Vibration of Piled Foundations in Liquefiable Soils During Strong Earthquakes

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Outline

- The need for advanced dynamic models
- Distributed model for a single pile
- Natural Frequencies of a pile
- Numerical examples
- Conclusions

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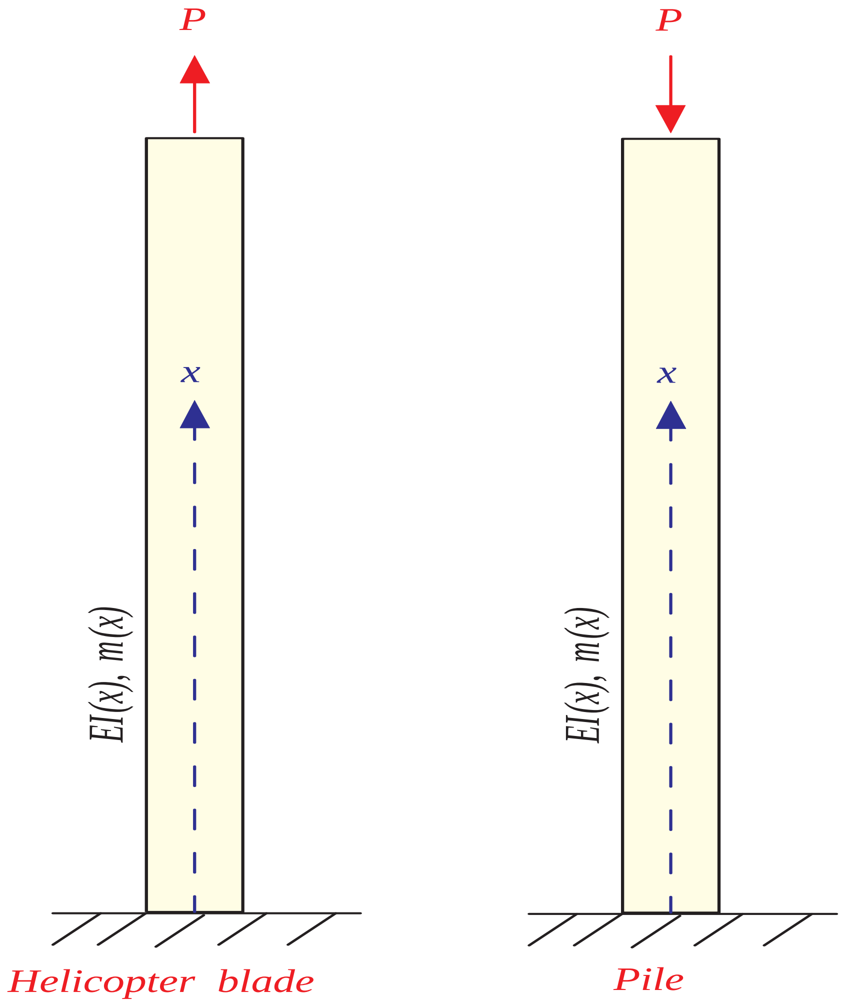
Helicopters & piles



Motivation

- The exact mechanism of pile a failure is difficult to obtain for real-life systems.
- In this lecture we address this problem using general theory of linear distributed dynamical systems.
- Our aim is to come up with an unified approach whereby different existing approaches can be identified as special cases.

Dynamic force modeling - I

During an earthquake there are three main types of forces/excitations acting on a pile:

- Bending force: arising due to the soil flow around the pile during an earthquake - may result in bending failure.
- Axial force: arising due to the load coming from the superstructure - may result in buckling failure.

Dynamic force modeling - II

- Point/distributed frequency dependent force: arising due to the ‘shaking’ of the bed-rock and the surrounding medium - may result in resonance failure.

The reality is perhaps some kind of nonlinear combination of the above three types of forces which is in general not known a priori!
We aim to address this issue by considering all of the above forces for generality.

Distributed dynamical system model

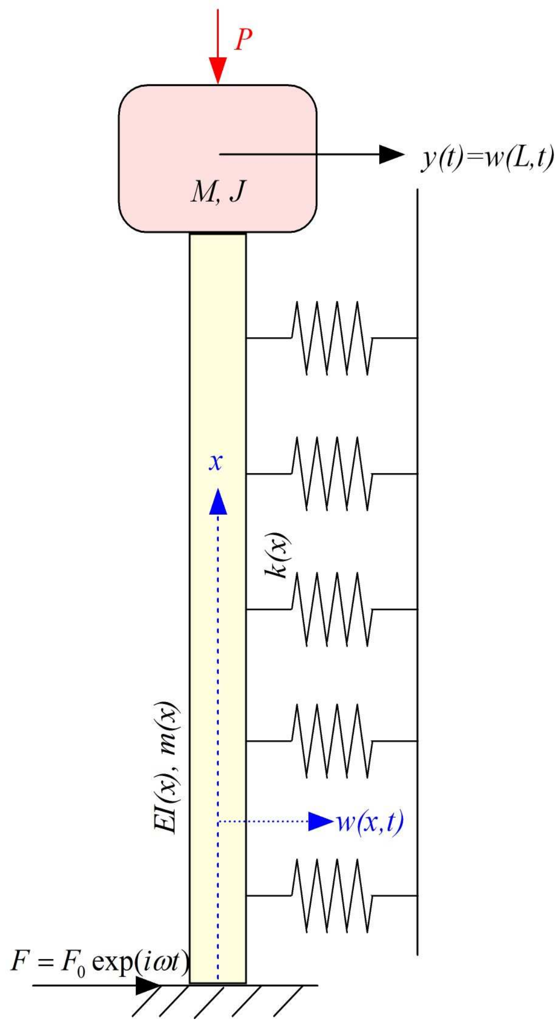
- Majority of existing works either consider a distributed static system (namely Euler Bernoulli beam equation) or a single degree-of-freedom (SDOF) dynamic model.
- The static model neglecters the resonance effect while the SDOF model neglects the buckling effect.
- We propose a distributed dynamical system model to include the neglected dynamics in the above models.

Scope of the proposed approach

- An Euler-Bernoulli beam model resting against an elastic support with axial force and tip mass with rotary inertia are considered.
- The elastic support is aimed at modeling the surrounding soil while the tip mass together with its rotary inertia is aimed at modeling the superstructure.
- Only free vibration analysis is considered in this work.

Equation of motion

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P(x)\frac{\partial w(x,t)}{\partial x}\right) - \frac{\partial}{\partial x}\left(mr^2(x)\frac{\partial \ddot{w}(x,t)}{\partial x}\right) + k(x)w(x,t) + m\ddot{w}(x,t) = f(x,t)$$



Boundary conditions

- Deflection at $x = 0$:

$$w(0, t) = 0 \tag{1}$$

- Rotation at $x = 0$:

$$\left.\frac{\partial w(x,t)}{\partial x} = 0\right|_{x=0} \quad \text{or} \quad w'(0,t) = 0 \tag{2}$$

- Bending moment at $x = L$:

$$\left. EI\frac{\partial^2 w(x,t)}{\partial x^2} + J\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\right|_{x=L} \quad \text{or} \quad EIw''(L,t) + J\frac{\partial \ddot{w}(L,t)}{\partial x} = 0 \tag{3}$$

- Shear force at $x = L$:

$$\left. EI\frac{\partial^3 w(x,t)}{\partial x^3} + P\frac{\partial w(x,t)}{\partial x} - M\ddot{w}(x,t) - mr^2\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\right|_{x=L}$$
$$\text{or} \quad EIw'''(L,t) + Pw'(L,t) - M\ddot{w}(L,t) - mr^2\frac{\partial \ddot{w}(L,t)}{\partial x} = 0 \tag{4}$$

Separation of variables

Assuming harmonic solution we have

$$w(x,t) = W(\xi)\exp\{\mathrm{i}\omega t\}, \quad \xi = x/L \tag{5}$$

Substituting in the equation of motion and boundary conditions

$$\frac{EI}{L^4}\frac{\partial^4 W(\xi)}{\partial \xi^4} + \frac{P}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} + k\,W(\xi) - m\omega^2 W(\xi) + \frac{mr^2\omega^2}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} = 0 \tag{6}$$

$$W(0) = 0 \tag{7}$$

$$W'(0) = 0 \tag{8}$$

$$\frac{EI}{L^2}\,W''(1) - \frac{\omega^2 J}{L}\,W'(1) = 0 \tag{9}$$

$$\frac{EI}{L^3}\,W'''(1) + \frac{P}{L}W'(1) + \omega^2 M\,W(1) + \frac{mr^2\omega^2}{L}W'(1) = 0 \tag{10}$$

Non-Dimensionalisation

We transform the previous equations as

$$\frac{\partial^4 W(\xi)}{\partial \xi^4} + \widetilde{\nu}\frac{\partial^2 W(\xi)}{\partial \xi^2} + \eta W(\xi) - \Omega^2 W(\xi) = 0 \tag{11}$$

$$W(0) = 0 \tag{12}$$

$$W'(0) = 0 \tag{13}$$

$$W''(1) - \beta\Omega^2 W'(1) = 0 \tag{14}$$

$$W'''(1) + \widetilde{\nu} W'(1) + \alpha\Omega^2 W(1) = 0 \tag{15}$$

where

$$\widetilde{\nu} = \nu + \mu^2\Omega^2 \tag{16}$$

Nondimensional parameters

$$\nu = \frac{PL^2}{EI} \quad \text{(nondimensional axial force)}, \;\; \nu = \frac{\pi^2}{4}(P/P_{cr}) \tag{17}$$

$$\eta = \frac{kL^4}{EI} \quad \text{(nondimensional support stiffness)} \tag{18}$$

$$\Omega^2 = \omega^2 \frac{mL^4}{EI} \quad \text{(nondimensional frequency parameter)} \tag{19}$$

$$\alpha = \frac{M}{mL} \quad \text{(mass ratio)} \tag{20}$$

$$\beta = \frac{J}{mL^3} \quad \text{(nondimensional rotary inertia)} \tag{21}$$

$$\mu = \frac{r}{L} \quad \text{(nondimensional radius of gyration).} \tag{22}$$

Solution of the boundary value problem

Assuming a solution of the form

$$W(\xi) = \exp\{\lambda\xi\} \tag{23}$$

and substituting in the equation of motion results

$$\lambda^4 + \widetilde{\nu}\lambda^2 - \left(\Omega^2 - \eta\right) = 0 \tag{24}$$

This is the equation governing the natural frequencies of the beam. Solving this equation for λ^2 we have

$$\begin{aligned} \lambda^2 &= -\frac{\widetilde{\nu}}{2} \pm \sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + (\Omega^2 - \eta)} \\ &= -\left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + (\Omega^2 - \eta)} + \frac{\widetilde{\nu}}{2}\right), \quad \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + (\Omega^2 - \eta)} - \frac{\widetilde{\nu}}{2}\right). \end{aligned} \tag{25}$$

Depending on whether $\Omega^2 - \eta > 0$ or not two cases arise.

Case 1

If $\widetilde{\nu} > 0$ and $\Omega^2 - \eta > 0$ or $\Omega^2 > \eta$ then both roots are real with one negative and one positive root. Therefore, the four roots can be expressed as

$$\lambda = \pm \mathrm{i}\lambda_1, \quad \pm\lambda_2 \tag{26}$$

where

$$\lambda_1 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + (\Omega^2 - \eta)} + \frac{\widetilde{\nu}}{2} \right)^{1/2} \tag{27}$$

$$\text{and} \quad \lambda_2 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + (\Omega^2 - \eta)} - \frac{\widetilde{\nu}}{2} \right)^{1/2}. \tag{28}$$

The solution $W(\xi)$ can be expressed as

$$W(\xi) = a_1 \sin\lambda_1\xi + a_2 \cos\lambda_1\xi + a_3 \sinh\lambda_2\xi + a_4 \cosh\lambda_2\xi \tag{29}$$

Case 1

Substituting $W(\xi)$ in the boundary conditions and eliminating the constants we have the frequency equation

$$
\begin{aligned}
&\left(-\sin(\lambda_1)\,{\lambda_1}^2\lambda_2\Omega^2\cosh(\lambda_2) + \lambda_1\Omega^2\cos(\lambda_1)\sinh(\lambda_2)\,{\lambda_2}^2 - \Omega^4\beta\,\sin(\lambda_1)\,{\lambda_1}^2\sinh(\lambda_2)\right.\\
&-\Omega^2\sin(\lambda_1)\cosh(\lambda_2)\,{\lambda_2}^3 + \Omega^4\sin(\lambda_1)\,\beta\,\sinh(\lambda_2)\,{\lambda_2}^2 + \cos(\lambda_1)\,{\lambda_1}^3\Omega^2\sinh(\lambda_2)\\
&\left.-2\,\lambda_1\Omega^4\cos(\lambda_1)\,\beta\,\cosh(\lambda_2)\,\lambda_2 + 2\,\Omega^4\lambda_2\beta\,\lambda_1\right)\alpha + \left(\lambda_1{\lambda_2}^3 - \cos(\lambda_1)\,\lambda_1\cosh(\lambda_2)\,{\lambda_2}^3\right.\\
&\left.-2\,\sin(\lambda_1)\,{\lambda_1}^2\sinh(\lambda_2)\,{\lambda_2}^2 - {\lambda_1}^3\lambda_2 + \cos(\lambda_1)\,{\lambda_1}^3\cosh(\lambda_2)\,\lambda_2\right)\widetilde{\nu} + {\lambda_1}^5\lambda_2 + \lambda_1{\lambda_2}^5\\
&+2\,\cos(\lambda_1)\,{\lambda_1}^3\cosh(\lambda_2)\,{\lambda_2}^3 + \sin(\lambda_1)\,{\lambda_1}^4\sinh(\lambda_2)\,{\lambda_2}^2 - \sin(\lambda_1)\,{\lambda_1}^2\sinh(\lambda_2)\,{\lambda_2}^4\\
&-\sin(\lambda_1)\,{\lambda_1}^4\Omega^2\beta\,\cosh(\lambda_2)\,\lambda_2 - \Omega^2\beta\,\sin(\lambda_1)\,{\lambda_1}^2\cosh(\lambda_2)\,{\lambda_2}^3 - \Omega^2\beta\,\cos(\lambda_1)\,\lambda_1\sinh(\lambda_2)\,{\lambda_2}^4\\
&-\cos(\lambda_1)\,{\lambda_1}^3\Omega^2\beta\,\sinh(\lambda_2)\,{\lambda_2}^2 = 0.
\end{aligned}
\tag{30}
$$

Case 2

If $\widetilde{\nu} > 0$ and $\Omega^2 - \eta < 0$ or $\Omega^2 < \eta$ then both the roots are real and negative. Therefore, all of the four roots can be expressed as

$$\lambda = \pm \mathrm{i}\lambda_1, \quad \pm \mathrm{i}\widehat{\lambda}_2 \tag{31}$$

where λ_1 is as in the previous case

$$\lambda_1 = \left(\frac{\widetilde{\nu}}{2} + \sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 - (\eta - \Omega^2)} \right)^{1/2} \tag{32}$$

and $\widehat{\lambda}_2$ is given by

$$\widehat{\lambda}_2 = \left(\frac{\widetilde{\nu}}{2} - \sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 - (\eta - \Omega^2)} \right)^{1/2} \tag{33}$$

In view of the roots in in Eq. (**??**) the solution $W(\xi)$ can be expressed as

$$W(\xi) = a_1 \sin \lambda_1 \xi + a_2 \cos \lambda_1 \xi + a_3 \sin \widehat{\lambda}_2 \xi + a_4 \cos \widehat{\lambda}_2 \xi \tag{34}$$

Case 2

Substituting $W(\xi)$ in the boundary conditions and eliminating the constants we have the frequency equation

$$\Big(\cos(\lambda_1)\,\lambda_1 \cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^3 - \lambda_1\widehat{\lambda}_2^3 - {\lambda_1}^3\widehat{\lambda}_2 + \cos(\lambda_1)\,{\lambda_1}^3 \cos\Big(\widehat{\lambda}_2\Big)\,\lambda_2 + 2\,\sin(\lambda_1)\,{\lambda_1}^2 \sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^2$$

$$+\Big(-\Omega^4\beta\,\sin(\lambda_1)\,{\lambda_1}^2 \sin\Big(\widehat{\lambda}_2\Big) + \cos(\lambda_1)\,{\lambda_1}^3\Omega^2 \sin\Big(\widehat{\lambda}_2\Big) - 2\,\lambda_1\Omega^4 \cos(\lambda_1)\,\beta\,\cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2 + 2\,\lambda_1\Omega^4$$

$$-\lambda_1\Omega^2\cos(\lambda_1)\sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^2 + \Omega^2\sin(\lambda_1)\cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^3 - \sin(\lambda_1)\,{\lambda_1}^2\widehat{\lambda}_2\Omega^2\cos\Big(\widehat{\lambda}_2\Big)$$

$$-\Omega^4\sin(\lambda_1)\,\beta\,\sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^2\Big)\,\alpha - 2\,\cos(\lambda_1)\,{\lambda_1}^3\cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^3 - \sin(\lambda_1)\,{\lambda_1}^4\sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^2$$

$$-\sin(\lambda_1)\,{\lambda_1}^2\sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^4 + {\lambda_1}^5\widehat{\lambda}_2 - \Omega^2\beta\,\cos(\lambda_1)\,\lambda_1\sin\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^4 + \Omega^2\beta\,\sin(\lambda_1)\,{\lambda_1}^2\cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2^3$$

$$-\sin(\lambda_1)\,{\lambda_1}^4\Omega^2\beta\,\cos\Big(\widehat{\lambda}_2\Big)\,\widehat{\lambda}_2 + \cos(\lambda_1)\,{\lambda_1}^3\Omega^2\beta\,\sin(\lambda_2)\,\widehat{\lambda}_2^2 + \lambda_1\widehat{\lambda}_2^5 = 0 \qquad (35)$$

Numerical Illustration

$$\nu = \frac{PL^2}{EI} = \frac{\pi^2}{4}(P/P_{cr}) \quad (36)$$

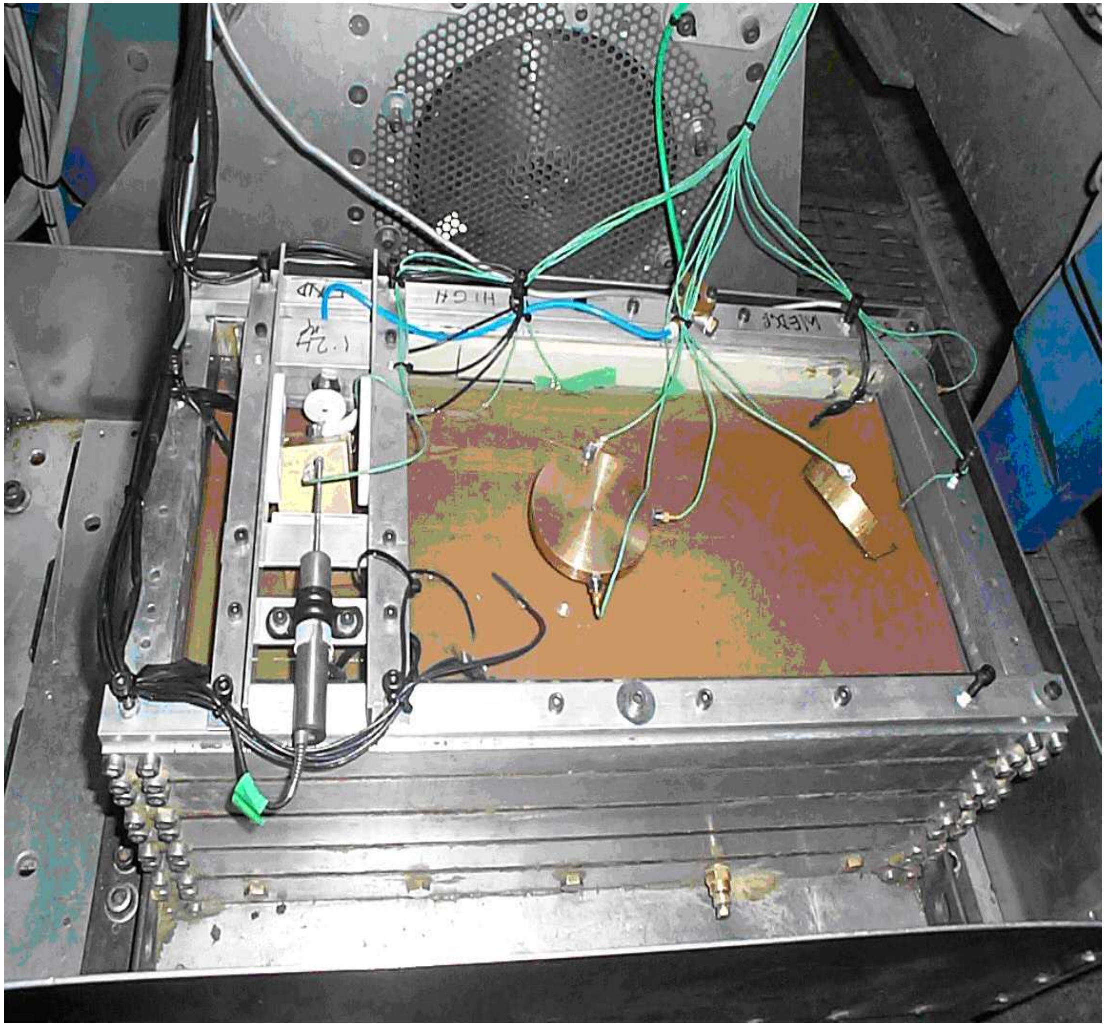
$$\eta = \frac{kL^4}{EI} \quad (37)$$

$$\Omega^2 = \omega^2 \frac{mL^4}{EI} = 4.7 \times 10^{-5}\omega^2 \quad (38)$$

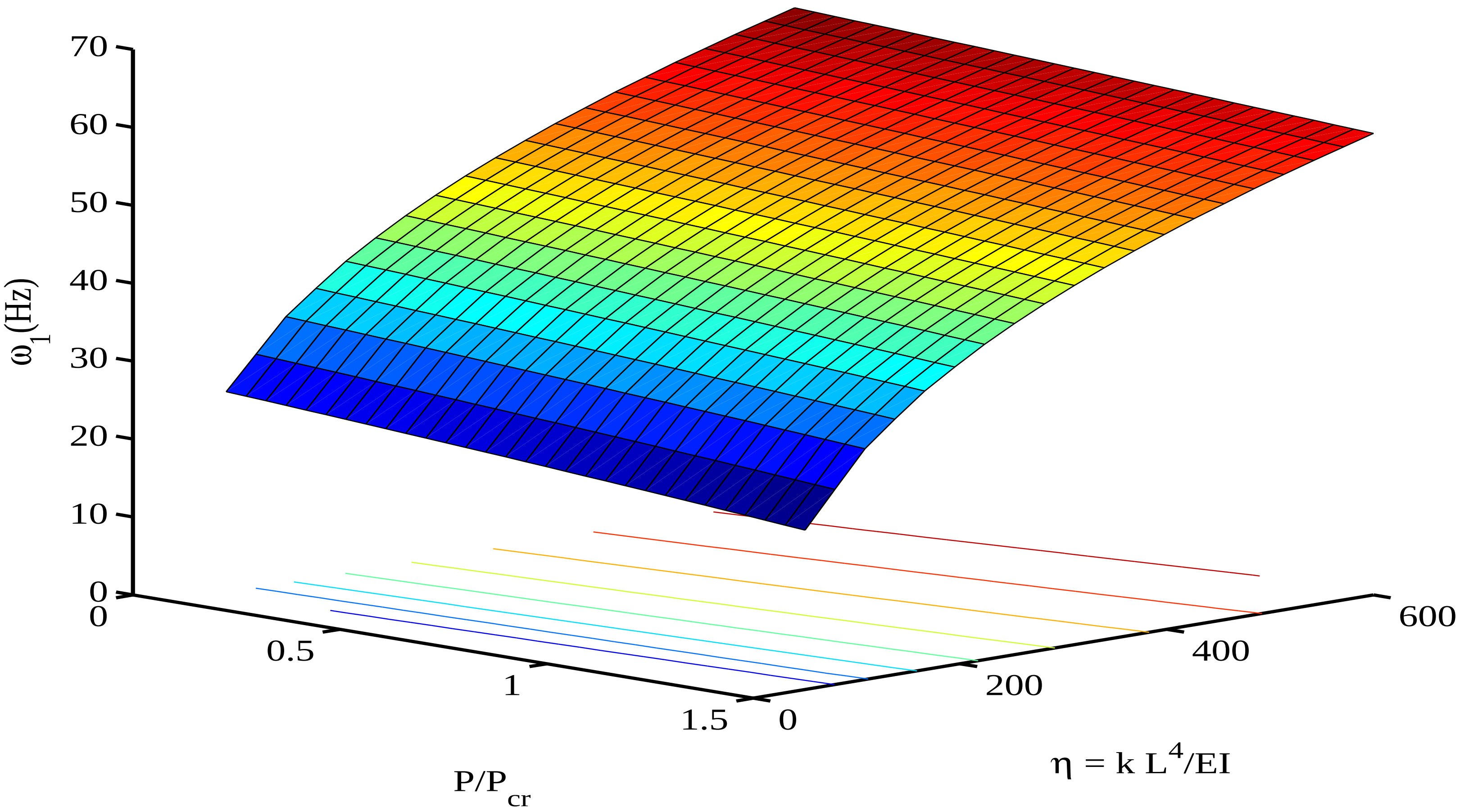
$$\alpha = \frac{M}{mL} = 10.18 \quad (39)$$

$$\beta = \frac{J}{mL^3} = 0.099 \quad (40)$$

$$\mu = \frac{r}{L} = 0.016 \quad (41)$$

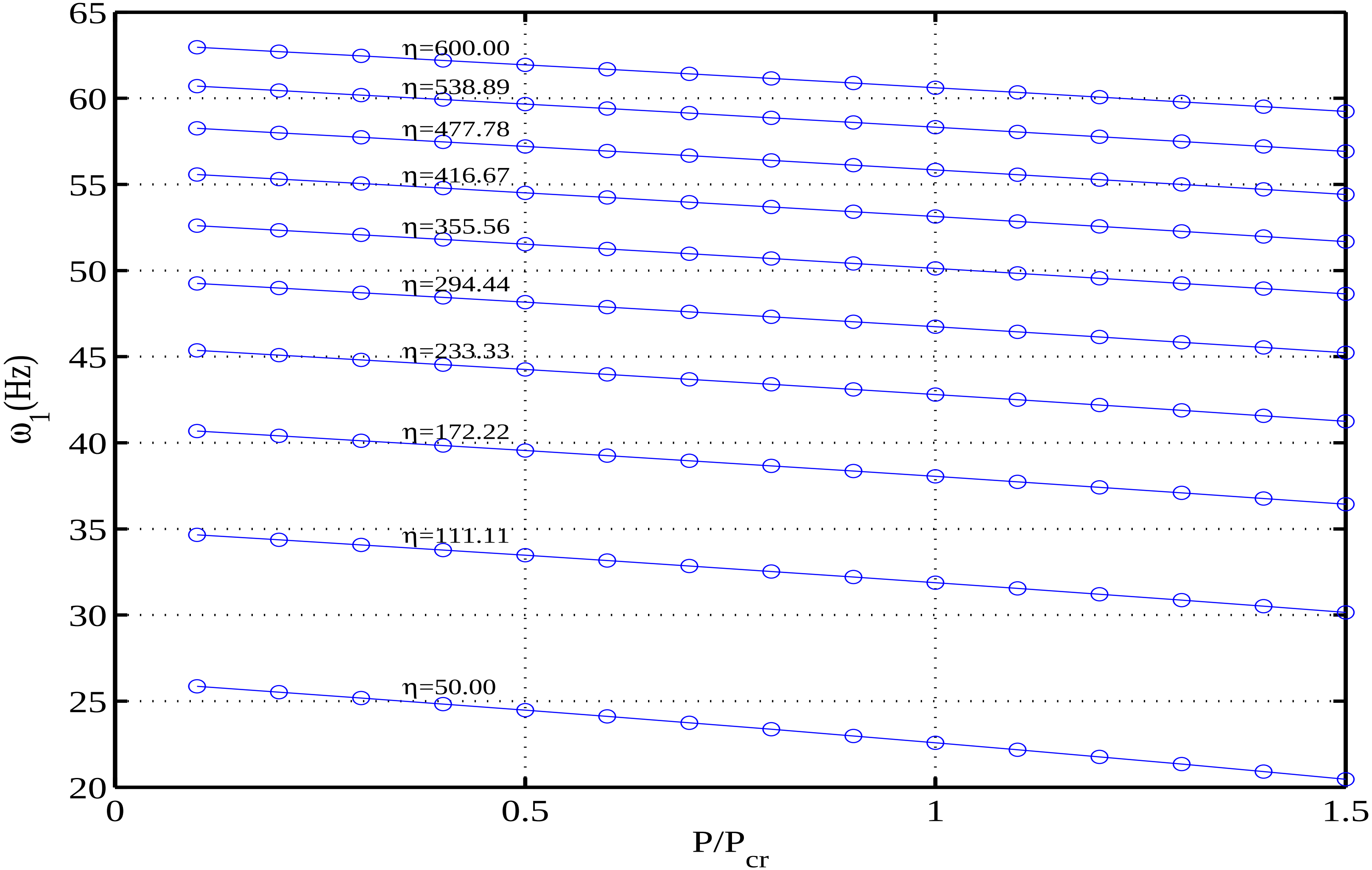


Parametric variation I



Variation of the resonance frequency

Parametric variation II



Variation of the resonance frequency for different values of η

Observations

- Natural frequency decreases with the increase in the axial load and with the decrease in the surrounding soil stiffness. When the soil is fully liquified, the natural frequency can come very close to the frequency of earthquake excitation (usually 1-10 Hz).
- Closed-form equation for the natural frequencies cannot be obtained due to the complex nature of the transcendental frequency equation. For pinned-pinned boundary conditions we have:

$$\Omega_n^2 = \frac{n^4\pi^4}{1+\mu^2 n^2\pi^2}\left(1 - \frac{P/P_{cr}}{n^2} + \frac{\eta}{n^4\pi^4}\right) \tag{42}$$

Conclusions & Outlook

- Pile dynamics is a coupled phenomenon: the response leading to failure is a fusion of bending, buckling and resonance. We are investigating the usefulness of a unified coupled dynamical system approach for the pile design.
- Our approach leads to a high-fidelity modeling (however, needs advanced mathematical/computational tools).
- Further work will look at the detailed dynamical response, optimal design and probabilistic reliability analysis.