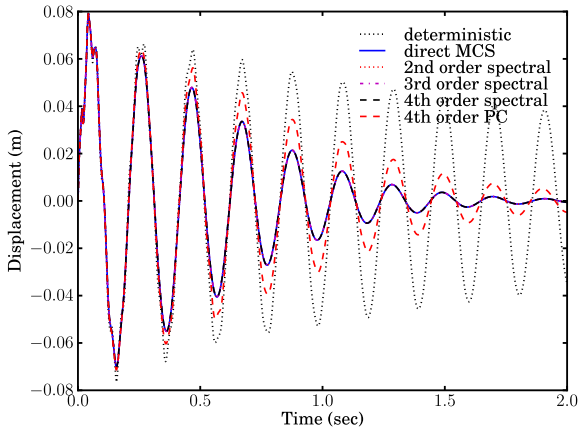
Mean of the response



(f) Mean, $\sigma_a = 0.05$.



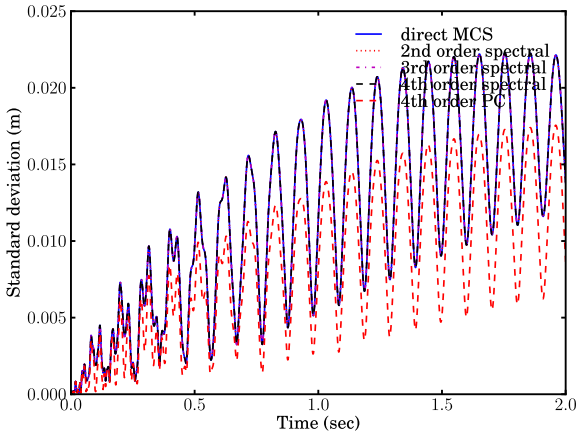
(g) Mean, $\sigma_a = 0.1$.



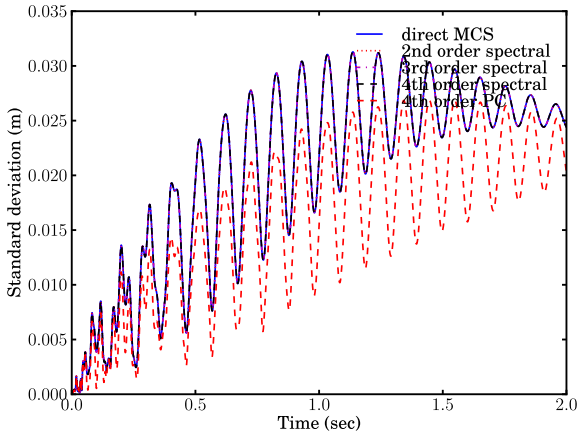
(h) Mean, $\sigma_a = 0.2$.

- Time domain response of the deflection of the tip of the cantilever for three values of standard deviation σ_a of the underlying random field.
- Spectral functions approach approximates the solution accurately.
- For long time-integration, the discrepancy of the 4^{th} order PC results increases.

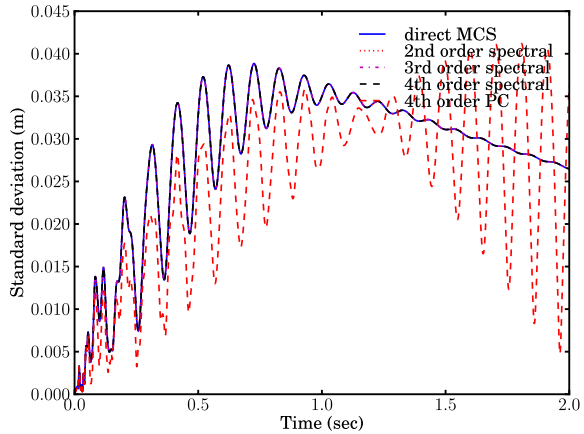
Standard deviation of the response



(i) Standard deviation of deflection, $\sigma_a = 0.05$.



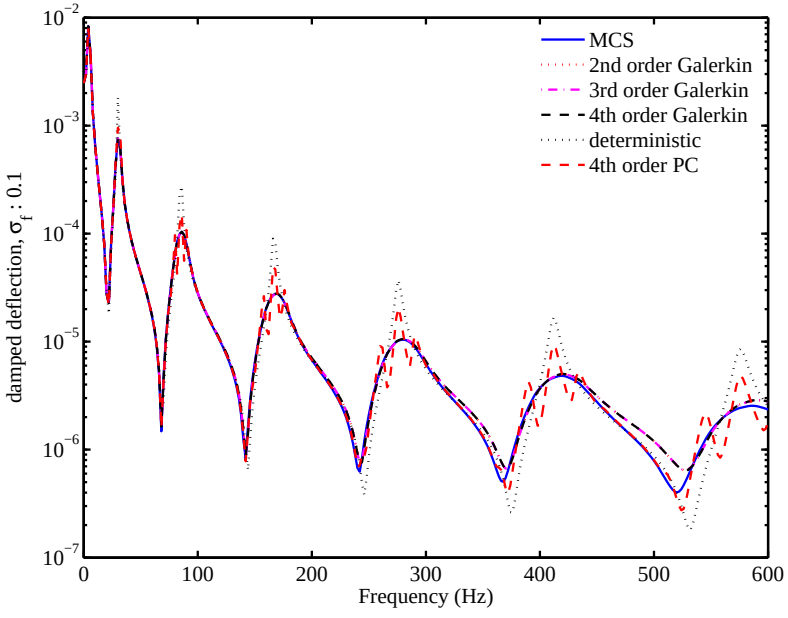
(j) Standard deviation of deflection, $\sigma_a = 0.1$.



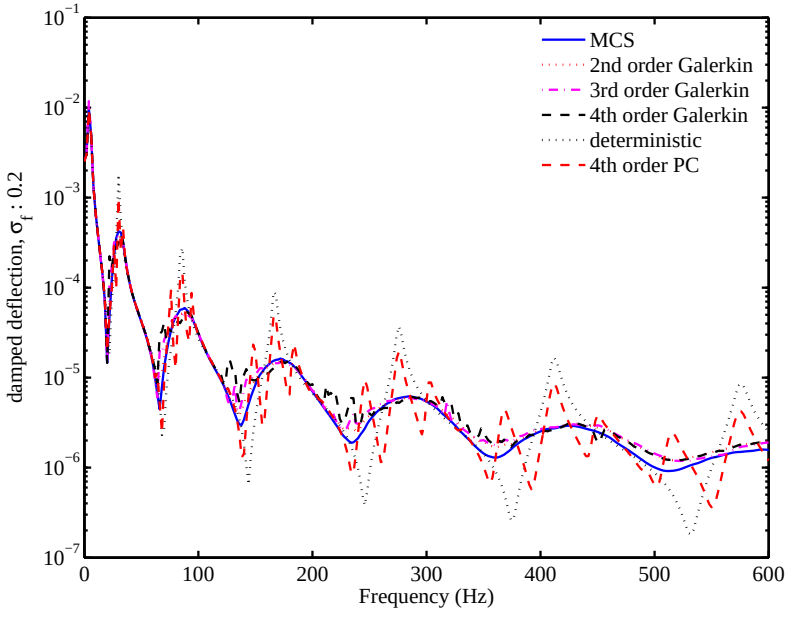
(k) Standard deviation of deflection, $\sigma_a = 0.2$.

- The standard deviation of the tip deflection of the beam.
- Since the standard deviation comprises of higher order products of the Hermite polynomials associated with the PC expansion, the higher order moments are less accurately replicated and tend to deviate more significantly.

Frequency domain response: mean



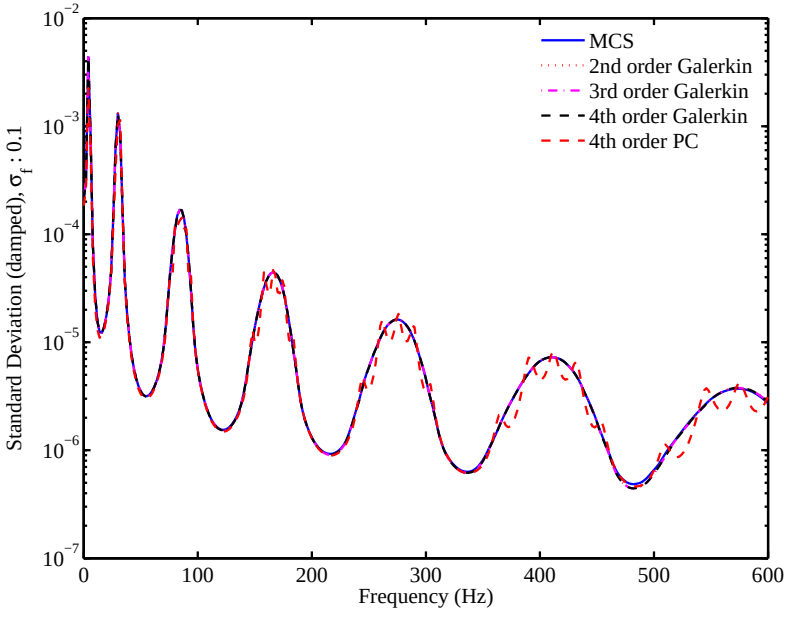
(l) Beam deflection for $\sigma_a = 0.1$.



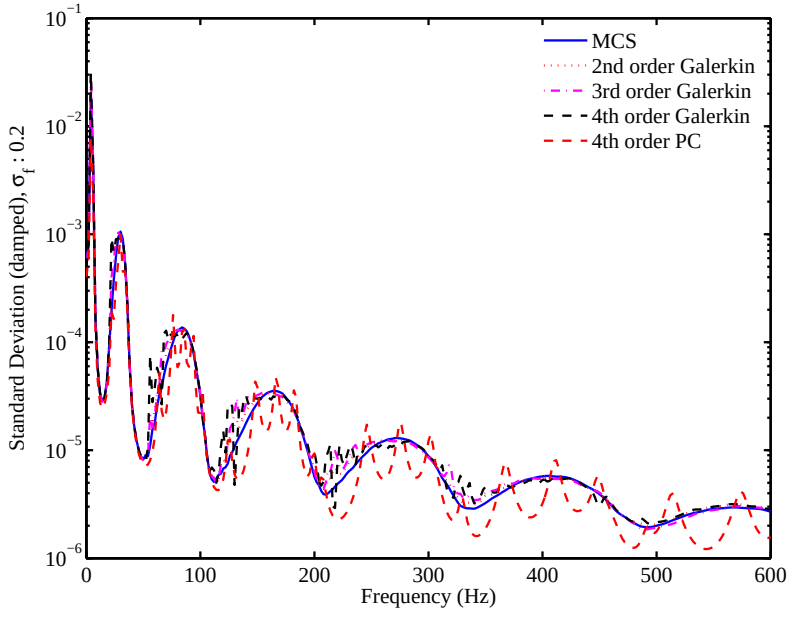
(m) Beam deflection for $\sigma_a = 0.2$.

The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Frequency domain response: standard deviation



(n) Standard deviation of the response for $\sigma_a = 0.1$.



(o) Standard deviation of the response for $\sigma_a = 0.2$.

The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Experimental investigations

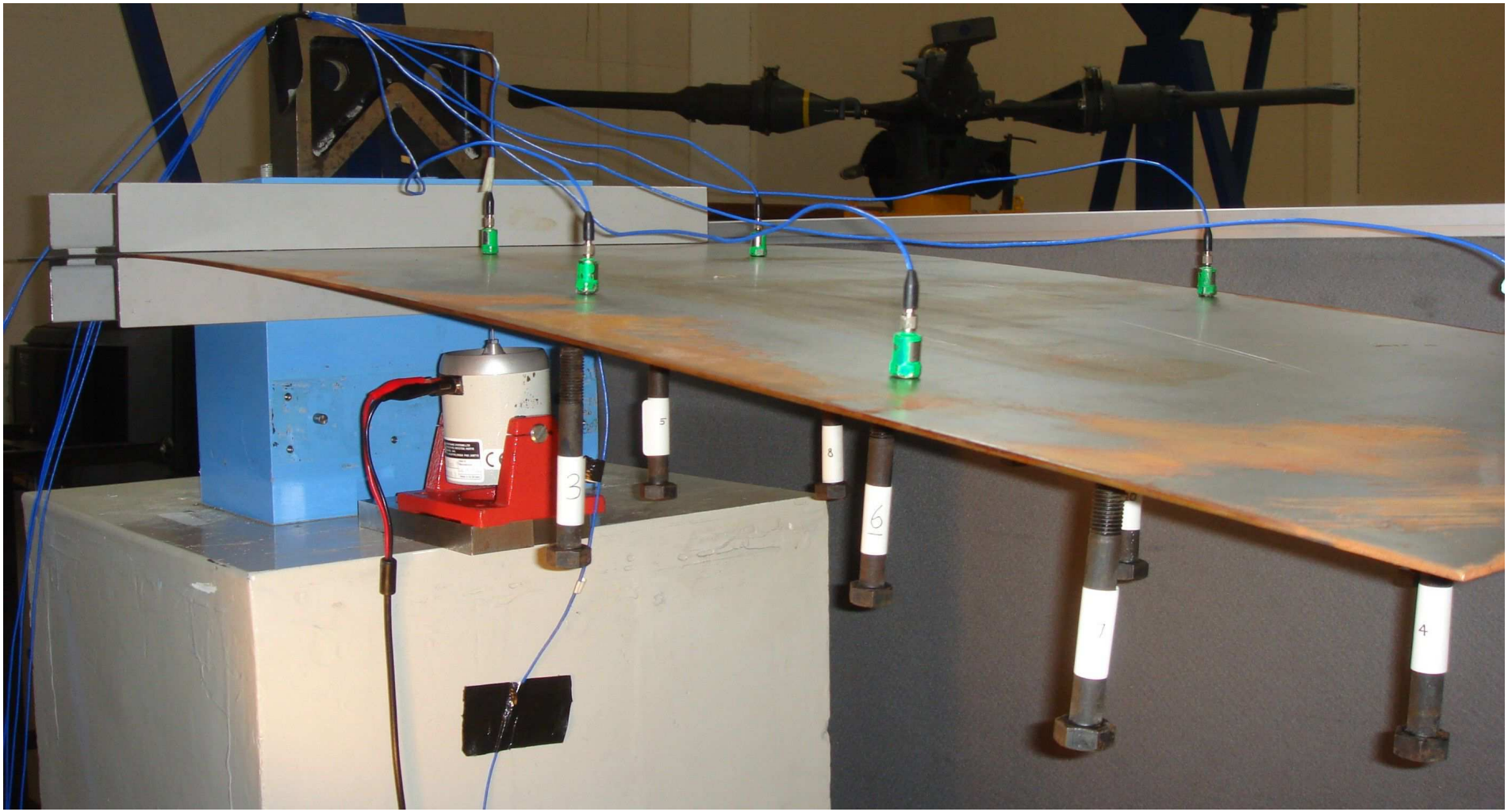
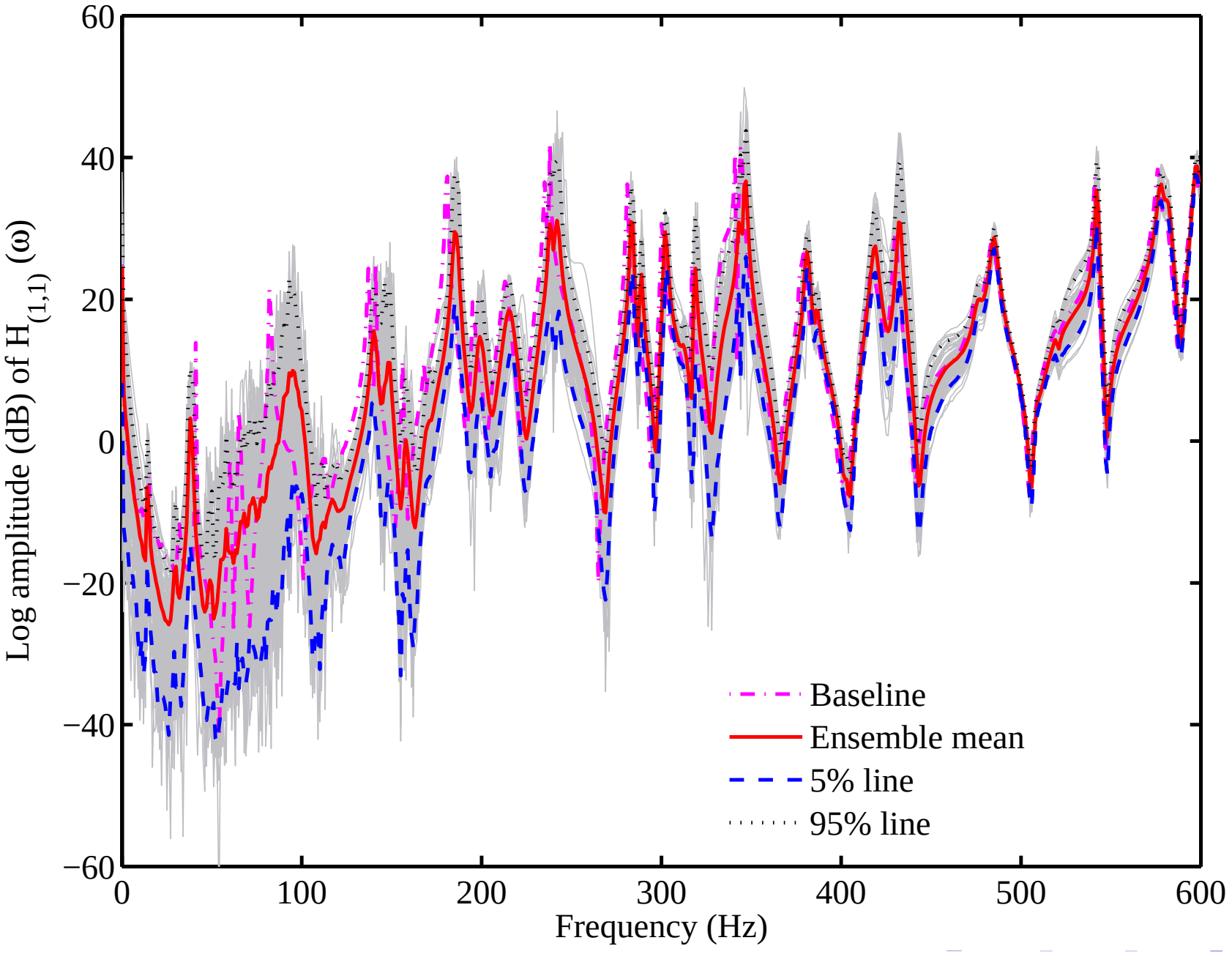


Figure : A cantilever plate with randomly attached oscillators - Probabilistic Engineering Mechanics, 24[4] (2009), pp. 473-492

Measured frequency response function



Summary

- The mean response of a damped stochastic system is more damped than the underlying baseline system
- For small damping, $\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi}$
- Care must be taken to apply random modal analysis to stochastic multiple degrees of freedom systems
- Conventional response surface based methods fails to capture the physics of damped dynamic systems
- Proposed spectral function approach uses the undamped modal basis and can capture the statistical trend of the dynamic response of stochastic damped MDOF systems

Summary

- The solution is projected into the modal basis and the associated stochastic coefficient functions are obtained at each frequency step (or time step).
- The coefficient functions, called as the spectral functions, are expressed in terms of the spectral properties (natural frequencies and mode shapes) of the system matrices.
- The proposed method takes advantage of the fact that for a given maximum frequency only a small number of modes are necessary to represent the dynamic response. This modal reduction leads to a significantly smaller basis.

Assimilation with experimental measurements

In the frequency domain, the response can be simplified as

$$\mathbf{u}(\omega,\theta) \approx \sum_{k=1}^{n_r} \frac{\phi_k^T \mathbf{f}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_{0_k} + \omega_{0_k}^2 + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)} \phi_k$$

Some parts can be obtained from experiments while other parts can come from stochastic modelling.

Introduction to Monte Carlo Methods

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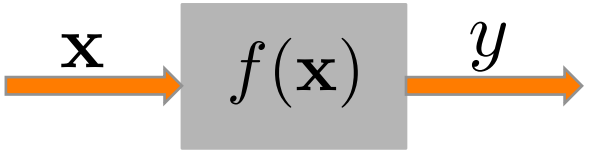
Monte Carlo Simulation

- Uncertainty analysis
- What are Monte Carlo methods?
- Steps in the Monte Carlo simulation procedure
- How to use Matlab for MCS?
- Examples:
  - Estimate of π
  - Expectation of a dice throw
  - Beam deflection with random properties
- Overview of practical aspects

Context

Deterministic computer model

$$\mathbb{R}^p \to \mathbb{R}$$



Uncertainty analysis

Task

Input uncertainty $\overset{\text{Map}}{\to}$ output uncertainty

Remark: understanding the impact of uncertainty on performance is a crucial step in engineering design

Measures of output uncertainty

- Mean
- Variance
- kth order moment
- $\text{Prob}[y \geq y^{\max}]$

Context (Contd.)

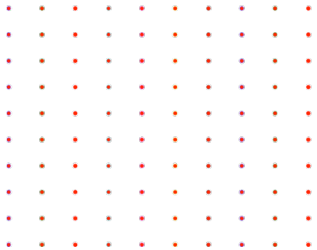
$$I = \langle \phi(\mathbf{x}) \rangle = \int \phi(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mathbf{x}$$

~~How about numerical quadrature schemes?~~

1D

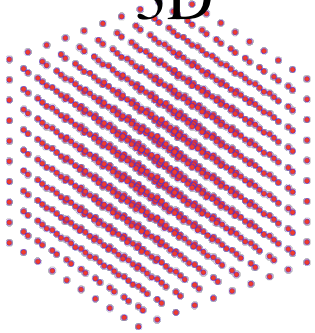
10

2D



100

3D



1000

Number of eval. points for p variables 10^p

The curse of dimensionality!

What are Monte Carlo Methods?

- A class of *non-deterministic* numerical methods that rely on random sampling
- Useful in situations where deterministic algorithms are difficult to formulate or implement
- The Monte Carlo method was coined in the 1940s by John von Neumann, Stanislaw Ulam and Nicholas Metropolis, while they were working on nuclear weapon projects (Manhattan Project) in the Los Alamos National Laboratory.
- It was named after the Monte Carlo Casino, a famous casino where Ulam's uncle often gambled away his money.

MCS for Uncertainty Analysis

$$I = \langle \phi(\mathbf{x}) \rangle = \int \phi(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mathbf{x}$$

1. Draw samples from the joint pdf of inputs

$$\mathbf{x}^{(i)}, i = 1, 2, \ldots, N$$

2. Compute outputs at corresponding points

$$\phi(\mathbf{x}^{(i)}), i = 1, 2, \ldots, N$$

3. Calculate Monte Carlo estimate of the integral

$$I \approx \widehat{I} = \frac{1}{N} \sum_{i=1}^{N} \phi(\mathbf{x}^{(i)})$$

Standard Error of the Monte Carlo Estimate

$$\phi(\mathbf{x}^{(i)}), i = 1, 2, \ldots, N$$

$$I \approx \widehat{I} = \frac{1}{N} \sum_{i=1}^{N} \phi(\mathbf{x}^{(i)})$$

$$\sigma_\phi^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left(\phi(\mathbf{x}^{(i)}) - \widehat{I} \right)^2$$

$$\text{Standard Error} = \frac{\sigma_\phi}{\sqrt{N}}$$

The MCS standard error is independent of the number of variables!

Example Problem Using Matlab

Compute the mean value of $f(X,Y) = 1 + X + Y + X^2 + Y^2 - 2XY$, given X and Y are uncorrelated Gaussian random variables with zero mean and unit variance.

```
N = 100000; % Define sample size
X=randn(N,1); Y=randn(N,1); % Draw samples of
inputs
F = 1 + X + Y + X.^2 + Y.^2 - 2*X.*Y; % Compute
function values
Muf = mean(F)  % Mean
stdf=std(F)  % Standard deviation
```

If the variables are Uniform with zero mean and unit variance?

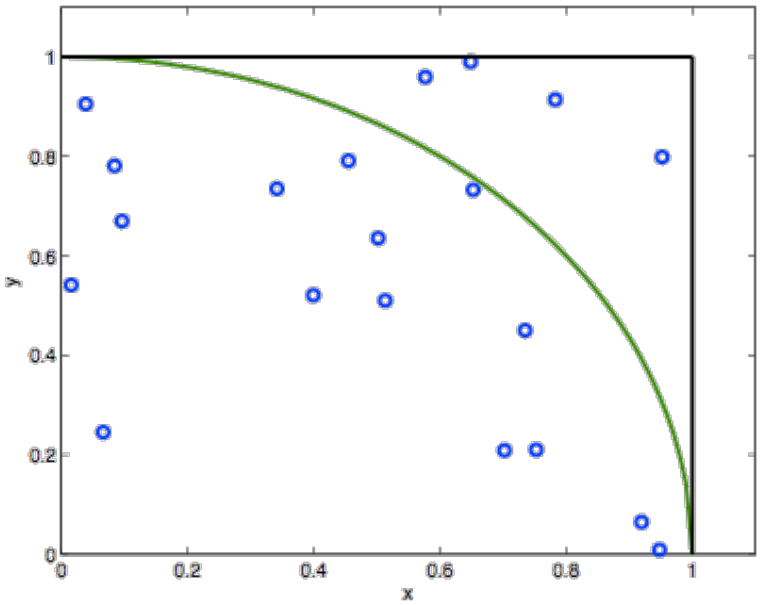
```
% Uniform RV
N = 10000; % Define sample size
a=-sqrt(3);b=sqrt(3);
X=a + (b-a).*rand(N,1); Y=a + (b-a).*rand(N,1);
F = 1 + X + Y + X.^2 + Y.^2 - 2*X.*Y; % Compute
function values
Muf = mean(F)  % Mean
stdf=std(F)  % Standard deviation
```

Monte Carlo Estimate of π

To start with, we generate random numbers X and Y that are uniform on the interval $[0, 1)$ (displayed with ∘ in figure 1.1) and make use of the algorithm $X^2 + Y^2 \leq 1 \rightarrow$ *"in circle"*.

experiment	number in circle	π_i
1	17	3.4
2	14	2.8
3	14	2.8
4	16	3.2

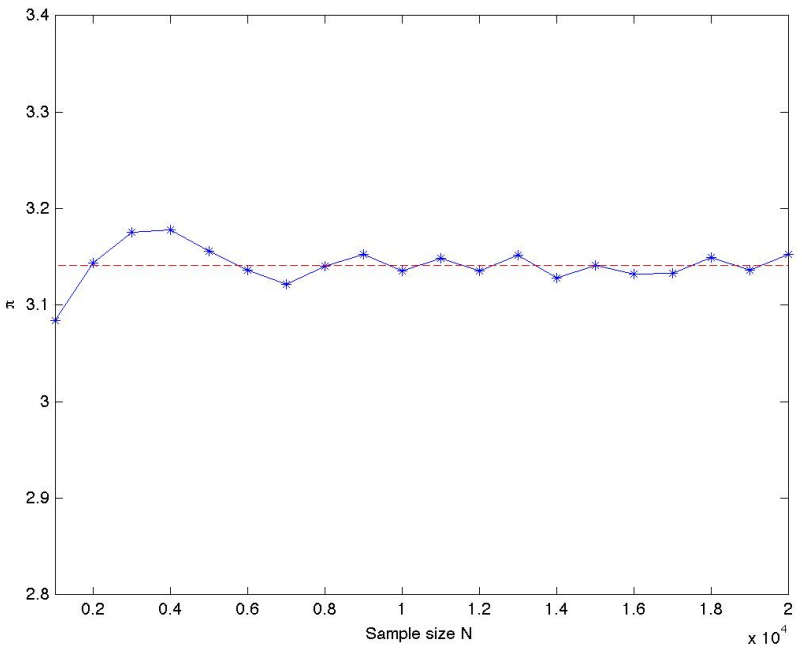
Table 1.1: Estimates of π



$$\pi \approx 4 \times \frac{\text{"\#in circle"}}{\text{"\#total"}} = 4\frac{16}{20}.$$

Monte Carlo Estimate of π

```
clear all;close all;
SampleSize = 1000; % Define sample
size
NInside = 0;
for i = 1:SampleSize
Xrand = rand; % Generate Random XY
Point
Yrand = rand;
if (Xrand^2 + Yrand^2 <= 1)
NInside = NInside + 1;
end
end
piapprox = 4*NInside/SampleSize
```



Throwing of a dice

Matlab command: `unidrnd(6)`
Or `unidrnd(6,SampleSize,1)`

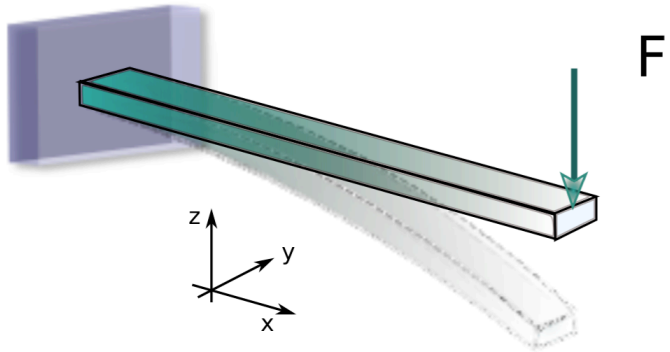
Mean and Variance of Dice Throw

	x	$P(X = x)$	px $[x \times P(X = x)]$	x^2	px^2 $[x^2 \times P(X = x)]$
	1	$\frac{1}{6}$	$\frac{1}{6}$	1	$\frac{1}{6}$
	2	$\frac{1}{6}$	$\frac{2}{6}$	4	$\frac{4}{6}$
	3	$\frac{1}{6}$	$\frac{3}{6}$	9	$\frac{9}{6}$
	4	$\frac{1}{6}$	$\frac{4}{6}$	16	$\frac{16}{6}$
	5	$\frac{1}{6}$	$\frac{5}{6}$	25	$\frac{25}{6}$
	6	$\frac{1}{6}$	$\frac{6}{6}$	36	$\frac{36}{6}$
Total		1	$\frac{21}{6}$		$\frac{91}{6}$
			$= 3.5$		$= + 15.17$

```
SampleSize = 1000;
AllSamples=unidrnd(6,Samp
leSize,1);
mean(AllSamples)
var(AllSamples)
```

$Var = 15.17 - 3.5^2 = 2.92$

Exercise: Deflection of a beam with random properties



- Bending rigidity (EI) is random: mean 9, standard deviation 20% of the mean
- Length(L) is random: mean 3, standard deviation 20% of the mean
- Unit force is applied

Obtain the mean and standard deviation of the tip deflection using MCS for the following two cases:

- Random variables are uniform
- Random variables are Gaussian – explain possible problems!
- Compare the pdfof the deflection for both cases and spot the difference!

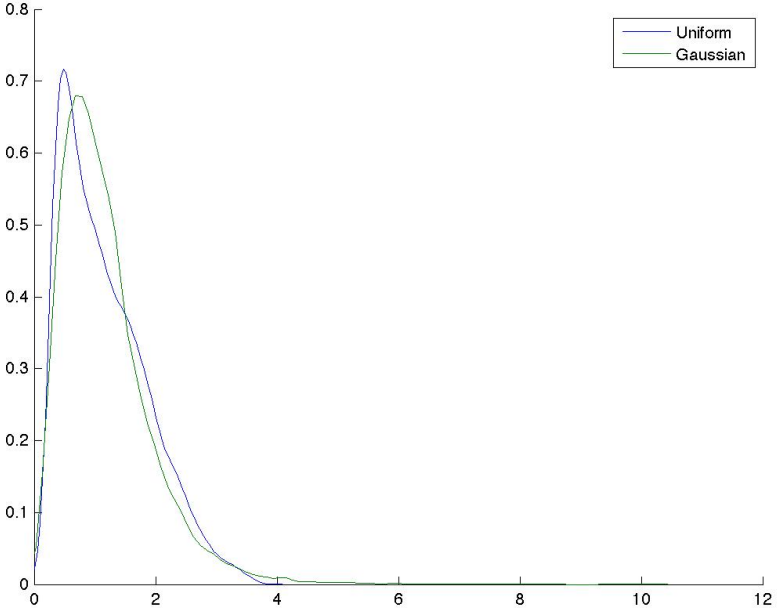
Exercise: Deflection of a beam with random properties

- When Random variables are uniform

Mean=1.1665, Std=0.70591, min=0.20852, max=3.7089

- Random variables are Gaussian

Mean=1.1725 , Std=0.7536, min=0.0143, max=10.124



Monte Carlo Methods in Practice

Good features

- Easy to implement
- Embarrassingly parallel
- Accuracy independent of number of variables

Bad feature

- Slow convergence rate makes it prohibitive for computationally expensive functions

Stochastic Methods in Structural Dynamics Part 3

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Overview of the course

The course is dived into eight topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

Outline of this talk

1. **Parametric sensitivity of eigensolutions**
   - Derivative of eigenvalues
   - Derivative of eigenvectors
2. **Statistics of the eigensolutions**
3. **Higher order perturbation**
   - Eigenvalue statistics using theory of quadratic forms
4. **Asymptotic integral method**
   - Multidimensional integrals in unbounded domains
   - Calculation of an arbitrary moment of the eigenvalues
   - Probability density function of the eigenvalues
     - Truncated Gaussian density function
     - Approximation by χ^2 probability density function
   - Application examples
     - A two DOF system
   - A three DOF system with closely spaced eigenvalues
     - Case 1: All eigenvalues are well separated
     - Case 2: Two eigenvalues are close

Parametric sensitivity of the eigensolutions

- Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest.
- Motivation for this kind of study arises, on one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualise the changes in the dynamic response with respect to system parameters.
- This kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods.
- Rates of change of eigenvalues and eigenvectors are useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration.
- Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems.

Parametric sensitivity of the eigensolutions

- The eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\phi_j = \lambda_j \mathbf{M}(\mathbf{x})\phi_j \tag{1}$$

- Here λ_j and ϕ_j are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$, the mass and stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random vector $\mathbf{x} \in \mathbb{R}^m$.
- The vector $\mathbf{x}$ may consist of material properties, e.g., mass density, Poisson's ratio, Young's modulus; geometric properties, e.g., length, thickness, and boundary conditions.
- The eigenvalues and eigenvectors are smooth differentiable functions of the random parameter vector $\mathbf{x}$.

Parametric sensitivity of the eigenvalues

- We rewrite the eigenvalue equation as

$$\left[\mathbf{K} - \lambda_j \mathbf{M}\right] \phi_j = \mathbf{0} \tag{2}$$

$$\text{or} \quad \phi_j^T \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \tag{3}$$

  The functional dependence of $\mathbf{x}$ is removed for notational convenience.

- Differentiating the eigenvalue equation (2) with respect to the element x_i of the parameter vector we have

$$\left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \phi_j + \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \frac{\partial \phi_j}{\partial x_i} = \mathbf{0} \tag{4}$$

- Premultiplying by ϕ_j^T we have

$$\phi_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \phi_j + \phi_j^T \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \frac{\partial \phi_j}{\partial x_i} = \mathbf{0} \tag{5}$$

Parametric sensitivity of the eigenvalues

- Using the identity in (3) we have

$$\boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i} \mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j = \mathbf{0} \tag{6}$$

$$\text{or} \quad \frac{\partial \lambda_j}{\partial x_i} = \frac{\boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j}{\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j} \tag{7}$$

- Note that when the modes are mass normalised $\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j = 1$
- The derivatives need to be evaluated at certain value $\mathbf{x}$. It is customary to evaluate this at the nominal value (which is normally the mean value if $\mathbf{x}$ is a random vector).
- Denote the mean of $\mathbf{x}$ as $\boldsymbol{\mu} \in \mathbb{R}^m$, and consider that

$$\mathbf{M}(\boldsymbol{\mu}) = \mathbf{M}_0, \quad \text{and} \quad \mathbf{K}(\boldsymbol{\mu}) = \mathbf{K}_0 \tag{8}$$

are the ‘deterministic parts’ of the mass and stiffness matrices respectively.

Parametric sensitivity of the eigenvalues

- The deterministic part of the eigenvalues:

$$\lambda_{0_j} = \lambda_j(\boldsymbol{\mu}) \tag{9}$$

  is obtained from the deterministic eigenvalue problem:

$$\mathbf{K}_0\, \boldsymbol{\phi}_{0_j} = \lambda_{0_j}\, \mathbf{M}_0\, \boldsymbol{\phi}_{0_j}. \tag{10}$$

- Using these, the derivative at the mean/nominal point can be obtained as

$$\frac{\partial \lambda_j}{\partial x_i} = \boldsymbol{\phi}_{0_j}{}^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_{0_j} \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_{0_j} \tag{11}$$

- Consider the standard expansion of the stiffness and mass matrices

$$\mathbf{K}(\mathbf{x}) = \mathbf{K}_0 + \sum_{i=1}^{m_K} x_i \mathbf{K}_i \quad \text{and} \quad \mathbf{M}(\mathbf{x}) = \mathbf{M}_0 + \sum_{i=1}^{m_M} x_i \mathbf{M}_i \tag{12}$$

  where $m = m_K + m_M$

Parametric sensitivity of the eigenvalues

- Therefore

$$\frac{\partial \mathbf{K}}{\partial x_i} = \mathbf{K}_i, i \in m_K \quad \text{and} \quad \frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{M}_i, i \in m_M \tag{13}$$

- Using these, the eigenvalue derive can be obtained succinctly as

$$\frac{\partial \lambda_j}{\partial x_i} = {\phi_{0_j}}^T \left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right] \phi_{0_j} \tag{14}$$

- Each eigenvalues can be expanded in a Taylor series about the mean of the parameter values as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\alpha})(\mathbf{x}-\boldsymbol{\alpha}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\alpha})^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})(\mathbf{x}-\boldsymbol{\alpha}) + \cdots \tag{15}$$

- Without any loss of generality, considering the mean of $\mathbf{x}$ is zero and retaining only the first order terms we have

$$\lambda_j(\mathbf{x}) \approx \lambda_{0_j} + \sum_{i=1}^{m} \left({\phi_{0_j}}^T \left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right] \phi_{0_j}\right) x_i \tag{16}$$

Parametric sensitivity of the eigenvectors

- Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of the eigenvectors

$$\frac{\partial \phi_j}{\partial x_i} = \sum_{r=1}^{n} \alpha_{jir} \phi_r \tag{17}$$

  It is necessary to find expressions for the constant α_{jir} for all $r = 1, 2, \cdots n$.
- Substituting this in Eq. (4) we have

$$\left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \phi_j + \sum_{r=1}^{n} \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \alpha_{jir} \phi_r = \mathbf{0} \tag{18}$$

- Premultiplying by ϕ_k^T we have

$$\phi_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \phi_j + \sum_{r=1}^{n} \phi_k^T \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \alpha_{jir} \phi_r = 0 \tag{19}$$

Parametric sensitivity of the eigenvectors

- We consider $r = k$ and the orthogonality of the eigenvectors:

$$\phi_k^T \mathbf{K} \phi_r = \lambda_k \delta_{kr} \quad \text{and} \quad \phi_k^T \mathbf{M} \phi_r = \delta_{kr} \tag{20}$$

- Using these we have

$$\phi_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \phi_j + \left(\lambda_k - \lambda_j \right) \alpha_{jik} = 0 \tag{21}$$

- From this we obtain

$$\alpha_{jik} = - \frac{\phi_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \phi_j}{\lambda_k - \lambda_j}, \quad \forall\, k \neq j \tag{22}$$

Parametric sensitivity of the eigenvectors

- To obtain the j-th term α_{jij} we differentiate the mass orthogonality relationship in (20) as

$$\frac{\partial(\phi_j^T \mathbf{M} \phi_j)}{\partial x_i} = 0 \quad \text{or} \quad \frac{\partial \phi_j^T}{\partial x_i} \mathbf{M} \phi_j + \phi_j^T \frac{\partial \mathbf{M}}{\partial x_i} \phi_j + \phi_j^T \mathbf{M} \frac{\partial \phi_j}{\partial x_i} = 0 \tag{23}$$

- Considering the symmetry of the mass matrix and using the expansion of the eigenvector derivative we have

$$\phi_j^T \frac{\partial \mathbf{M}}{\partial x_i} \phi_j + 2\phi_j^T \mathbf{M} \frac{\partial \phi_j}{\partial x_i} = 0 \quad \text{or} \quad \sum_{r=1}^{n} 2\phi_j^T \mathbf{M} \alpha_{jir} \phi_r = -\phi_j^T \frac{\partial \mathbf{M}}{\partial x_i} \phi_j \tag{24}$$

- Utilising the othonormality of the mode shapes we have

$$\alpha_{jij} = -\frac{1}{2} \phi_j^T \frac{\partial \mathbf{M}}{\partial x_i} \phi_j \tag{25}$$

Parametric sensitivity of the eigenvectors

- The complete eigenvector derivative is therefore given by

$$\frac{\partial \phi_j}{\partial x_i} = -\frac{1}{2}\left(\phi_j^T \frac{\partial \mathbf{M}}{\partial x_i}\phi_j\right)\phi_j + \sum_{k=1\neq j}^{n} \frac{\phi_k^T\left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right]\phi_j}{\lambda_j - \lambda_k}\phi_k \tag{26}$$

- Considering the conventional expansion of the mass and stiffness matrices, the derivative at the mean values of the parameter can be obtained as

$$\frac{\partial \phi_j}{\partial x_i} = -\frac{1}{2}\left(\phi_{0_j}^T \mathbf{M}_i \phi_{0_j}\right)\phi_{0_j} + \sum_{k=1\neq j}^{n} \frac{\phi_{0_k}^T\left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right]\phi_{0_j}}{\lambda_{0_j} - \lambda_{0_k}}\phi_{0_k} \tag{27}$$

- Considering the mean of $\mathbf{x}$ is zero and retaining only the first order terms we have

$$\phi_j(\mathbf{x}) \approx \phi_{0_j} + \sum_{i=1}^{m} \frac{\partial \phi_j}{\partial x_i} x_i = \phi_{0_j} + \sum_{i=1}^{m}\left(\sum_{k=1}^{n} \alpha_{jik}\phi_{0_k}\right) x_i \tag{28}$$

Covariance of the eigensolutions

- Suppose Σ_{ir} is the ir-th element of the covariance matrix, that is

$$\Sigma_{ir} = \mathrm{cov}\left(x_i, x_r\right) \tag{29}$$

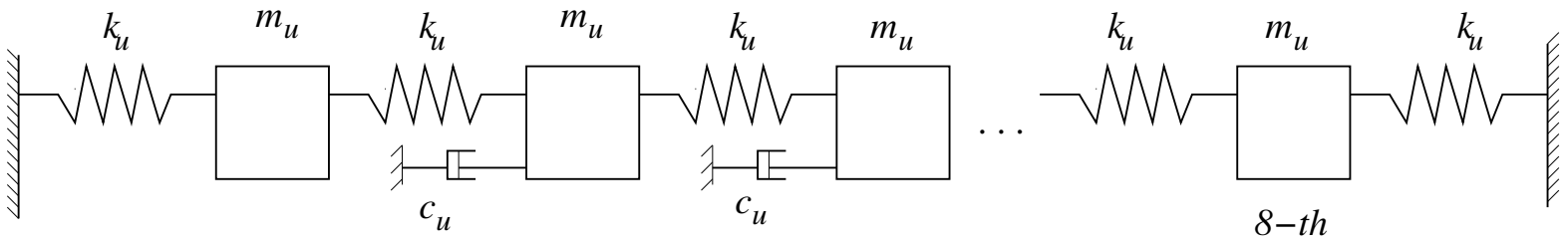
- The covariance of eigenvalue j and s can be obtained as

$$\begin{aligned}&\mathrm{E}\left[\left(\lambda_j - \lambda_{0_j}\right)\left(\lambda_s - \lambda_{0_s}\right)\right] \\ &\quad = \sum_{i=1}^{m}\sum_{r=1}^{m}\left(\phi_{0_j}{}^T\left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right]\phi_{0_j}\right)\left(\phi_{0_s}{}^T\left[\mathbf{K}_r - \lambda_{0_s}\mathbf{M}_r\right]\phi_{0_s}\right)\Sigma_{ir}\end{aligned} \tag{30}$$

- The covariance matrix of eigenvector j and s can be obtained as

$$\mathrm{E}\left[\left(\phi_j - \phi_{0_j}\right)\left(\phi_s - \phi_{0_s}\right)^T\right] = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{0_k}\phi_{0_l}^T\sum_{i=1}^{m}\sum_{r=1}^{m}\alpha_{jir}\alpha_{srl}\Sigma_{ir} \tag{31}$$

Numerical example

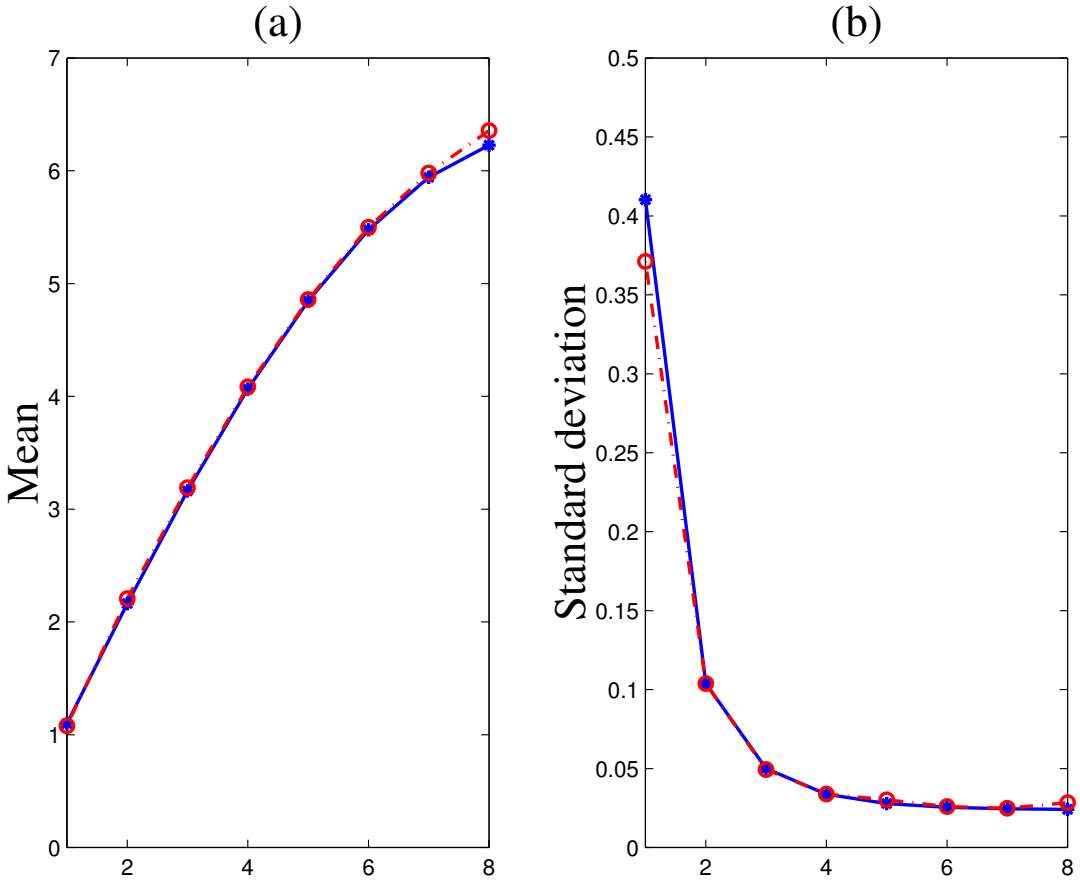


- An eight DOF system consisting of a linear array of spring-mass oscillators is considered
- Eight masses, each of nominal mass $m_u = 1$ kg, are connected by springs of nominal stiffness $k_u = 10$ N/m
- It is assumed that the mass and stiffness associated with all the units are random. Randomness associated with each unit has the following form

$$m_{u_j} = m_u \left(1 + \epsilon_{m_j} g_j\right), \quad k_{u_j} = k_u \left(1 + \epsilon_{k_j} g_j\right) \tag{32}$$

- Here $g_j, \forall j$ are assumed to be uncorrelated, identically distributed, zero-mean, unit-standard-deviation Gaussian random variables ($N(0,1)$). For this assumption, the joint covariance matrix Σ becomes a diagonal matrix.

Numerical example

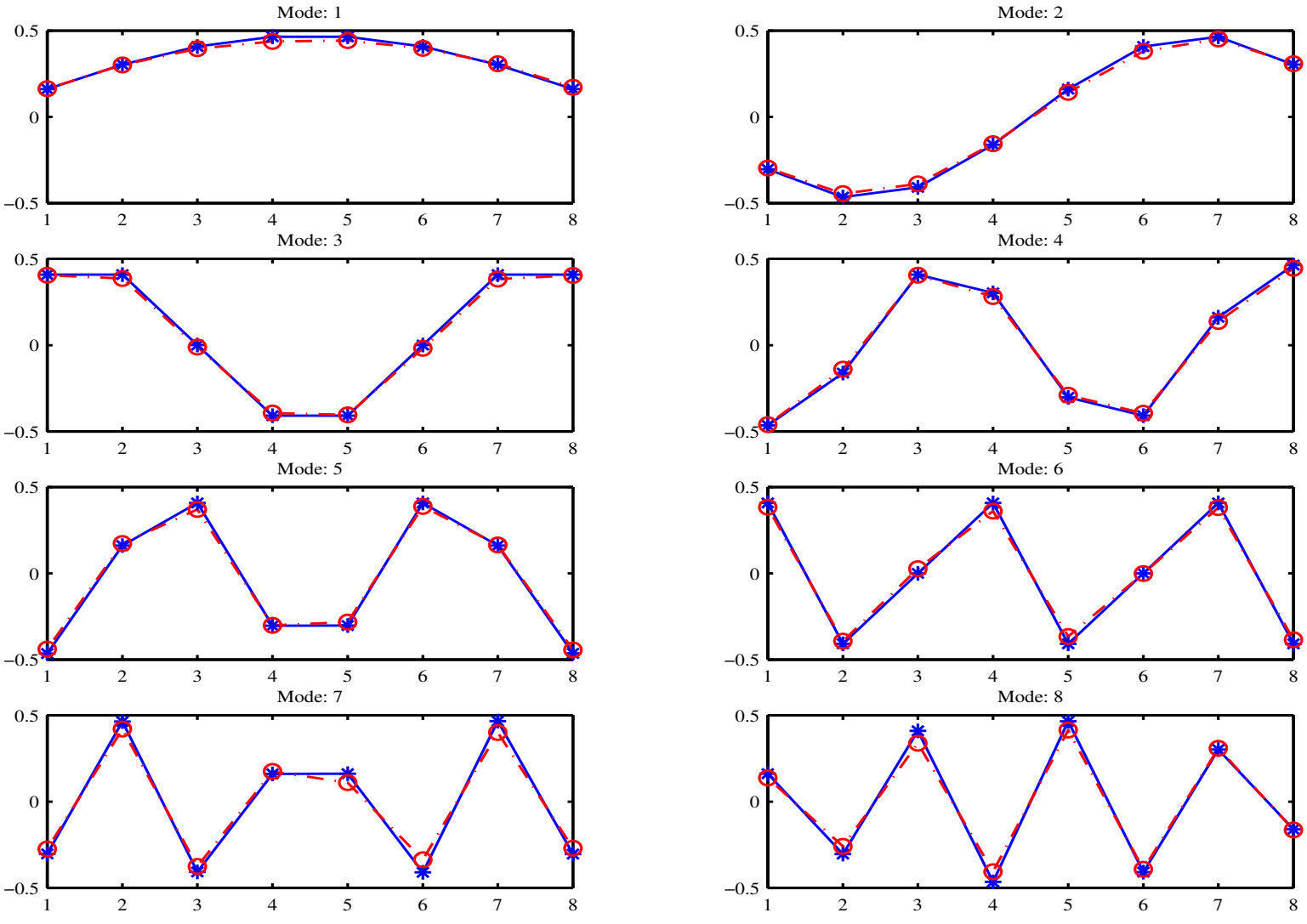


(a) The mean of the natural frequencies (b) Standard deviation of the natural frequencies; 'X-axis' Mode number; '—' Analytical; '-.-.-' MCS

Numerical example

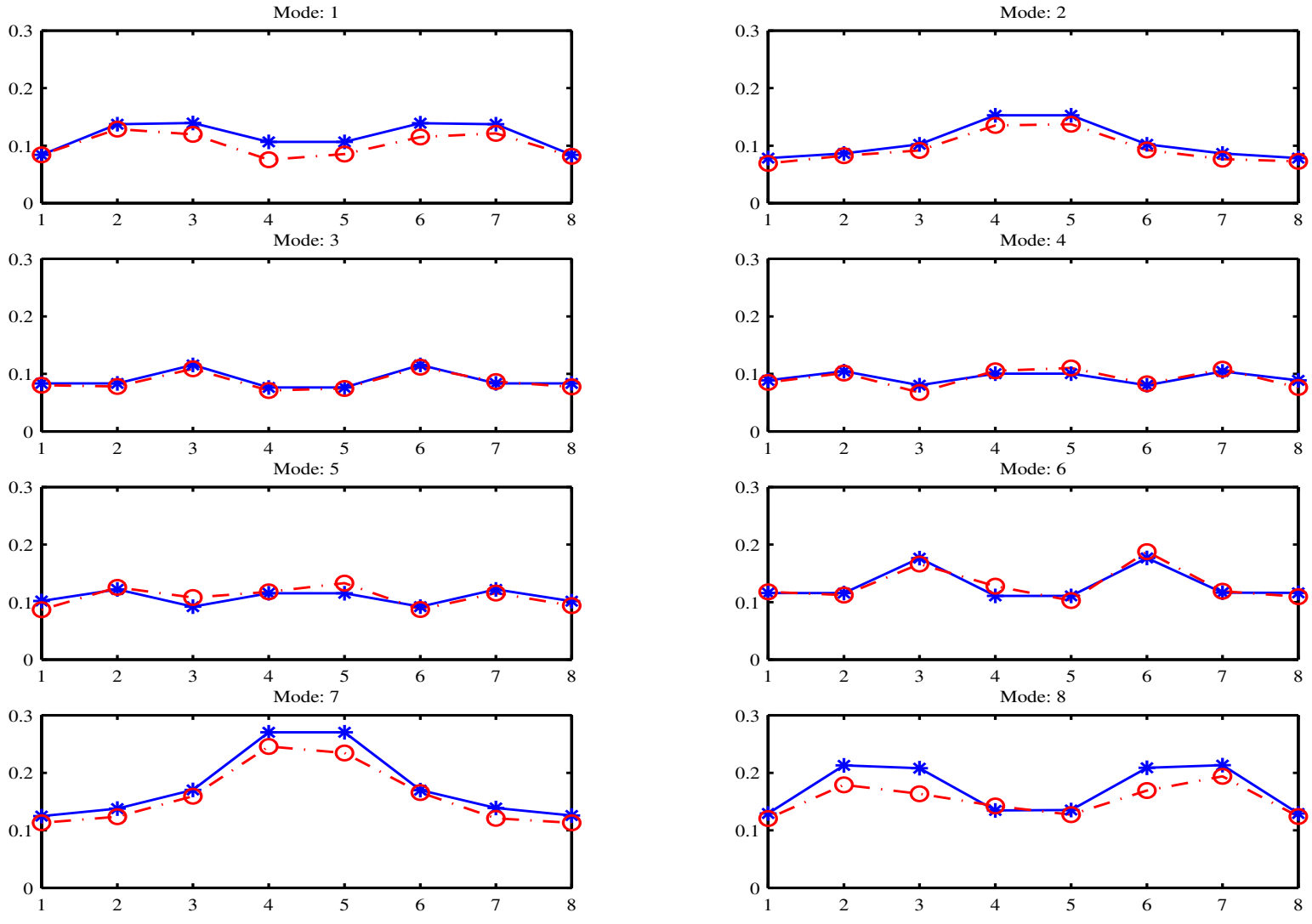
- Numerical values of the 'strength parameters', ϵ_{m_j} and ϵ_{k_j} are assumed to be 0.1, that is, we consider 10% randomness for all the parameter values.
- Because the random variables describing the system properties are assumed to be Gaussian, the mean values are the same as the nominal values.
- In the same figure, the mean values obtained from the proposed theory are compared with the results obtained from an independent Monte Carlo simulation (MCS) using 5000 samples. Both the curves follow each other very closely.

Mode shape statistics



The mean of the mode shapes; '—' Analytical; '-.-.-' MCS

Mode shape statistics



The standard deviation of the mode shapes; '—' Analytical; '-.-.-' MCS

Higher-order perturbation

- Higher-order perturbation method can be used to improve upon the results obtained from the first-order perturbation method
- Statistical properties of the system are completely described by the joint probability density function $p_{\mathbf{X}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. For mathematical convenience we express

$$p_{\mathbf{X}}(\mathbf{x}) = \exp\{-L(\mathbf{x})\} \tag{33}$$

  where $-L(\mathbf{x})$ is often known as the log-likelihood function.
- For example, if $\mathbf{x}$ is a m-dimensional multivariate Gaussian random vector with mean $\boldsymbol{\mu} \in \mathbb{R}^m$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m \times m}$ then

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\det\{\boldsymbol{\Sigma}\} + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}). \tag{34}$$

- It is assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive.

Higher-order perturbation

- The eigenvalues, $\lambda_j(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ are non-linear functions of the parameter vector $\mathbf{x}$.
- If the eigenvalues are not repeated, then each $\lambda_j(\mathbf{x})$ is expected to be a smooth and twice differentiable function since the mass and stiffness matrices are smooth and twice differentiable functions of the random parameter vector.
- In the mean-centered perturbation approach the function $\lambda_j(\mathbf{x})$ is expanded by its Taylor series about the point $\mathbf{x} = \boldsymbol{\mu}$ as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\mu}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu}) + \frac{1}{2}\,(\mathbf{x}-\boldsymbol{\mu})^T\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu})\,. \tag{35}$$

- Here $\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^m$ and $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\mu}$, that is

$$\left\{\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\right\}_k = \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}|\mathbf{x}=\boldsymbol{\mu} \tag{36}$$

$$\text{and} \quad \left\{\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\}_{kl} = \frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l|\mathbf{x}=\boldsymbol{\mu}. \tag{37}$$

- Providing the eigenvalues are distinct, the element of the Hessian matrix can be explicitly obtained as

$$\begin{aligned}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k\, \partial x_l} = \phi_j(\mathbf{x})^T \left[\frac{\partial^2 \mathbf{K}(\mathbf{x})}{\partial {x_k}^2} x_l - \lambda_j(\mathbf{x}) \frac{\partial^2 \mathbf{M}(\mathbf{x})}{\partial {x_k}^2} x_l\right] \phi_j(\mathbf{x}) \\ - \left(\phi_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \phi_j(\mathbf{x})\right) \left(\phi_j(\mathbf{x})^T \mathcal{G}_{jl}(\mathbf{x}) \phi_j(\mathbf{x})\right) \\ - \left(\phi_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_l} \phi_j(\mathbf{x})\right) \left(\phi_j(\mathbf{x})^T \mathcal{G}_{jk}(\mathbf{x}) \phi_j(\mathbf{x})\right) \\ + 2 \sum_{r=1}^{N} \frac{\left(\phi_r(\mathbf{x})^T \mathcal{G}_{jk}(\mathbf{x}) \phi_j(\mathbf{x})\right) \left(\phi_r(\mathbf{x})^T \mathcal{G}_{jl}(\mathbf{x}) \phi_j(\mathbf{x})\right)}{\lambda_j(\mathbf{x}) - \lambda_r(\mathbf{x})}. \end{aligned} \tag{38}$$

- The elements of the gradient vector and Hessian matrix of the eigenvalues are therefore completely defined in closed-form.

- When $\mathbf{x}$ is a multivariate Gaussian random vector, the moment generating function of $\lambda_j(\mathbf{x})$, for any $s \in \mathbb{C}$, can be obtained from (35) as

$$M_{\lambda_j}(s) = \mathrm{E}\left[\exp\left\{s\lambda_j(\mathbf{x})\right\}\right] = \int_{\mathbb{R}^m} \exp\Big\{s\lambda_j(\boldsymbol{\mu}) + s\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) \tag{39}$$

$$+\frac{s}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{D}_{\lambda_j}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) - L(\mathbf{x})\Big\}\, d\mathbf{x} \tag{40}$$

  where $L(\mathbf{x})$ is given by equation (34).
- Using the transformation

$$\mathbf{y} = (\mathbf{x}-\boldsymbol{\mu}) \tag{41}$$

  the integral in (39) can be evaluated exactly as

$$M_{\lambda_j}(s) = (2\pi)^{-m/2}\|\boldsymbol{\Sigma}\|^{-1/2} \int_{\mathbb{R}^m} \exp\Big\{s\overline{\lambda}_j + s\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\mathbf{y}$$

$$-\frac{1}{2}\mathbf{y}^T\left[\boldsymbol{\Sigma}^{-1} - s\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]\mathbf{y}\Big\}\, d\mathbf{y}$$

$$= \frac{\exp\left\{s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\left[\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\right\}}{\sqrt{\left\|\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\|}}. \tag{42}$$

- To obtain the pdf of $\lambda_j(\mathbf{x})$, the inverse Laplace transform of equation (42) is required. T
- If the mean-centered first-order perturbation is used then $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) = \mathbf{O}$ and from equation (42) we obtain

$$M_{\lambda_j}(s) \approx \exp\left\{ s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \right\}. \tag{43}$$

- This implies that $\lambda_j(\mathbf{x})$ is a Gaussian random variable with mean $\overline{\lambda}_j$ and variance $\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})$.
- However, for second-order perturbations in general the mean of the eigenvalues is not the deterministic value. The cumulants of $\lambda_j(\mathbf{x})$ can be obtained from

$$\kappa_j^{(r)} = \frac{d^r}{ds^r} \ln M_{\lambda_j}(s)|_{s=0}. \tag{44}$$

- Here $\kappa_j^{(r)}$ is the rth order cumulant of jth eigenvalue and from equation (42) we have

$$\ln M_{\lambda_j}(s) = s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\left[\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) - \frac{1}{2}\ln\left\|\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\|. \quad (45)$$

- Using this expression and after some simplifications it can be shown that

$$\kappa_j^{(r)} = \overline{\lambda}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right), r = 1, \quad (46)$$

$$\kappa_j^{(r)} = \frac{r!}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{r-2}\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \quad (47)$$

$$+ \frac{(r-1)!}{2}\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^r\right), r \geq 2. \quad (48)$$

- The mean and first few cumulants of the eigenvalues can be explicitly obtained as

$$\widehat{\lambda}_j = \kappa_j^{(1)} = \overline{\lambda}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right) \tag{49}$$

$$\mathrm{Var}\left[\lambda_j\right] = \kappa_j^{(2)} = \mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \frac{1}{2}\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^2\right), \tag{50}$$

$$\kappa_j^{(3)} = 3\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^3\right), \tag{51}$$

$$\text{and } \kappa_j^{(4)} = 12\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^2\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + 3\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^4\right). \tag{52}$$

- From the cumulants, the raw moments $\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r\right]$ and the central moments $\mu_j^{\prime(r)} = \mathrm{E}\left[(\lambda_j - \bar{\lambda}_j)^r\right]$ can be obtained using standard formulae.

Asymptotic integral method

- The moments of the eigenvalues are obtained based on an asymptotic approximation of the multidimensional integral.
- Consider a function $f(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ which is smooth and at least twice differentiable. Suppose we want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} \exp\{-f(\mathbf{x})\}\, d\mathbf{x}. \tag{53}$$

  This is a m-dimensional integral over the unbounded domain $\mathbb{R}^m$.
- The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum. Suppose that $f(\mathbf{x})$ reaches its global minimum at a *unique* point $\boldsymbol{\theta} \in \mathbb{R}^m$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0}. \tag{54}$$

Higher-order perturbation

- Using this, $f(\mathbf{x})$ is expanded in a Taylor series about θ and equation (53) is rewritten as

$$\begin{aligned}\mathcal{J} &= \int_{\mathbb{R}^m} \exp\left\{-\left\{f(\theta) + \frac{1}{2}(\mathbf{x}-\theta)^T \mathbf{D}_f(\theta)(\mathbf{x}-\theta) + \varepsilon(\mathbf{x},\theta)\right\}\right\} d\mathbf{x} \\ &= \exp\{-f(\theta)\} \int_{\mathbb{R}^m} \exp\left\{-\frac{1}{2}(\mathbf{x}-\theta)^T \mathbf{D}_f(\theta)(\mathbf{x}-\theta) - \varepsilon(\mathbf{x},\theta)\right\} d\mathbf{x}\end{aligned} \tag{55}$$

  where $\varepsilon(\mathbf{x},\theta)$ is the error if only the terms up to second-order were retained in the Taylor series expansion.
- With suitable scaling of $\mathbf{x}$ the integral in (53) can be transformed to the so called 'Laplace integral'. Under special conditions such integrals can be well approximated using asymptotic methods.
- We neglect the error $\varepsilon(\mathbf{x},\theta)$ considering the higher-order derivatives are small.

Higher-order perturbation

- The integral in (55) can be approximated as

$$\mathcal{J} \approx \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \int_{\mathbb{R}^m} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\theta}\right)^T \mathbf{D}_f\left(\boldsymbol{\theta}\right)\left(\mathbf{x}-\boldsymbol{\theta}\right)\right\} d\mathbf{x}. \tag{56}$$

  If $\boldsymbol{\theta}$ is the global minimum of $f(\mathbf{x})$ in $\mathbb{R}^m$, the symmetric Hessian matrix $\mathbf{D}_f\left(\boldsymbol{\theta}\right) \in \mathbb{R}^{m\times m}$ is also expected to be positive definite.
- Using the coordinate transformation

$$\boldsymbol{\xi} = \left(\mathbf{x}-\boldsymbol{\theta}\right)\mathbf{D}_f^{-1/2}(\boldsymbol{\theta}). \tag{57}$$

  The Jacobian of this transformation is

$$\det\left\{\mathbf{J}\right\} = \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2}. \tag{58}$$

- Using equation (57), the integral in equation (56) can be evaluated as

$$\mathcal{J} \approx \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \int_{\mathbb{R}^m} \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\xi}^T\boldsymbol{\xi}\right)\right\} d\boldsymbol{\xi} \tag{59}$$

  or $$\mathcal{J} \approx (2\pi)^{m/2} \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2}. \tag{60}$$

- This approximation is expected to yield good result if the minimum of $f(\mathbf{x})$ around $\mathbf{x} = \boldsymbol{\theta}$ is sharp. Equation (60) will now be used to obtain moments of the eigenvalues.

Moments of the eigenvalues

- An arbitrary rth order moment of the eigenvalues can be obtained from

$$\begin{aligned}\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\right] &= \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x})\right)\right\}\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots .\end{aligned} \tag{61}$$

- The equation can be expressed in the form of equation (53) by choosing

$$f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}). \tag{62}$$

- Differentiating the above equation with respect to x_k we obtain

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \frac{\partial L(\mathbf{x})}{\partial x_k} - \frac{r}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}. \tag{63}$$

Moments of the eigenvalues

- The optimal point θ can be obtained from (54) by equating the above expression to zero. Therefore at $\mathbf{x} = \theta$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \quad \forall\, k \tag{64}$$

$$\text{or} \quad \frac{r}{\lambda_j(\boldsymbol{\theta})} \frac{\partial \lambda_j(\boldsymbol{\theta})}{\partial x_k} = \frac{\partial L(\boldsymbol{\theta})}{\partial x_k}, \quad \forall\, k \tag{65}$$

$$\text{or} \quad \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta}). \tag{66}$$

- Equation (66) needs to be solved numerically to obtain θ. It implies that at the optimal point the gradient vectors of the eigenvalues and log-likelihood function are parallel.
- The non-linear set of equations (66) have to be solved numerically. Due to the explicit analytical expression of $\mathbf{d}_{\lambda_j}$ in terms of the derivative of the mass and stiffness matrices, expensive numerical differentiation of $\lambda_j(\mathbf{x})$ at each step is not needed.

Moments of the eigenvalues

- For most $p_{\mathbf{X}}(\mathbf{x})$, a closed-form expression of $\mathbf{d}_L(\mathbf{x})$ is available.
- For example, when $\mathbf{x}$ has multivariate Gaussian distribution, $L(\mathbf{x})$ is given by equation (34). By differentiating this we obtain

$$\mathbf{d}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}). \tag{67}$$

Substituting this in equation (66), the optimal point $\boldsymbol{\theta}$ can be obtained as

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \frac{1}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}). \tag{68}$$

- This equation also gives a recipe for an iterative algorithm to obtain $\boldsymbol{\theta}$. One starts with an initial $\boldsymbol{\theta}$ in the right-hand side and obtains an updated $\boldsymbol{\theta}$ in the left-hand side.
- This procedure can be continued until the difference between the values of $\boldsymbol{\theta}$ obtained from both sides of (68) is less than (l_2 vector norm can be used to measure the difference) a predefined small value.
- A good value to start the iteration process is $\boldsymbol{\theta} = \boldsymbol{\mu}$, as in the case of mean-centred approach. Note that the solution of a deterministic eigenvalue problem is needed at each step of the iteration process.

Moments of the eigenvalues

- The elements of the Hessian matrix $\mathbf{D}_f(\theta)$ can be obtained by differentiating equation (63) with respect to x_l:

$$\begin{aligned}\frac{\partial^2 f(\mathbf{x})}{\partial {x_k}^2} x_l &= \frac{\partial^2 L(\mathbf{x})}{\partial {x_k}^2} x_l - r\left(-\frac{1}{\lambda_j^2(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} + \frac{1}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l\right) \\ &= \frac{\partial^2 L(\mathbf{x})}{\partial {x_k}^2} x_l + \frac{1}{r}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}\right\}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\right\} - \frac{r}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l.\end{aligned} \tag{69}$$

- At $\mathbf{x} = \theta$ we can use equation (65) so that equation (69) reads

$$\frac{\partial^2 f(\mathbf{x})}{\partial {x_k}^2} x_l|_{\mathbf{x}=\boldsymbol{\theta}} = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial {x_k}^2} x_l + \frac{1}{r}\frac{\partial L(\boldsymbol{\theta})}{\partial x_k}\frac{\partial L(\boldsymbol{\theta})}{\partial x_l} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\frac{\partial^2 \lambda_j(\boldsymbol{\theta})}{\partial {x_k}^2} x_l. \tag{70}$$

- Combining this equation for all k and l we have

$$\mathbf{D}_f(\theta) = \mathbf{D}_L(\theta) + \frac{1}{r}\mathbf{d}_L(\theta)\mathbf{d}_L(\theta)^T - \frac{r}{\lambda_j(\theta)}\mathbf{D}_{\lambda_j}(\theta). \tag{71}$$

where $\mathbf{D}_{\lambda_j}(\bullet)$ is defined in equation (37).

Moments of the eigenvalues

- Using the asymptotic approximation (60), the rth moment of the eigenvalues can be obtained as

$$\mu_j^{(r)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) \exp\{-L(\boldsymbol{\theta})\} \left\| \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}. \quad (72)$$

  This is perhaps the most general formula to obtain the moments of the eigenvalues of linear stochastic dynamic systems. The optimal point $\boldsymbol{\theta}$ needs to be calculated by solving non-linear set of equations equation (66) for each λ_j and r. Several special cases arising from equation (72) are of practical interest:

- *Mean of the eigenvalues:* The mean of the eigenvalues can be obtained by substituting $r = 1$ in equation (72), that is

$$\widehat{\lambda}_j = \mu_j^{(1)} = (2\pi)^{m/2} \lambda_j(\boldsymbol{\theta}) \exp\{-L(\boldsymbol{\theta})\} \det\left\{ \mathbf{D}_L(\boldsymbol{\theta}) + \mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}) \right\}^{-1/2}. \quad (73)$$

Moments of the eigenvalues

- *Central moments of the eigenvalues:* Once the mean in known, the central moments can be expressed in terms of the raw moments $\mu_j^{(r)}$ using the binomial transform

$$\mu_j'^{(r)} = \mathrm{E}\left[\left(\lambda_j - \widehat{\lambda}_j\right)^r\right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu_j^{(k)} \widehat{\lambda}_j^{r-k}. \tag{74}$$

- *Random vector* $\mathbf{x}$ *has multivariate Gaussian distribution:* In this case $L(\mathbf{x})$ is given by equation (34) and by differentiating equation (67) we obtain

$$\text{and} \quad \mathbf{D}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}. \tag{75}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from equation (66) as

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \frac{r}{\lambda_j(\boldsymbol{\theta})} \boldsymbol{\Sigma}\, \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}). \tag{76}$$

Moments of the eigenvalues

- Using equation (67) and equation (75), the Hessian matrix can be derived from equation (71) as

$$\begin{aligned}\mathbf{D}_f(\boldsymbol{\theta}) &= \boldsymbol{\Sigma}^{-1} + \frac{1}{r}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \\ &= \boldsymbol{\Sigma}^{-1}\left(\mathbf{I} + \frac{1}{r}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}\right) - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}).\end{aligned} \tag{77}$$

- Therefore, the rth moment of the eigenvalues can be obtained from Eq. (72) as

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\right\} \det\{\boldsymbol{\Sigma}\}^{-1/2} \det\{\mathbf{D}_f(\boldsymbol{\theta})\}^{-1/2} \tag{78}$$

Moments of the eigenvalues

- Using Eq. (77) and recalling that for any two matrices **A** and **B**, $\det\{\mathbf{A}\}\det\{\mathbf{B}\} = \det\{\mathbf{AB}\}$ we have

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\right\} \det\left\{\mathbf{I} + \widetilde{\mathbf{D}}_f(\boldsymbol{\theta})\right\}^{-1/2} \quad (79)$$

  where

$$\widetilde{\mathbf{D}}_f(\boldsymbol{\theta}) = \frac{1}{r}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \quad (80)$$

- The probability density function of the eigenvalues can be obtained from these moments.

Maximum entropy probability density function

- Once the cumulants/moments of the eigenvalues are known, the pdf of the eigenvalues can be obtained using the Maximum Entropy Method (MEM). Because equations (46), (47) and (72) can be used to calculate any arbitrary order cumulant and moment, the pdf can be obtained accurately by taking higher order terms.
- Since **M** and **K** are symmetric and positive definite random matrices, all the eigenvalues are real and positive. Suppose the pdf of λ_j is given by $p_{\lambda_j}(u)$ where $u \in \mathbb{R}$ is positive, that is $u \in [0, \infty]$. Considering that only first n moments are used, the pdf of each eigenvalue must satisfy the following constraints:

$$\int_0^\infty p_{\lambda_j}(u)du = 1 \tag{81}$$

$$\text{and} \quad \int_0^\infty u^r p_{\lambda_j}(u)du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n. \tag{82}$$

Maximum entropy probability density function

- Using Shannon's measure of entropy

$$\mathcal{S} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du \tag{83}$$

we construct the Lagrangian

$$\mathcal{L} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du - (\rho_0 - 1)\left[\int_0^\infty p_{\lambda_j}(u) du - 1\right] - \sum_{r=1}^{n} \rho_r \left[\int_0^\infty u^r p_{\lambda_j}(u) du - \mu_j^{(r)}\right]. \tag{84}$$

where $\rho_r, r = 0, 1, 2, \cdots, n$ are Lagrange multipliers.

- The function $p_{\lambda_j}(u)$ which maximizes $\mathcal{L}$ can be obtained using the calculus of variations. Using the Euler-Lagrange equation the solution is given by

$$p_{\lambda_j}(u) = \exp\left\{-\rho_0 - \sum_{i=1}^{n} \rho_i u^i\right\} = \exp\{-\rho_0\} \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\}, \quad u \geq 0. \tag{85}$$

Maximum entropy probability density function

- The Lagrange multipliers can be obtained from the constraint equations (81) and (82) as

$$\exp\{\rho_0\} = \int_0^\infty \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} du$$

$$\text{and} \quad \exp\{\rho_0\}\,\mu_j^{(r)} = \int_0^\infty u^r \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} du, \quad \text{for} \quad r = 0, 1, 2, \cdots n.$$

- Closed-form expressions for ρ_r are in general not possible for all n. If we take $n = 2$, then the resulting pdf can be expressed as the truncated Gaussian density function

$$p_{\lambda_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j\,\Phi\left(\widehat{\lambda}_j/\sigma_j\right)} \exp\left\{-\frac{\left(u-\widehat{\lambda}_j\right)^2}{2\sigma_j^2}\right\}, \quad u \geq 0. \tag{86}$$

where σ_j is given by

$$\sigma_j^2 = \mu_j^{(2)} - \widehat{\lambda}_j^2. \tag{87}$$

- The truncated Gaussian density function derived here ensures that the probability of any eigenvalues becoming negative is zero.

Maximum entropy probability density function

- We use an approximation analogous to Pearson's three moment central χ^2 approximation to the distribution of a noncentral χ^2 . The eigenvalues are approximated as

$$\lambda_j \approx \eta_j + \gamma_j \chi^2_{\nu_j}(u) \tag{88}$$

where $\chi^2_{\nu_j}(u)$ is a central χ^2 density function with ν_j degrees-of-freedom.

- The constants η_j, γ_j, and ν_j are obtained such that the first three moments of λ_j are equal to that of the approximated χ^2 pdf. The moment generating function of the approximated χ^2 pdf is given by

$$\mathrm{E}\left[\exp\left\{-s\left(\eta_j + \gamma_j \chi^2_{\nu_j}\right)\right\}\right] = \exp\left\{-s\eta_j\right\}\left(1 + 2s\gamma_j\right)^{-\nu_j/2}. \tag{89}$$

- Equating the first three moments we have

$$\eta_j + \nu_j\gamma_j = \mu_j^{(1)}, \tag{90}$$

$$\eta_j{}^2 + 2\eta_j\nu_j\gamma_j + \nu_j{}^2\gamma_j{}^2 + 2\nu_j\gamma_j{}^2 = \mu_j^{(2)} \tag{91}$$

and $$\eta_j{}^3 + 3\eta_j{}^2\nu_j\gamma_j + 3\eta_j\nu_j{}^2\gamma_j{}^2 + 6\eta_j\nu_j\gamma_j{}^2 + \nu_j{}^3\gamma_j{}^3 + 6\nu_j{}^2\gamma_j{}^3 + 8\nu_j\gamma_j{}^3 = \mu_j^{(3)}. \tag{92}$$

Maximum entropy probability density function

- This set of coupled non-linear equations can be solved exactly in closed-form to obtain η_j, γ_j, and ν_j:

$$\eta_j = \frac{\mu_j^{(1)^2}\mu_j^{(2)} - 2\,\mu_j^{(2)^2} + \mu_j^{(1)}\mu_j^{(3)}}{2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_j^{(3)}} \tag{93}$$

$$\gamma_j = \frac{2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_j^{(3)}}{4\left(\mu_j^{(2)} - \mu_j^{(1)^2}\right)}, \tag{94}$$

$$\text{and } \nu_j = 8\,\frac{\left(\mu_j^{(2)} - \mu_j^{(1)^2}\right)^3}{\left(2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_3'\right)^2}. \tag{95}$$

Maximum entropy probability density function

- Moments of $\lambda_j(\mathbf{x})$ obtained in equation (72), can be used directly in the right-hand side of these equations. Alternatively, this approach can also be used in conjunction with the perturbation methods by transforming the cumulants obtained from equations (46) and (47) to moments.
- Using the transformation in equation (88) the approximate probability density function of $\lambda_j(\mathbf{x})$ is given by

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi^2_{\nu_j}}\left(\frac{u-\eta_j}{\gamma_j}\right) = \frac{(u-\eta_j)^{\nu_j/2-1}\exp\left\{-(u-\eta_j)/2\gamma_j\right\}}{(2\gamma_j)^{\nu_j/2}\Gamma(\nu_j/2)}. \quad (96)$$

- The two approximated pdf proposed here have simple forms but it should be noted that they are not exhaustive. Given the moments/cumulants, different probability density functions can be fitted using different methods. The application of the approximate pdfs derived here is illustrated in the next section.

A two DOF system

- A simple two-degree-of-freedom undamped system has been considered to illustrate a possible application of the expressions developed so far. The main purpose of this example is to understand how the proposed methods compare with the existing methods. 1 shows the example, together with the numerical values of the masses and spring stiffnesses.

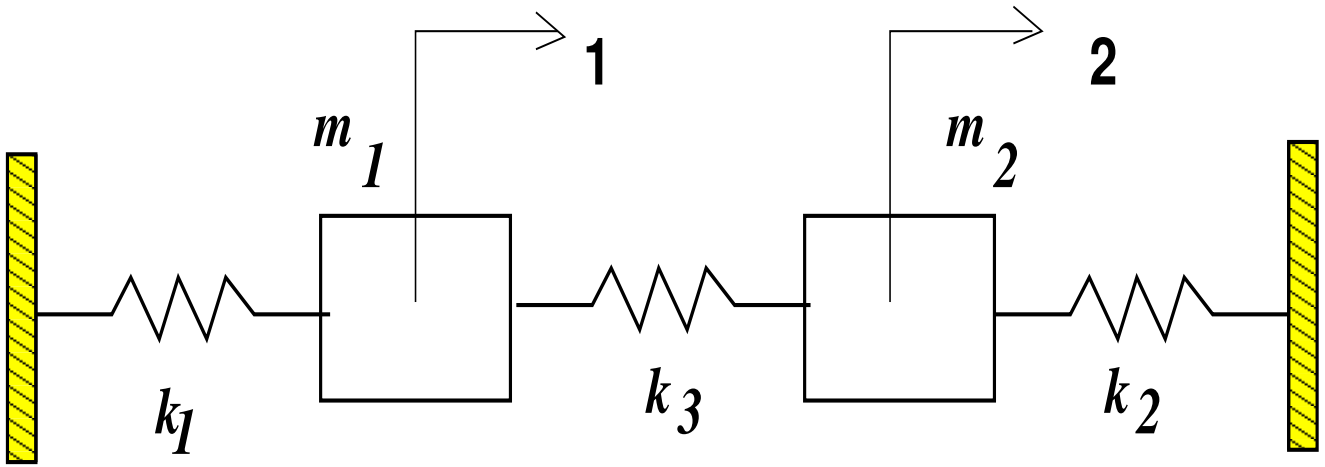


Figure : The undamped two degree-of-system system, $m_1 = 1$ kg, $m_2 = 1.5$ kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.

A two DOF system

- The system matrices for the example are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}. \tag{97}$$

- It is assumed that only the stiffness parameters k_1 and k_2 are uncertain so that $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$ and $\bar{k}_i$ denote the deterministic values of the spring constants. Here $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ is a vector of standard Gaussian random variables, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$.
- The numerical values of the 'strength parameters' are considered as $\epsilon_1 = \epsilon_2 = 0.25$. The strength parameters are selected so that the system matrices are almost surely positive definite.

A two DOF system

- Noting that **M** is independent of **x** and **K** is a linear function of **x**, the derivative of the system matrices with respect to the random vector **x** can be obtained as

$$\frac{\partial \mathbf{K}}{\partial x_1} = \epsilon_1 \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_2} = \epsilon_2 \begin{bmatrix} 0 & 0 \\ 0 & \bar{k}_2 \end{bmatrix}, \tag{98}$$

$$\frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{O} \quad \text{and} \quad \frac{\partial^2 \mathbf{K}}{\partial {x_i}^2} x_j = \mathbf{O}. \tag{99}$$

- We calculate the raw moments and the probability density functions of the two eigenvalues of the system. Recall that the eigenvalues obtained from equation (1) are the square of the natural frequencies ($\lambda_j = \omega_j^2$). Several methods are used to obtain the moments and the pdfs.

Mean-centered first-order perturbation

- This case arises when $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})$ in the Taylor series expansion (35) is assumed to be a null matrix so that only the first-order terms are retained.
- This is the simplest approximation, and as mentioned earlier, results in a Gaussian distribution of the eigenvalues.
- Recalling that for this problem $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$, the resulting statistics for this special case can be obtained from equations (49) and (50) as

$$\widehat{\lambda}_j = \overline{\lambda}_j \tag{100}$$

$$\text{and} \quad \mathrm{Var}\left[\lambda_j\right] = \mathbf{d}_{\lambda_j}^T(\mathbf{0})\mathbf{d}_{\lambda_j}(\mathbf{0}). \tag{101}$$

- The gradient vector $\mathbf{d}_{\lambda_j}(\mathbf{0})$ can be obtained from equation (11) using the system derivative matrices (98) and (99).

Mean-centered second-order perturbation

- In this case all the terms in equation (35) are retained. This approximation results in a quadratic form in the Gaussian random variables. The resulting statistics can be obtained from equations (46) and (47) by substituting $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$.
- The elements of the Hessian matrix $\mathbf{D}_{\lambda_j}(\mathbf{0})$ can be obtained from equation (38) and using the system derivative matrices (98) and (99).

Monte Carlo Simulation

- The samples of two independent Gaussian random variables x_1 and x_2 are generated and the eigenvalues are computed directly from equation (1).
- A total of 15000 samples are used to obtain the statistical moments and pdf of both the eigenvalues.
- The results obtained from the Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the five analytical methods described above.
- The percentage error for an arbitrary kth moment of an eigenvalue obtained using any one of the five analytical methods is given by

$$\text{Error}_{i\text{th method}} = \frac{\left| \{\mu_j^{(r)}\}_{i\text{th method}} - \{\mu_j^{(r)}\}_{\text{MCS}} \right|}{\{\mu_j^{(r)}\}_{\text{MCS}}} \times 100. \qquad (102)$$

A two DOF system

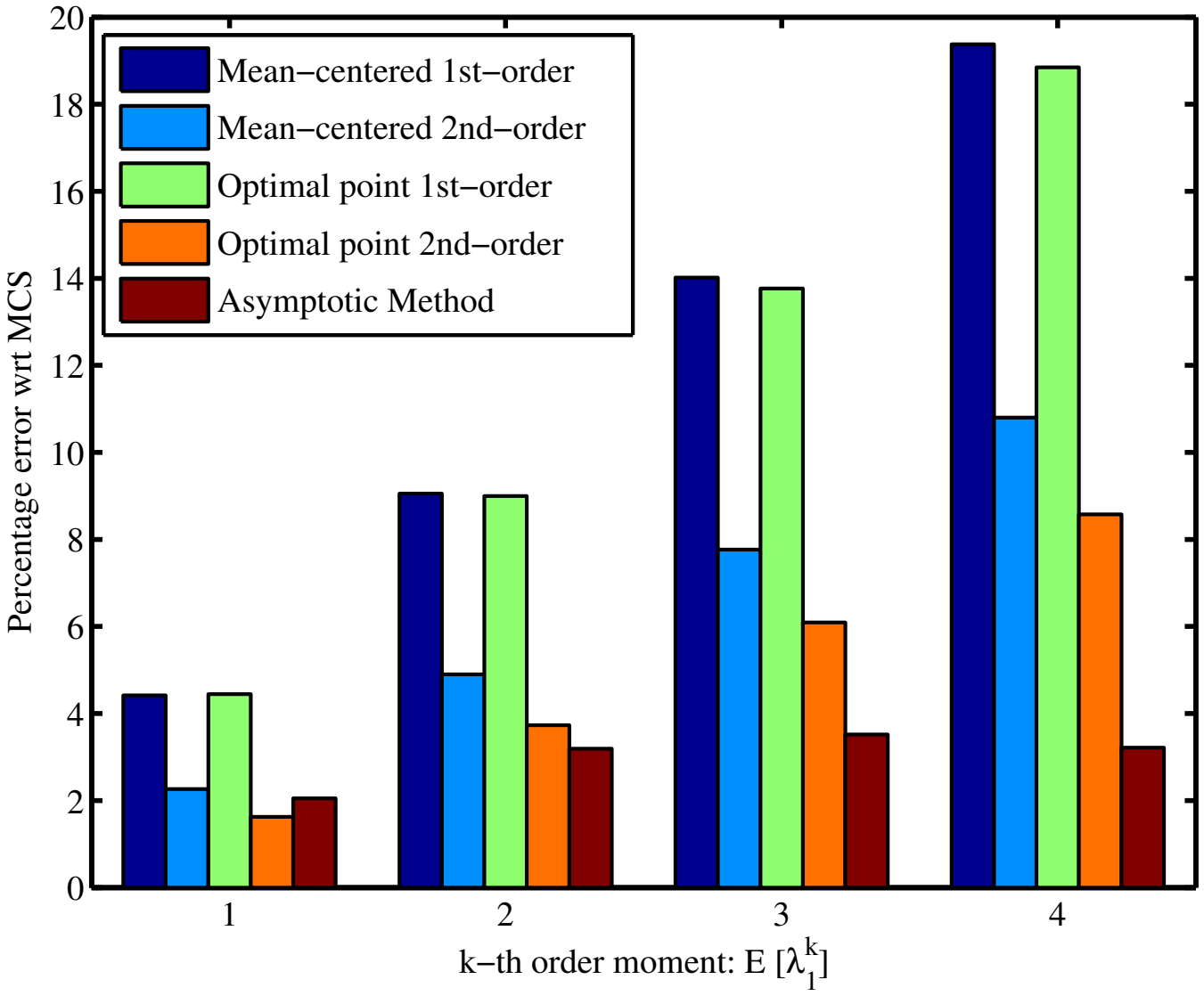


Figure : Percentage error for the first eigenvalue.

A two DOF system

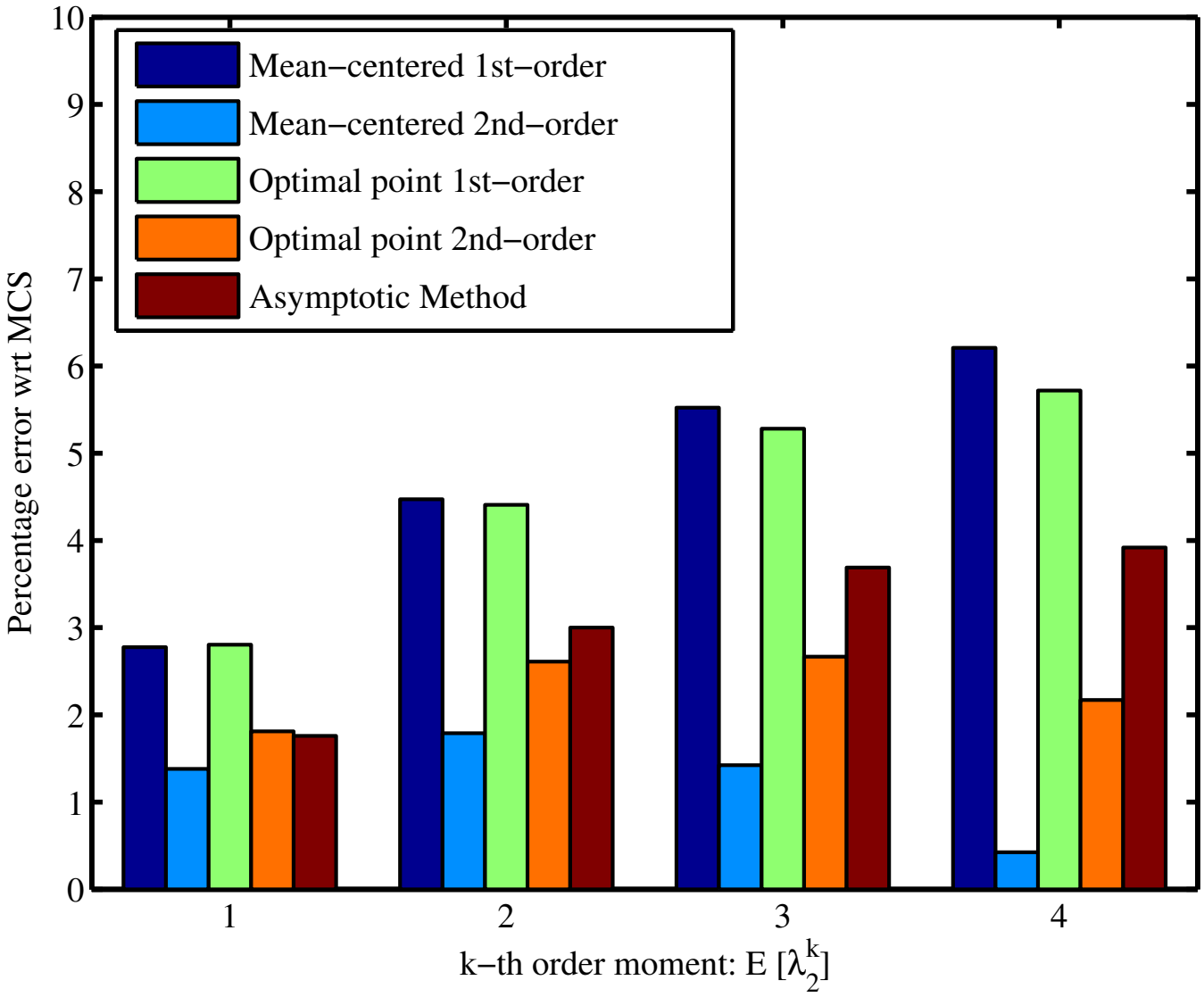


Figure : Percentage error for the second eigenvalue.

A two DOF system

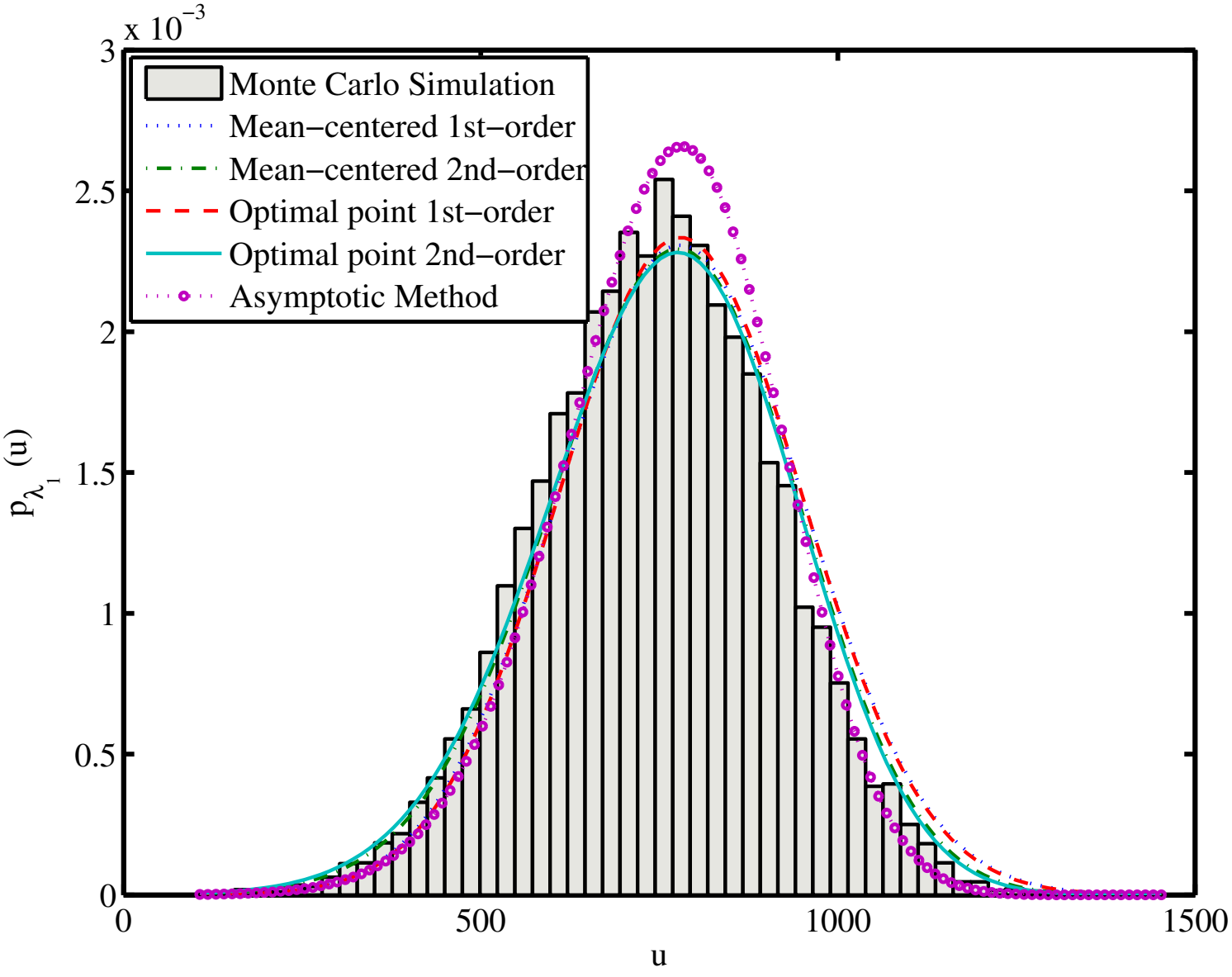


Figure : Probability density function of the first eigenvalue.

A two DOF system

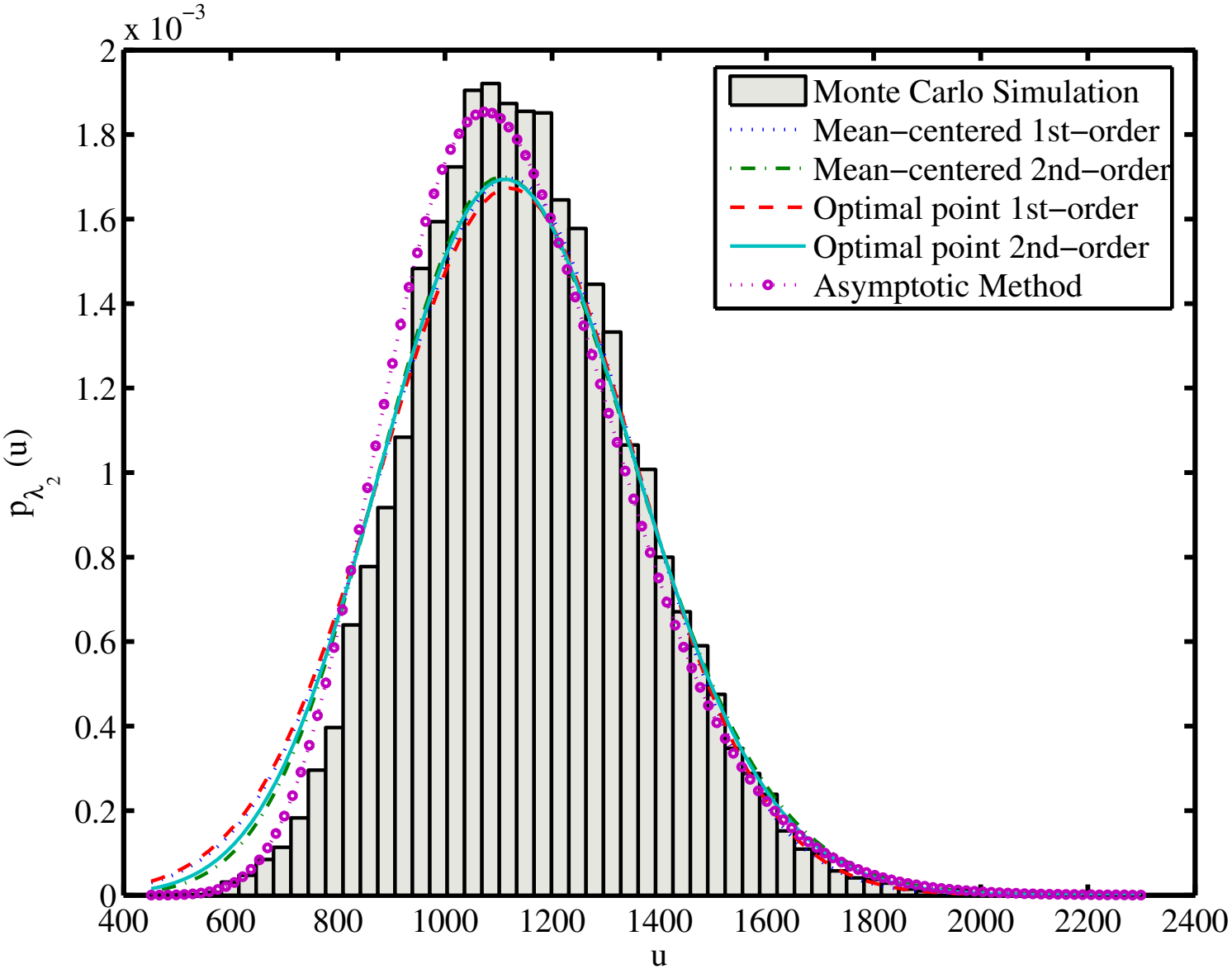


Figure : Probability density function of the second eigenvalue.

A three DOF system

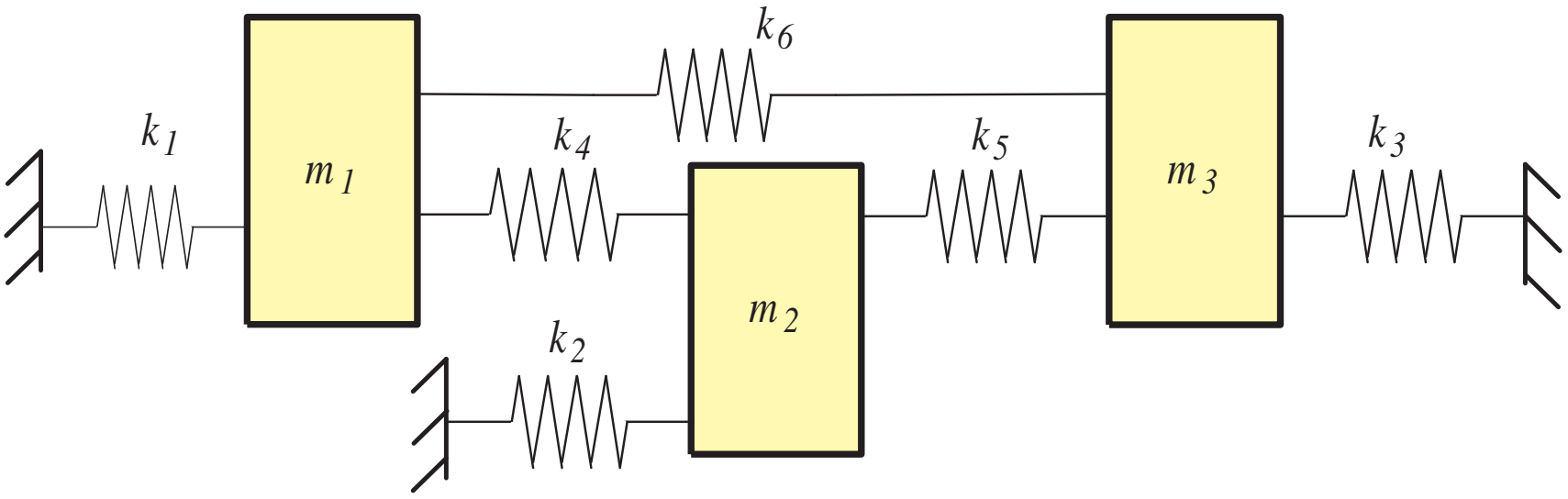


Figure : The three degree-of-freedom random system.

- The main purpose of this example is to understand how the proposed methods work when some of the system eigenvalues are closely spaced.

A three DOF system

- This is an interesting case because it is well known that closely spaced eigenvalues are parameter sensitive. We will investigate how the parameter uncertainty affects the eigenvalue distribution in such cases. This study has particular relevance to the dynamics of nominally symmetric rotating machines, for example, turbine blades with random imperfections. The mass and stiffness matrices of the example system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_4 + k_5 + k_2 & -k_5 \\ -k_6 & -k_5 & k_5 + k_3 + k_6 \end{bmatrix}. \tag{103}$$

  It is assumed that all mass and stiffness constants are random.

A three DOF system

- The randomness in these parameters are assumed to be of the following form:

$$m_i = \overline{m}_i \left(1 + \epsilon_m x_i\right), \quad i = 1, 2, 3 \tag{104}$$

$$k_i = \overline{k}_i \left(1 + \epsilon_k x_{i+3}\right), \quad i = 1, \cdots, 6. \tag{105}$$

  Here $\mathbf{x} = \{x_1, \cdots, x_9\}^T \in \mathbb{R}^9$ is the vector of random variables. It is assumed that all random variables are Gaussian and uncorrelated with zero mean and unit standard deviation, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$. Therefore, the mean values of m_i and k_i are given by $\overline{m}_i$ and $\overline{k}_i$. The numerical values of both of the 'strength parameters' ϵ_m and ϵ_k are fixed at 0.15.
- In order to obtain statistics of the eigenvalues using the methods developed in this paper the gradient vector and the Hessian matrix of the eigenvalues are required. This in turn requires the derivative of the system matrices with respect to the entries of $\mathbf{x}$. For most practical problems, which usually involve Finite Element modeling, these derivatives need to be determined numerically.

A three DOF system

- The derivatives of $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ with respect to elements of $\mathbf{x}$ can be obtained from equation (103) together with equations (104) and (105). For the mass matrix we have

$$\frac{\partial \mathbf{M}}{\partial x_1} = \begin{bmatrix} \overline{m}_1 \epsilon_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{m}_2 \epsilon_m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{m}_3 \epsilon_m \end{bmatrix}. \tag{106}$$

  All other $\frac{\partial \mathbf{M}}{\partial x_i}$ are null matrices.

A three DOF system

- The derivatives of the stiffness matrix are

$$\frac{\partial \mathbf{K}}{\partial x_4} = \begin{bmatrix} \overline{k}_1\epsilon_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_5} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{k}_2\epsilon_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{k}_3\epsilon_k \end{bmatrix},$$

$$\frac{\partial \mathbf{K}}{\partial x_7} = \begin{bmatrix} \overline{k}_4\epsilon_k & -\overline{k}_4\epsilon_k & 0 \\ -\overline{k}_4\epsilon_k & \overline{k}_4\epsilon_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_8} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{k}_5\epsilon_k & -\overline{k}_5\epsilon_k \\ 0 & -\overline{k}_5\epsilon_k & \overline{k}_5\epsilon_k \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_9} = \begin{bmatrix} \overline{k}_6\epsilon \\ 0 \\ -\overline{k}_6 \end{bmatrix} \tag{107}$$

  and all other $\frac{\partial \mathbf{K}}{\partial x_i}$ are null matrices.
- Also note that all of the first-order derivative matrices are independent of $\mathbf{x}$. For this reason, all the higher order derivatives of the $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ matrices are null matrices.

A three DOF system

- We calculate the moments and the probability density functions of the three eigenvalues of the system. The following two sets of physically meaningful parameter values are considered:
  - *Case 1: All eigenvalues are well separated*
    For this case $\overline{m}_i = 1.0$ kg for $i = 1, 2, 3$; $\overline{k}_i = 1.0$ N/m for $i = 1, \cdots, 5$ and $k_6 = 3.0$ N/m.
  - *Case 2: Two eigenvalues are close*
    All parameter values are the same except $k_6 = 1.275$ N/m.
- The moments of the eigenvalues for the above two cases are calculated first.

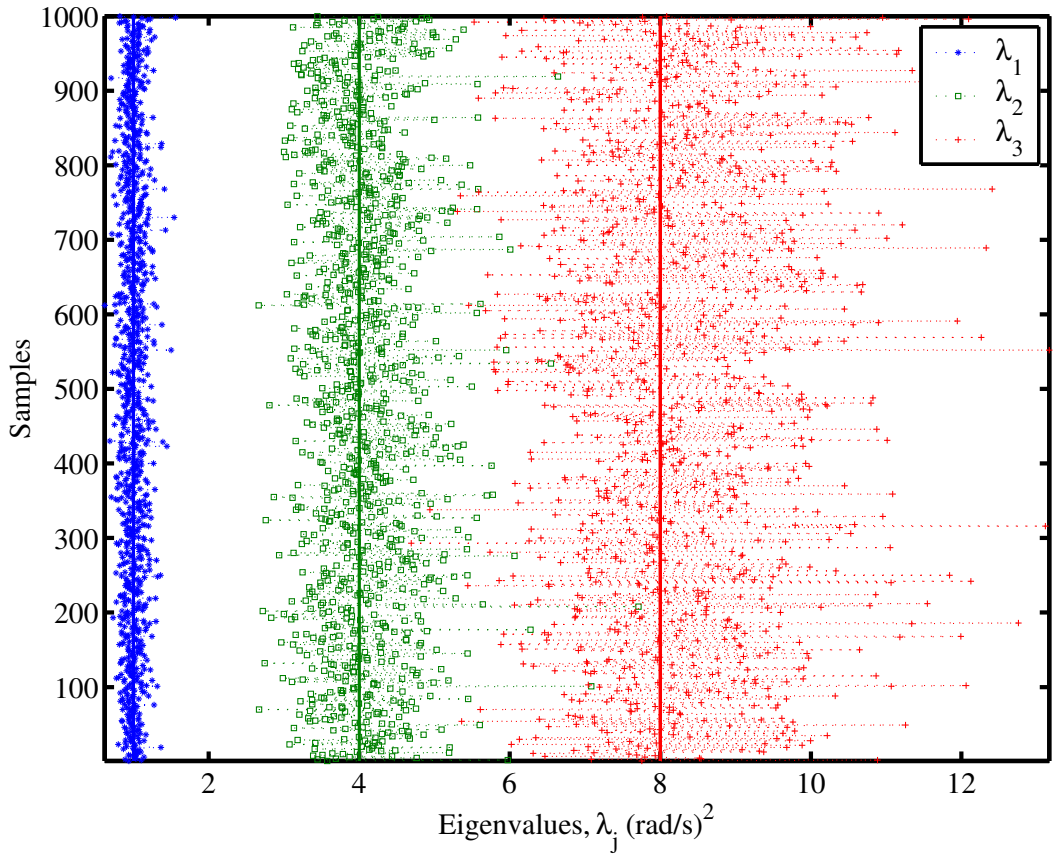
A three DOF system

- The moments are then used to obtain σ_j from equation (87) and the constants in equations (93)–(95).
- Using these constants the truncated Gaussian pdf and the χ^2 pdf of the eigenvalues are obtained from equations (86) and (96) respectively. These results are compared with Monte Carlo simulation.
- The samples of the nine independent Gaussian random variables $x_i, i = 1, \cdots, 9$ are generated and the eigenvalues are computed directly from equation (1). A total of 15000 samples are used to obtain the statistical moments and histograms of the pdf of the eigenvalues.
- The results obtained from Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the analytical methods.
- For the purpose of determining the accuracy, we again calculate the percentage error associated with an arbitrary *r*th moment using equation (102).

All eigenvalues are well separated

- When all of the eigenvalues are well separated their derivatives with respect to the system parameters generally behave well.
- For the given parameter values the eigenvalues of the corresponding deterministic system is given by

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4, \quad \text{and} \quad \overline{\lambda}_3 = 8. \tag{108}$$



All eigenvalues are well separated

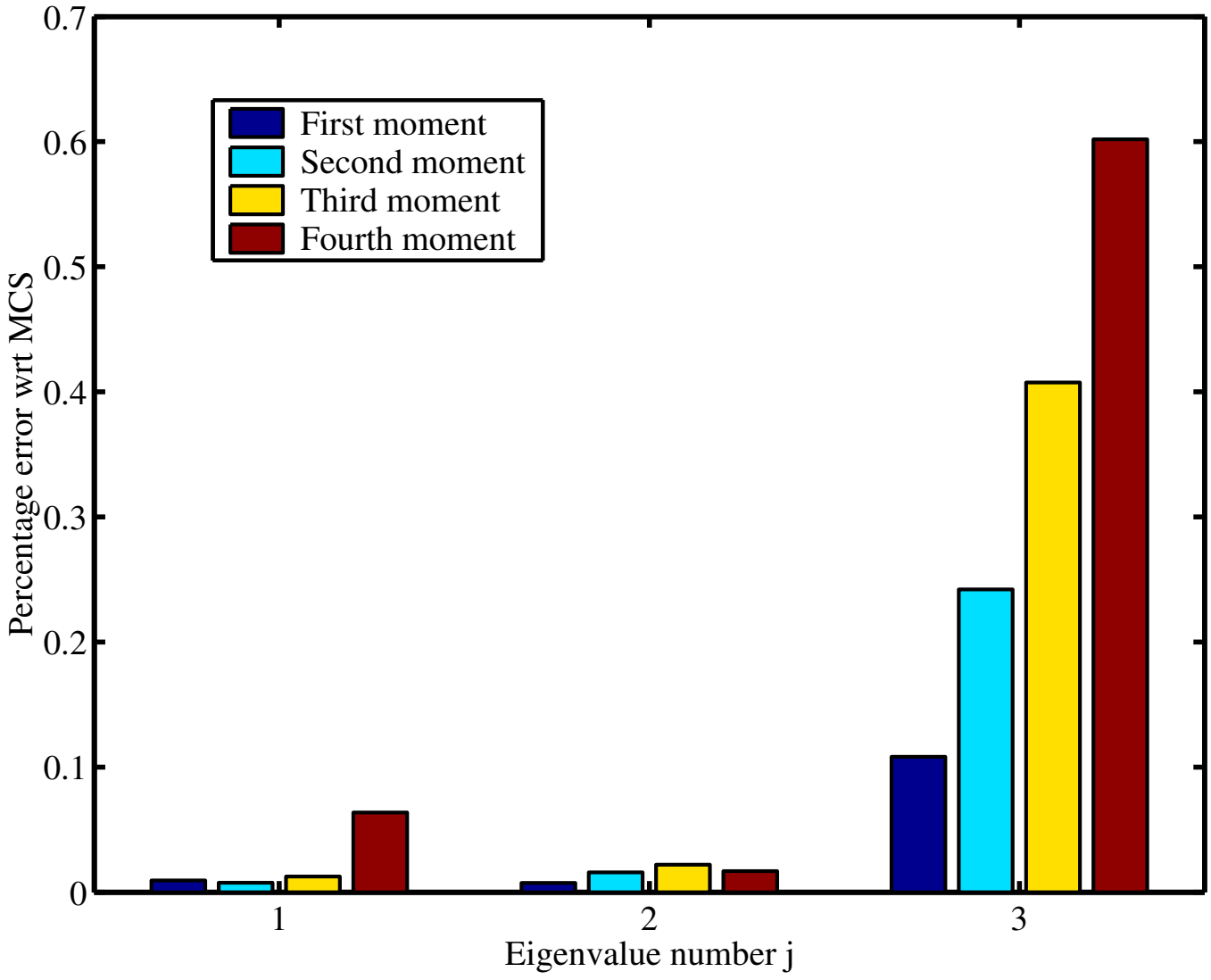


Figure : Percentage error for first four moments of the eigenvalues; Case 1.

All eigenvalues are well separated

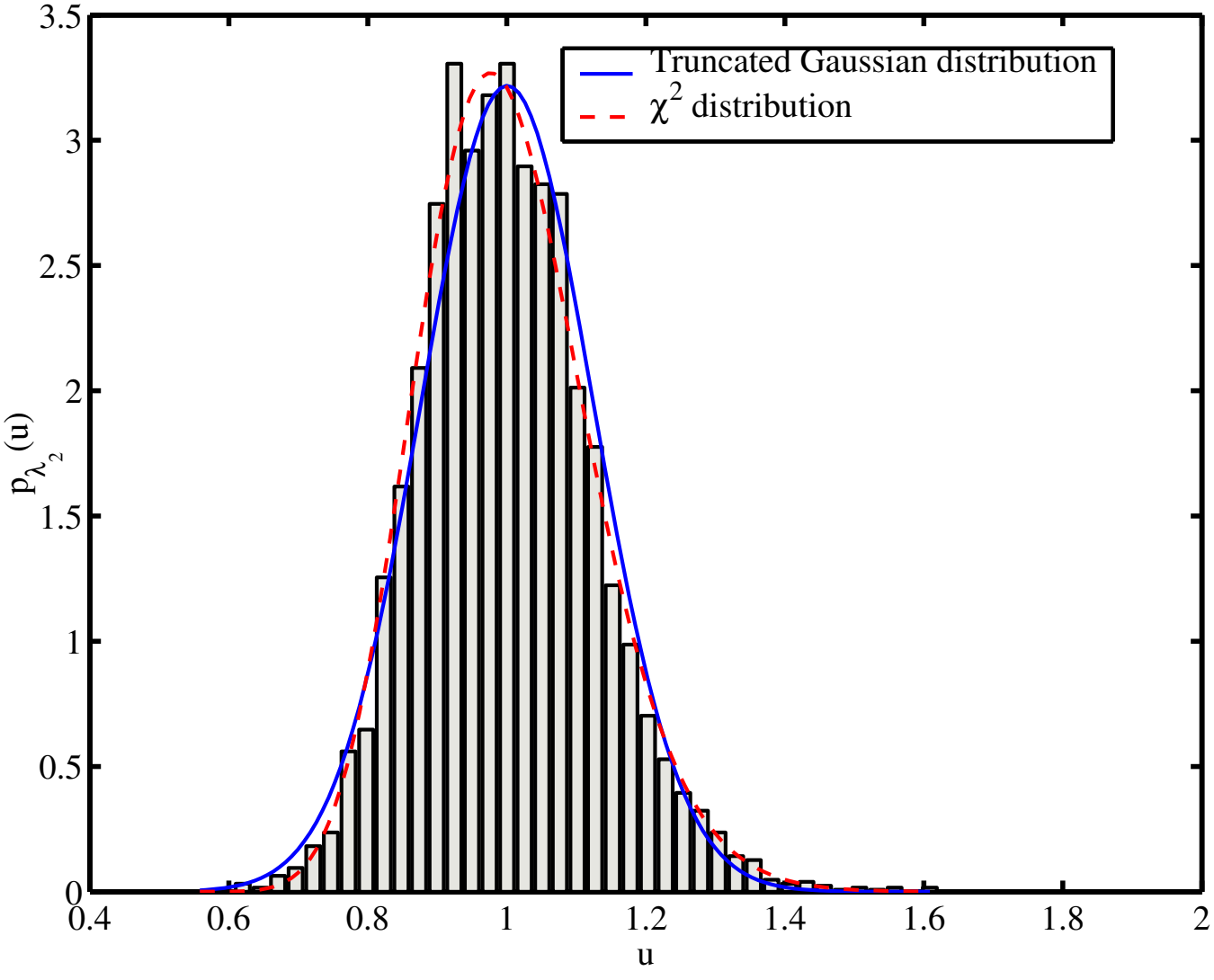


Figure : Probability density function of the first eigenvalue; Case 1.

All eigenvalues are well separated

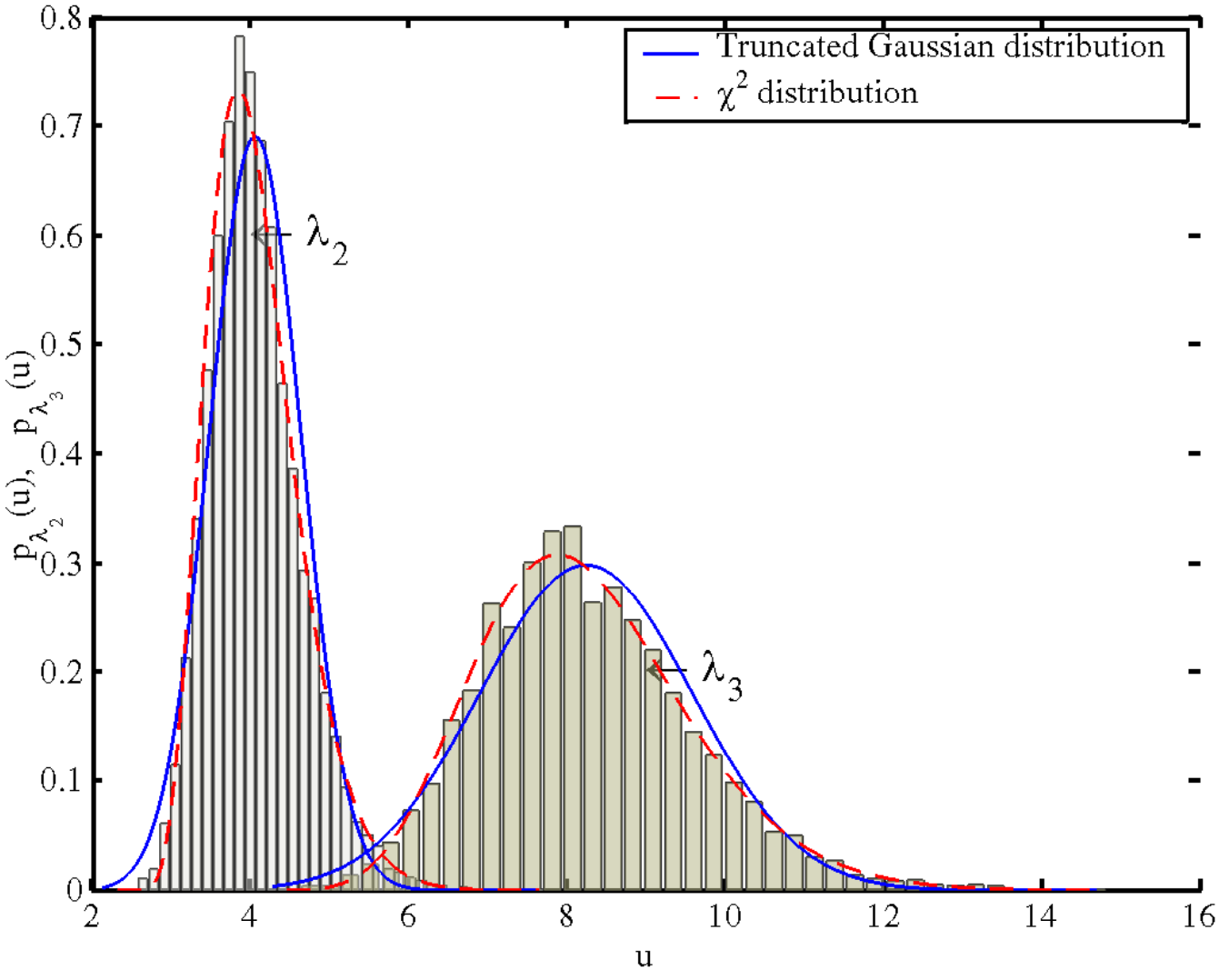


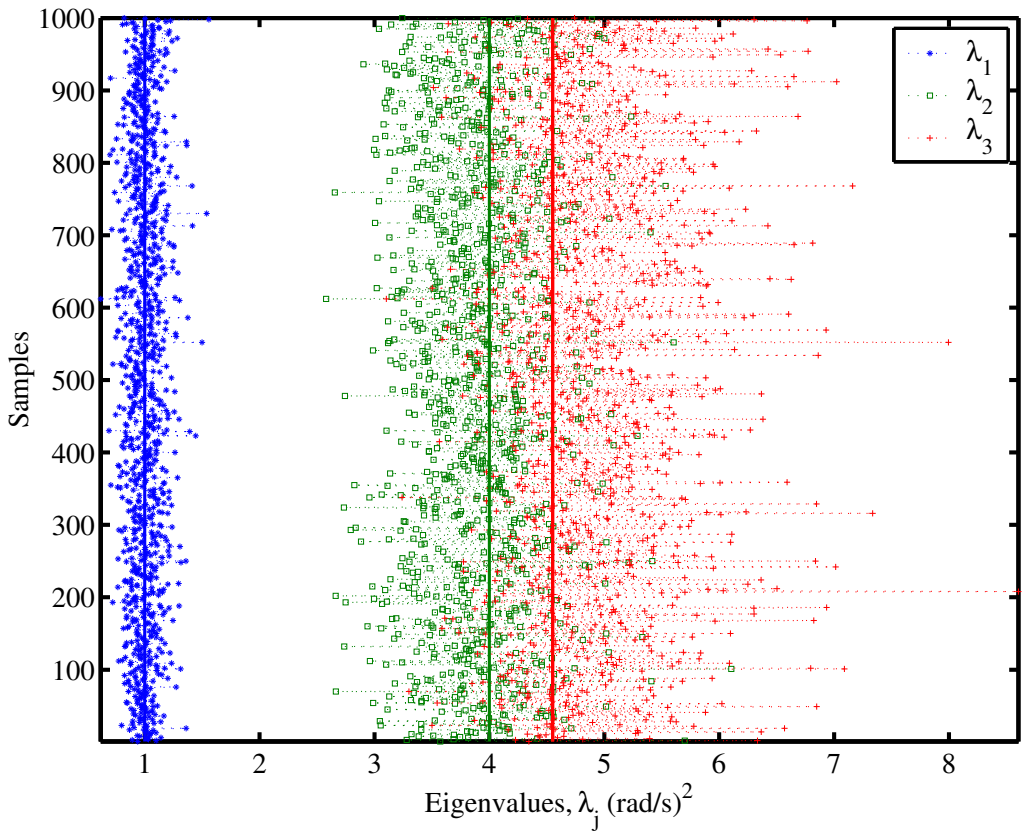
Figure : Probability density functions of the second and third eigenvalues; Case 1.

Two eigenvalues are close

- When some eigenvalues are closely spaced, their derivatives with respect to the system parameters may not behave well
- For the given parameter values the eigenvalues of the corresponding deterministic system are calculated as

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4, \quad \text{and} \quad \overline{\lambda}_3 = 4.55. \tag{109}$$

Clearly $\overline{\lambda}_2$ and $\overline{\lambda}_3$ are close to each other.



Two eigenvalues are close

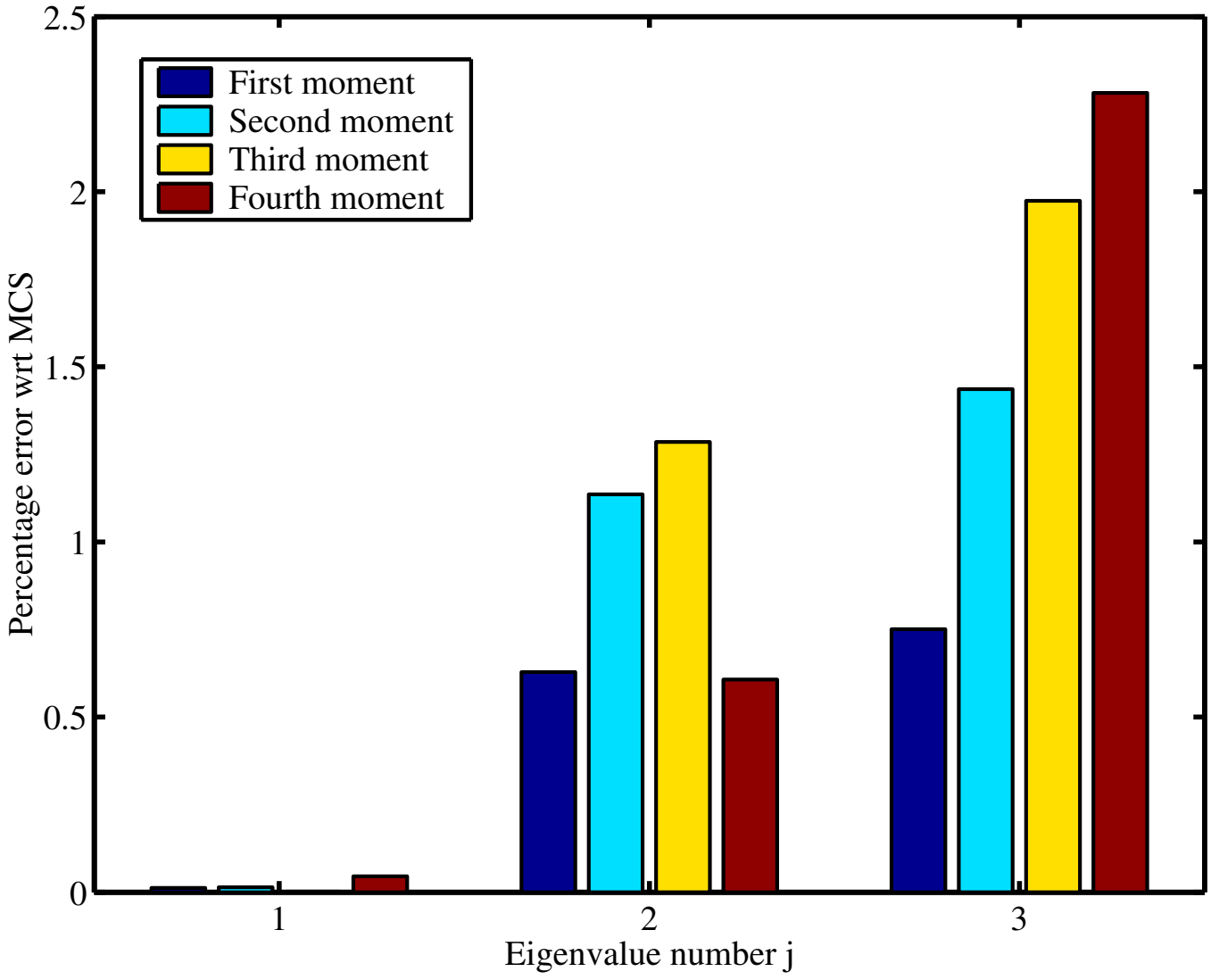


Figure : Percentage error for first four moments of the eigenvalues; Case 2.

Two eigenvalues are close

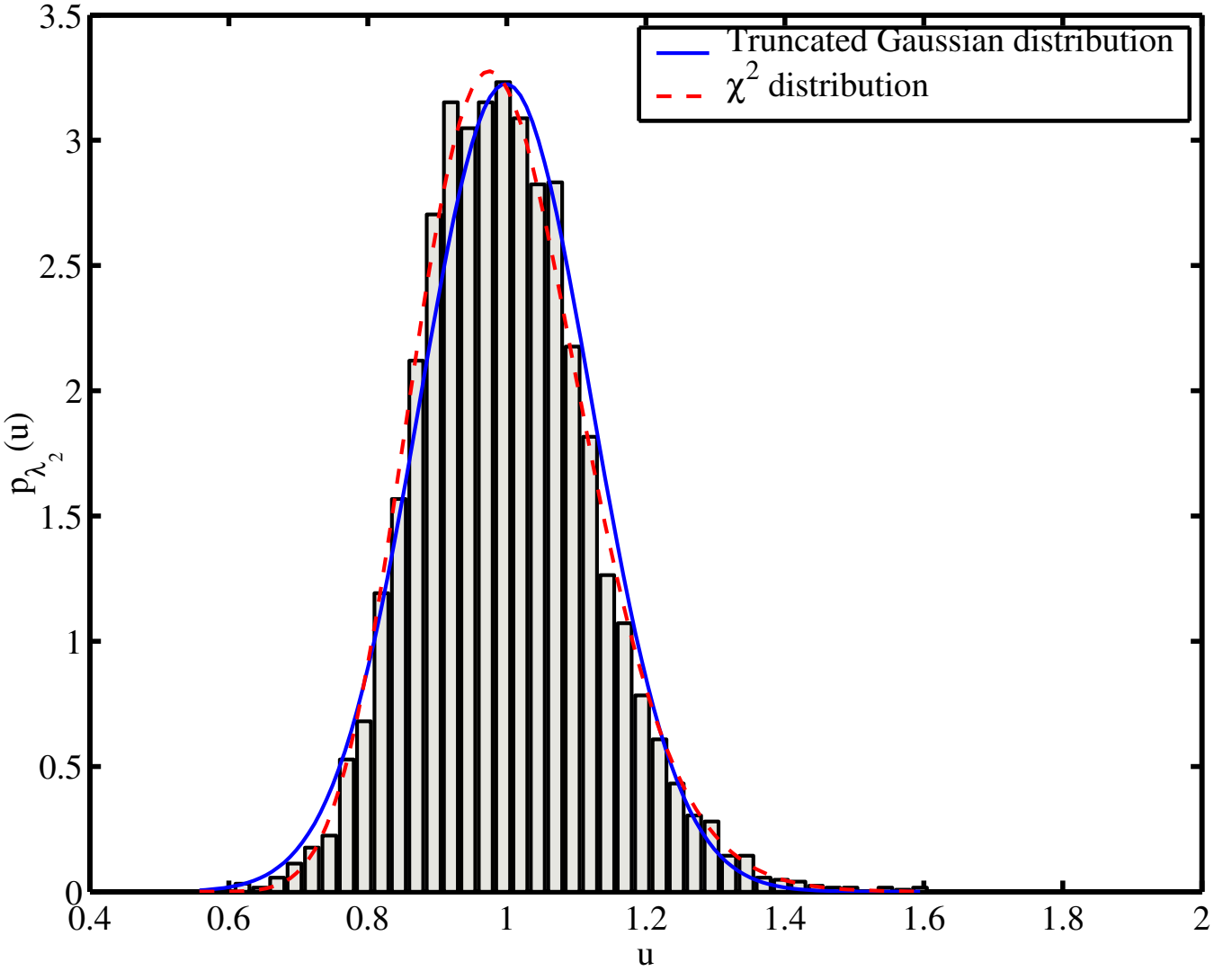


Figure : Probability density function of the first eigenvalue; Case 2.

Two eigenvalues are close

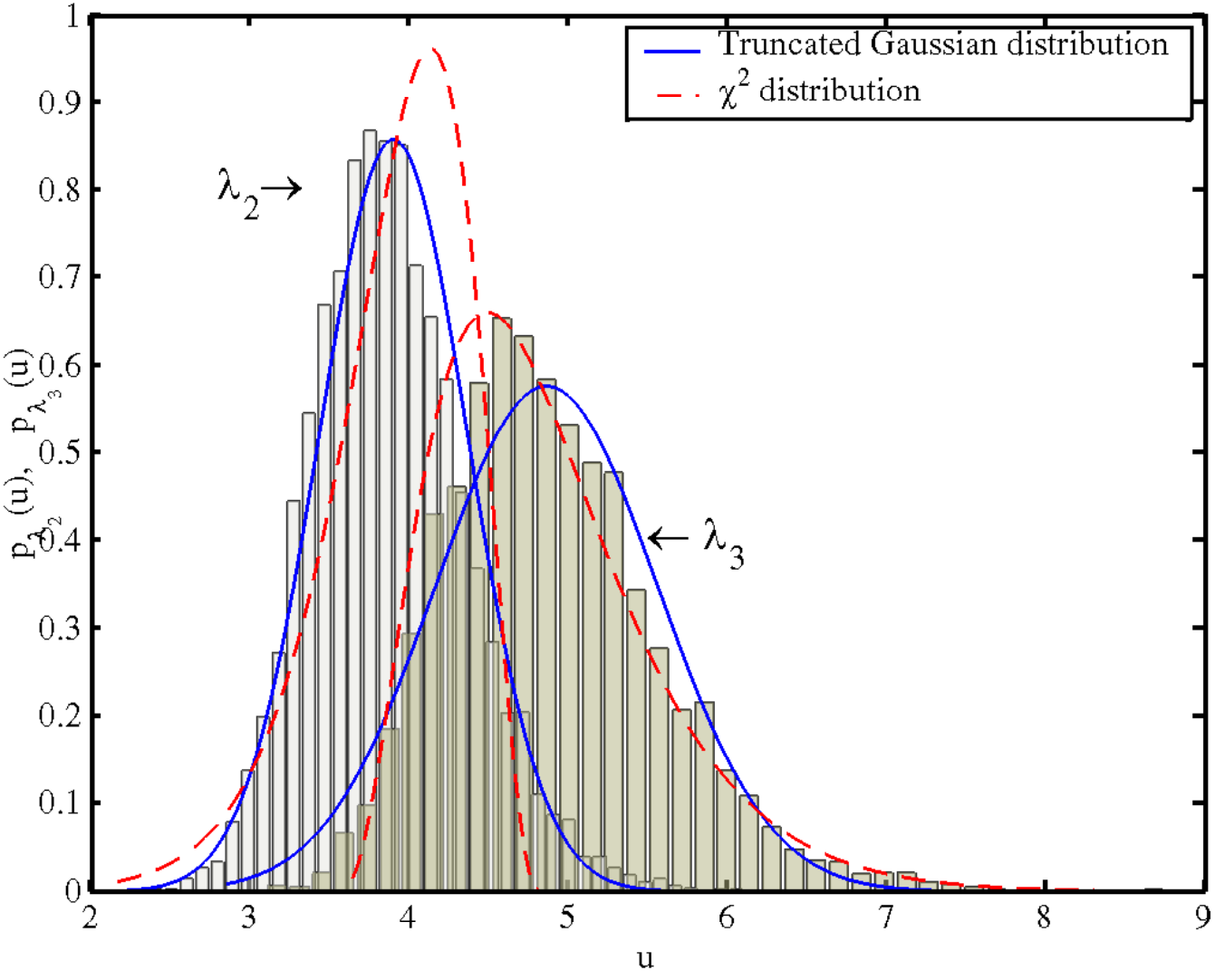


Figure : Probability density functions of the second and third eigenvalues; Case 2.

Stochastic Methods in Structural Dynamics
Part 4

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Overview of the course

The course is dived into eight topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

Outline of this talk

1. **Matrix variate distributions**
2. **Matrix distributions for system matrices**
   - Wishart random matrices
   - Parameter selection
   - Reduced computational modelling
3. **Identification of the dispersion parameters**
4. **Examples applications for random matrix theory**
5. **Experimental validations**
6. **Hybrid uncertainty formulations**
7. **Domain decomposition for multi-frequency scale problems**
   - Domain decomposition for two domains
   - Computational approach for uncertainty propagation
     - Stochastic interface problem
     - Stochastic interior problems
   - Numerical example

Introduction

- Broadly speaking, there are two complimentary approaches to quantify uncertainties in a model. The first is the *parametric approach* and the second is the *non-parametric approach*.
- In the parametric approach the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element.
- This type of approach is suitable to quantify aleatoric uncertainties.
- Epistemic uncertainty on the other hand do not explicitly depend on the systems parameters. For example, there can be unquantified errors associated with the equation of motion (linear on non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods (e.g, discretisation of displacement fields, truncation and roundoff errors, tolerances in the optimization and iterative algorithms, step-sizes in the time-integration methods).
- The parametric approach is not suitable to quantify this type of uncertainties and a non-parametric approach is needed for this purpose.
-

Non-parametric uncertainty

- In majority of practical problems, the complete information regarding uncertainties is not available.
- In some ceases, for example, cars manufactured from a production chain and soil property distribution in a construction site, it may be possible to obtain probabilistic descriptions of the system parameters experimentally.
- However, obtaining such probabilistic information may be prohibitively expensive for many problems.
- In another class of problems, for example, dynamic analysis of a space vehicle, even 'in principle' it may not be possible to obtain probabilistic information because there may be just 'only one sample'. However, there will still be some uncertainties in the model.
- Regardless of what type of uncertainties exist in the model of a linear dynamical system, it must be characterized by the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$.
- These We obtain the probability density function of the random matrices based on the maximum entropy principle.
- It will be shown that Wishart random matrix is the simplest physically realistic random matrix model for the system matrices appearing in linear structural dynamical systems.

Dynamics of a general linear system

The equation of motion is given by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty **M**, **C** and **K** become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector **q**
- We aim to derive the probability density function of the system matrices directly.

Random Matrix Method (RMM)

- Derive the matrix variate probability density functions of **M**, **C** and **K**[1] using available information.
- The main assumption is that a reliable model of the baseline system matrices is known.
- Additionally we need to assume that 'some information' on the dispersion of the system matrices from the 'mean' model is available.
- Once the distribution is identified, we propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)
- Several ways the parameters of the distribution can be identified.

[1] AIAA Journal, 45[7] (2007), pp. 1748-1762

Matrix variate distributions

- A random matrix can be considered as an observable phenomenon representable in the form of a matrix which under repeated observation yields different non-deterministic outcomes.
- Therefore, a random matrix is simply a collection of random variables which may satisfy certain rules (for example symmetry, positive definiteness etc). Random matrices were introduced by Wishart in 1928in the context of multivariate statistics.
- However, the Random Matrix Theory (RMT) was not used in other branches until 1950s whenWigner published his works (leading to the Nobel prize in Physics in 1963) on the eigenvalues of random matrices arising in high-energy physics.
- Using an asymptotic theory for large dimensional matrices, Wigner was able to bypass the Schrödinger equation and explain the statistics of measured atomic energy levels in terms of the limiting eigenvalues of these random matrices.
- Since then research on random matrices has continued to attract interests in multivariate statistics, physics, number theory and more recently in mechanical and electrical engineering.

Gaussian random matrix

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.
- Density of a random matrix is effectively the joint density function of all its elements
  The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \det\{\mathbf{\Sigma}\}^{-p/2} \det\{\mathbf{\Psi}\}^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (2)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Symmetric Gaussian random matrix

- Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ be a symmetric random matrix and $\mathbf{M}, \mathbf{\Sigma}$ and $\mathbf{\Psi}$ are $n \times n$ constant matrices such that the commutative relation $\mathbf{\Sigma\Psi} = \mathbf{\Psi\Sigma}$ holds. If the $n(n+1)/2 \times 1$ vector $\mathrm{vecp}\,(\mathbf{Y})$ formed from $\mathbf{Y}$ is distributed as $N_{n(n+1)/2,1}\left(\mathrm{vecp}\,(\mathbf{M}), \mathbf{B}_n^T(\mathbf{\Sigma}\otimes\mathbf{\Psi})\mathbf{B}_n\right)$, then $\mathbf{Y}$ is said to have a symmetric matrix variate Gaussian distribution with mean $\mathbf{M}$ and covariance matrix $\mathbf{B}_n^T(\mathbf{\Sigma}\otimes\mathbf{\Psi})\mathbf{B}_n$ and its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4}\det\left\{\mathbf{B}_n^T(\mathbf{\Sigma}\otimes\mathbf{\Psi})\mathbf{B}_n\right\}^{-1/2}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{Y} \tag{3}$$

- This distribution is usually denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}\left(\mathbf{M}, \mathbf{B}_n^T(\mathbf{\Sigma}\otimes\mathbf{\Psi})\mathbf{B}_n\right)$.
- For a symmetric matrix $\mathbf{Y} \in \mathbb{R}^{n\times n}$, $\mathrm{vecp}\,(\mathbf{Y})$ is a $n(n+1)/2$-dimensional column vector formed from the elements above and including the diagonal of $\mathbf{Y}$ taken columnwise. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2\times n(n+1)/2}$ are given by

$$(B_n)_{ij,gh} = \frac{1}{2}\left(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}\right), \quad i \le n, j \le n, g \le h \le n \tag{4}$$

where δ_{ij} is the usual Kronecker's delta.

Wishart matrix

- A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2} p \right) \det \{\mathbf{\Sigma}\}^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr} \left\{ -\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{S} \right\} \quad (5)$$

  This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.
  Note: If $p = n + 1$, then the matrix is non-negative definite.
- Wishart distribution is *the* most important distribution for structural dynamics due to it symmetry and nonnegative definite property.

Matrix variate gamma distribution

- A $n \times n$ symmetric positive definite random matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \det\{\mathbf{\Psi}\}^{-a}\right\}^{-1} \det\{\mathbf{W}\}^{a-\frac{1}{2}(n+1)} \operatorname{etr}\{-\mathbf{\Psi}\mathbf{W}\};$$
$$\Re(a) > \frac{1}{2}(n-1) \quad (6)$$

  This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$.
- Comparing this distribution with the Wishart distribution, we have $G_n(a, \mathbf{\Psi}) = W_n(2a, \mathbf{\Psi}^{-1}/2)$. The main difference between the gamma and the Wishart distribution is that originally only integer values were considered for the shape parameter p in the Wishart distribution.
- From an analytical point of view the gamma and the Wishart distributions are identical.

Distribution of the system matrices

The distribution of the random system matrices **M**, **C** and **K** should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\, \omega$. This ensures that the moments of the response exist for all frequency values.

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\mathbf{M}_0$, $\mathbf{C}_0$ and $\mathbf{K}_0$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{7}$$

$$\text{and} \quad \int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \mathbf{G}_0 \quad \text{(the mean matrix)} \tag{8}$$

Further constraints

- Suppose that the inverse moments up to order ν of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}} = \left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^{T}\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln \det\left\{\mathbf{G}\right\}^{-\nu}\right] < \infty$$

- We extend the maximum entropy principle to matrix variate distribution to obtain the density of the matrices
- This requires calculus of variation on matrix quantities

Maximum entropy distribution

- The Lagrangian becomes:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\}\, d\mathbf{G} + \\ \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right)\, d\mathbf{G} - 1\right) - \nu \int_{\mathbf{G}>0} \ln\det\left\{\mathbf{G}\right\}\, p_{\mathbf{G}}\, d\mathbf{G} \\ + \operatorname{Trace}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}\left(\mathbf{G}\right)\, d\mathbf{G} - \mathbf{G}_0\right]\right) \quad (9)$$

  Note: ν cannot be obtained uniquely!

- Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

$$\text{or } -\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \operatorname{Trace}\left(\mathbf{\Lambda}_1\mathbf{G}\right) - \ln\det\left\{\mathbf{G}\right\}^{\nu}$$

$$\text{or } p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\}\det\left\{\mathbf{G}\right\}^{\nu}\operatorname{etr}\left\{-\mathbf{\Lambda}_1\mathbf{G}\right\}$$

Maximum entropy distribution

- Using the matrix variate Laplace transform
  ($\mathbf{T} \in \mathbb{R}_{n,n}, \mathbf{S} \in \mathbb{C}_{n,n}, a > (n+1)/2$)

$$\int_{\mathbf{T}>0} \operatorname{etr}\{-\mathbf{ST}\} \det\{\mathbf{T}\}^{a-(n+1)/2} \, d\mathbf{T} = \Gamma_n(a) \det\{\mathbf{S}\}^{-a}$$

  and substituting $p_{\mathbf{G}}(\mathbf{G})$ into the constraint equations it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = r^{-nr} \{\Gamma_n(r)\}^{-1} \det\{\mathbf{G}_0\}^{-r} \det\{\mathbf{G}\}^{\nu} \operatorname{etr}\left\{-r\mathbf{G}_0{}^{-1}\mathbf{G}\right\} \qquad (10)$$

  where $r = \nu + (n+1)/2$.
- Comparing it with the Wishart distribution we have: If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\mathbf{G}_0$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\mathbf{\Sigma} = \mathbf{G}_0/(2\nu + n + 1)$, that is $\mathbf{G} \sim W_n(2\nu + n + 1, \mathbf{G}_0/(2\nu + n + 1))$.

Properties of the distribution

- Covariance tensor of **G**:

$$\operatorname{cov}\left(G_{ij}, G_{kl}\right) = \frac{1}{2\nu + n + 1}\left(G_{0_{ik}} G_{0_{jl}} + G_{0_{il}} G_{0_{jk}}\right)$$

- Normalized standard deviation matrix

$$\delta_G^2 = \frac{\mathrm{E}\left[\left\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}}^2\right]}{\left\|\mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}}^2} = \frac{1}{2\nu + n + 1}\left\{1 + \frac{\left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2}{\mathrm{Trace}\left(\mathbf{G}_0{}^2\right)}\right\}$$

- $\delta_G^2 \leq \dfrac{1+n}{2\nu + n + 1}$ and $\nu \uparrow \Rightarrow \delta_{\mathbf{G}}^2 \downarrow$.

Wishart random matrix approach

- Suppose we 'know' (e.g, by measurements or stochastic finite element modeling) the mean ($\mathbf{G}_0$) and the (normalized) standard deviation (δ_G) of the system matrices:

$$\delta_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{11}$$

  This is known as the *dispersion parameter*.
- The parameters of the Wishart distribution can be identified using the expressions derived before.
- Samples from the Wishart distribution can be drawn and MCS can be sued to obtain system response and eigensolutions.
- We consider some strategies for reduced computational approaches.

Stochastic dynamic response

- Taking the Laplace transform of the equation of motion:

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{12}$$

  The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\phi_j = \omega_j^2 \mathbf{M}\phi_j, \quad j = 1, 2, \ldots, n \tag{13}$$

  where ω_j^2 and ϕ_j are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- We form the truncated undamped modal matrices $m \leq n$

$$\mathbf{\Omega} = \operatorname{diag}\left[\omega_1, \omega_2, \ldots, \omega_m\right] \quad \text{and} \quad \mathbf{\Phi} = \left[\phi_1, \phi_2, \ldots, \phi_m\right]. \tag{14}$$

$$\text{so that} \quad \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}^2 \quad \text{and} \quad \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}_m \tag{15}$$

Stochastic dynamic response

- Transforming it into the reduced modal coordinates:

$$\left[s^2\mathbf{I}_m + s\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{16}$$

- Here

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = 2\zeta\mathbf{\Omega}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{17}$$

- When we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ will be a random matrix of dimension m. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{m\times m}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{18}$$

Stochastic dynamic response

- Since $\boldsymbol{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\boldsymbol{\Psi}_r$ such that

$$\boldsymbol{\Psi}_r^T \boldsymbol{\Xi} \boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \tag{19}$$

  Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\boldsymbol{\Xi}$.
- Recalling that $\boldsymbol{\Psi}_r^T \boldsymbol{\Psi}_r = \mathbf{I}_m$ we obtain

$$\bar{\mathbf{q}}' = \left[s^2 \mathbf{I}_m + s\mathbf{C}' + \boldsymbol{\Omega}^2 \right]^{-1} \bar{\mathbf{f}}' \tag{20}$$

$$= \boldsymbol{\Psi}_r \left[s^2 \mathbf{I}_m + 2s\zeta\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2 \right]^{-1} \boldsymbol{\Psi}_r^T \bar{\mathbf{f}}' \tag{21}$$

Stochastic dynamic response

- The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \mathbf{\Phi}\bar{\mathbf{q}}'(s) = \mathbf{\Phi}\mathbf{\Psi}_r \left[s^2\mathbf{I}_n + 2s\zeta\mathbf{\Omega}_r + \mathbf{\Omega}_r^2\right]^{-1} (\mathbf{\Phi}\mathbf{\Psi}_r)^T \bar{\mathbf{f}}(s)$$
$$= \sum_{j=1}^{m} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j}.$$

  Here

$$\mathbf{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_m}\right], \quad \mathbf{X}_r = \mathbf{\Phi}\mathbf{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}\right]$$

  are respectively the matrices containing random eigenvalues and eigenvectors of the system.

- Conventional modal truncation has been applied to reduce the system. This will lead to a smaller random eigenvalue problem to be solved.

Parameter-selection of Wishart matrices

Approach 1: M and K are fully correlated Wishart (most complex). For this case $\mathbf{M} \sim W_n(p_M, \mathbf{\Sigma}_M)$, $\mathbf{K} \sim W_n(p_K, \mathbf{\Sigma}_K)$ with $\mathrm{E}[\mathbf{M}] = \mathbf{M}_0$ and $\mathrm{E}[\mathbf{K}] = \mathbf{K}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices.
Here

$$\mathbf{\Sigma}_M = \mathbf{M}_0/p_M, p_M = \frac{\gamma_M + 1}{\delta_M} \tag{22}$$

$$\text{and} \quad \mathbf{\Sigma}_K = \mathbf{K}_0/p_K, p_K = \frac{\gamma_K + 1}{\delta_K} \tag{23}$$

$$\gamma_G = \{\mathrm{Trace}(\mathbf{G}_0)\}^2/\mathrm{Trace}\left(\mathbf{G}_0{}^2\right) \tag{24}$$

Approach 2: Scalar Wishart (most simple) In this case it is assumed that

$$\Xi \sim W_m\left(p, \frac{a^2}{n}\mathbf{I}_m\right) \tag{25}$$

Considering $\mathrm{E}[\Xi] = \mathbf{\Omega}_0^2$ and $\delta_{\Xi} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1+\gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \mathrm{Trace}\left(\mathbf{\Omega}_0^2\right)/p \tag{26}$$

Parameter-selection of Wishart matrices

Approach 3: Diagonal Wishart with different entries (something in the middle). For this case $\boldsymbol{\Xi} \sim W_m\left(p, \boldsymbol{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\boldsymbol{\Xi}^{-1}\right] = \boldsymbol{\Omega}_0^{-2}$ and $\delta_{\boldsymbol{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem.
The parameters can be obtained as

$$p = m + 1 + \theta \quad \text{and} \quad \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1) \tag{27}$$

Parameter-selection of Wishart matrices

- Defining $\mathbf{H}_0 = {\mathbf{M}_0}^{-1}\mathbf{K}_0$, the constant γ_H:

$$\gamma_H = \frac{\{\mathrm{Trace}\,(\mathbf{H}_0)\}^2}{\mathrm{Trace}\left({\mathbf{H}_0}^2\right)} = \frac{\left\{\mathrm{Trace}\left(\mathbf{\Omega}_0^2\right)\right\}^2}{\mathrm{Trace}\left(\mathbf{\Omega}_0^4\right)} = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{28}$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\begin{aligned}\delta_H^2 = &\frac{\left({p_M}^2 + (p_K - 2 - 2n)\,p_M + (-n-1)\,p_K + n^2 + 1 + 2n\right)\gamma_H}{p_K\,(-p_M + n)\,(-p_M + n + 3)} \\ &+ \frac{{p_M}^2 + (p_K - 2n)\,p_M + (1-n)\,p_K - 1 + n^2}{p_K\,(-p_M + n)\,(-p_M + n + 3)}\end{aligned} \tag{29}$$

Summary of the method

A step-by-step method for implementing the new computational approach in conjunction with any general purpose finite element software is given below:

- Form the deterministic mass and stiffness matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ using the standard finite element method and the modal damping factors ζ_j. Select the number of modes $m < n$. The number of modes to be retained, m should be selected based on the frequency of excitation.
- Solve the deterministic undamped eigenvalue problem

$$\mathbf{K}_0 \phi_{0_j} = \omega_{0_j}^2 \mathbf{M}_0 \phi_{0_j}, \quad j = 1, 2, \ldots, m \tag{30}$$

and create the matrix

$$\mathbf{\Phi}_0 = \left[\phi_{0_1}, \phi_{0_2}, \ldots, \phi_{0_m}\right] \in \mathbb{R}^{n \times m} \tag{31}$$

Calculate the ratio

$$\gamma_H = \left(\sum_{j=1}^{m} \omega_{0_j}^2\right)^2 / \sum_{j=1}^{m} \omega_{0_j}^4 \tag{32}$$

- Obtain the dispersion parameters δ_M and δ_K corresponding to the mass and stiffness matrices. This can be obtained from physical or computer experiments.

Summary of the method

- Obtain the dispersion parameter of the generalized Wishart matrix **H**

$$\delta_H^2 = \frac{\left({p_M}^2 + \left(p_K - 2 - 2n\right) p_M + \left(-n-1\right) p_K + n^2 + 1 + 2n\right) \gamma_H}{p_K \left(-p_M + n\right) \left(-p_M + n + 3\right)} + \frac{{p_M}^2 + \left(p_K - 2n\right) p_M + \left(1-n\right) p_K - 1 + n^2}{p_K \left(-p_M + n\right) \left(-p_M + n + 3\right)} \tag{33}$$

where

$$p_M = \frac{1}{\delta_M^2} \left\{ 1 + \left\{ \text{Trace}\left(\mathbf{M}_0\right) \right\}^2 / \text{Trace}\left({\mathbf{M}_0}^2 \right) \right\} \tag{34}$$

$$\text{and} \quad p_K = \frac{1}{\delta_K^2} \left\{ 1 + \left\{ \text{Trace}\left(\mathbf{K}_0\right) \right\}^2 / \text{Trace}\left({\mathbf{K}_0}^2 \right) \right\} \tag{35}$$

- Calculate the parameters

$$\theta = \frac{\left(1 + \gamma_H\right)}{\delta_H^2} - \left(m + 1\right) \quad \text{and} \quad p = \left[m + 1 + \theta\right] \tag{36}$$

where p is approximated to the nearest integer of $m + 1 + \theta$.

Summary of the method

- Create an $m \times p$ matrix $\mathbf{Y}$ such that

$$Y_{ij} = \omega_{0_i} \widehat{Y}_{ij} / \sqrt{\theta}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, p \tag{37}$$

  where $\widehat{Y}_{ij}$ are independent and identically distributed (i.i.d.) Gaussian random numbers with zero mean and unit standard deviation.

- Simulate the $m \times m$ Wishart random matrix

$$\boldsymbol{\Xi} = \mathbf{Y}\mathbf{Y}^T \quad \text{or} \quad \Xi_{ij} = \frac{\omega_{0_i}\omega_{0_j}}{\theta} \sum_{k=1}^{p} \widehat{Y}_{ik}\widehat{Y}_{jk}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, m \tag{38}$$

  Since $\boldsymbol{\Xi}$ is symmetric, only the upper or lower triangular part need to be simulated.

- Solve the symmetric eigenvalue problem ($\boldsymbol{\Omega}_r, \boldsymbol{\Psi}_r \in \mathbb{R}^{m \times m}$) for every sample

$$\boldsymbol{\Xi}\boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \boldsymbol{\Psi}_r \tag{39}$$

  and obtain the random eigenvector matrix

$$\mathbf{X}_r = \boldsymbol{\Phi}_0 \boldsymbol{\Psi}_r = [\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_m}] \in \mathbb{R}^{n \times m} \tag{40}$$

Summary of the method

- Finally calculate the dynamic response in the frequency domain as

$$\bar{\mathbf{q}}_r(\mathrm{i}\omega) = \sum_{j=1}^{m} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{-\omega^2 + 2\mathrm{i}\omega\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j} \tag{41}$$

The samples of the response in the time domain can also be obtained from the random eigensolutions as

$$\mathbf{q}_r(t) = \sum_{j=1}^{m} a_{r_j}(t)\mathbf{x}_{r_j},$$

$$\text{where } a_{r_j}(t) = \frac{1}{\omega_{r_j}} \int_0^t \mathbf{x}_{r_j}^T \mathbf{f}(\tau) e^{-\zeta_j\omega_{r_j}(t-\tau)} \sin\left(\omega_{r_j}(t-\tau)\right) d\tau \tag{42}$$

Identification of the dispersion parameters - 1

- The dispersion parameter is related to the first and second moments of eigenvalues

$$\delta_G^2 = \frac{\sum_{j=1}^{n} \mathrm{E}\left[\lambda_j^2\right]}{\sum_{j=1}^{n} \mathrm{E}\left[\lambda_j\right]^2} - 1 \tag{43}$$

  so that, if information on the eigenvalues of the system is available, the dispersion parameter can be retrieved.
- Suppose the standard deviation of each eigenvalue is σ_j and mean of each eigenvalue is $\overline{\lambda}_j$. Therefore $\mathrm{E}\left[\lambda_j\right] = \overline{\lambda}_j$ and $\mathrm{E}\left[\lambda_j^2\right] = \sigma_j^2 + \overline{\lambda}_j^2$
- Suppose the standard deviation is expressed as a fraction of the respective mean values $\sigma_j = \epsilon_j \overline{\lambda}_j$.
- Using these and applying modal truncation, from Eq. (43) we have

$$\delta_G^2 = \frac{\sum_{j=1}^{m} \overline{\lambda}_j^2 \epsilon_j^2}{\sum_{j=1}^{m} \overline{\lambda}_j^2} \tag{44}$$

- This can be measured from experiments.

Identification of the dispersion parameters- 1

- The dispersion parameter for the mass matrix can be obtained as

$$p_M = \frac{(n+1)\sum_{j=1}^{n} \mathrm{E}\left[\lambda_j\right]}{\sum_{j=1}^{n} \mathrm{E}\left[\lambda_j\right] - \mathrm{Trace}\left((\overline{\mathbf{M}})^{-1/2}\overline{\mathbf{K}}(\overline{\mathbf{M}})^{-1/2}\right)} \tag{45}$$

- The dispersion parameter for the stiffness matrix can be obtained as

$$p_K = \frac{(p_M - n - 1)^3 T_1^2 + (p_M - n + 3)(T_2)}{(n_1 - p_M) T_1^2 + (p_M - n) T_2((p_M - n)(p_M - n - 3)(\delta_G^2 - 1) - 1)} \tag{46}$$

with

$$T_1 = \mathrm{Trace}\left((\overline{\mathbf{M}})^{-1}\overline{\mathbf{K}}\right) \quad \text{and} \quad T_2 = \mathrm{Trace}\left(((\overline{\mathbf{M}})^{-1}\overline{\mathbf{K}})^2\right). \tag{47}$$

Identification of the dispersion parameters- 2

- We assume that the matrix $\mathbf{G}(\theta) = \{\mathbf{K}(\theta), \mathbf{M}(\theta)\}$ can be expanded as

$$\mathbf{G}(\theta) = \mathbf{G}_0 + \epsilon_G \sum_{j=1}^{M} \xi_{G_j}(\theta)\mathbf{G}_j \tag{48}$$

- Substituting this expansion in the expression of the dispersion parameter one has

$$\delta_G^2 = \frac{\mathrm{E}\left[\left\| \epsilon_G \sum_{j=1}^{M} \xi_{G_j}(\theta)\mathbf{G}_j \right\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2} \tag{49}$$

- Because the matrices $\mathbf{G}_j$ are symmetric, using the definition of Frobenius norm, from Eq. (49) we have

$$\delta_G^2 = \frac{\mathrm{E}\left[\mathrm{Trace}\left((\epsilon_G \sum_{j=1}^{M} \xi_{G_j}(\theta)\mathbf{G}_j)(\epsilon_G \sum_{k=1}^{M} \xi_{G_k}(\theta)\mathbf{G}_k)\right)\right]}{\|\mathbf{G}_0\|_{\mathrm{F}}^2} \tag{50}$$

Identification of the dispersion parameters- 2

- Since both trace and expectation operators are linear they can be swaped. Doing this we obtain

$$\delta_G^2 = \frac{\epsilon_G^2 \mathrm{Trace}\left(\mathrm{E}\left[\left(\sum_{j=1}^{M}\sum_{k=1}^{M}\xi_{G_j}(\theta)\xi_{G_k}(\theta)\mathbf{G}_j\mathbf{G}_k\right)\right]\right)}{\|\mathbf{G}_0\|_{\mathrm{F}}^2} \tag{51}$$

- Recalling that the matrices $\mathbf{G}_j$ are not random and $\{\xi_{G_1}(\theta), \xi_{G_2}(\theta), \dots\}$ is a set of uncorrelated random variables with zero mean and $\mathrm{E}\left[\xi_{G_j}(\theta)\xi_{G_k}(\theta)\right] = \delta_{jk}$, we have

$$\begin{aligned}\delta_G^2 &= \frac{\epsilon_G^2 \mathrm{Trace}\left(\left(\sum_{j=1}^{M}\sum_{k=1}^{M}\mathrm{E}\left[\xi_{G_j}(\theta)\xi_{G_k}(\theta)\right]\mathbf{G}_j\mathbf{G}_k\right)\right)}{\|\mathbf{G}_0\|_{\mathrm{F}}^2} \\ &= \frac{\epsilon_G^2 \mathrm{Trace}\left(\left(\sum_{j=1}^{M}\mathbf{G}_j^2\right)\right)}{\|\mathbf{G}_0\|_{\mathrm{F}}^2}\end{aligned} \tag{52}$$

Identification of the dispersion parameters- 2

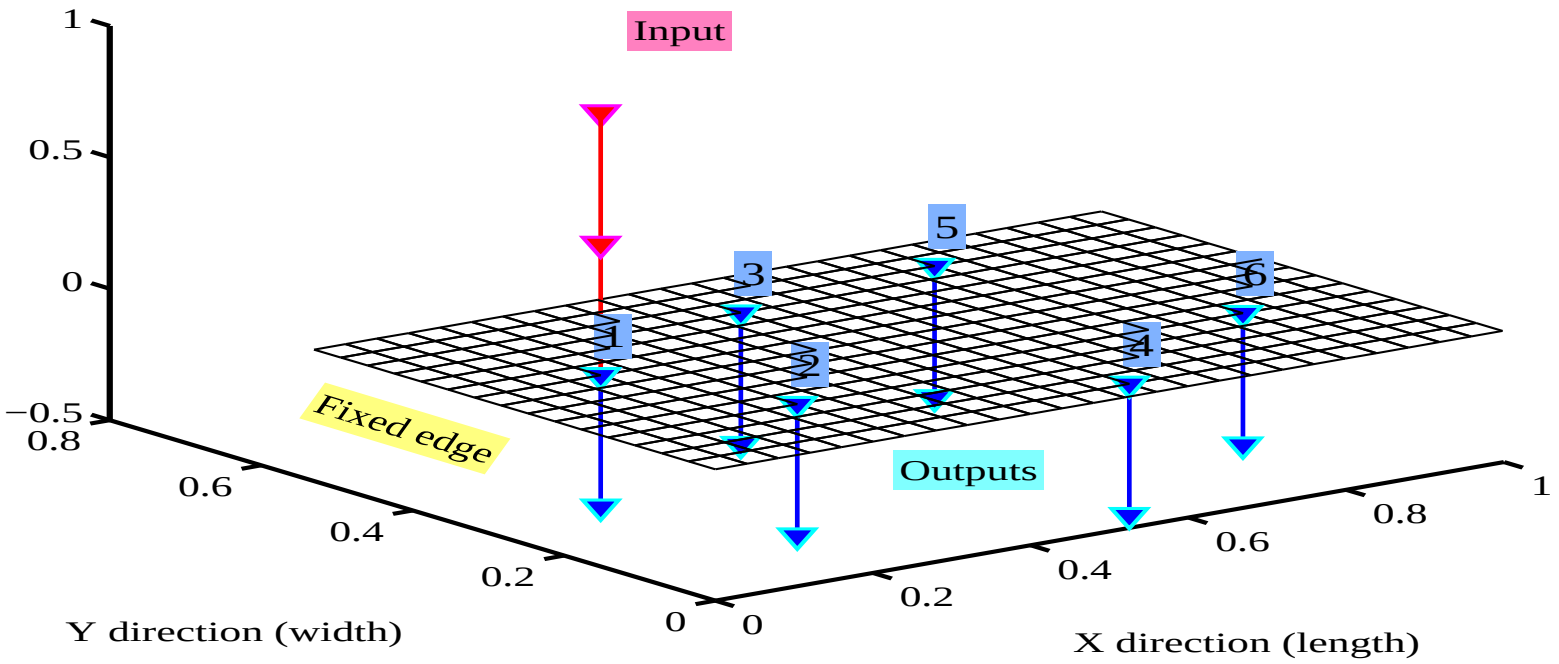
- In the above expressions, the uncorrelated nature of the random variables allowed us to transform the double summation into a single summation. Finally, as trace and sum operators can be interchanged and $\mathrm{Trace}\left(\mathbf{G}_j^2\right) = \left\|\mathbf{G}_j\right\|_{\mathrm{F}}^2$, we have

$$\delta_G^2 = \epsilon_G^2 \frac{\sum_j^M \left\|(\mathbf{G}_j)\right\|_{\mathrm{F}}^2}{\left\|\mathbf{G}_0\right\|_{\mathrm{F}}^2} \tag{53}$$

  This result allows one to obtain the dispersion parameter using the stochastic finite element, therefore avoiding the direct Monte Carlo simulation. This expression also relates the stochastic finite element and random matrix theory.

- δ_M and δ_K obtained in this way can be used in (34) and (35) for the simulation of the random matrices.

A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/$\sim$adhikaris/uq/.

Uncertainty type 1: random fields

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\,(1 + \epsilon_E f_1(\mathbf{x})) \tag{54}$$

$$\mu(\mathbf{x}) = \bar{\mu}\,(1 + \epsilon_\mu f_2(\mathbf{x})) \tag{55}$$

$$\rho(\mathbf{x}) = \bar{\rho}\,(1 + \epsilon_\rho f_3(\mathbf{x})) \tag{56}$$

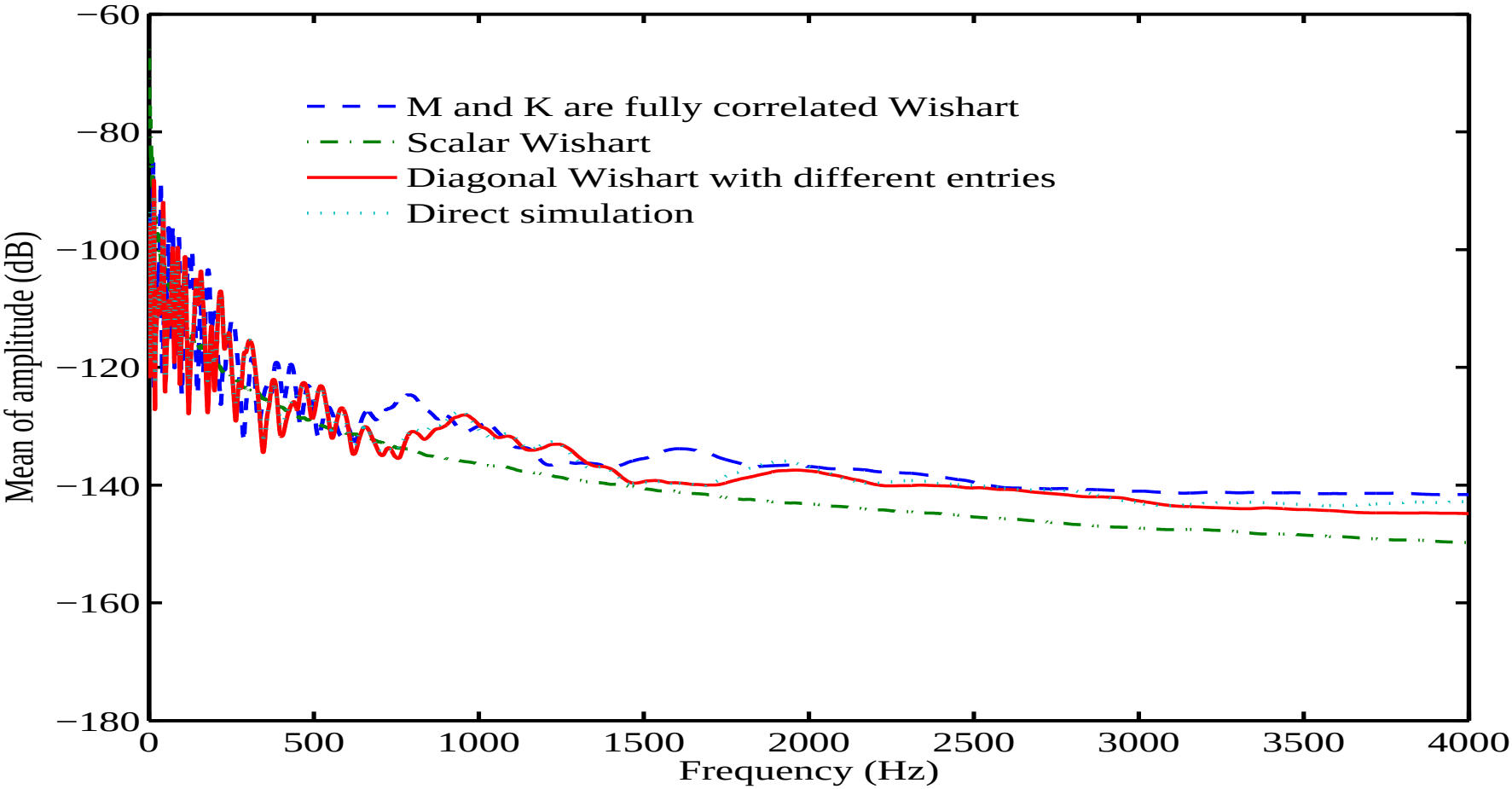
$$\text{and} \quad t(\mathbf{x}) = \bar{t}\,(1 + \epsilon_t f_4(\mathbf{x})) \tag{57}$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

Uncertainty type 2: random attached oscillators

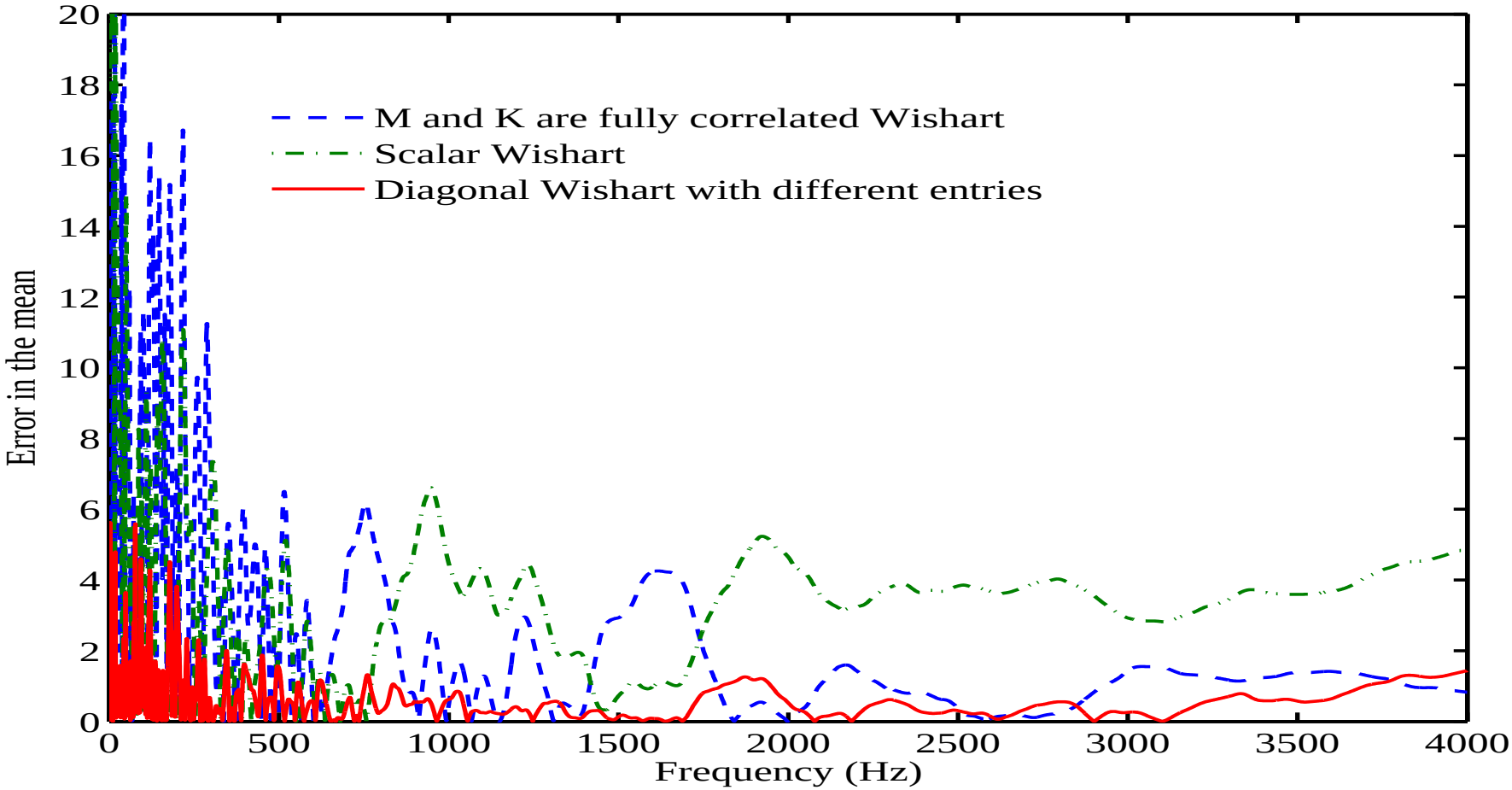
- Here we consider that the baseline plate is ‘perturbed’ by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty.
- This case will be investigated experimentally later.

Mean of cross-FRF: Utype 1



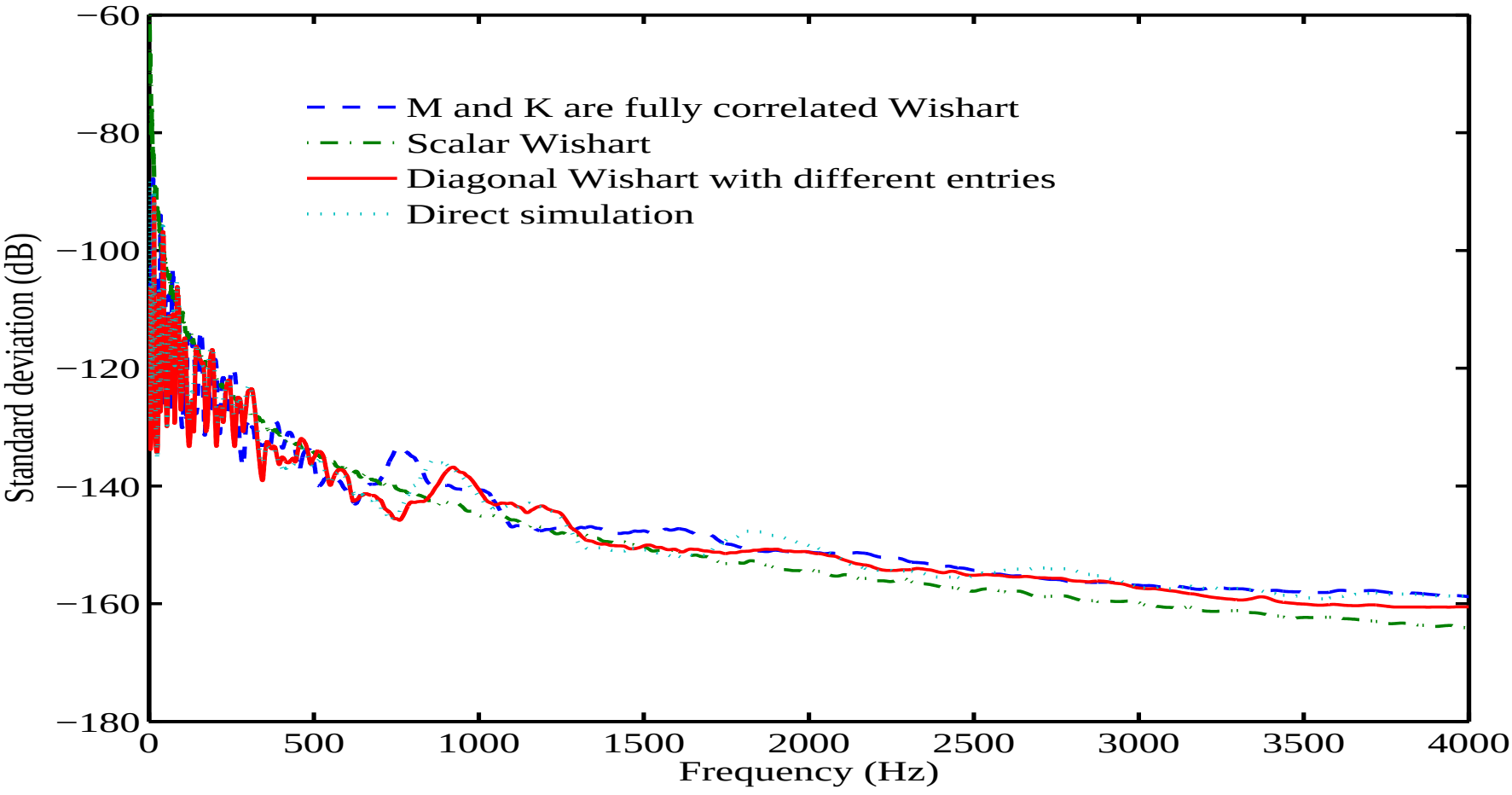
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the mean of cross-FRF: Utype 1



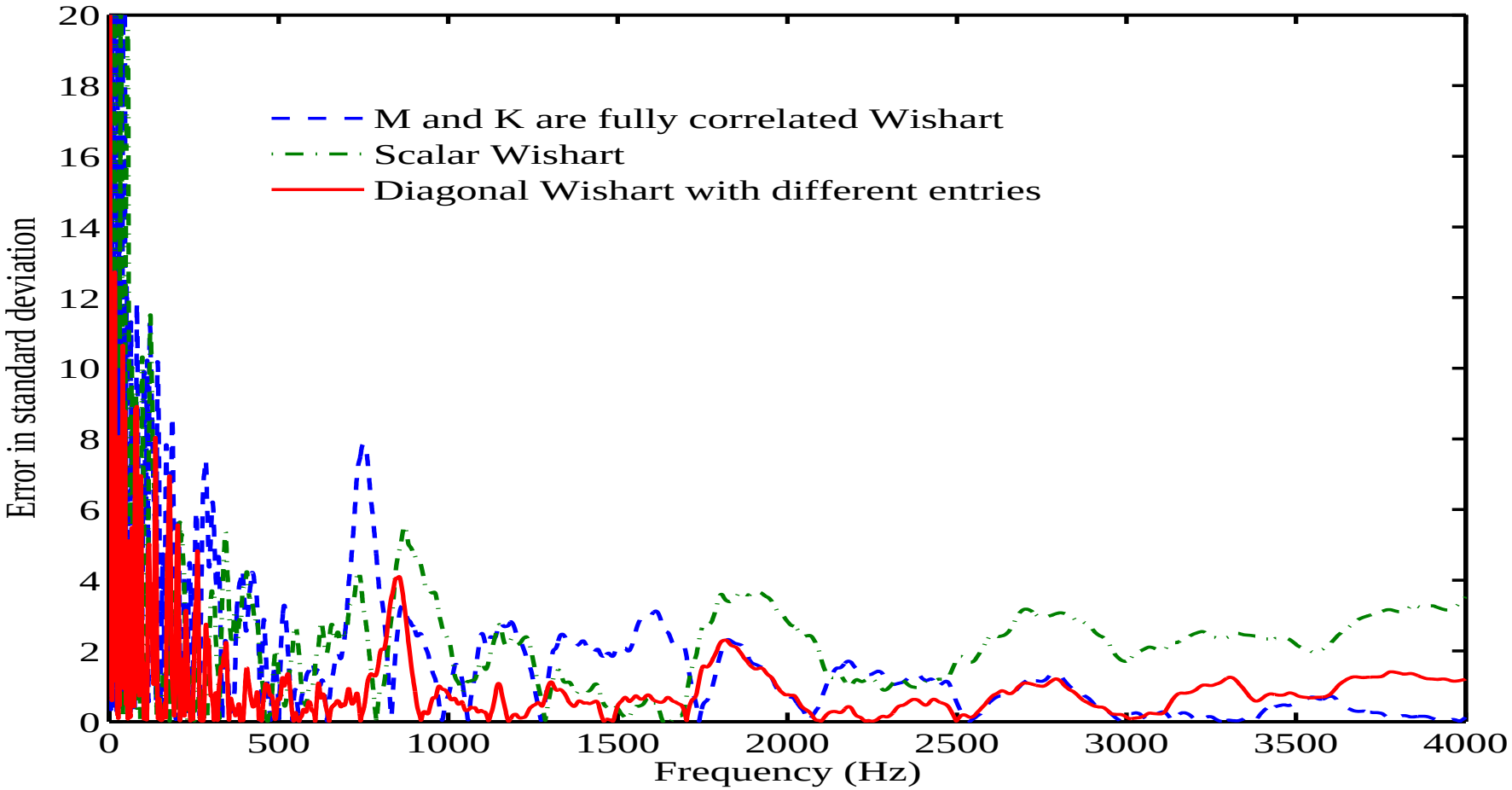
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Standard deviation of driving-point-FRF: Utype 1



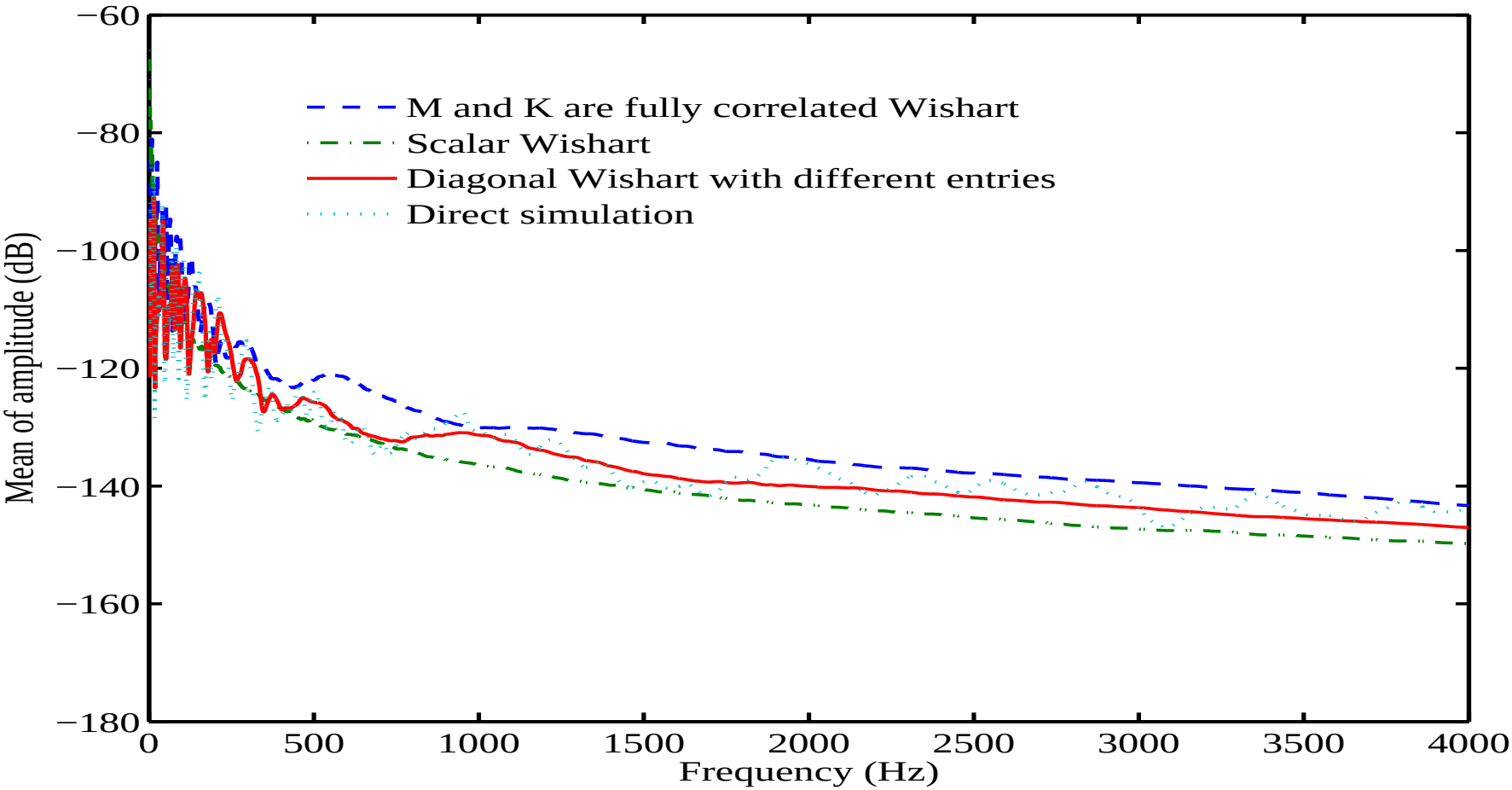
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the standard deviation of driving-point-FRF: Utype 1



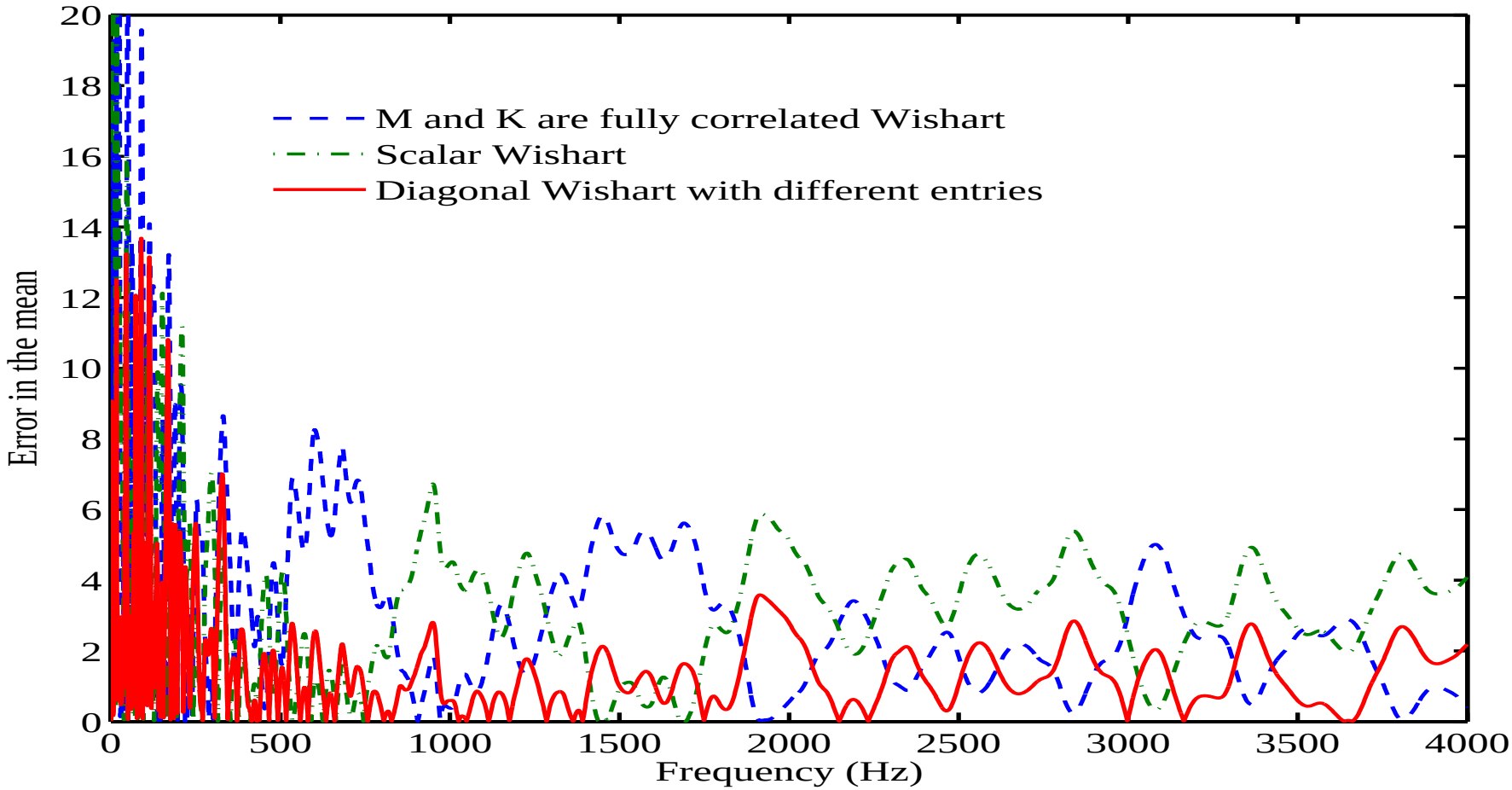
Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Mean of cross-FRF: Utype 2



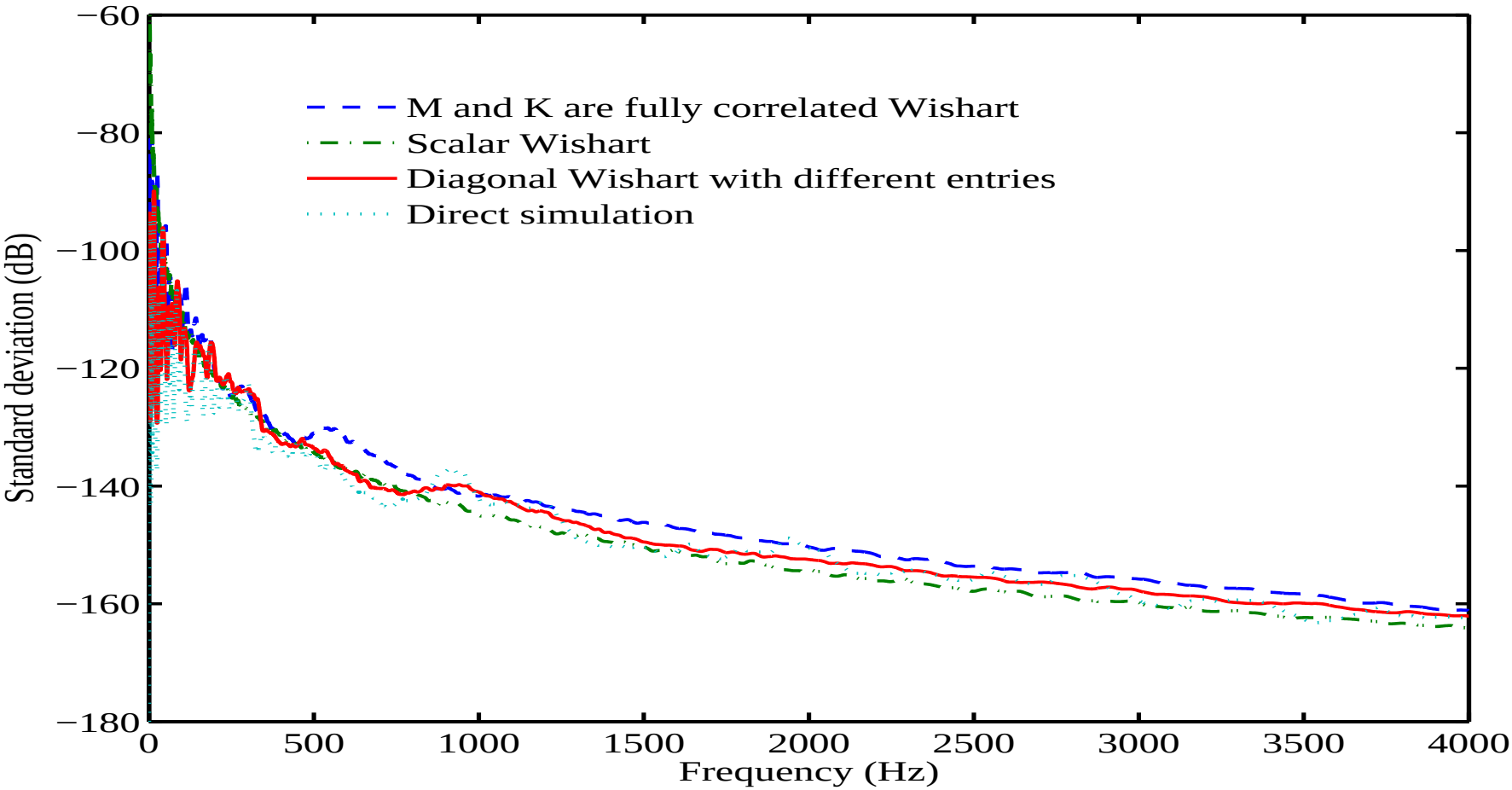
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the mean of cross-FRF: Utype 2



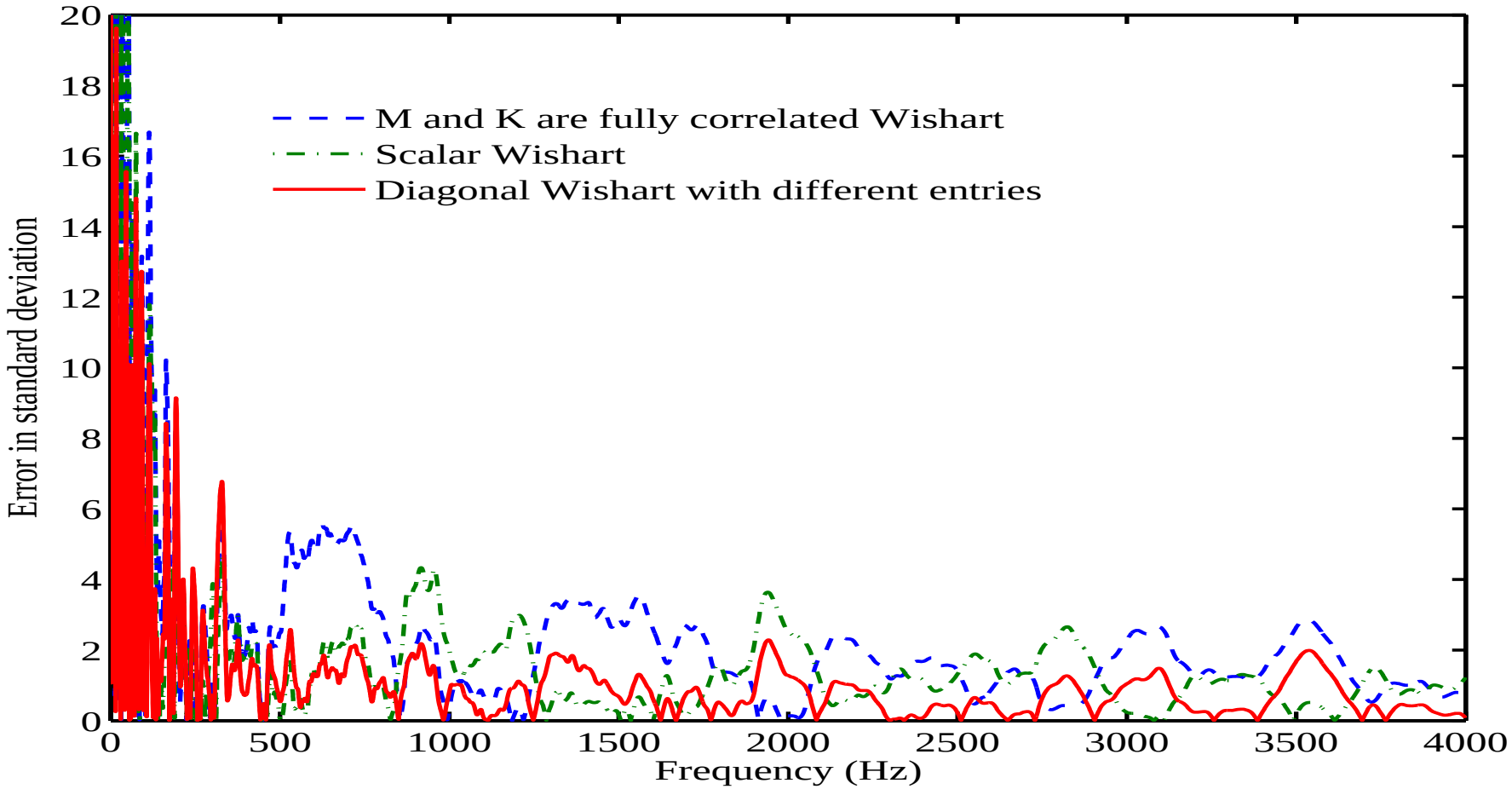
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Standard deviation of driving-point-FRF: Utype 2



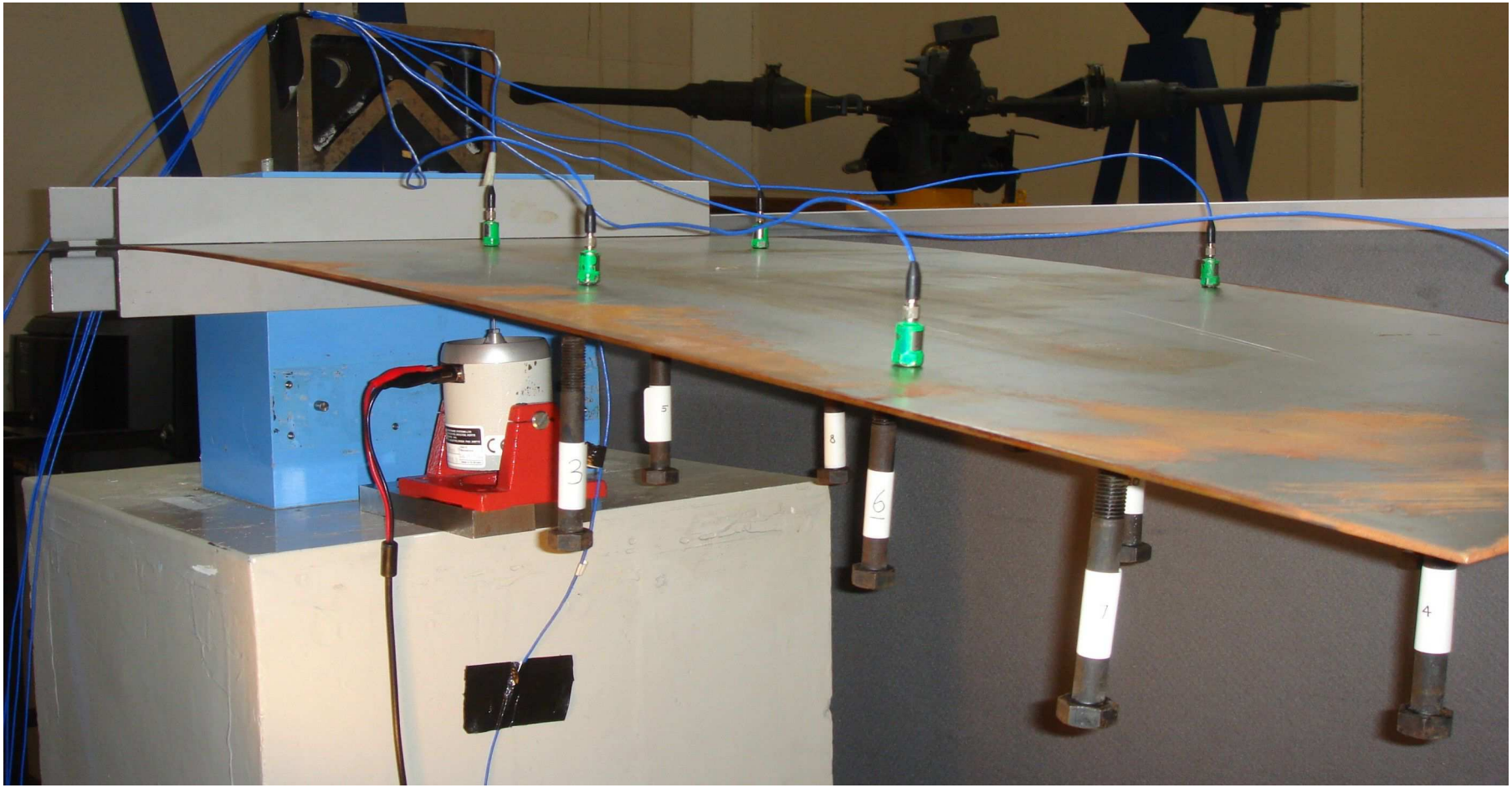
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the standard deviation of driving-point-FRF: Utype 2



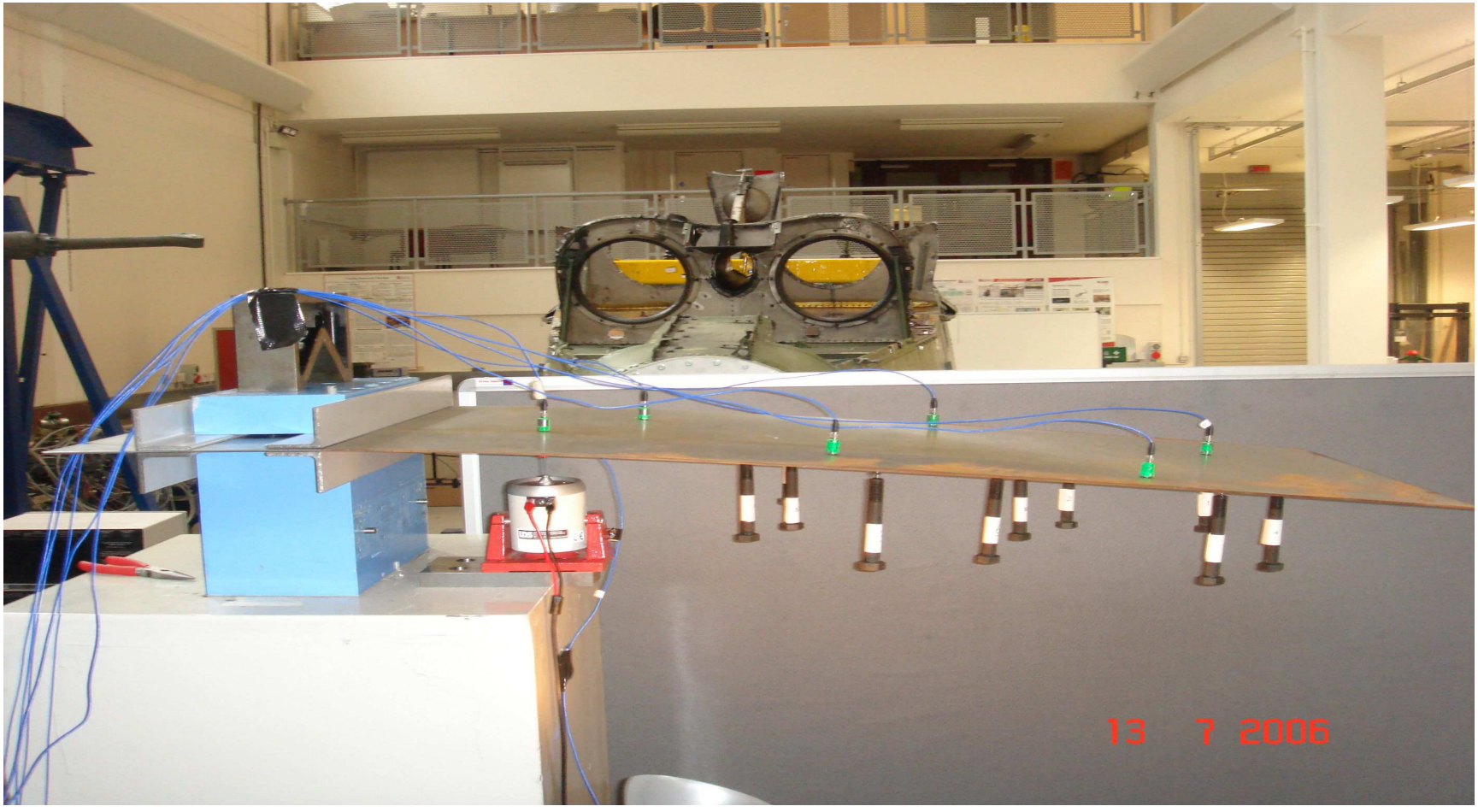
Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

A cantilever plate: front view



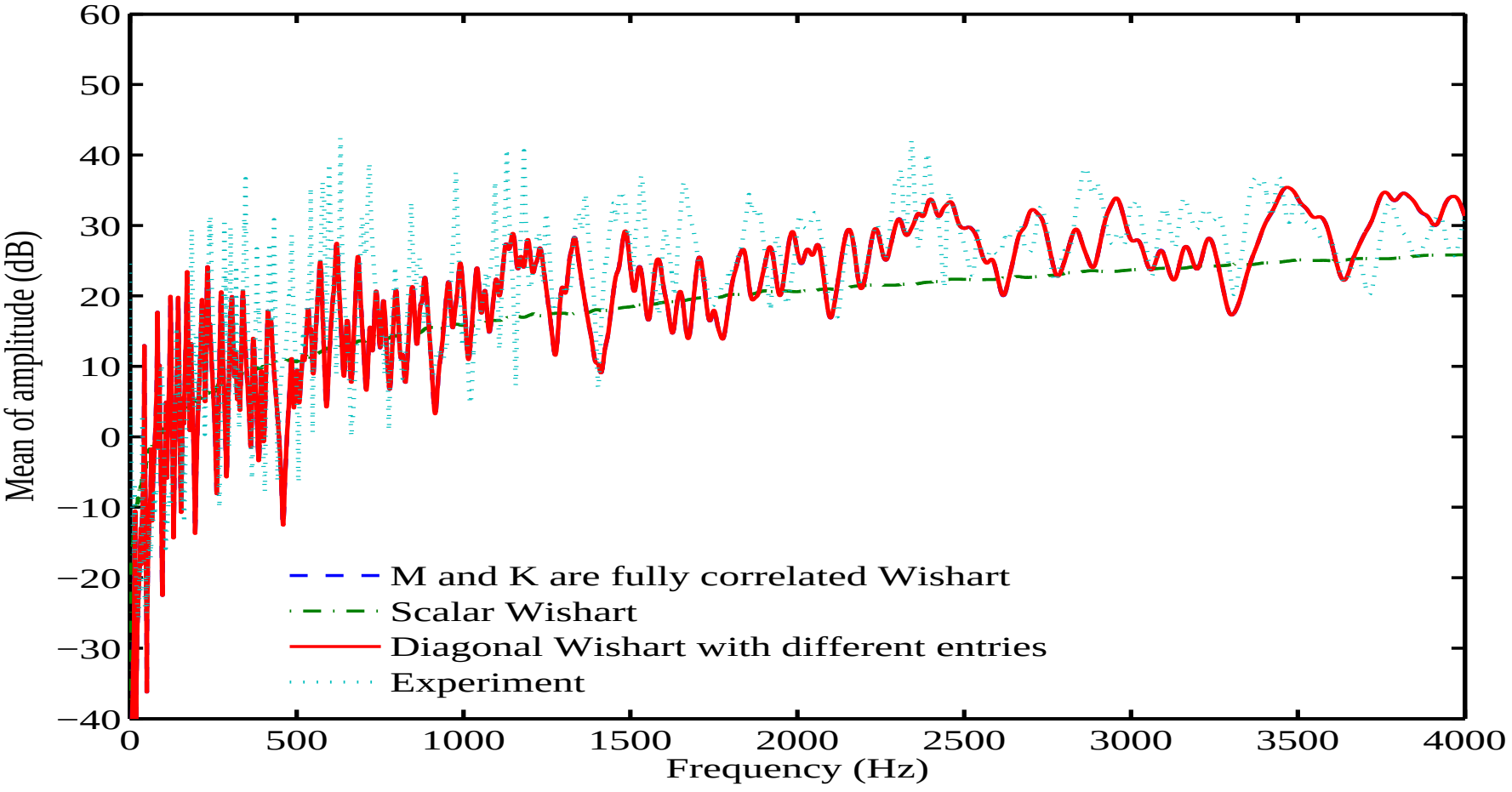
The test rig for the cantilever plate; front view.

A cantilever plate: side view



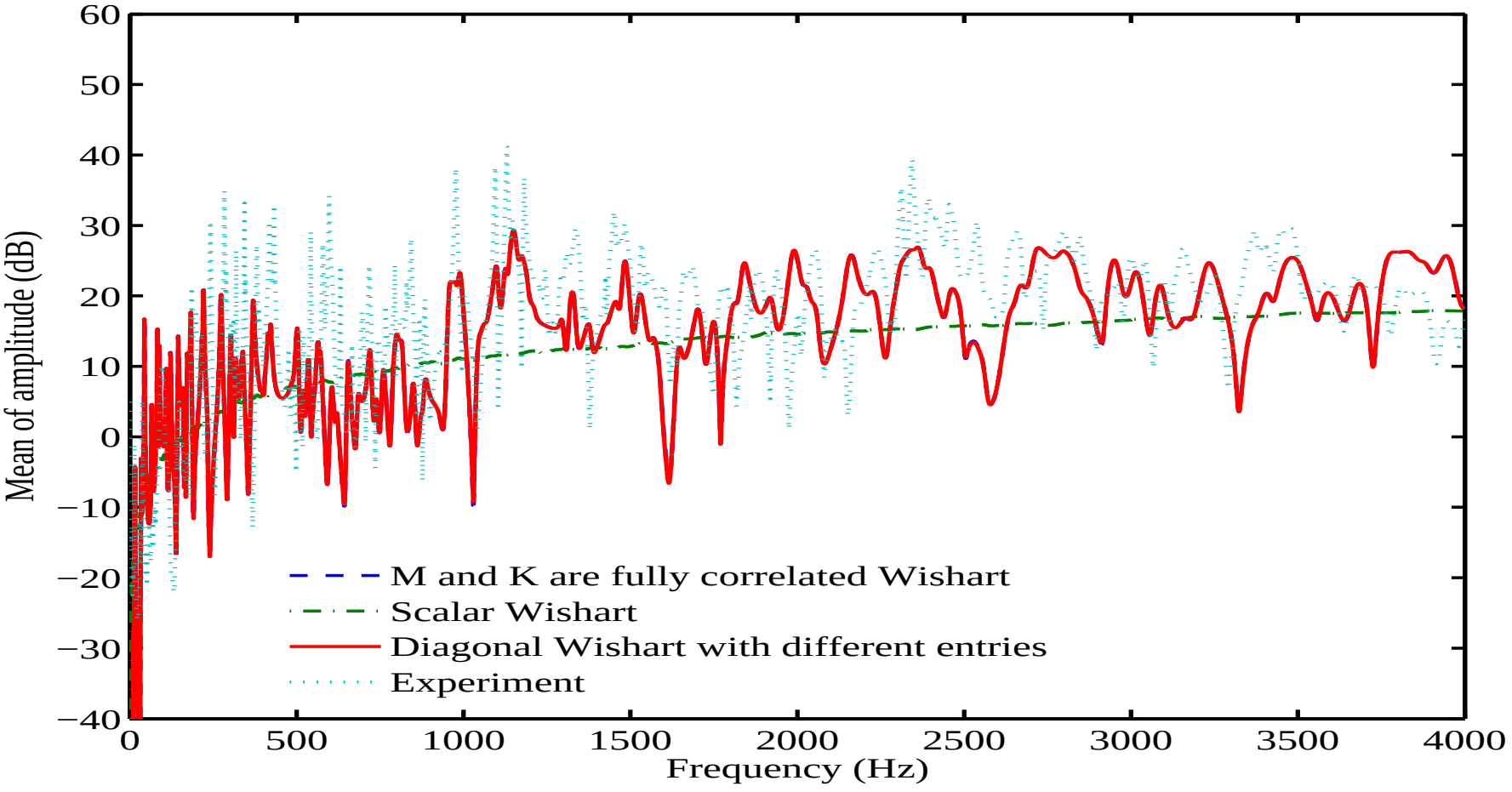
The test rig for the cantilever plate; side view.

Comparison of driving-point-FRF



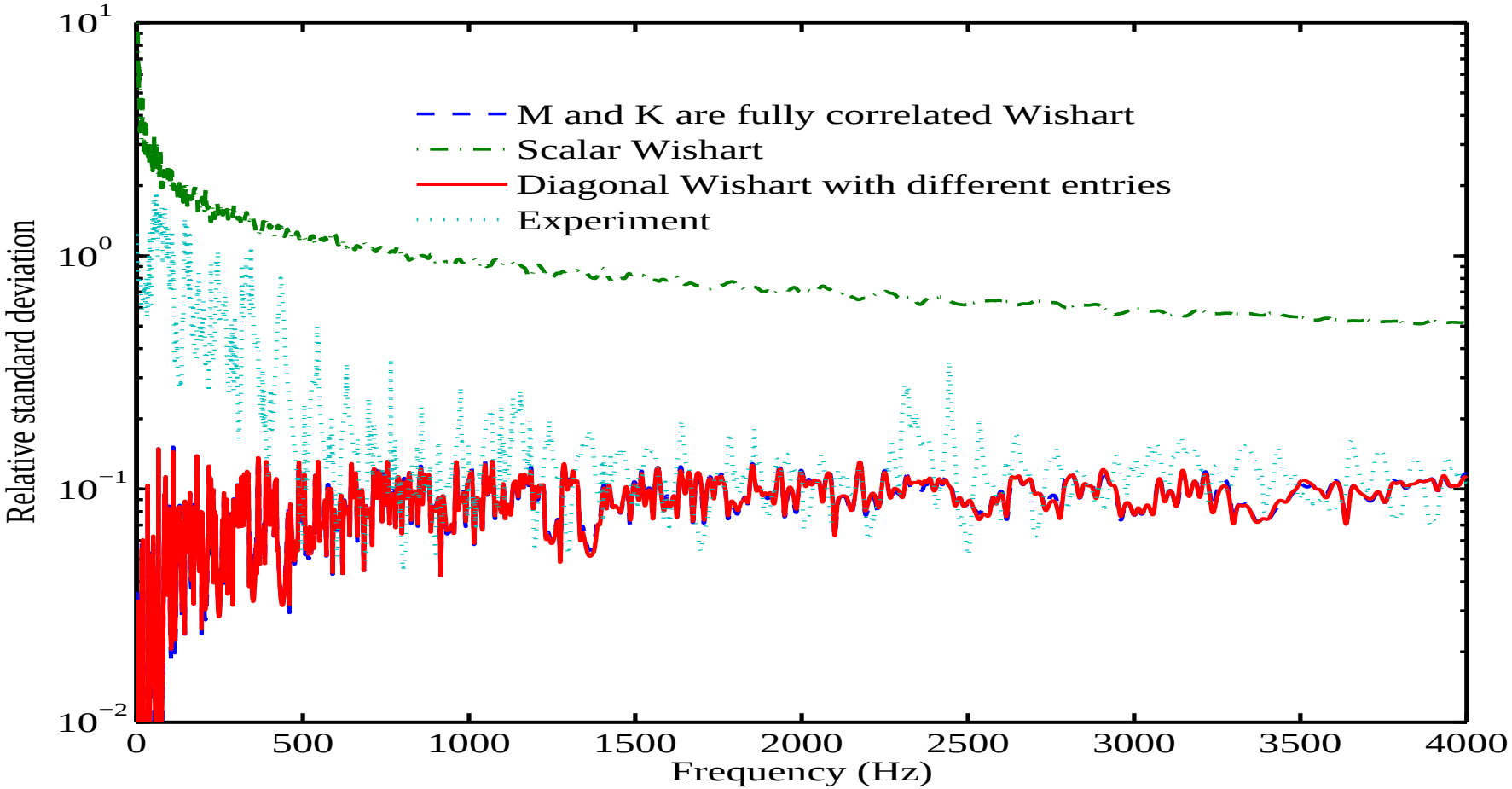
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



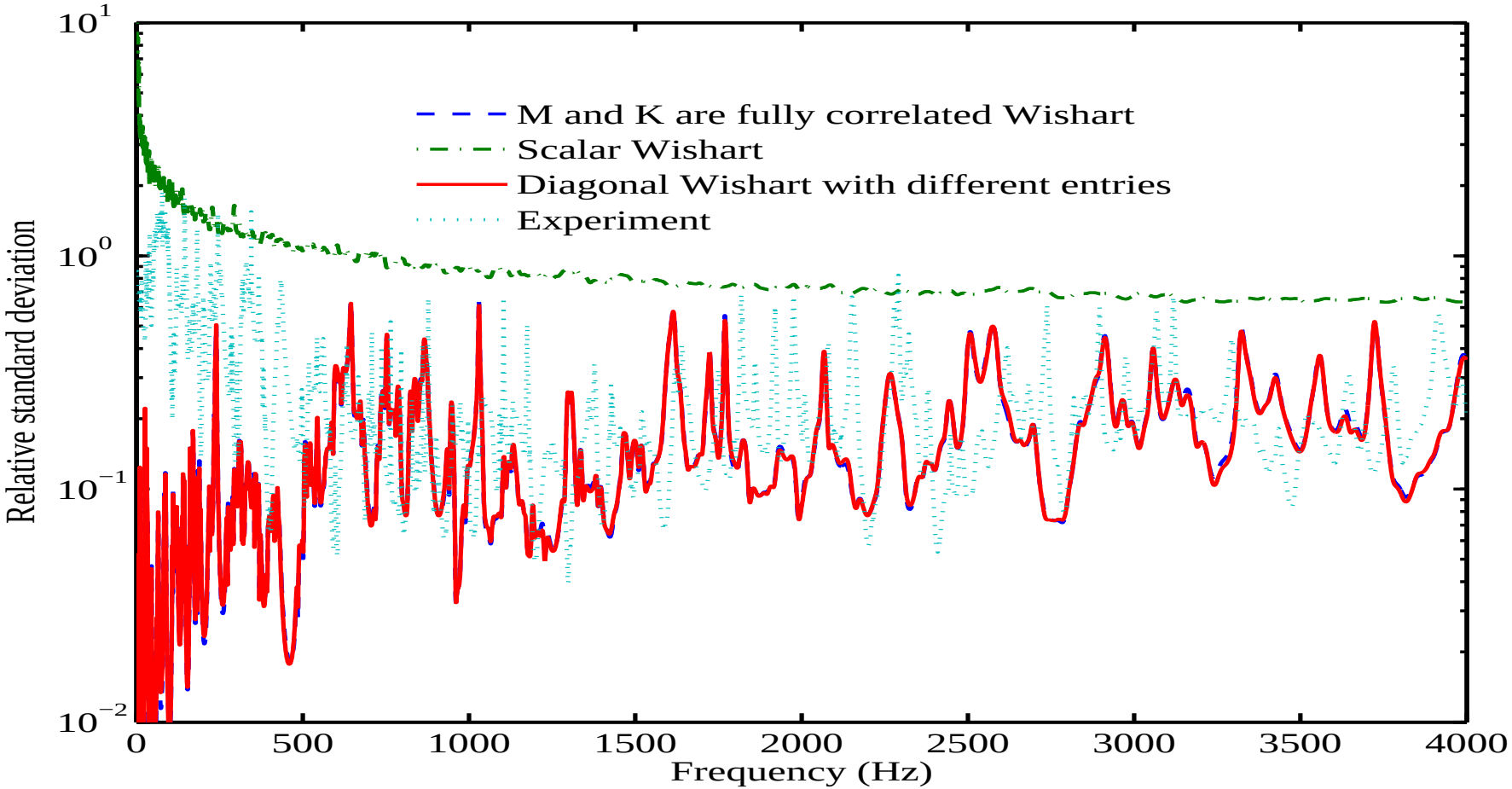
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of driving-point-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Role of vibration frequency on uncertainty modelling

- For low- frequency vibration problems (longer wavelength), parametric uncertainty model is normally used.
- Random field or random variables can be used to model uncertain parameters and stochastic finite element method can be used to propagate uncertainty.
- For high-frequency vibration problems (shorter wavelength), nonparametric uncertainty model is normally used.
- Random matrix model, such as those based on Wishart random matrices, can be used for this purpose.
- In majority of practical engineering problems, one expects a mixture of wavelengths.

Multifrequency dynamics

- Complex dynamic structures such as aircrafts, helicopters contain several substructures.
- For a given frequency of excitation, the wavelength of vibration in different substructures can be significantly different.
- For example, in the context of an aircraft fuselage, the ring girders will have significantly longer wavelength of vibration compared to the thin panel for a given frequency of excitation.

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Multifrequency dynamics

(a) Aircraft fusulage

(b) Car body

Parametric uncertainty: low-frequency vibration problem

- Fist few vibration modes (typically few tens) are participating in the dynamical response of interest
- Uncertainty models aim to characterise parametric uncertainty (type 'a')
- Random variable or random field models are used to represent uncertain parameters
- Well established methods such as stochastic finite element method (polynomial chaos, perturbation methods, spectral method) exist in literature
- A system matrix can be expressed as

$$\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}_i$$

$\mathbf{A}_0$: baseline model, $\xi_i(\theta_1)$: random variables

Non-parametric uncertainty: high-frequency vibration problem

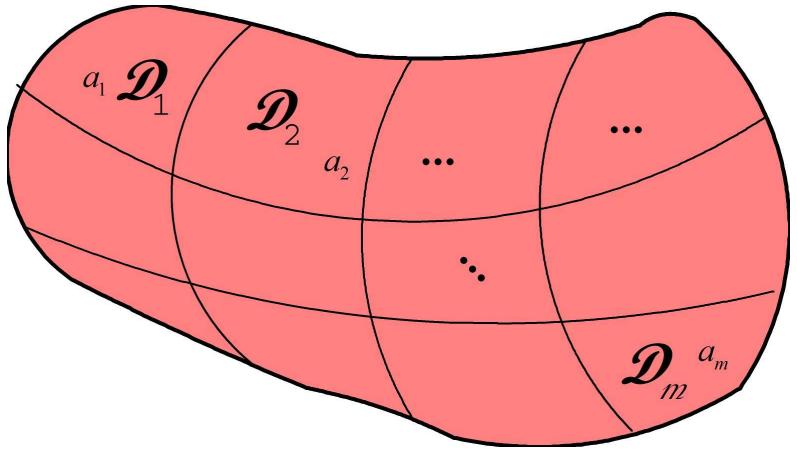
- Many vibration modes are (in hundreds) participating in the dynamical response of interest
- Uncertainty models aim to characterise non-parametric uncertainties (type 'b-d')
- Random matrix models can be used to represent uncertain system matrices
- A system matrix can be expressed as

$$\mathbf{A} = W_n(\delta_A, \mathbf{A}_0)$$

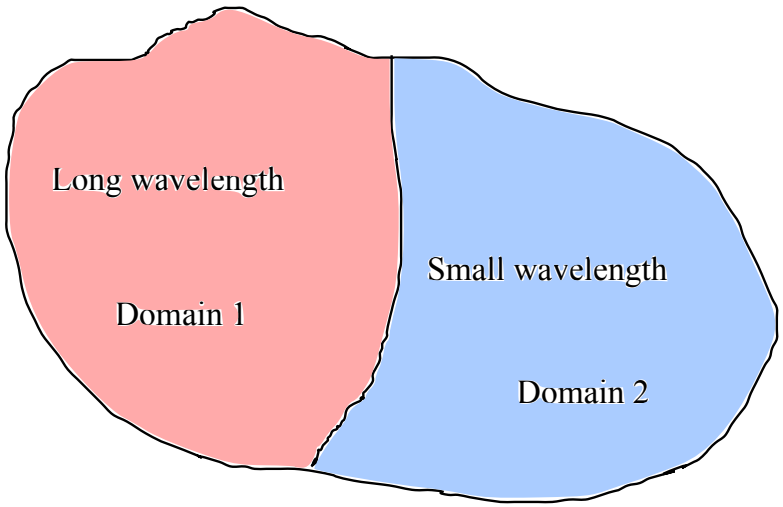
$\mathbf{A}_0$: baseline model, δ_A: dispersion parameter, W_n: Wishart random matrix of dimension n.

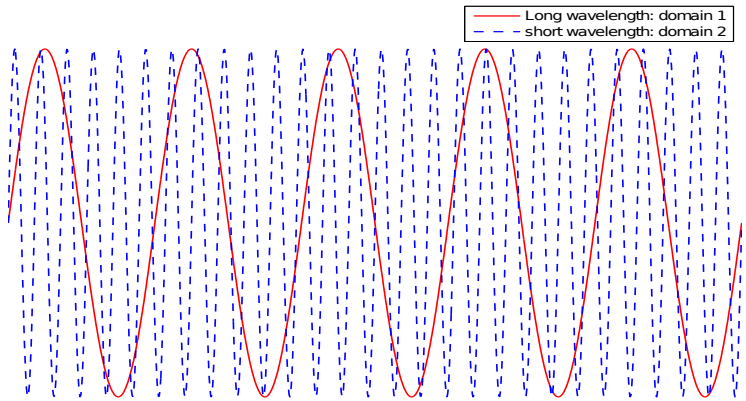
Domain decomposition method

- Developed to solve a boundary value problem by splitting it into smaller boundary value problems on subdomains
- The problems on the subdomains are independent, which makes domain decomposition methods suitable for parallel computing
- Originally developed for numerical solution of partial differential equations (not explicitly for uncertainty quantification)
- Excellent and powerful computational tools are available



Domain decomposition method





Domain 1: $\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}_i$ (dimension n_1) - parametric uncertainty

Domain 2: $\mathbf{A}(\theta_2) = W_{n_2}(\delta_A, \mathbf{A}_0)$ (dimension n_2) - nonparametric uncertainty

Two subdomains

The equation of motion of a linear dynamic system in the frequency domain is

$$\mathbf{A}(\omega)\mathbf{u} = \mathbf{f} \tag{58}$$

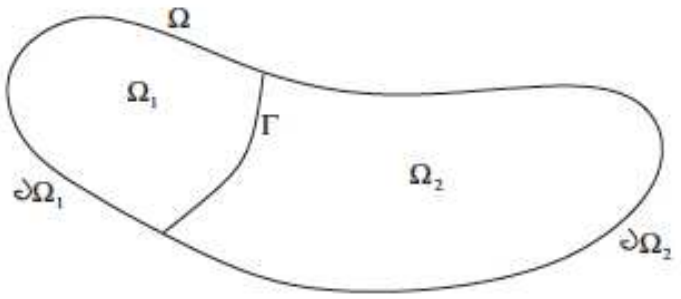
where the dynamic stiffness matrix over the whole domain Ω, $\mathbf{A}(\omega)$ is given by

$$\mathbf{A}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \in \mathbb{C}^n \tag{59}$$

We aim to subdivide the domain Ω into two nonoverlapping domains.

Two subdomains

The region Ω partitioned into two nonoverlapping subdomains Ω_1 and Ω_2 as below



The equilibrium equation of the system can be partitioned as

$$\begin{bmatrix} [\mathbf{A}^1_{II}]_{m_1\times m_1} & 0 & [\mathbf{A}^1_{I\Gamma}]_{m_1\times m_\Gamma} \\ 0 & [\mathbf{A}^2_{II}]_{m_2\times m_2} & [\mathbf{A}^2_{I\Gamma}]_{m2\times m_\Gamma} \\ [\mathbf{A}^1_{\Gamma I}]_{m_\Gamma\times m_1} & [\mathbf{A}^2_{\Gamma I}]_{m_\Gamma\times m_2} & [\mathbf{A}^1_{\Gamma\Gamma}+\mathbf{A}^2_{\Gamma\Gamma}]_{m_2\times m_2} \end{bmatrix} \times \tag{60}$$

$$\begin{Bmatrix} \mathbf{u}^1_I \\ \mathbf{u}^2_I \\ \mathbf{u}_\Gamma \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^1_I \\ \mathbf{f}^2_I \\ \mathbf{f}^1_\Gamma + \mathbf{f}^2_\Gamma \end{Bmatrix}$$

Equilibrium equations

The above equilibrium equation can be rearranged into following explicit forms (interior and interface problems):

$$[\mathbf{A}^1_{II}]\{\mathbf{u}^1_I\} = \{\mathbf{f}^1_I\} - [\mathbf{A}^1_{I\Gamma}]\{\mathbf{u}_\Gamma\} \tag{61}$$

$$[\mathbf{A}^2_{II}]\{\mathbf{u}^2_I\} = \{\mathbf{f}^2_I\} - [\mathbf{A}^2_{I\Gamma}]\{\mathbf{u}_\Gamma\} \tag{62}$$

$$\begin{aligned}[\underbrace{[\mathbf{A}^1_{\Gamma\Gamma}] - [\mathbf{A}^1_{\Gamma I}][\mathbf{A}^1_{II}]^{-1}[\mathbf{A}^1_{I\Gamma}]}_{\mathbf{S}_1} + \underbrace{[\mathbf{A}^2_{\Gamma\Gamma}] - [\mathbf{A}^2_{\Gamma I}][\mathbf{A}^2_{II}]^{-1}[\mathbf{A}^2_{I\Gamma}]}_{\mathbf{S}_2}]\{\mathbf{u}_\Gamma\} \\ = \underbrace{[\{\mathbf{f}^1_\Gamma\} - [\mathbf{A}^1_{\Gamma I}][\mathbf{A}^1_{II}]^{-1}]\{\mathbf{f}^1_I\}]}_{\mathbf{F}_1} + \underbrace{[\{\mathbf{f}^2_\Gamma\} - [\mathbf{A}^2_{\Gamma I}][\mathbf{A}^2_{II}]^{-1}]\{\mathbf{f}^2_I\}]}_{\mathbf{F}_2}\end{aligned} \tag{63}$$

The coefficient matrix $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is known as the *Schur complement* matrix.

We have two system matrices. For the domain with parametric uncertainty (long wavelength scale):

$$[\mathbf{A}^1(\theta_1)]_{n_1 \times n_1} = \begin{bmatrix} \mathbf{A}^1_{II}(\theta_1) & \mathbf{A}^1_{I\Gamma}(\theta_1) \\ \mathbf{A}^1_{\Gamma I}(\theta_1) & \mathbf{A}^1_{\Gamma\Gamma}(\theta_1) \end{bmatrix} = \mathbf{A}^1_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}^1_i \tag{64}$$

with $n_1 = m_1 + m_\Gamma$.
For the domain with nonparametric uncertainty (short wavelength scale):

$$\mathbf{A}^2(\theta_2)_{n_2 \times n_2} = \begin{bmatrix} \mathbf{A}^2_{II}(\theta_2) & \mathbf{A}^2_{I\Gamma}(\theta_2) \\ \mathbf{A}^2_{\Gamma I}(\theta_2) & \mathbf{A}^2_{\Gamma\Gamma}(\theta_2) \end{bmatrix} = W_{n_2}(\delta_{A_2}, \mathbf{A}^2_0) \tag{65}$$

with $n_2 = m_2 + m_\Gamma$.

Stochastic interface problem

For the stochastic interface problem we have a system of (densely) coupled m_Γ complex stochastic equations

$$[\mathbf{S}_1(\theta_1) + \mathbf{S}_2(\theta_2)]\mathbf{u}_\Gamma(\theta_1, \theta_2) = \mathbf{F}_1(\theta_1) + \mathbf{F}_2(\theta_2) \tag{66}$$

where

$$\mathbf{S}_1(\theta_1) = \mathbf{A}^1_{\Gamma\Gamma}(\theta_1) - \mathbf{A}^1_{\Gamma I}(\theta_1)[\mathbf{A}^1_{II}(\theta_1)]^{-1}\mathbf{A}^1_{I\Gamma}(\theta_1) \tag{67}$$

$$\mathbf{F}_1(\theta_1) = \mathbf{f}^1_\Gamma - \mathbf{A}^1_{\Gamma I}(\theta_1)[\mathbf{A}^1_{II}(\theta_1)]^{-1}\mathbf{f}^1_I \tag{68}$$

and

$$\mathbf{S}_2(\theta_2) = \mathbf{A}^2_{\Gamma\Gamma}(\theta_2) - \mathbf{A}^2_{\Gamma I}(\theta_2)[\mathbf{A}^2_{II}(\theta_2)]^{-1}\mathbf{A}^2_{I\Gamma}(\theta_2) \tag{69}$$

$$\mathbf{F}_2(\theta_2) = \mathbf{f}^2_\Gamma - \mathbf{A}^2_{\Gamma I}(\theta_2)[\mathbf{A}^2_{II}(\theta_2)]^{-1}\mathbf{f}^2_I \tag{70}$$

Stochastic interior problems

Solving the interface problem we have $\mathbf{u}_\Gamma(\theta_1, \theta_2)$. This can used to obtain the interior solutions as

$$\mathbf{u}_I^1(\theta_1, \theta_2) = [\mathbf{A}_{II}^1(\theta_1)]^{-1}[\mathbf{f}_I^1 - \mathbf{A}_{I\Gamma}^1(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \tag{71}$$

$$\mathbf{u}_I^2(\theta_1, \theta_2) = [\mathbf{A}_{II}^2(\theta_1)]^{-1}[\mathbf{f}_I^2 - \mathbf{A}_{I\Gamma}^2(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \tag{72}$$

The most computationally intensive parts of the solution process is obtaining $[\mathbf{A}_{II}^1(\theta_1)]^{-1}$ and $[\mathbf{A}_{II}^2(\theta_1)]^{-1}$ which involves the solution of m_1 and m_2 number of coupled complex stochastic equations.
Existing computational methods for uncertainty propagation can be used.

Stochastic interior problems

Recall that in the frequency domain

$$\mathbf{A}_{II}(\omega,\theta) = -\omega^2\mathbf{M}_{II}(\theta) + i\omega\mathbf{C}_{II}(\theta) + \mathbf{K}_{II}(\theta) \tag{73}$$

Assuming proportional damping model, we have

$$[\mathbf{A}_{II}(\omega,\theta)]^{-1} = \sum_{k=1}^{m} \frac{\phi_k(\theta)\phi_k^T(\theta)}{\omega_k^2(\theta) - \omega^2 + 2i\zeta_k\omega_k(\theta)} \tag{74}$$

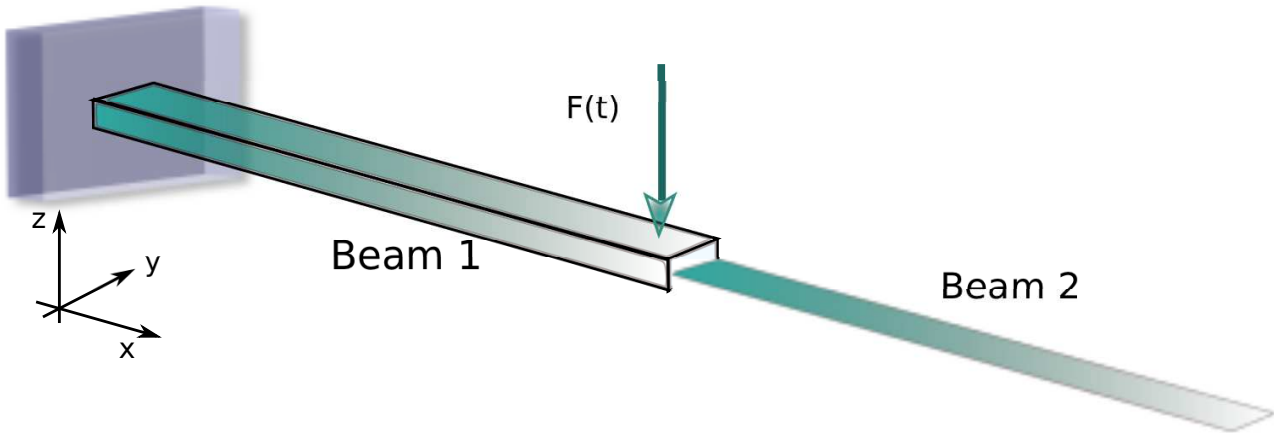
Here ζ_k are the modal damping factors and the eigenvalues are eigenvectors are obtained from

$$\mathbf{K}_{II}(\theta)\phi_k(\theta) = \omega_k^2\mathbf{M}_{II}(\theta)\phi_k(\theta), \quad k = 1, 2, \cdots \tag{75}$$

Any existing methods for random eigenvalue problem can be used (perturbation, polynomial chaos, Neumann series ...).

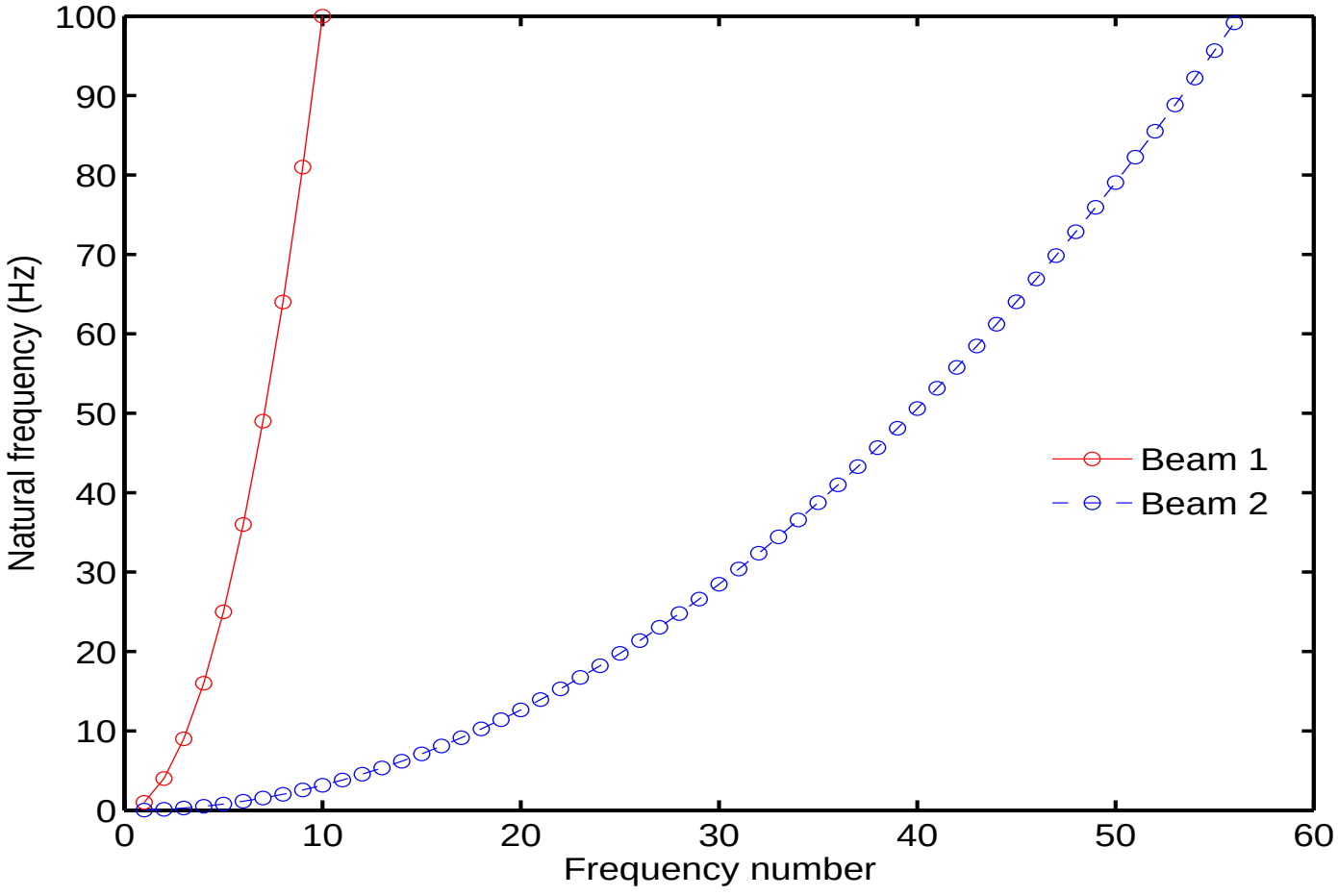
An Euler-Bernoulli beam example

- Two coupled Euler-Bernoulli beams with stochastic elasticity are considered



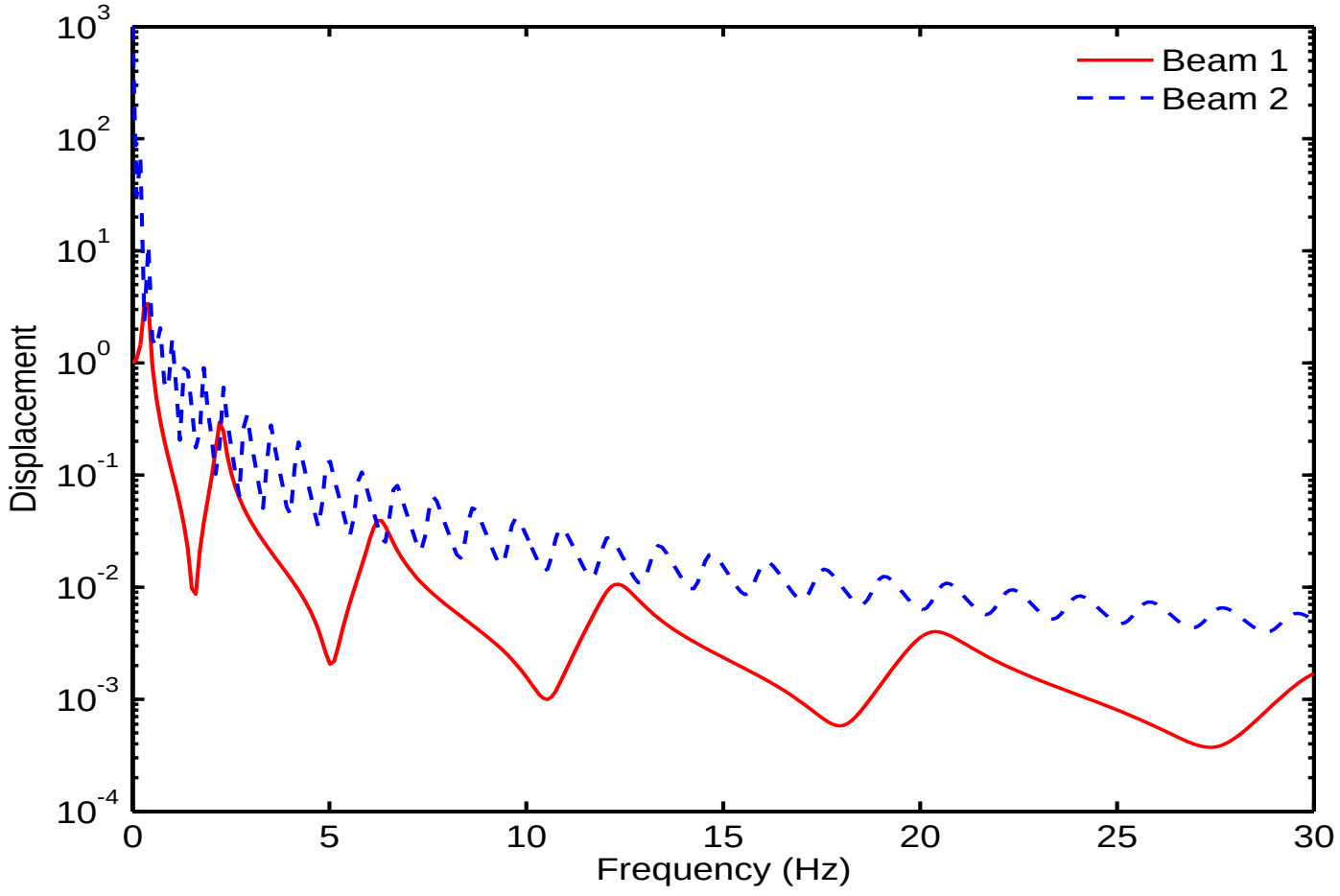
- $L_1 = 1$, $EI_{1_0} = 1/3$, $\rho A_1 = \pi^2/12$, $\zeta_1 = 0.04$
- $L_2 = L_1$, $EI_{2_0} = EI_{1_0}/10^3$, $\rho A_2 = \rho A_1$, $\zeta_2 = \zeta_1/2$
- We study the deflection of the beam under the action of a point harmonic load on the interior of beam 1.

Natural frequencies



Due to the difference in the stiffness values, beam 1 has less number of frequencies compared to beam 2 within a given frequency range.

Frequency response



Frequency response functions of the two beams in isolation (in cantilever configuration with a point load at the end).

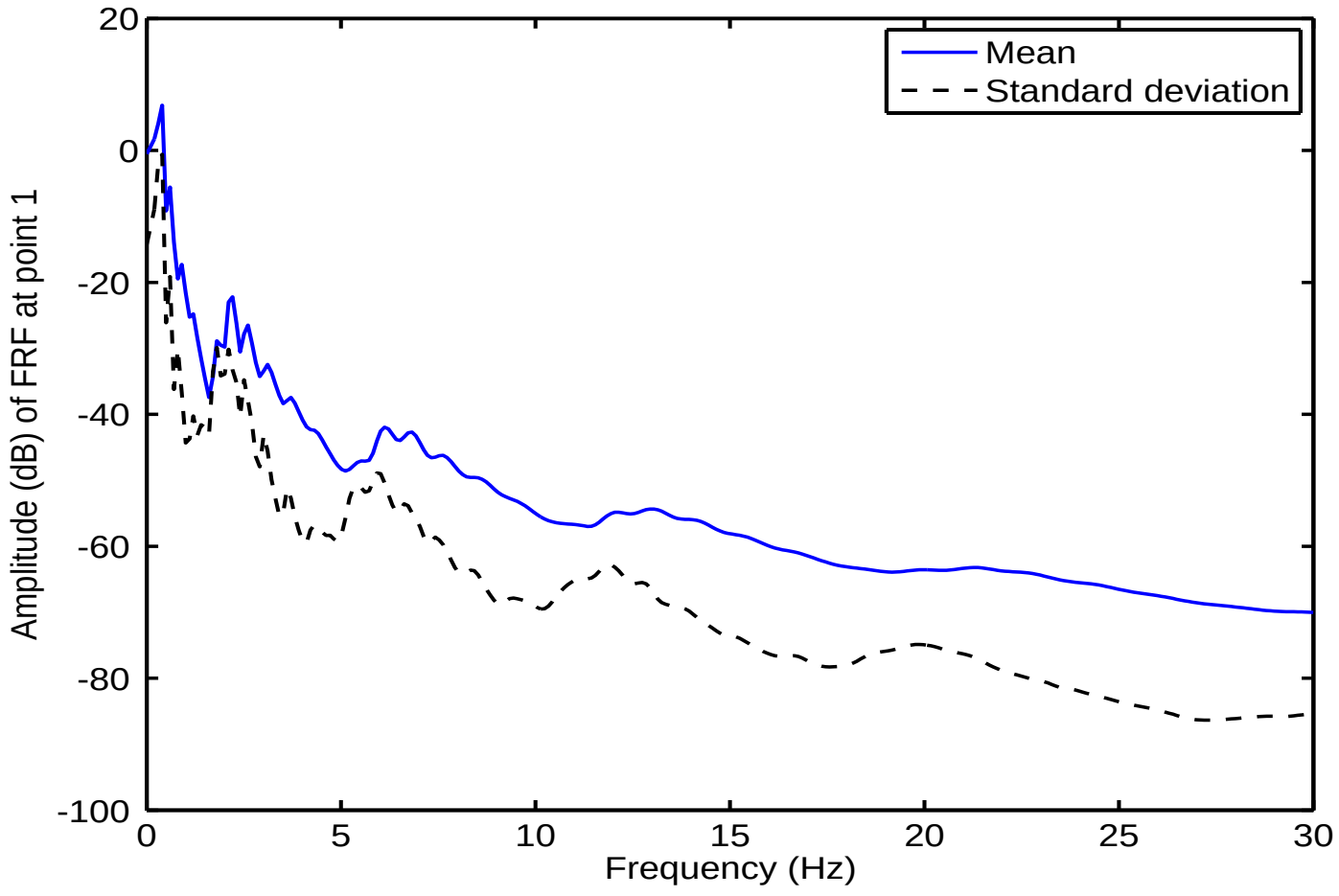
Stochastic models

- The bending modulus of the first beam is modelled by two Gaussian random variables (a discretised random field with standard deviation $\sigma_a = 0.2$). The stiffness matrix is of the form

$$\mathbf{K}^1(\theta_1) = \mathbf{K}_0 + \xi_1(\theta_1)\mathbf{K}_1^1 + \xi_2(\theta_1)\mathbf{K}_2^1$$

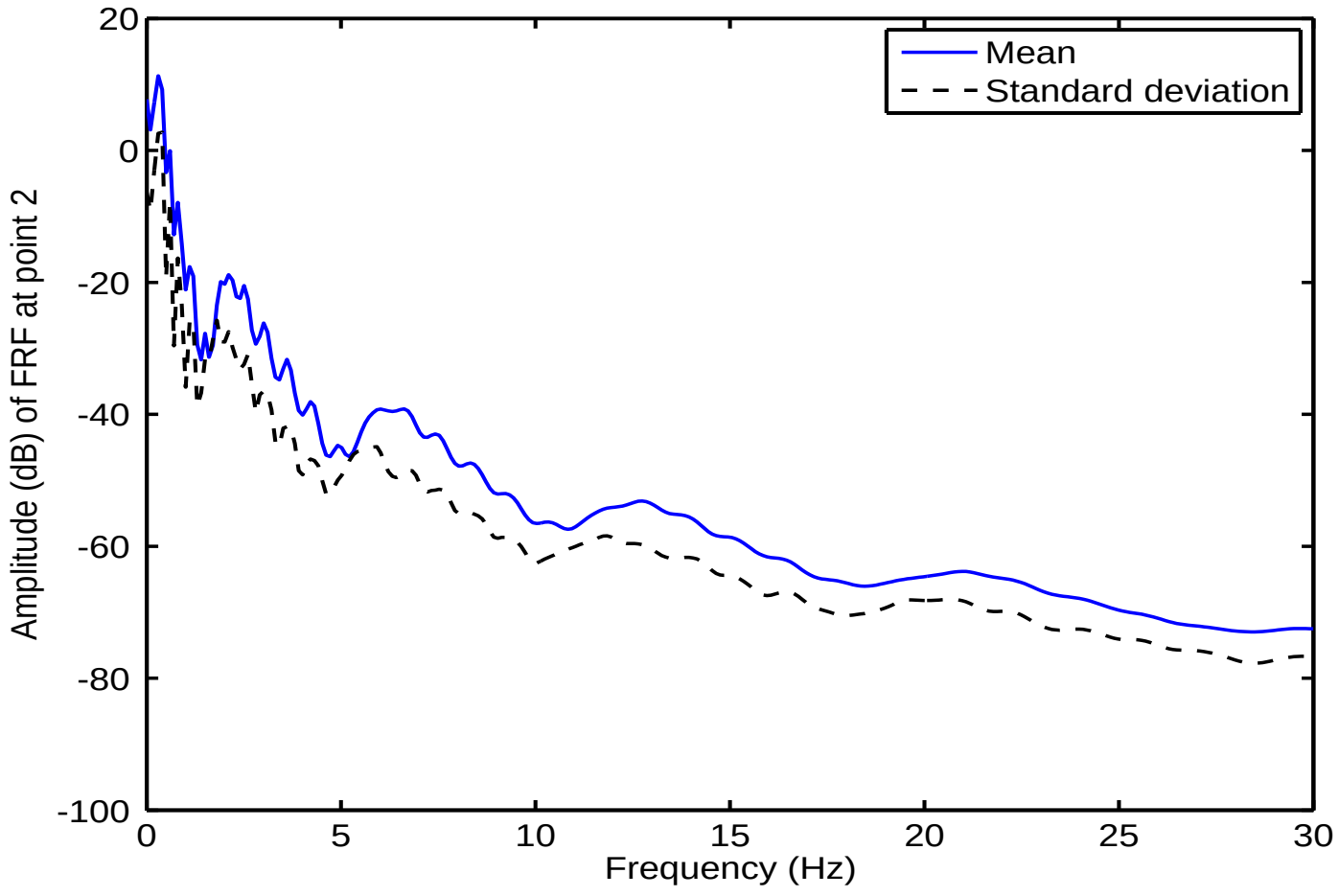
- For the second beam, an Wishart random matrix model with $\delta = 0.2$ is considered.
- The mass matrix and the damping factors are deterministic for both the beams.
- First-order perturbation is used for the interior random eigenvalue problems. 1000-sample Monte Carlo sample is used to for the interface problem.
- For the numerical calculation we used $n_1 = 60$, $n_2 = 328$. In the domain decomposition approach, $m_1 = 58$, $m_2 = 336$ and $m_\Gamma = 2$.

Stochastic response - driving point



Response statistics of the stochastic multiscale system at the driving point.

Stochastic response - tip point



Response statistics of the stochastic multiscale system at the tip.

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