Uncertainty Quantification and Propagation in Aerospace Structural Dynamical Models

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Outline of the presentation

- Probabilistic structural dynamics
- Random matrix model for aerospace systems
- Wishart random matrices
- Uncertainty propagation
- Random eigenvalue problems
- Experimental validation
- Open problems & discussions

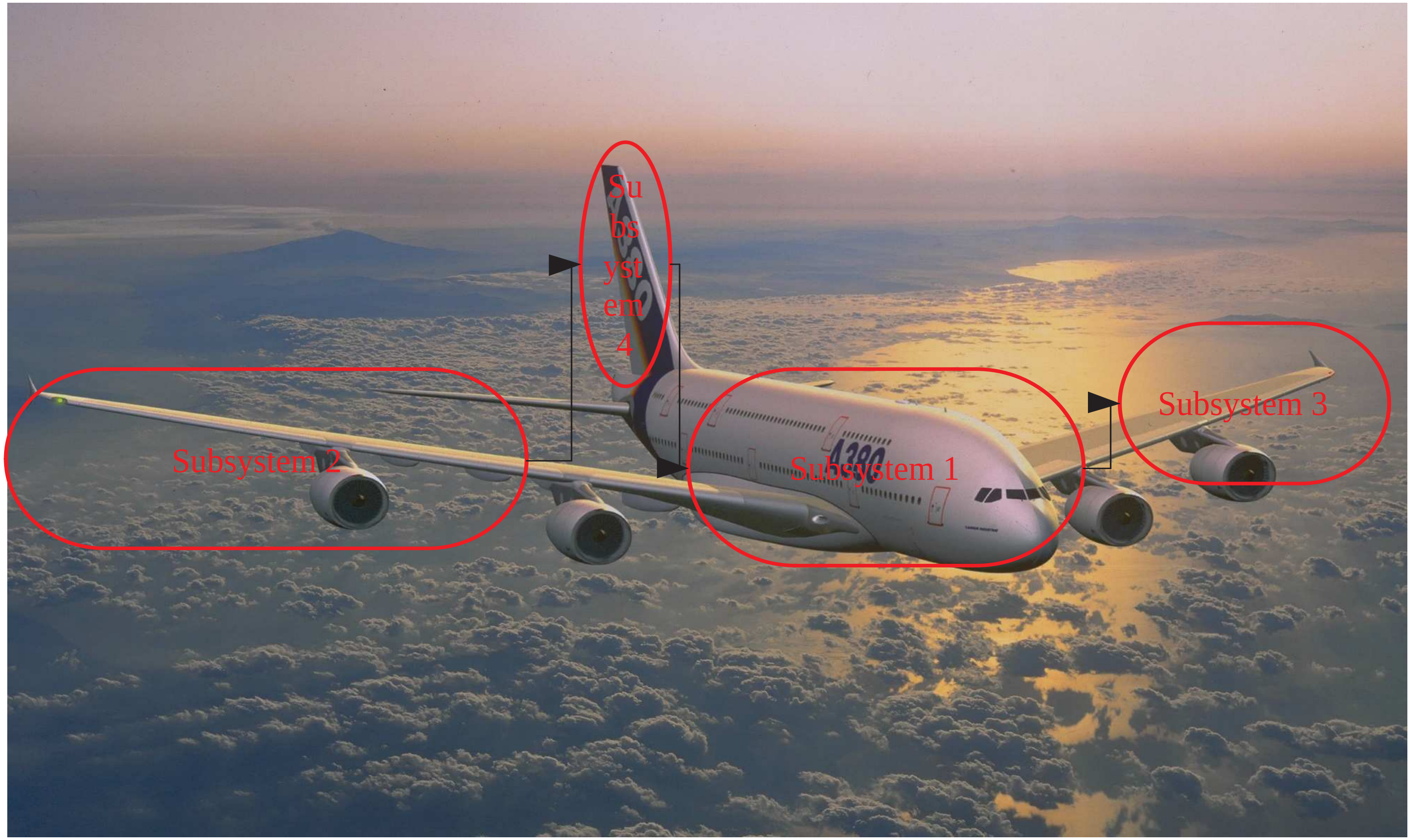
Overview of Predictive Methods in Engineering

There are five key steps:

- Physics (mechanics) model building
- Uncertainty Quantification (UQ)
- Uncertainty Propagation (UP)
- Model Verification & Validation (V & V)
- Prediction

Tools are available for each of these steps. Currently, our focus is mainly on UQ and UP in linear dynamical systems.

Complex aerospace model



Possible subsystem models for an aircraft

Why uncertainty?

Different sources of uncertainties in the modeling and simulation of dynamic systems may be attributed, but not limited, to the following factors:

- Mathematical models: equations (linear, non-linear), geometry, damping model (viscous, non-viscous, fractional derivative), boundary conditions/initial conditions, input forces;
- Model parameters: Young's modulus, mass density, Poisson's ratio, damping model parameters (damping coefficient, relaxation modulus, fractional derivative order)

Why uncertainty?

- Numerical algorithms: weak formulations, discretisation of displacement fields (in finite element method), discretisation of stochastic fields (in stochastic finite element method), approximate solution algorithms, truncation and roundoff errors, tolerances in the optimization and iterative methods, artificial intelligent (AI) method (choice of neural networks)
- Measurements: noise, resolution (number of sensors and actuators), experimental hardware, excitation method (nature of shakers and hammers), excitation and measurement point, data processing (amplification, number of data points, FFT), calibration

Structural dynamics

- The equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{x}$

Current Methods

Two different approaches are currently available

- Low frequency : Stochastic Finite Element Method (SFEM) - assumes that stochastic fields describing parametric uncertainties are known in details
- High frequency : Statistical Energy Analysis (SEA) - do not consider parametric uncertainties in details

Random Matrix Method (RMM)

- The objective: To have an unified method which will work across the frequency range.
- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \left|\mathbf{\Sigma}\right|^{-p/2} \left|\mathbf{\Psi}\right|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (1)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n \left(\frac{1}{2} p \right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{S} \right\} \tag{2}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \left|\mathbf{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\}; \quad \Re(a) > \frac{1}{2}(n- \tag{3}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \tag{4}$$

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$

Distribution of the system matrices

- The exact application of the last constraint requires the derivation of the joint probability density function of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, which is quite difficult to obtain.
- We consider a simpler problem where it is required that the inverse moments of each of the system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ must exist.
- Provided the system is damped, this will guarantee the existence of the moments of the frequency response function matrix.

Maximum Entropy Distribution

Soize (2000,2006) used this approach and obtained the matrix variate Gamma distribution.
Since Gamma and Wishart distribution are similar we have:

Theorem 1. *If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/(2\nu + n + 1)$, that is*
$\mathbf{G} \sim W_n\left(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1)\right)$.

Matrix Factorization Approach (MFA)

- Because $\mathbf{G}$ is a symmetric and positive-definite random matrix, it can be always factorized as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^T \tag{5}$$

where $\mathbf{X} \in \mathbb{R}^{n \times p}, p \geq n$ is in general a rectangular matrix.

- The simplest case is when the mean of $\mathbf{X}$ is $\mathbf{O} \in \mathbb{R}^{n \times p}, p \geq n$ and the covariance tensor of $\mathbf{X}$ is given by $\mathbf{\Sigma} \otimes \mathbf{I}_p \in \mathbb{R}^{np \times np}$ where $\mathbf{\Sigma} \in \mathbb{R}_n^+$.
- $\mathbf{X}$ is a Gaussian random matrix with mean $\mathbf{O} \in \mathbb{R}^{n \times p}, p \geq n$ and covariance $\mathbf{\Sigma} \otimes \mathbf{I}_p \in \mathbb{R}^{np \times np}$.

Wishart Pdf

After some algebra it can be shown that $\mathbf{G}$ is a $W_n(p, \mathbf{\Sigma})$ Wishart random matrix, whose pdf is given given by

$$p_{\mathbf{G}}\left(\mathbf{G}\right) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)\left|\mathbf{\Sigma}\right|^{\frac{1}{2}p}\right\}^{-1}\left|\mathbf{G}\right|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{G}\right\} \tag{6}$$

Parameter Estimation of Wishart Distribution

- The distribution of $\mathbf{G}$ must be such that $\mathrm{E}\left[\mathbf{G}\right]$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right]$ should be closest to $\overline{\mathbf{G}}$ and $\overline{\mathbf{G}}^{-1}$ respectively.
- Since $\mathbf{G} \sim W_n\left(p, \mathbf{\Sigma}\right)$, there are two unknown parameters in this distribution, namely, p and $\mathbf{\Sigma}$. This implies that there are in total $1 + n(n+1)/2$ number of unknowns.
- We define and subsequently minimize 'normalized errors':

$$\varepsilon_1 = \left\|\overline{\mathbf{G}} - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}\right\|_{\mathrm{F}}$$

$$\varepsilon_2 = \left\|\overline{\mathbf{G}}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}^{-1}\right\|_{\mathrm{F}}$$

MFA Distribution

Solving the optimization problem we have:

Theorem 2. *If* ν*-th order inverse-moment of a system matrix* $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ *exists and only the mean of* $\mathbf{G}$ *is available, say* $\overline{\mathbf{G}}$*, then the distribution of* $\mathbf{G}$ *follows the Wishart distribution with parameters* $p = (2\nu + n + 1)$ *and* $\mathbf{\Sigma} = \overline{\mathbf{G}} / \sqrt{2\nu(2\nu + n + 1)}$*, that is*

$$\mathbf{G} \sim W_n \left(2\nu + n + 1, \overline{\mathbf{G}} / \sqrt{2\nu(2\nu + n + 1)} \right).$$

Response statistics - 1

- The equation of motion is $\mathbf{D}\mathbf{x} = \mathbf{p}$, $\mathbf{D}$ is in general $n \times n$ complex random matrix.
- The response is given by

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{p}$$

- Consider static problems so that all matrices/vectors are real.

Response statistics - 2

- We may want the statistics of few elements or some linear combinations of the elements in $\mathbf{x}$. So the quantify of interest is

$$\mathbf{y} = \mathbf{R}\mathbf{x} = \mathbf{R}\mathbf{D}^{-1}\mathbf{p} \tag{7}$$

  Here $\mathbf{R}$ is in general $r \times n$ rectangular matrix. For the special case when $\mathbf{R} = \mathbf{I}_n$, we have $\mathbf{y} = \mathbf{x}$.

- Eq. (7) arises in SFEM. There are many papers on its solution. Mainly perturbation methods are used.

Response statistics - 3

Suppose $\mathbf{D} = \mathbf{D}_0 + \mathbf{\Delta D}$, where $\mathbf{D}_0$ is the deterministic part and $\mathbf{\Delta D}$ is the (small) random part. It can be shown that

$$\mathbf{D}^{-1} = \mathbf{D}_0 - \mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1} + \mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1} + \cdots$$

From, this

$$\mathbf{y} = \mathbf{y}_0 - \mathbf{R}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{x}_0 + \mathbf{R}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{D}_0^{-1}\mathbf{\Delta D}\mathbf{x}_0 + \cdots \quad (8)$$

where $\mathbf{x}_0 = \mathbf{D}_0^{-1}\mathbf{p}$ and $\mathbf{y}_0 = \mathbf{R}\mathbf{x}_0$.

Response statistics - 4

The statistics of $\mathbf{y}$ can be calculated from Eq. (8). However,

- The calculation is difficult if $\mathbf{\Delta D}$ is non-Gaussian.
- Even if $\mathbf{\Delta D}$ is Gaussian, inclusion of higher-order terms results very messy calculations (I have not seen any published work for more than second-order)
- For these reasons, the response statistics will be inaccurate for large randomness.

Response statistics - 5

Response moments can be obtained exactly using RMT. Suppose $\mathbf{D} \sim W_n\left(n+1+\theta, \mathbf{\Sigma}\right)$.

$$\mathrm{E}\left[\mathbf{y}\right] = \mathrm{E}\left[\mathbf{R}\mathbf{D}^{-1}\mathbf{p}\right] = \mathbf{R}\,\mathrm{E}\left[\mathbf{D}^{-1}\right]\mathbf{p} = \mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{p}/\theta \quad (9)$$

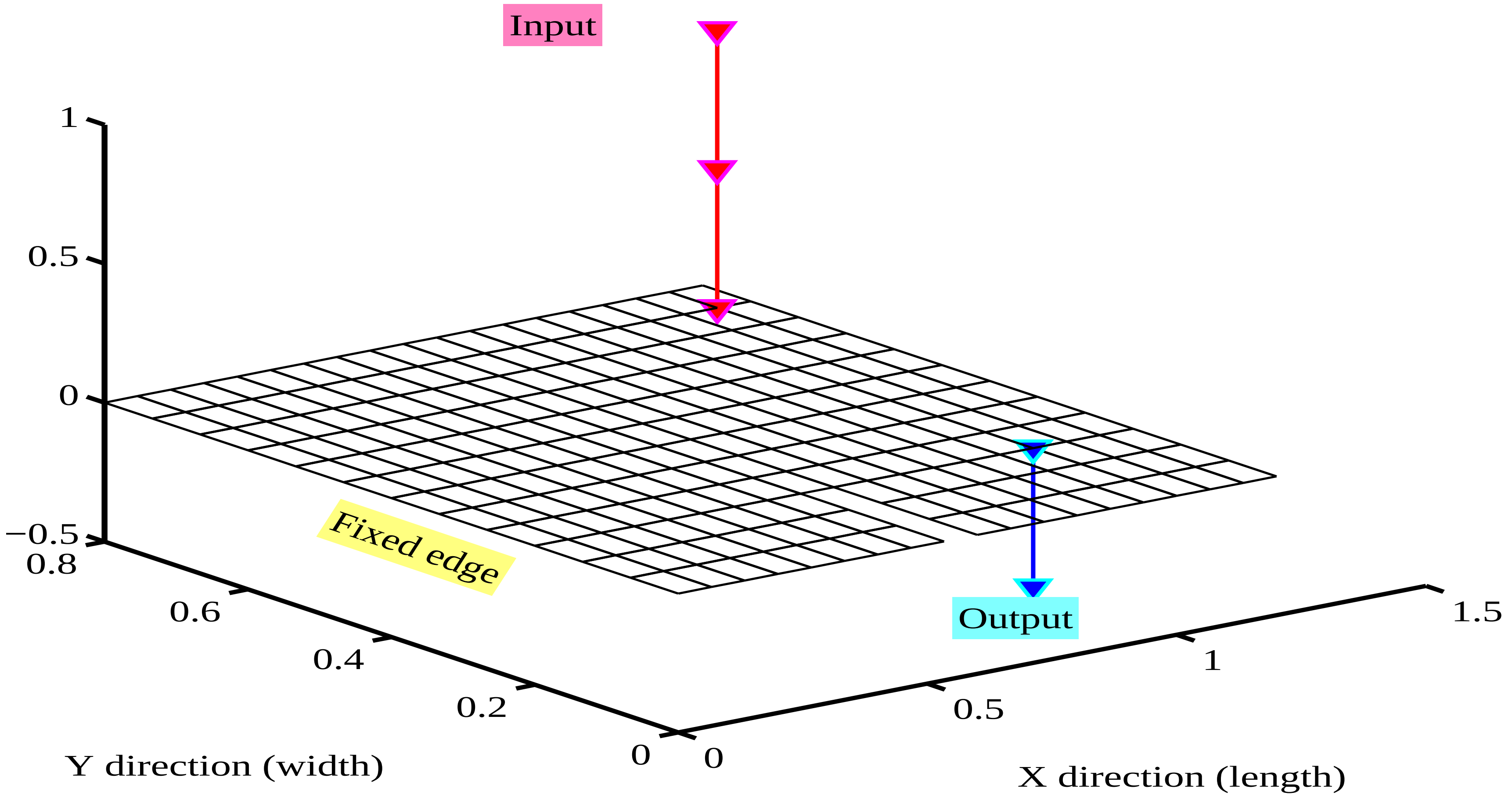
The complete covariance matrix of $\mathbf{y}$

$$\mathrm{E}\left[(\mathbf{y}-\mathrm{E}\left[\mathbf{y}\right])(\mathbf{y}-\mathrm{E}\left[\mathbf{y}\right])^T\right]$$
$$= \mathbf{R}\,\mathrm{E}\left[\mathbf{D}^{-1}\mathbf{p}\mathbf{p}^T\mathbf{D}^{-1}\right]\mathbf{R}^T - \mathrm{E}\left[\mathbf{y}\right]\left(\mathrm{E}\left[\mathbf{y}\right]\right)^T$$
$$= \frac{\mathrm{Trace}\left(\mathbf{\Sigma}^{-1}\mathbf{p}\mathbf{p}^T\right)\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{R}^T}{\theta(\theta+1)(\theta-2)} + \frac{(\theta+2)\mathbf{R}\mathbf{\Sigma}^{-1}\mathbf{p}\mathbf{p}^T\mathbf{\Sigma}^{-1}\mathbf{R}^T}{\theta^2(\theta+1)(\theta-2)} \quad (10)$$

Simulation Algorithm: Dynamical Systems

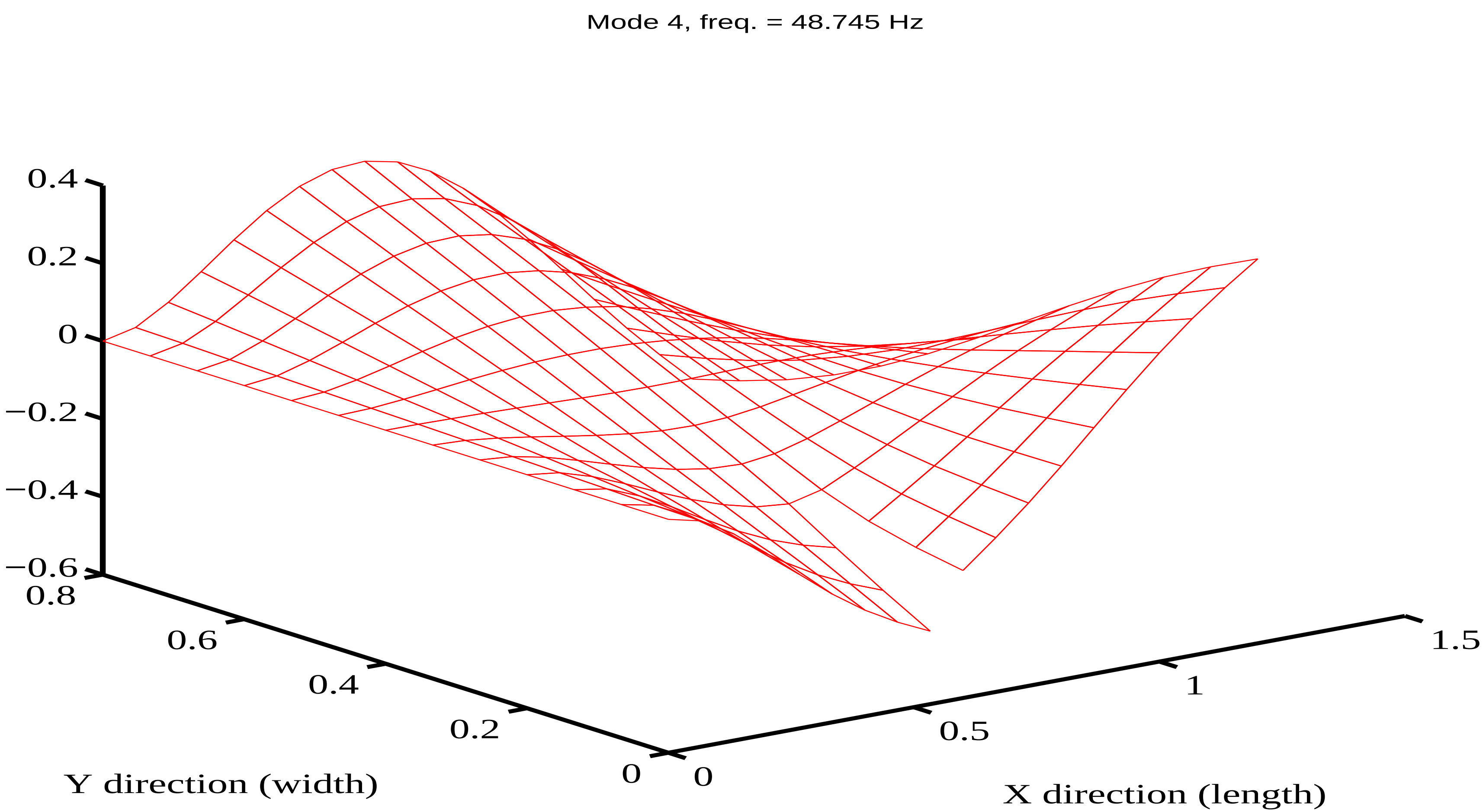
- Obtain $\theta = \frac{1}{\delta_{\mathbf{G}}^2}\left\{1+\frac{\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}\right\}-(n+1)$
- If $\theta < 4$, then select $\theta = 4$.
- Calculate $\alpha = \sqrt{\theta(n+1+\theta)}$
- Generate samples of $\mathbf{G} \sim W_n\left(n+1+\theta, \overline{\mathbf{G}}/\alpha\right)$ (MATLAB® command `wishrnd` can be used to generate the samples)
- Repeat the above steps for all system matrices and solve for every samples

Example 1: A cantilever Plate



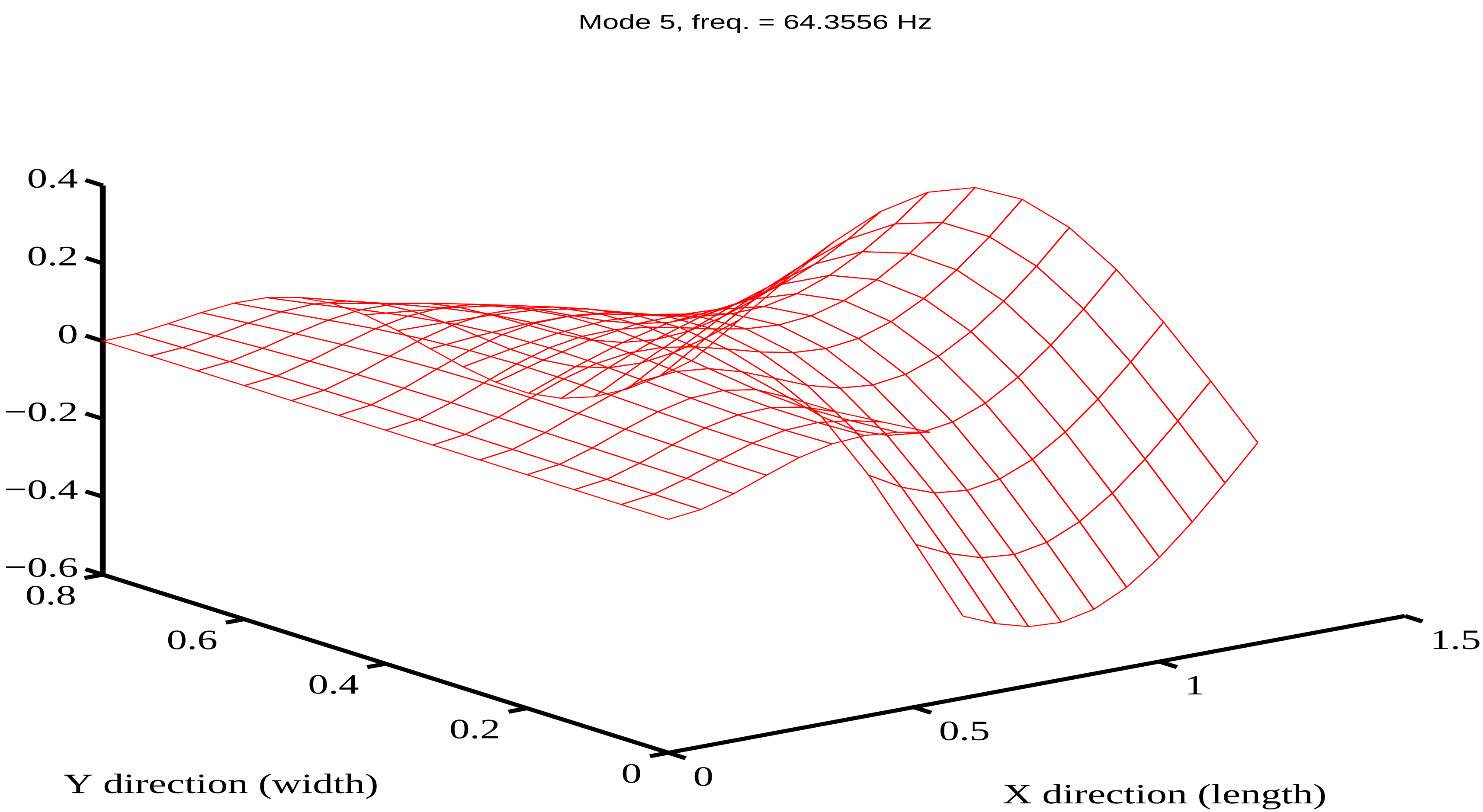
A Cantilever plate with a slot: $\bar{E} = 200 \times 10^9$N/m^2, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860$kg/m^3, $\bar{t} = 7.5$mm, $L_x = 1.2$m, $L_y = 0.8$m.

Plate Mode 4



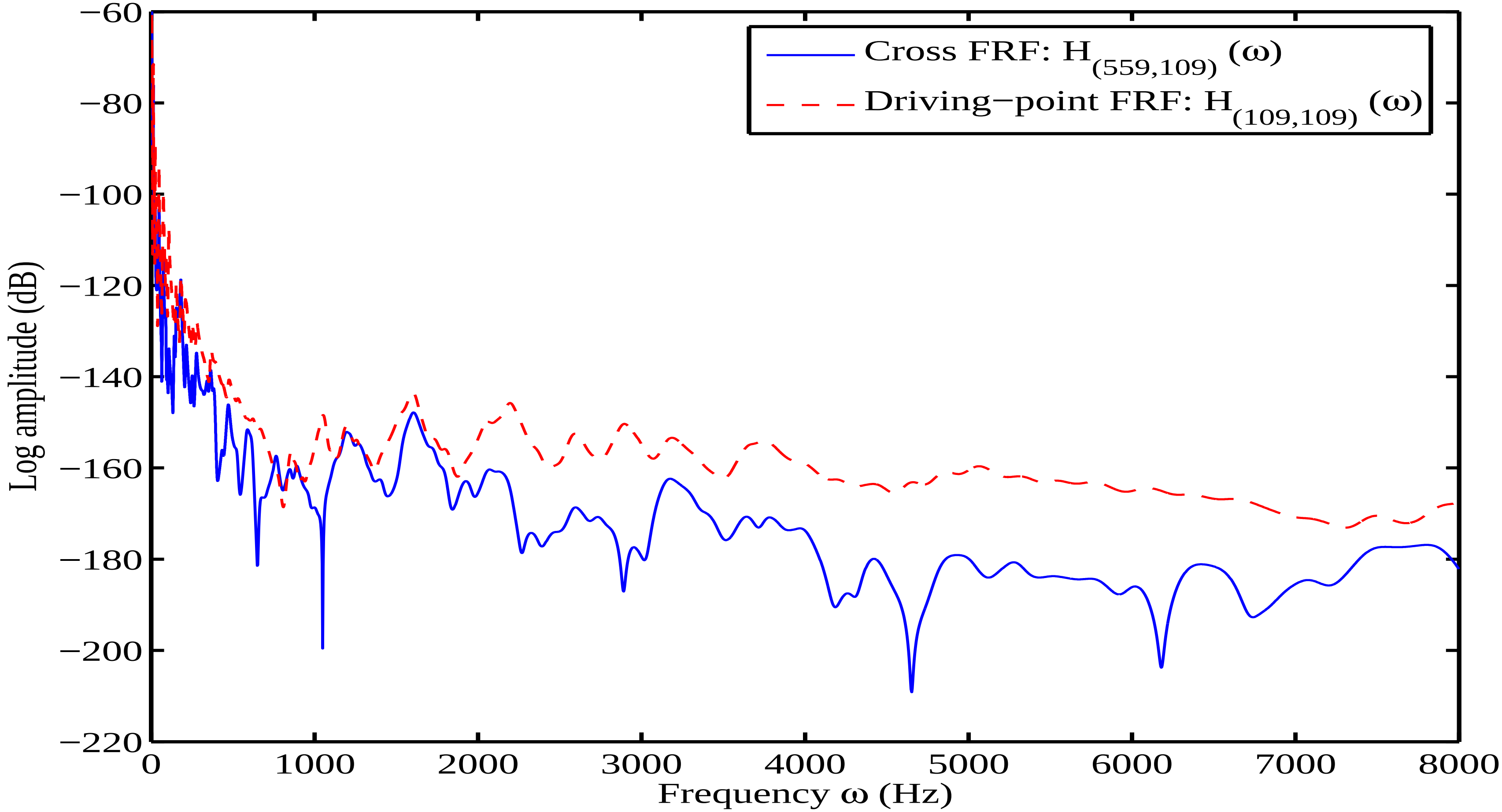
Fourth Mode shape

Plate Mode 5



Fifth Mode shape

Deterministic FRF



FRF of the deterministic plate

Stochastic Properties

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \quad (11)$$

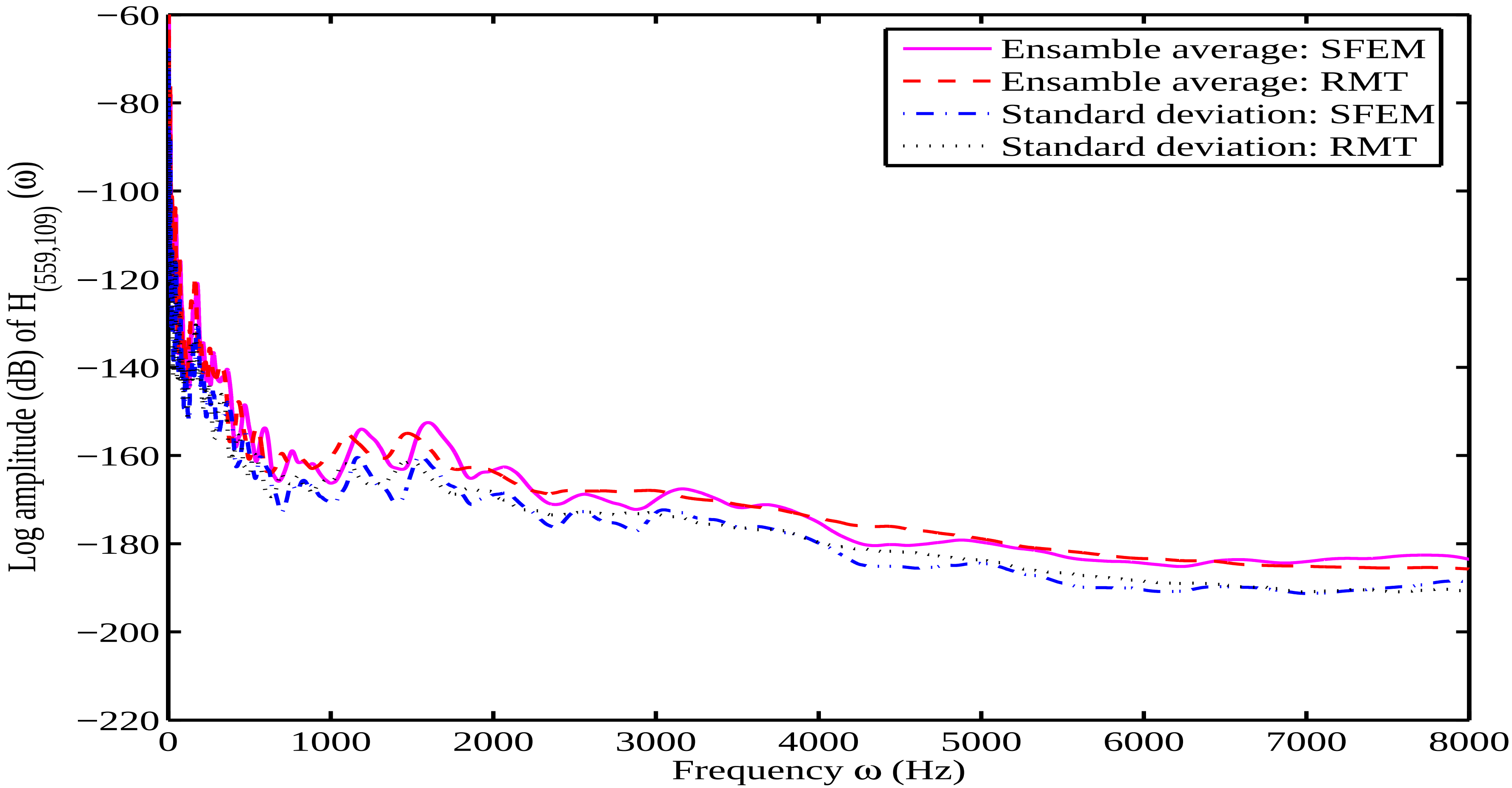
$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \quad (12)$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \quad (13)$$

and $$t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \quad (14)$$

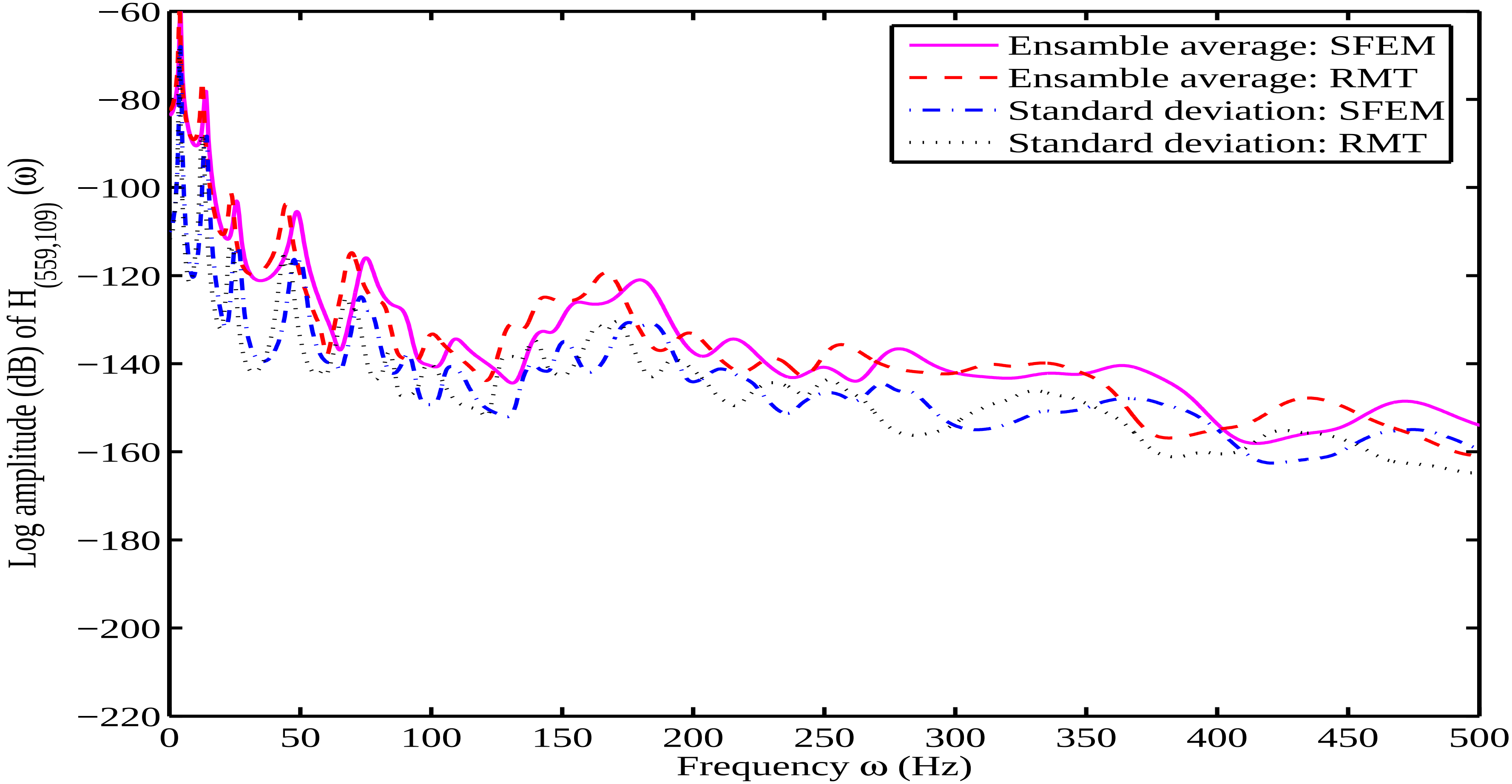
- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

Comparison of cross-FRF



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 702$, $\delta_M = 0.1166$ and $\delta_K = 0.2622$.

Comparison of cross-FRF: Low Freq



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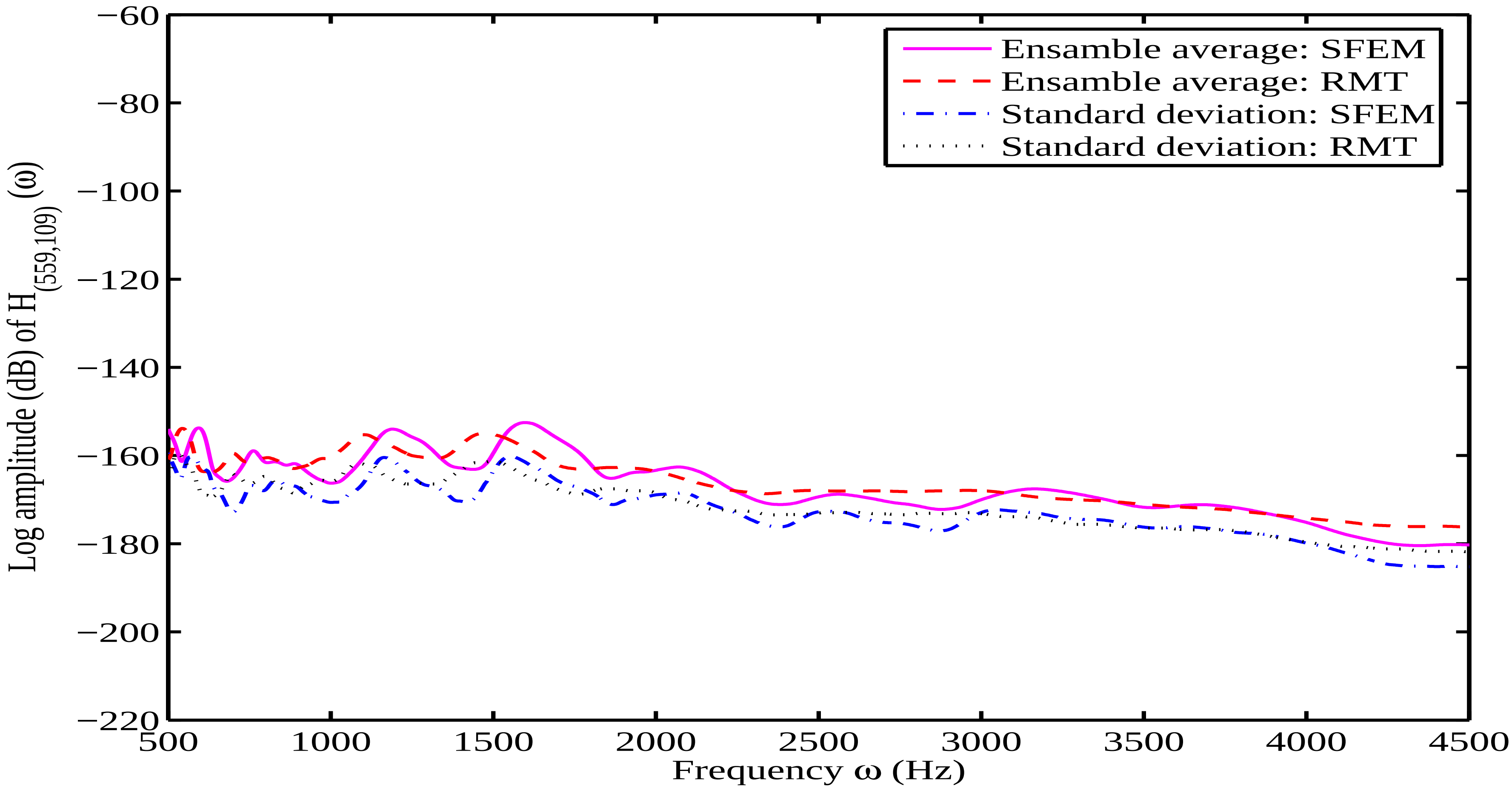
Comparison of cross-FRF: Mid Freq



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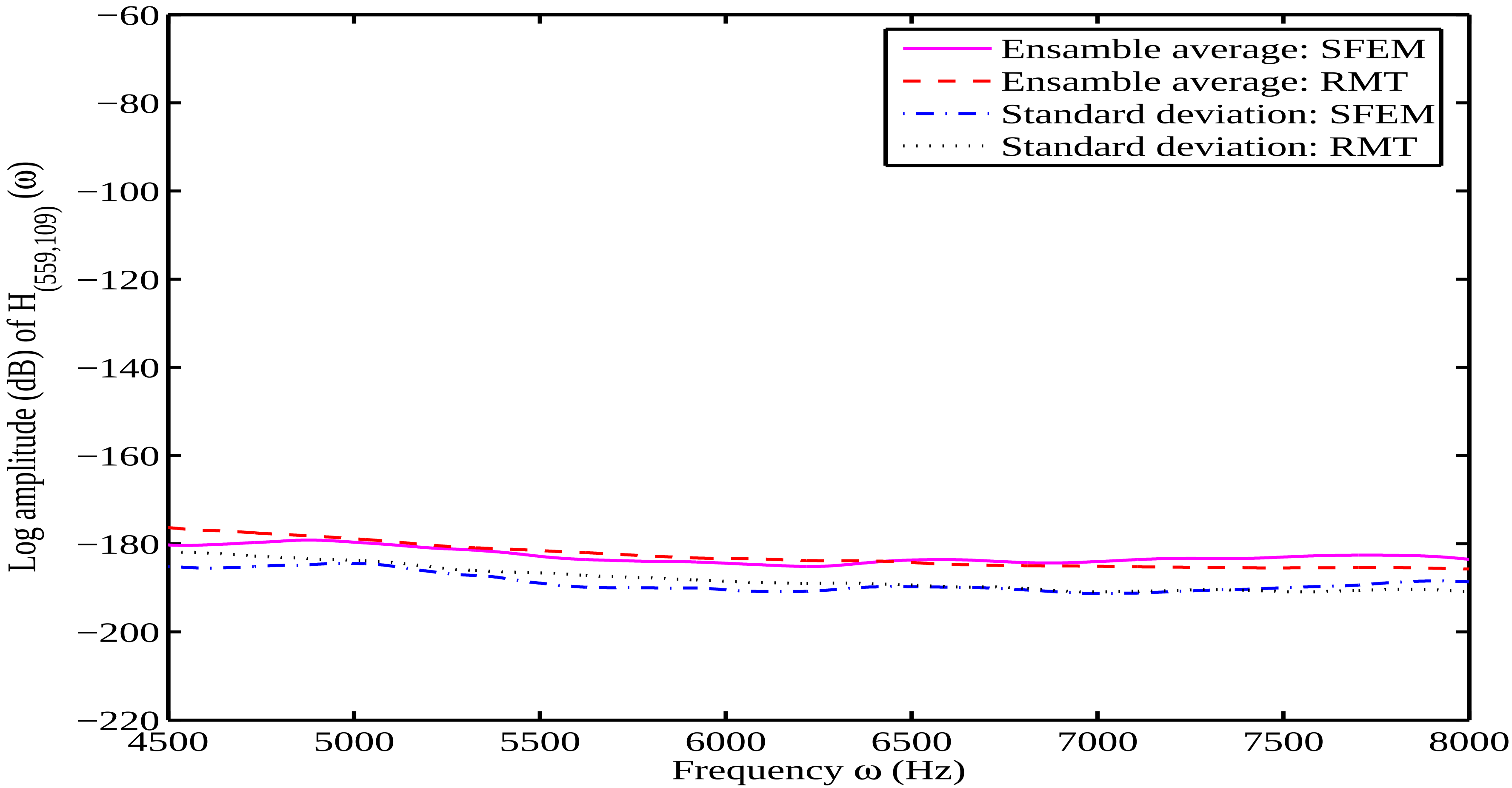
Comparison of cross-FRF: High Freq



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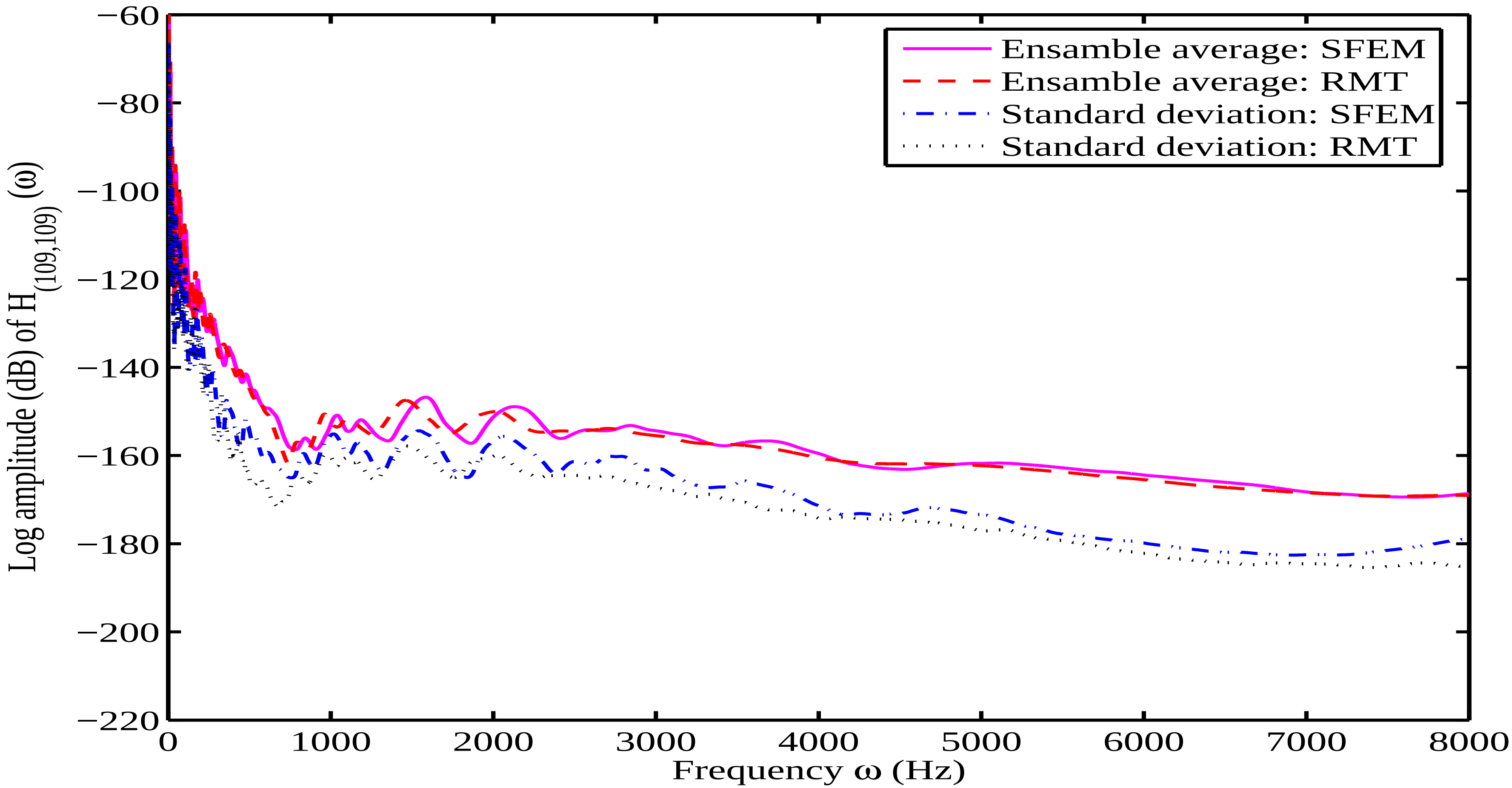
Comparison of driving-point-FRF



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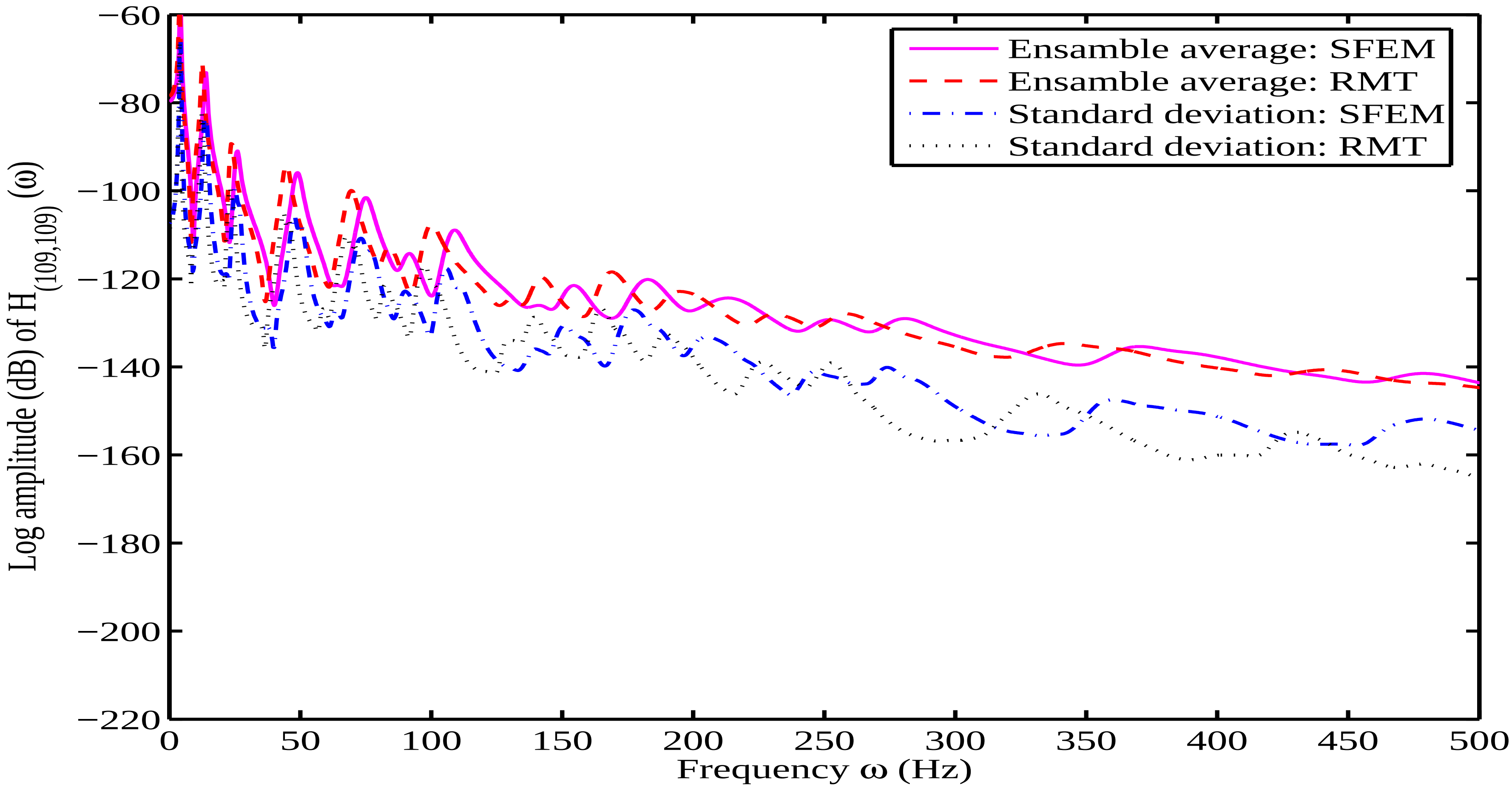
Comparison of driving-point-FRF: Low Freq



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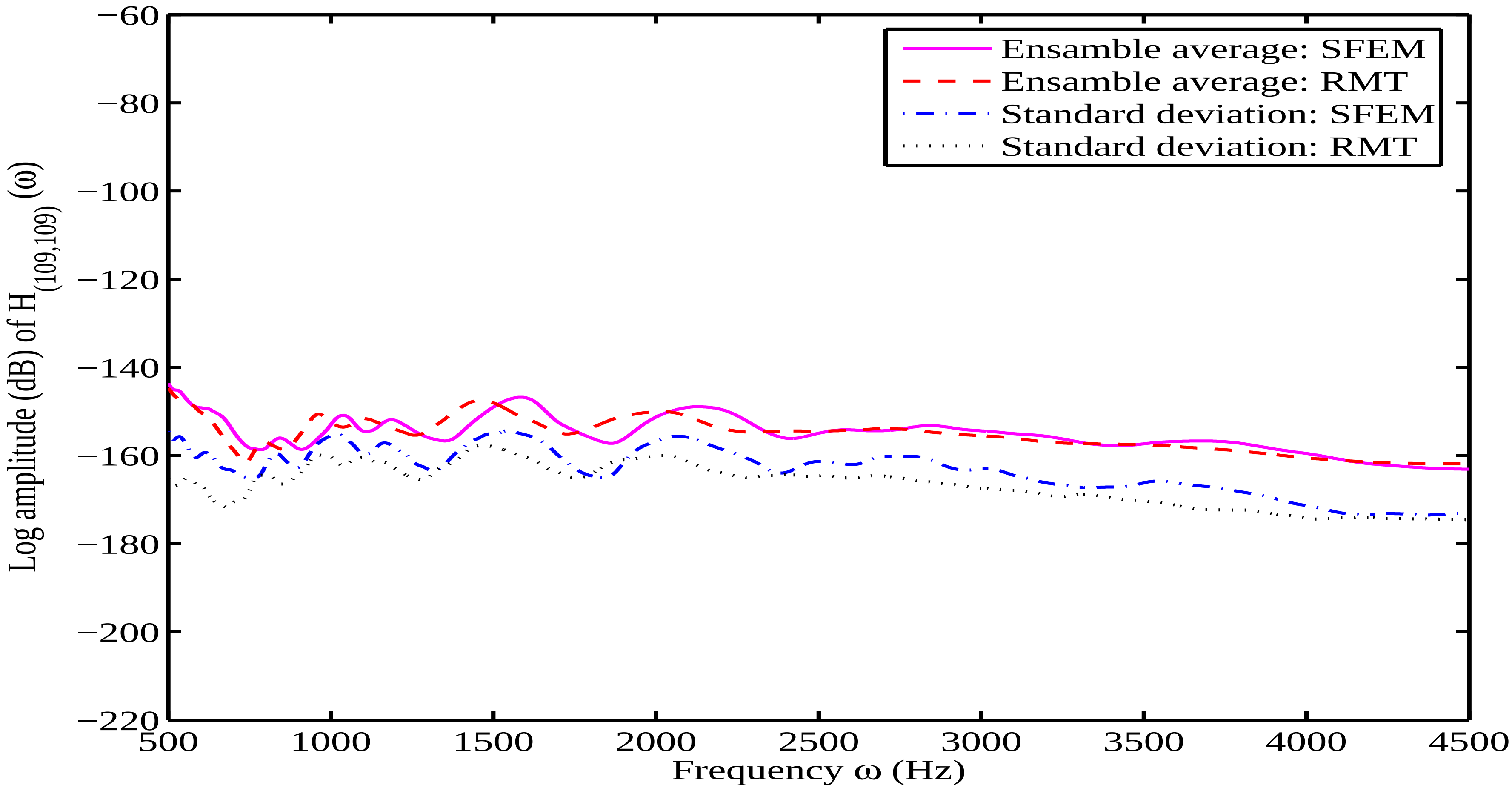
Comparison of driving-point-FRF: Mid Freq



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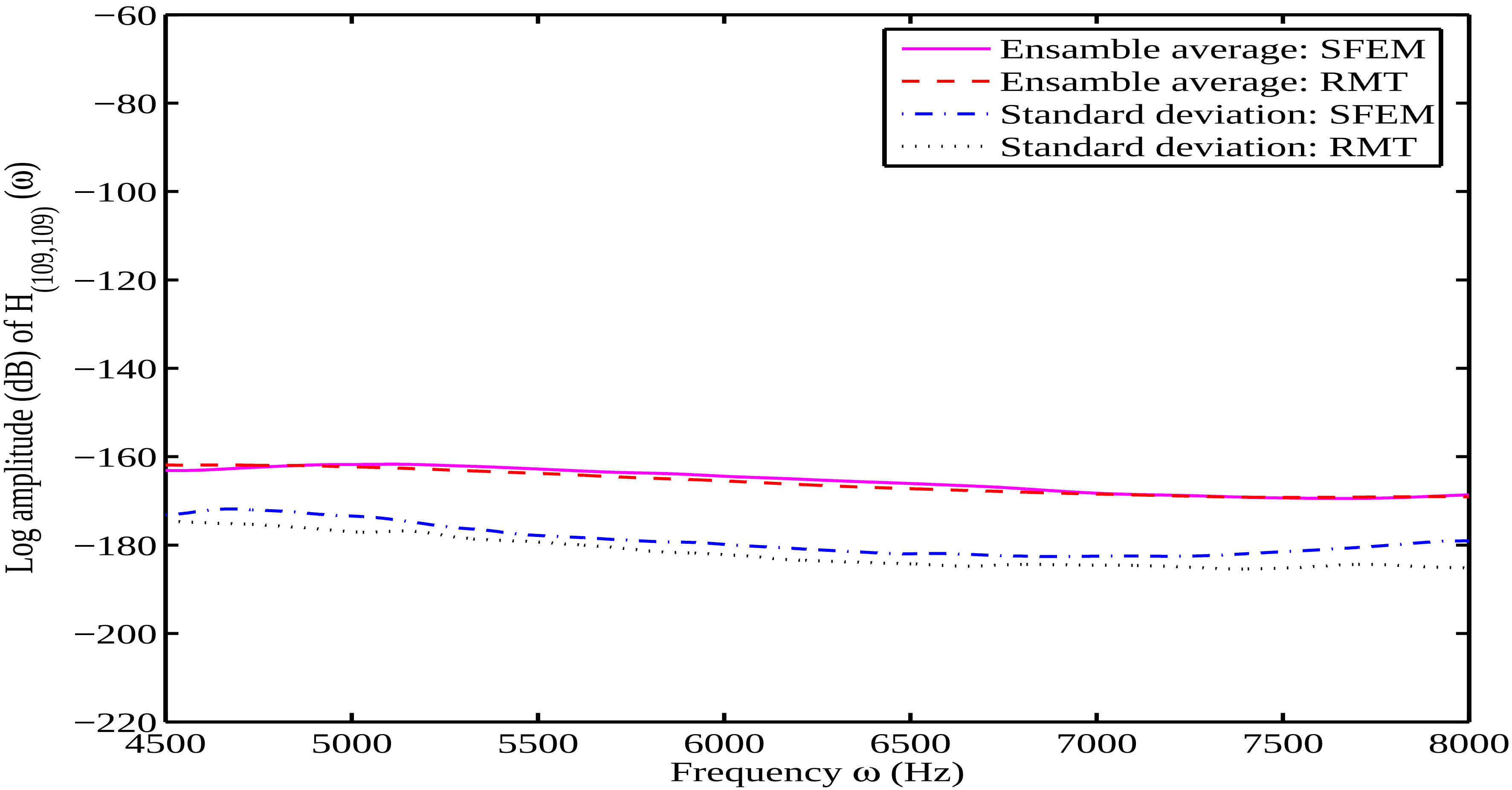
Comparison of driving-point-FRF: High Freq



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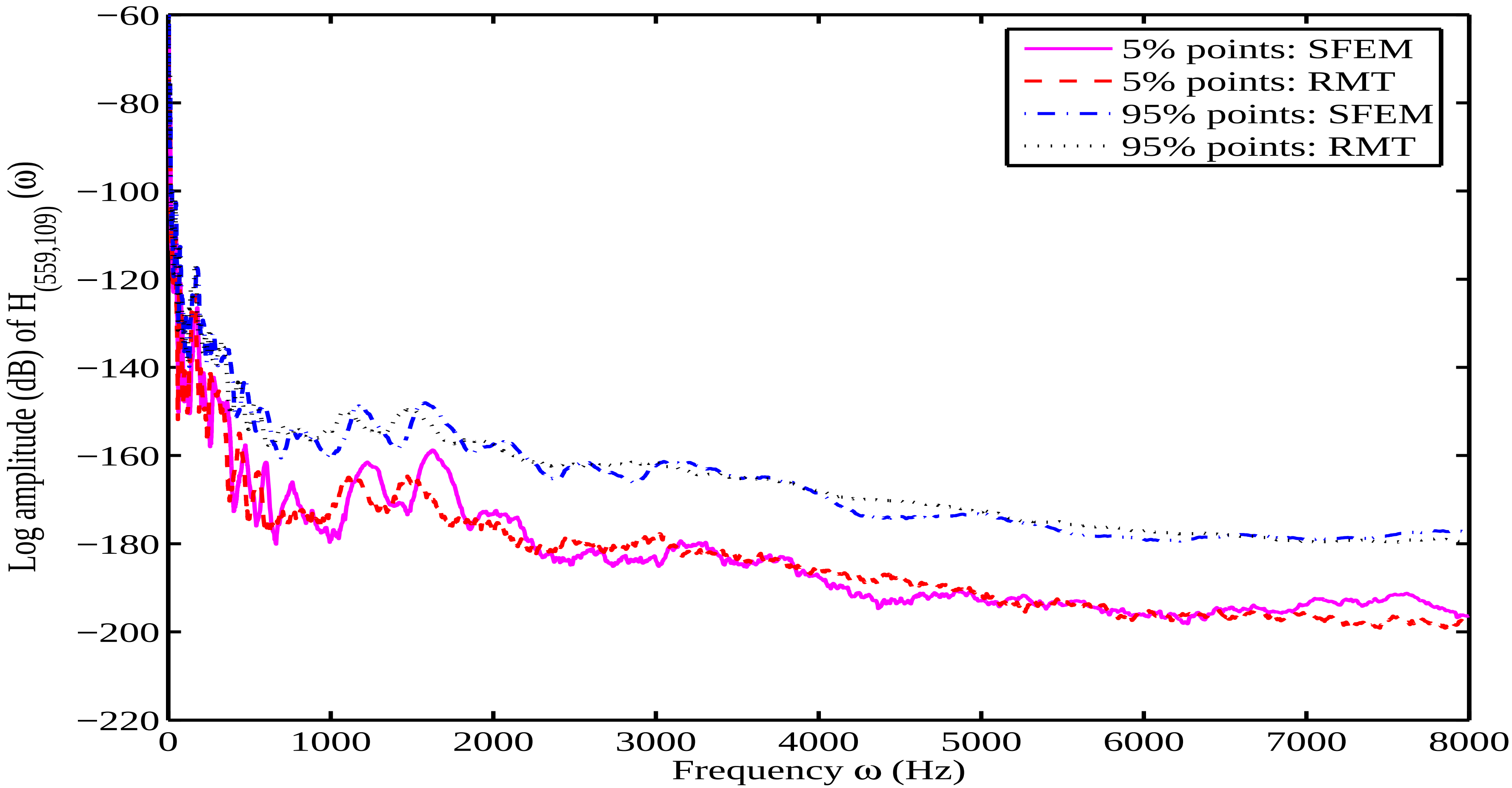
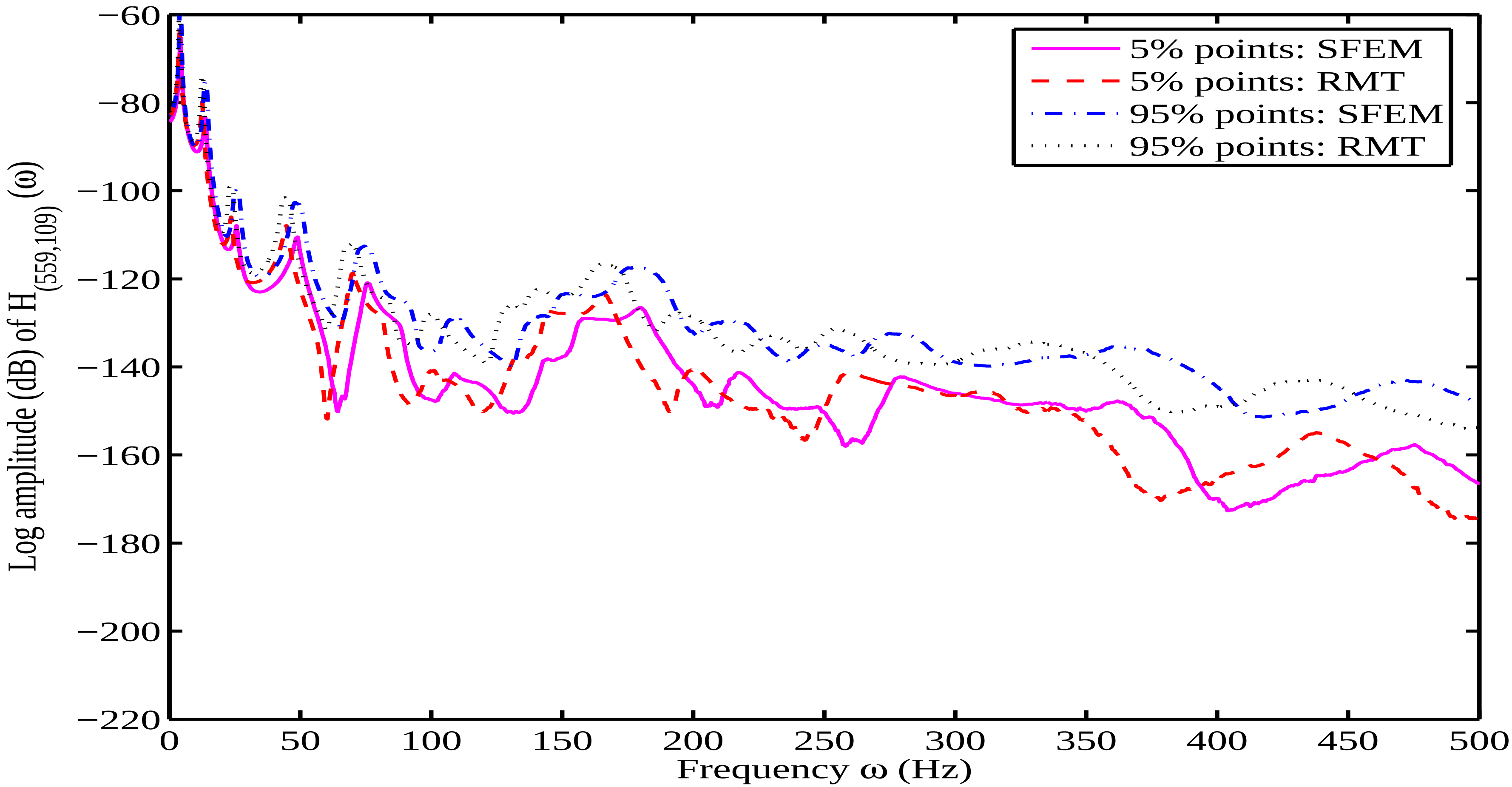
Comparison of cross-FRF



Comparison of the 5% and 95% probability points of the amplitude of the cross-FRF, $n = 702$, $\delta_M = 0.1166$ and $\delta_K = 0.2622$.

Comparison of cross-FRF: Low Freq



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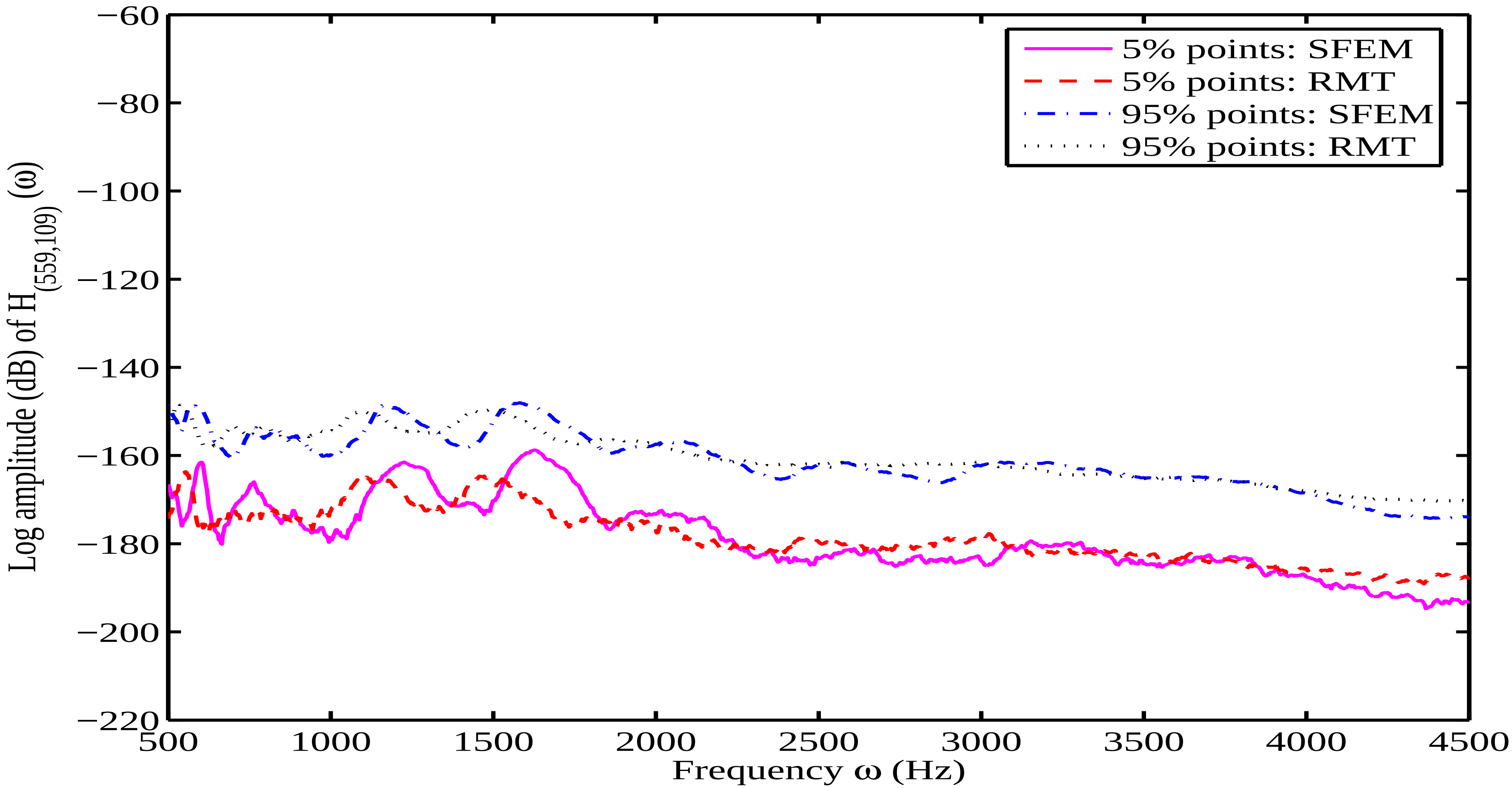
Comparison of cross-FRF: Mid Freq



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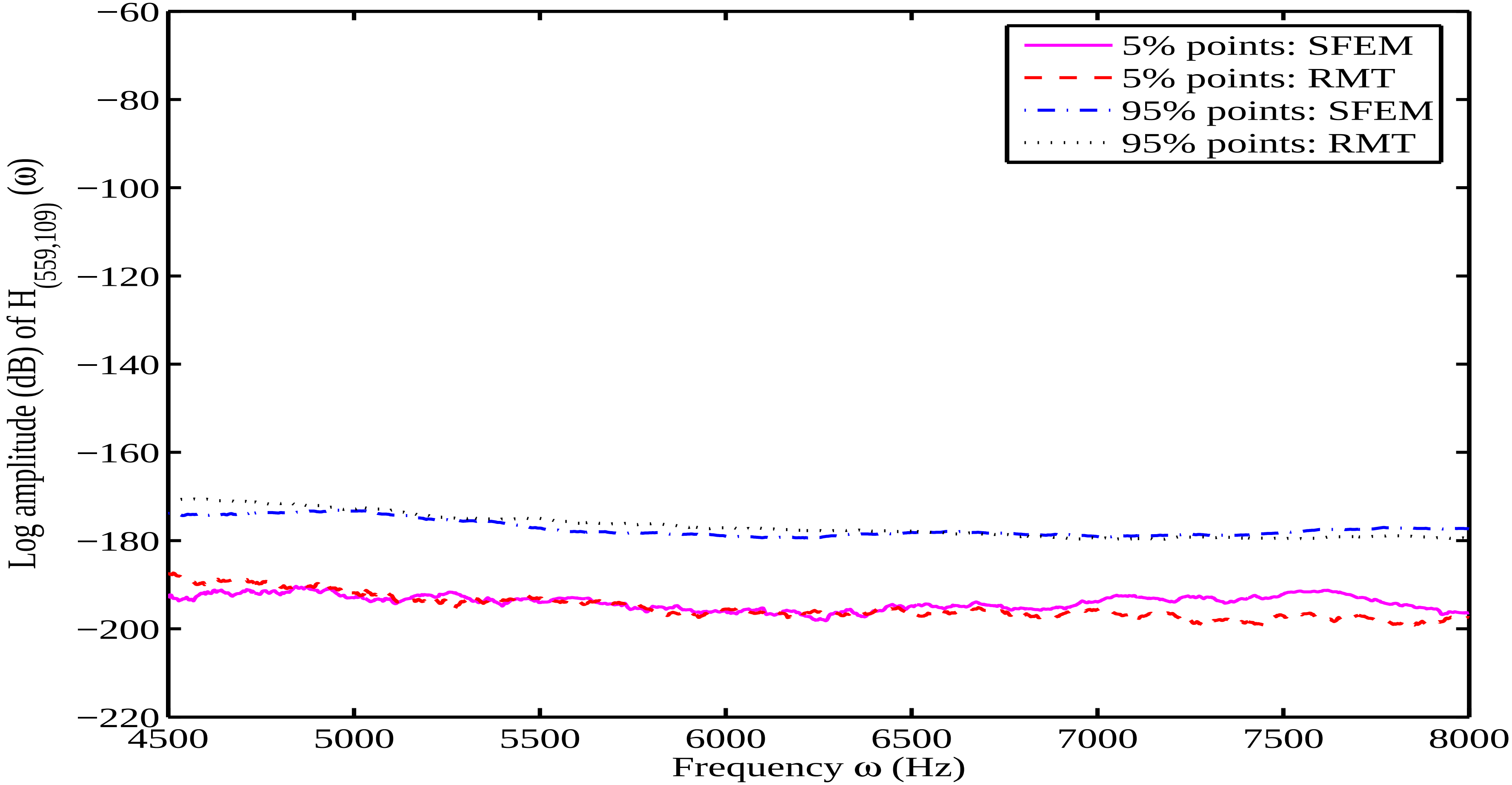

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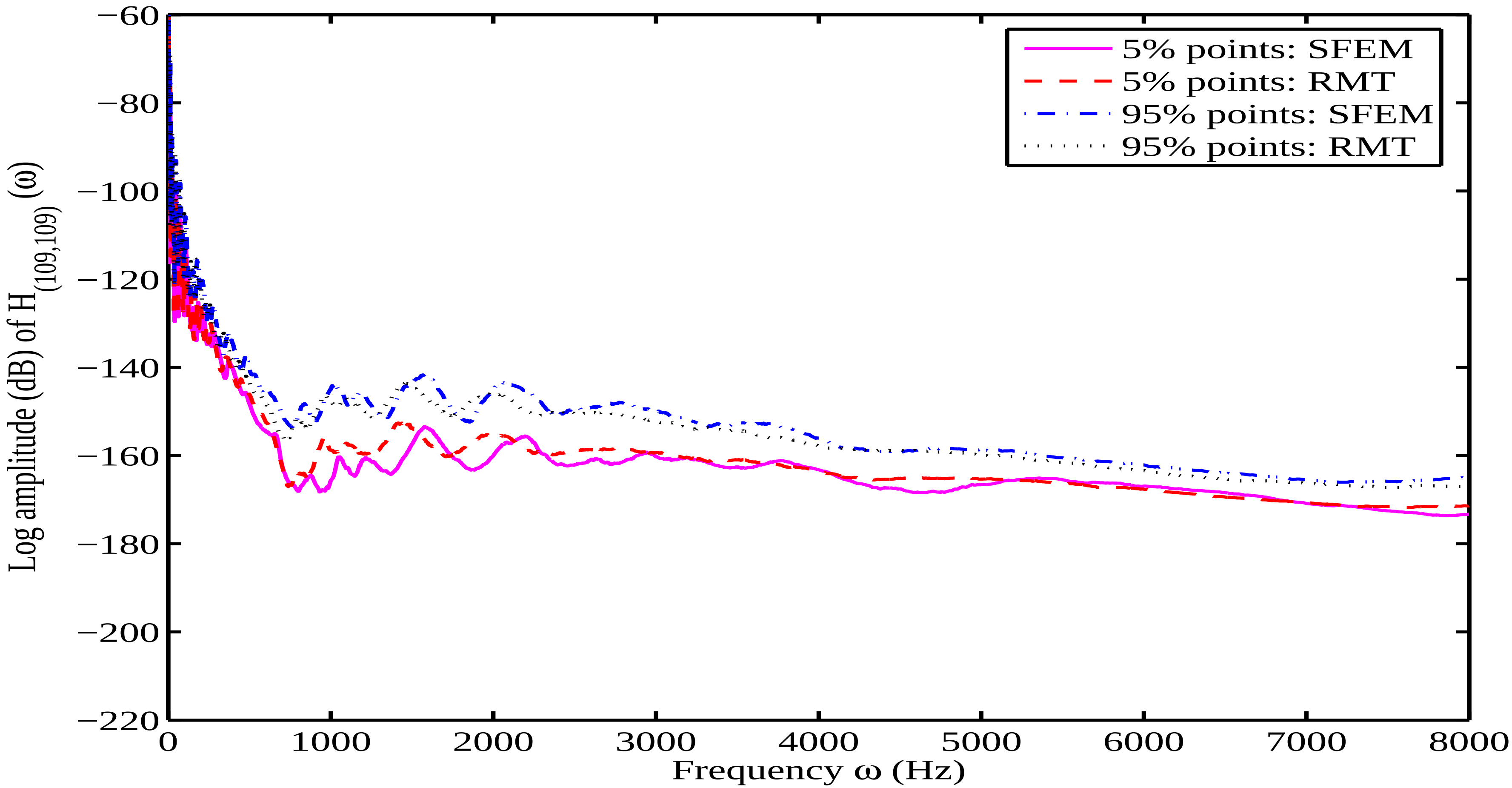
Comparison of driving-point-FRF



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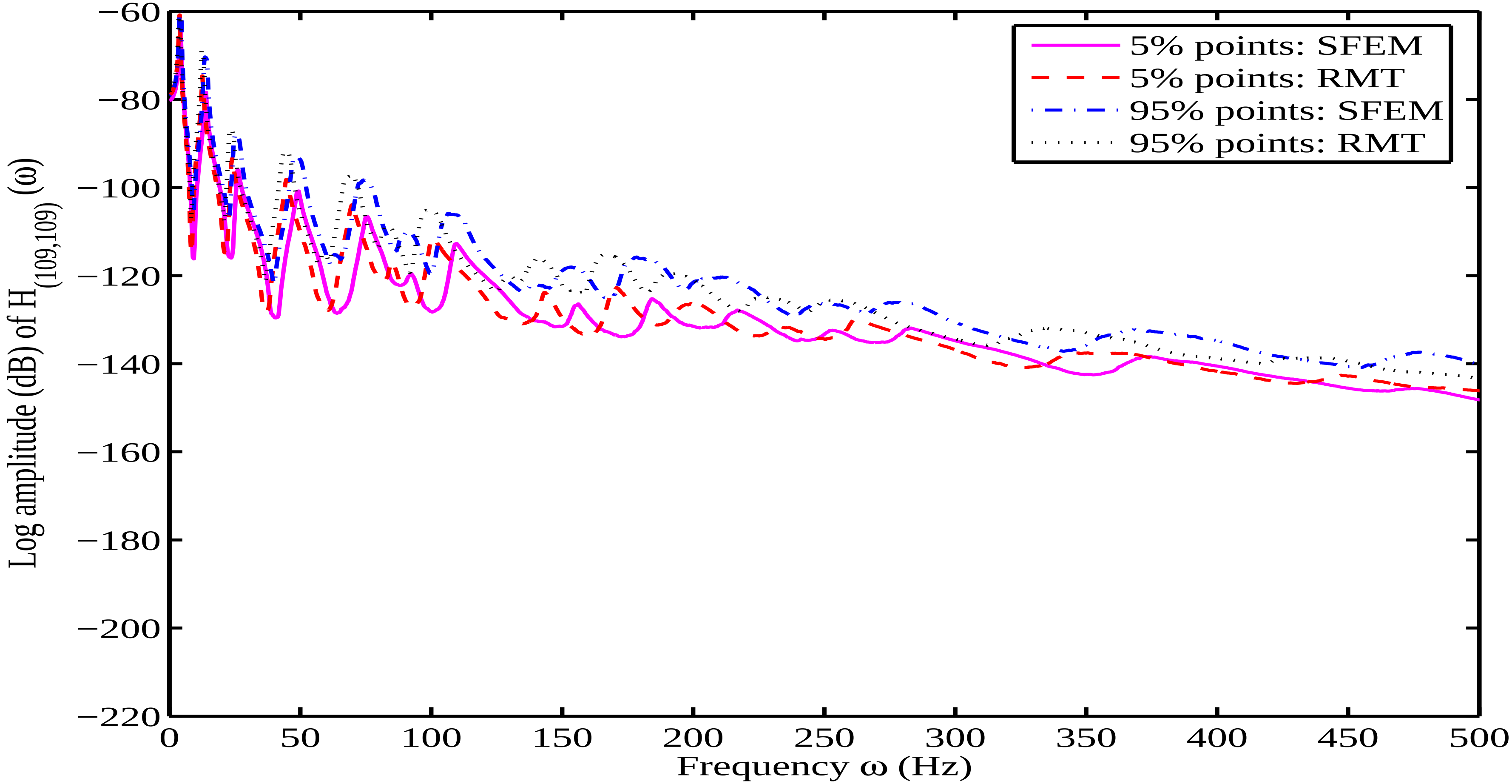
Comparison of driving-point-FRF: Low Freq



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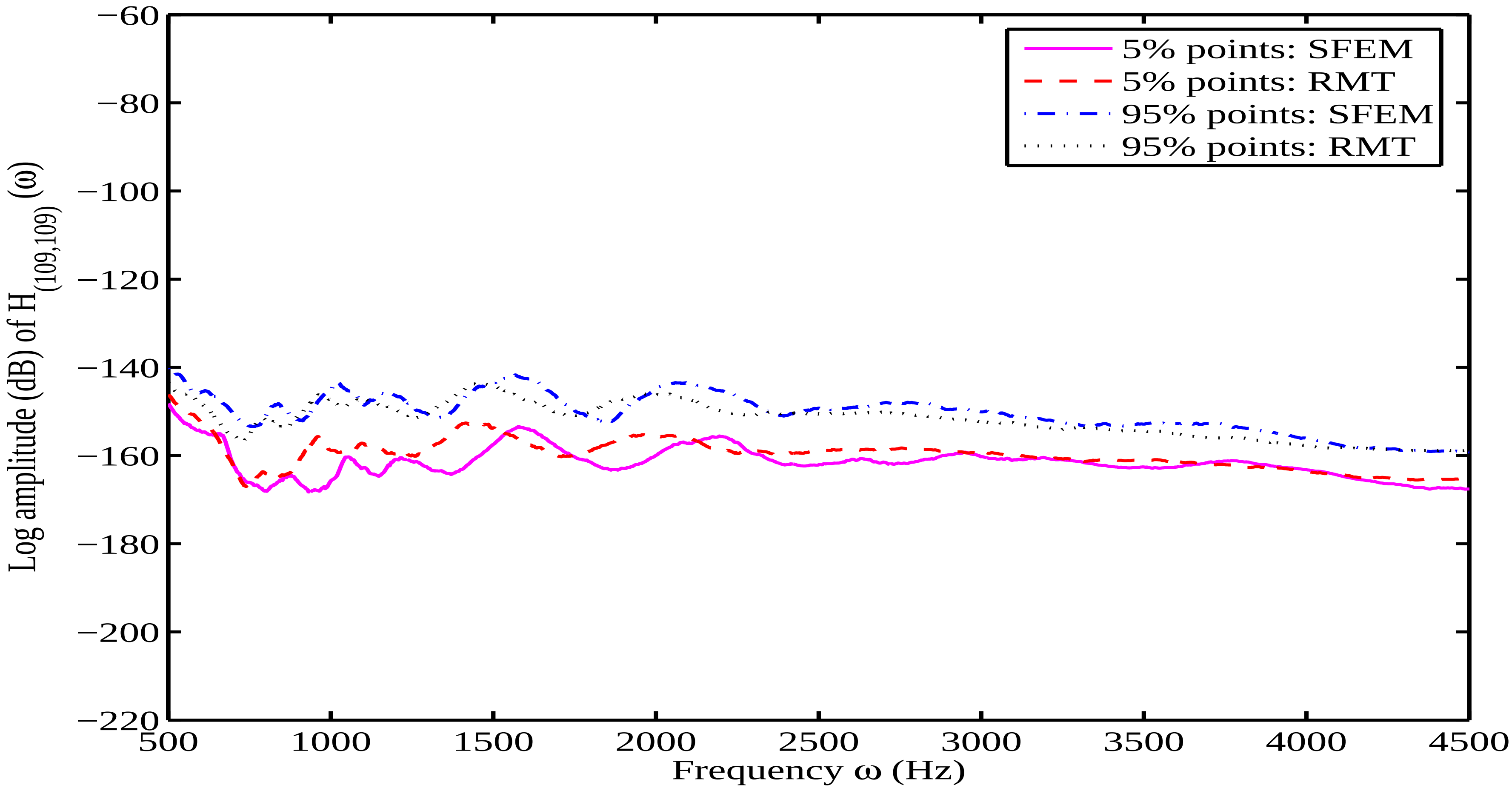
Comparison of driving-point-FRF: Mid Freq



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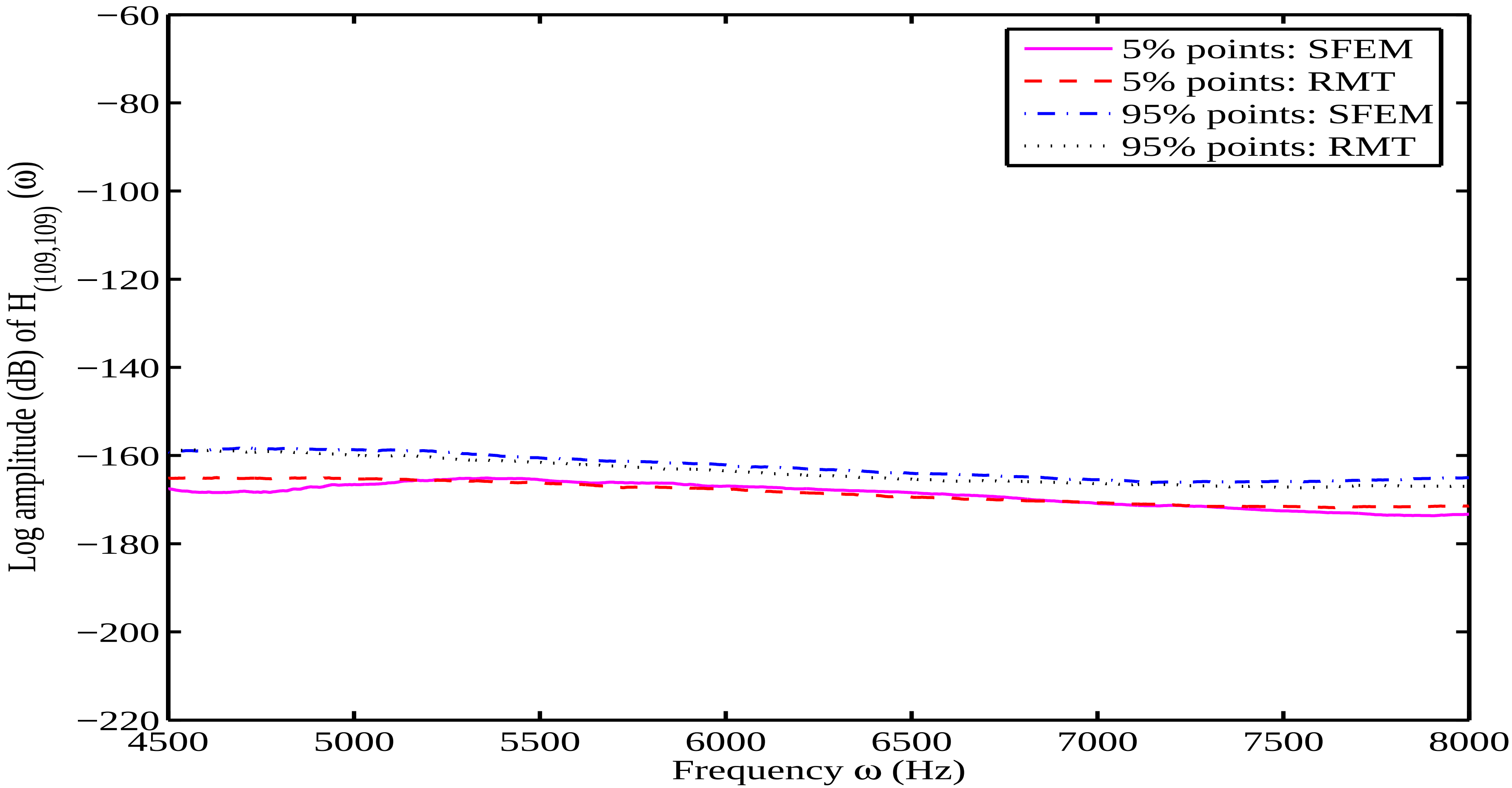
Comparison of driving-point-FRF: High Freq



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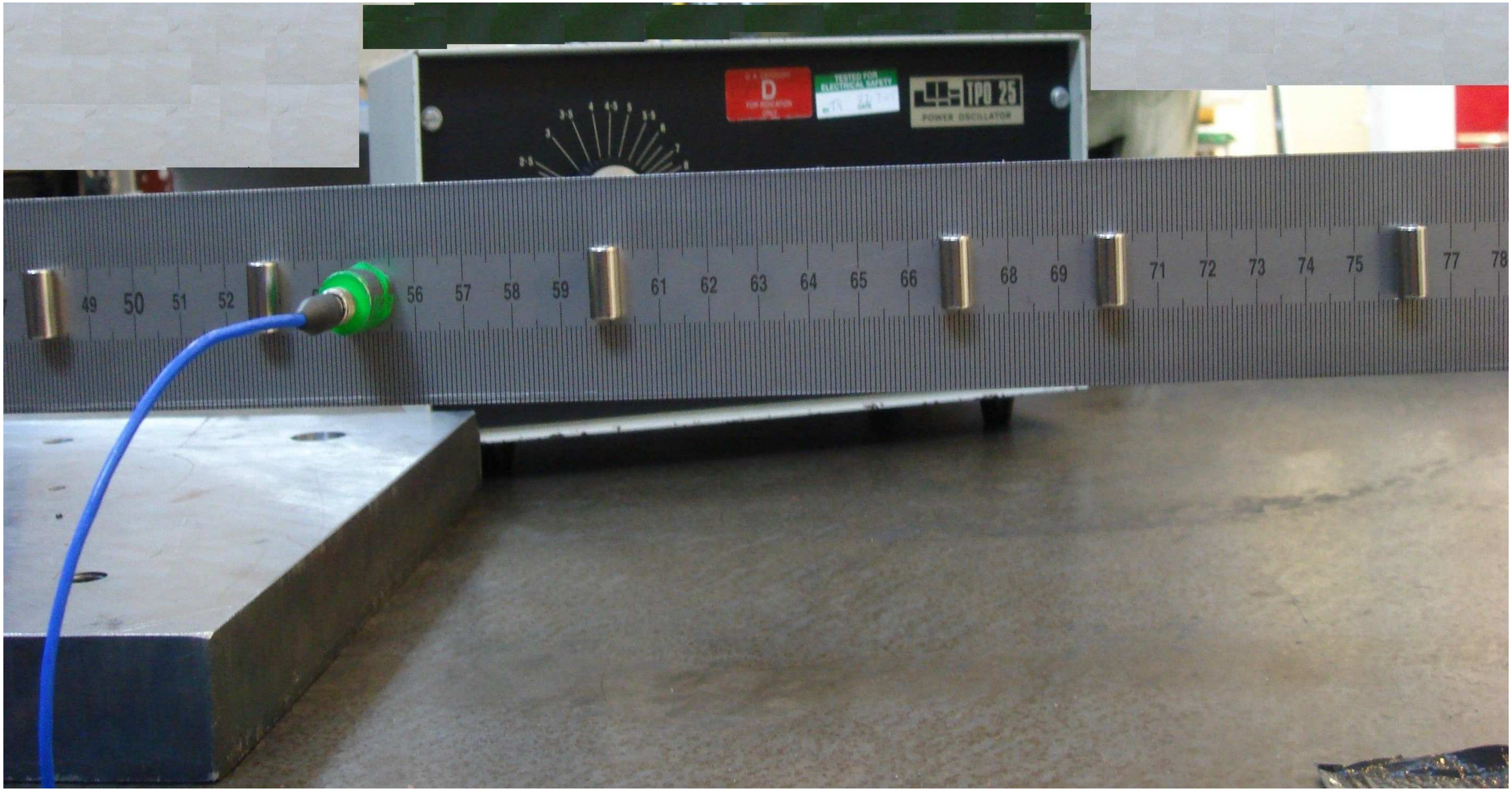
Experimental Study - 1

A fixed-fixed beam: Length: 1200 mm, Width: 40.06 mm, Thickness: 2.05 mm,

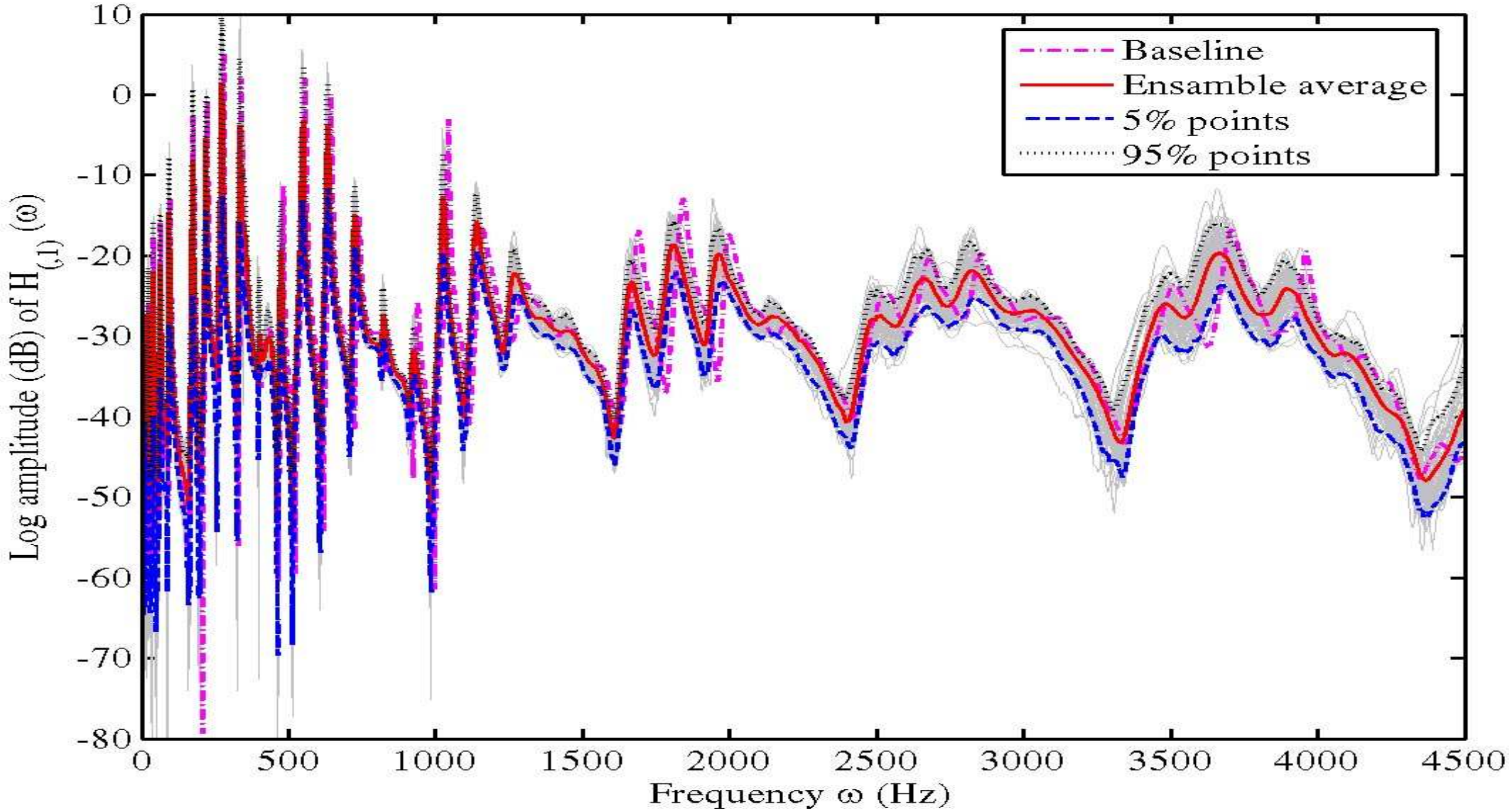
Density: 7800 kg/m3, Young's Modulus: 200 GPa

Experimental Study - 1



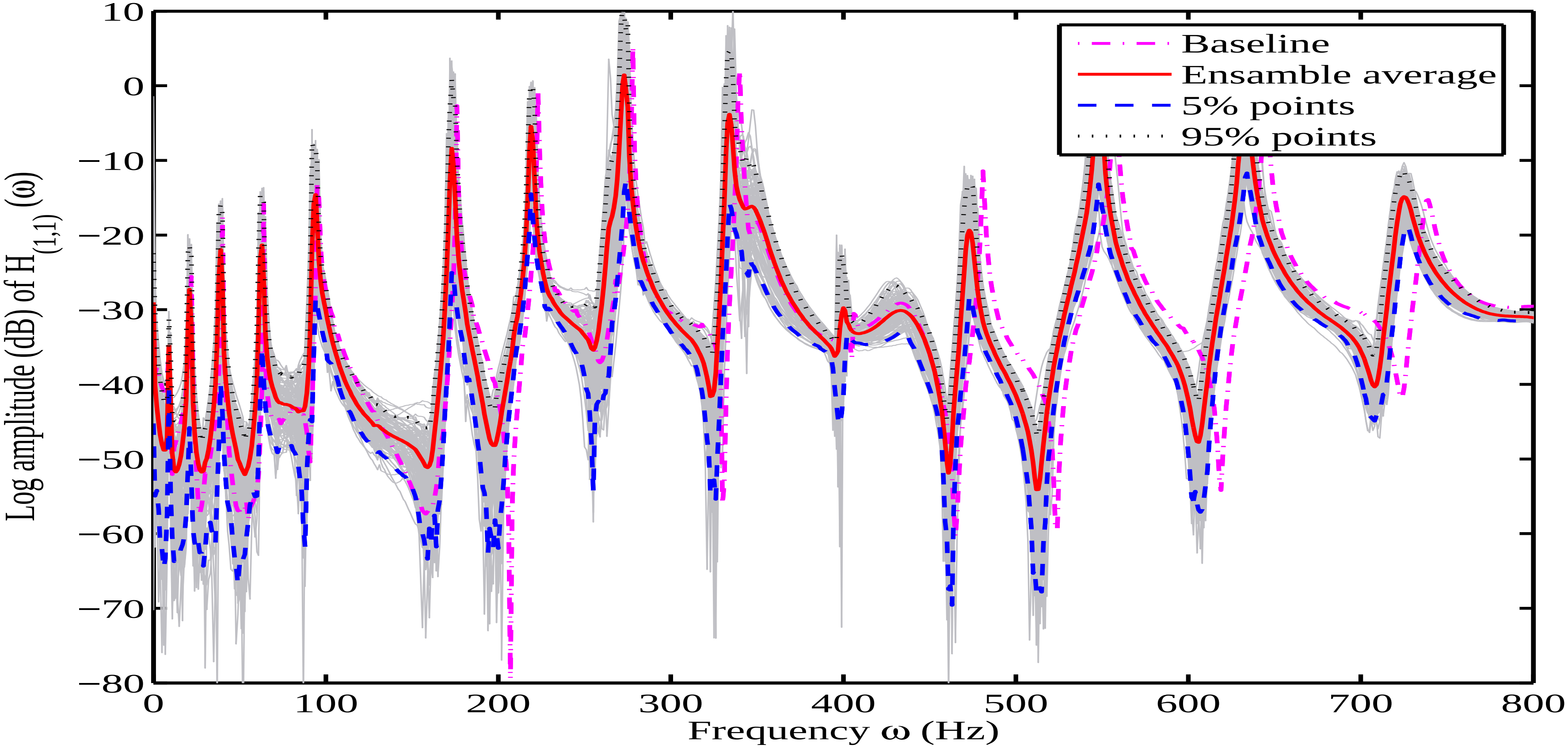
12 randomly placed masses (magnets), each weighting 2 g (total variation: 3.2%): mass locations are generated using uniform distribution

FRF Variability: complete spectrum



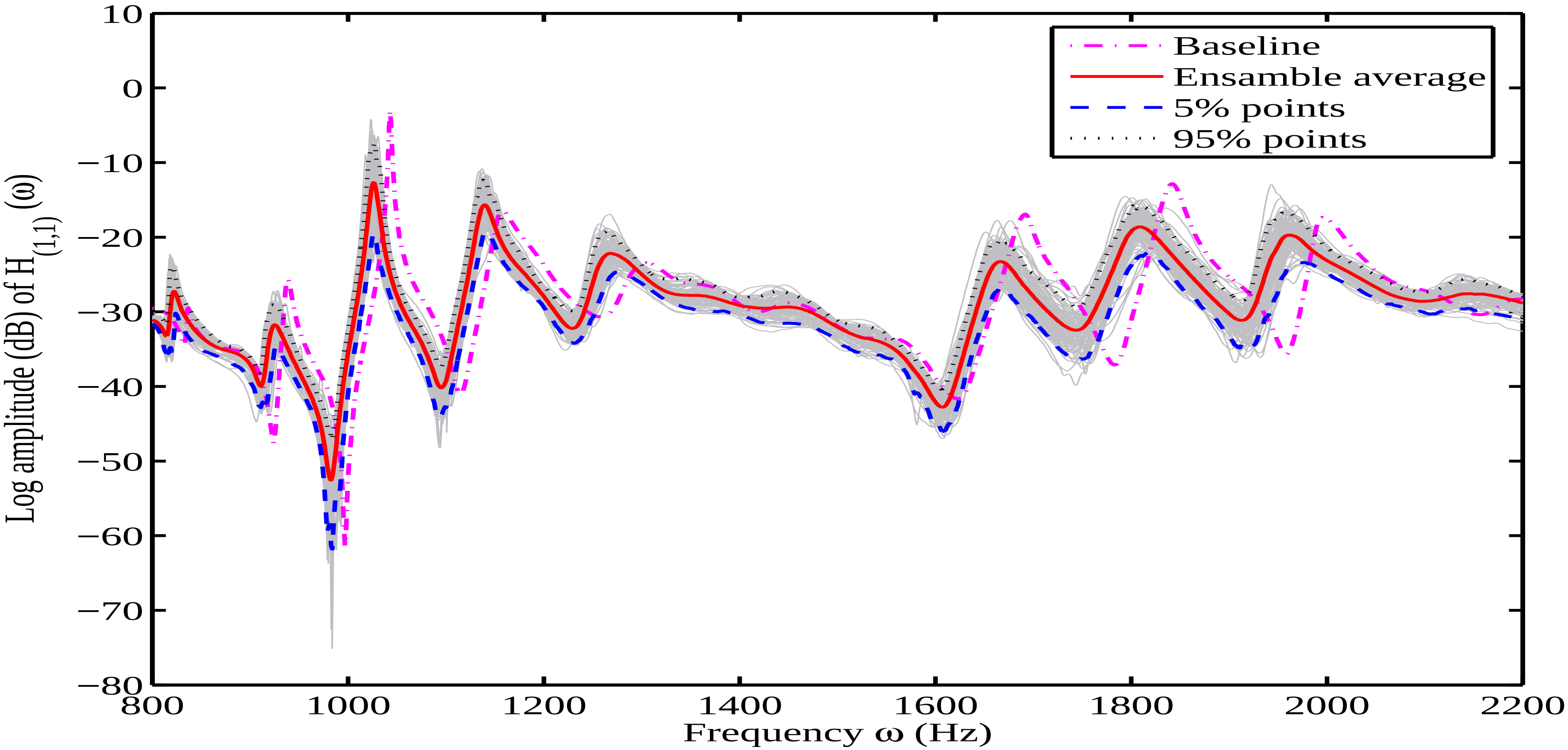
Variability in the amplitude of the driving-point-FRF.

FRF Variability: Low Freq



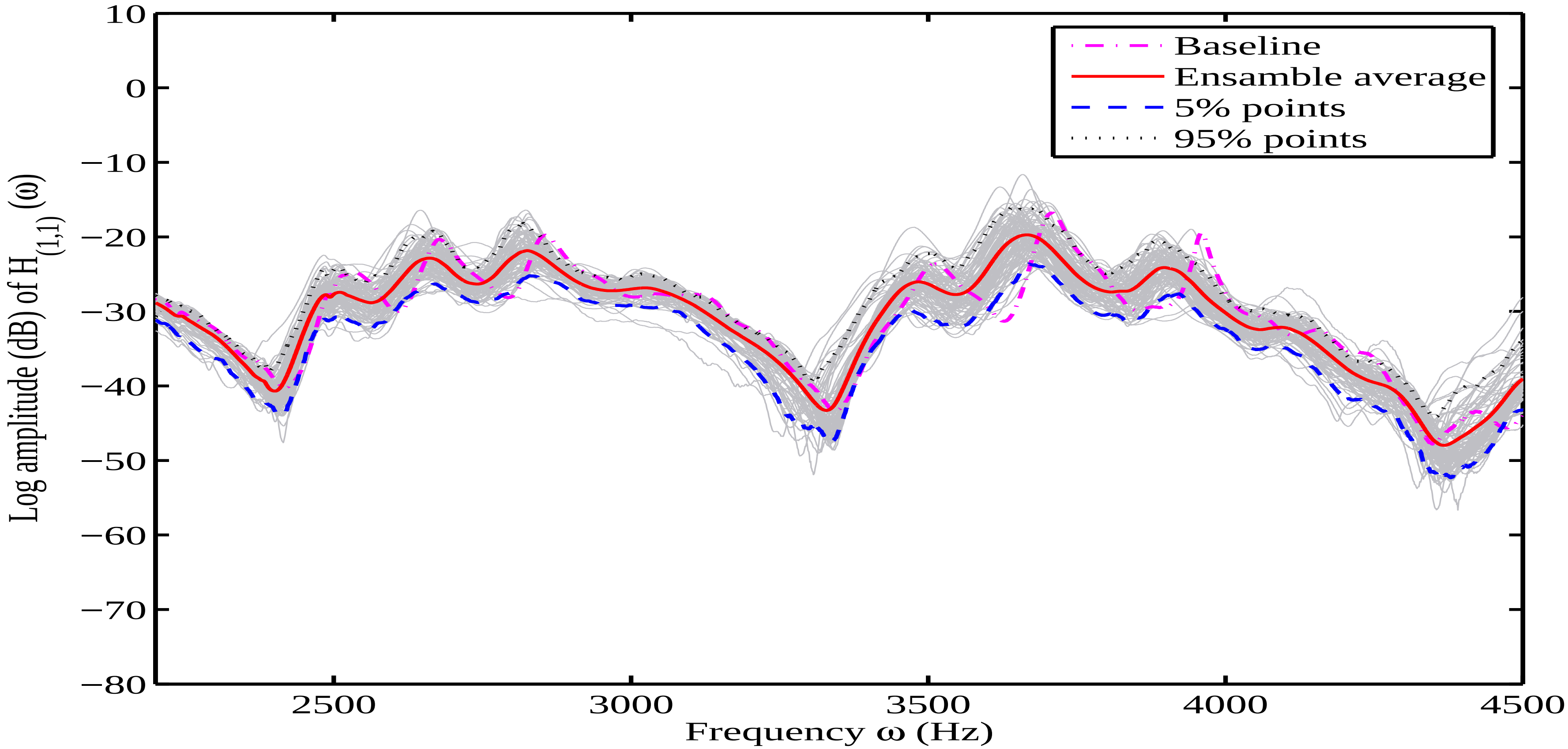
Variability in the amplitude of the driving-point-FRF.

FRF Variability: Mid Freq



Variability in the amplitude of the driving-point-FRF.

FRF Variability: High Freq



Variability in the amplitude of the driving-point-FRF.

Other applications of RMT

- Mid-frequency vibration problem
- Modelling random unmodelled dynamics
- Damping model uncertainty
- Flow through porous media
- Localized uncertainty modeling
- Stochastic domain decomposition method

Experimental Study: cantilever plate

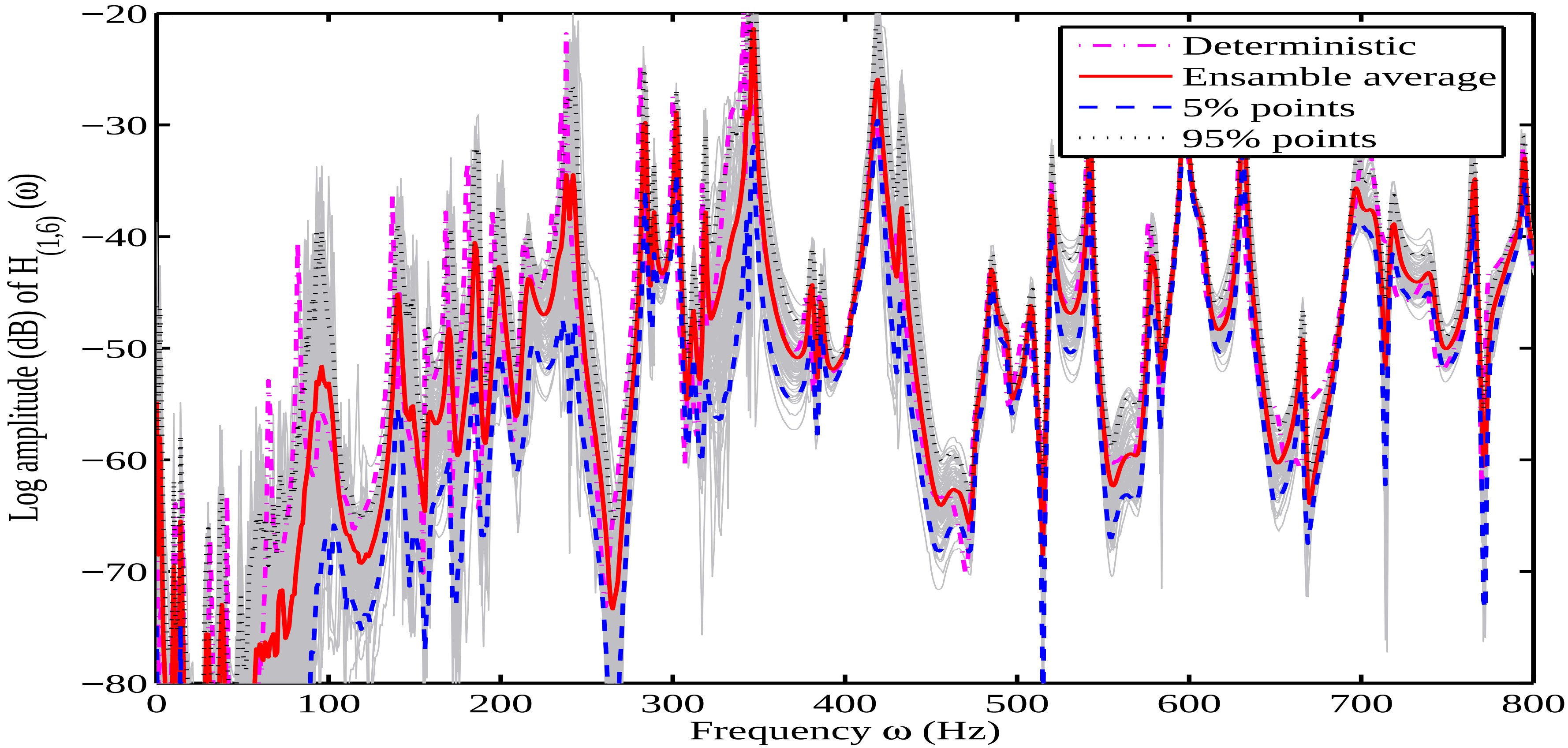
A cantilever plate: Length: 998 mm, Width: 530 mm, Thickness: 3 mm,

Density: 7860 kg/m3, Young's Modulus: 200 GPa

Unmodelled dynamics

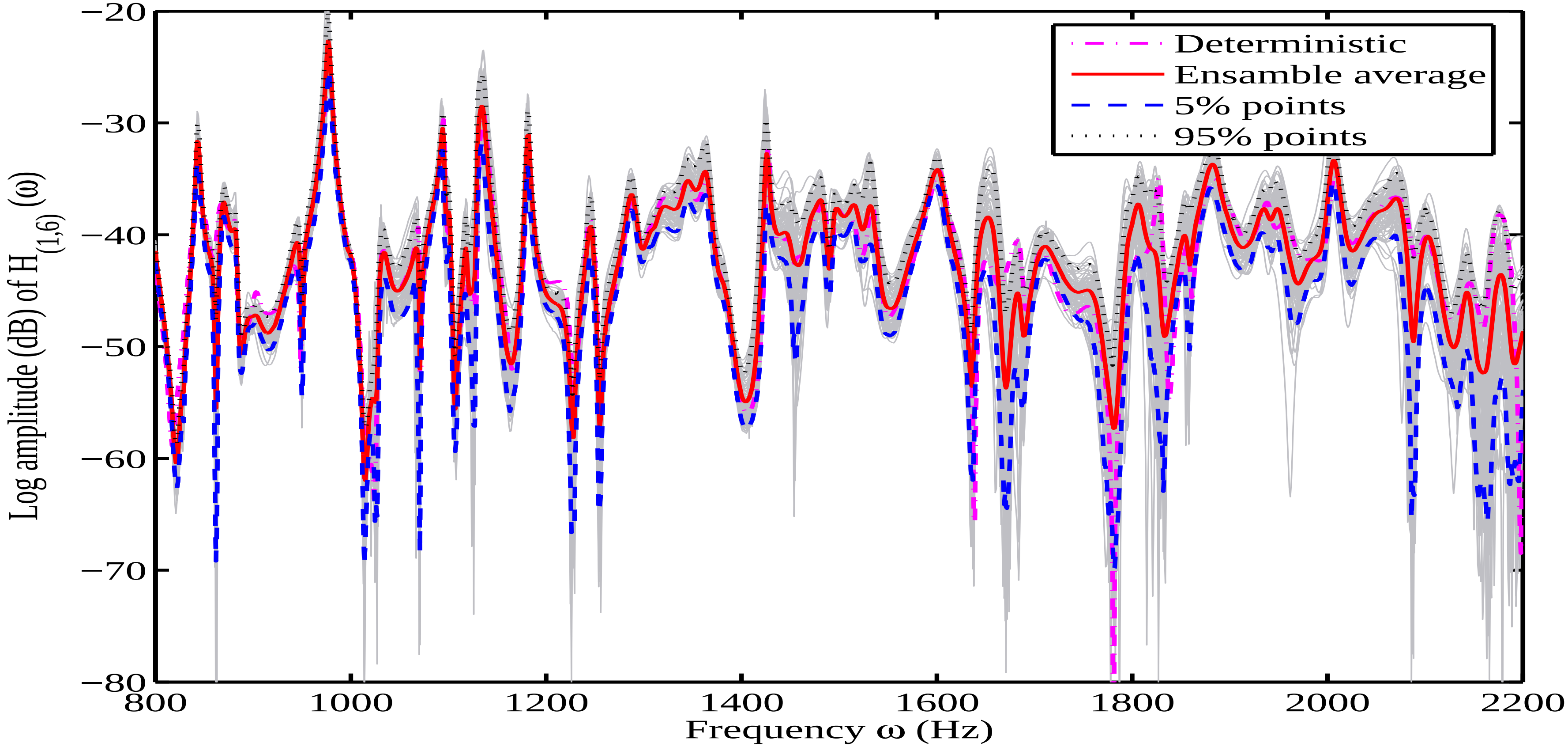
10 randomly placed oscillator; oscillatory mass: 121.4 g, fixed mass: 2 g, spring stiffness vary from 10 - 12 KN/m

FRF Variability: Low Freq



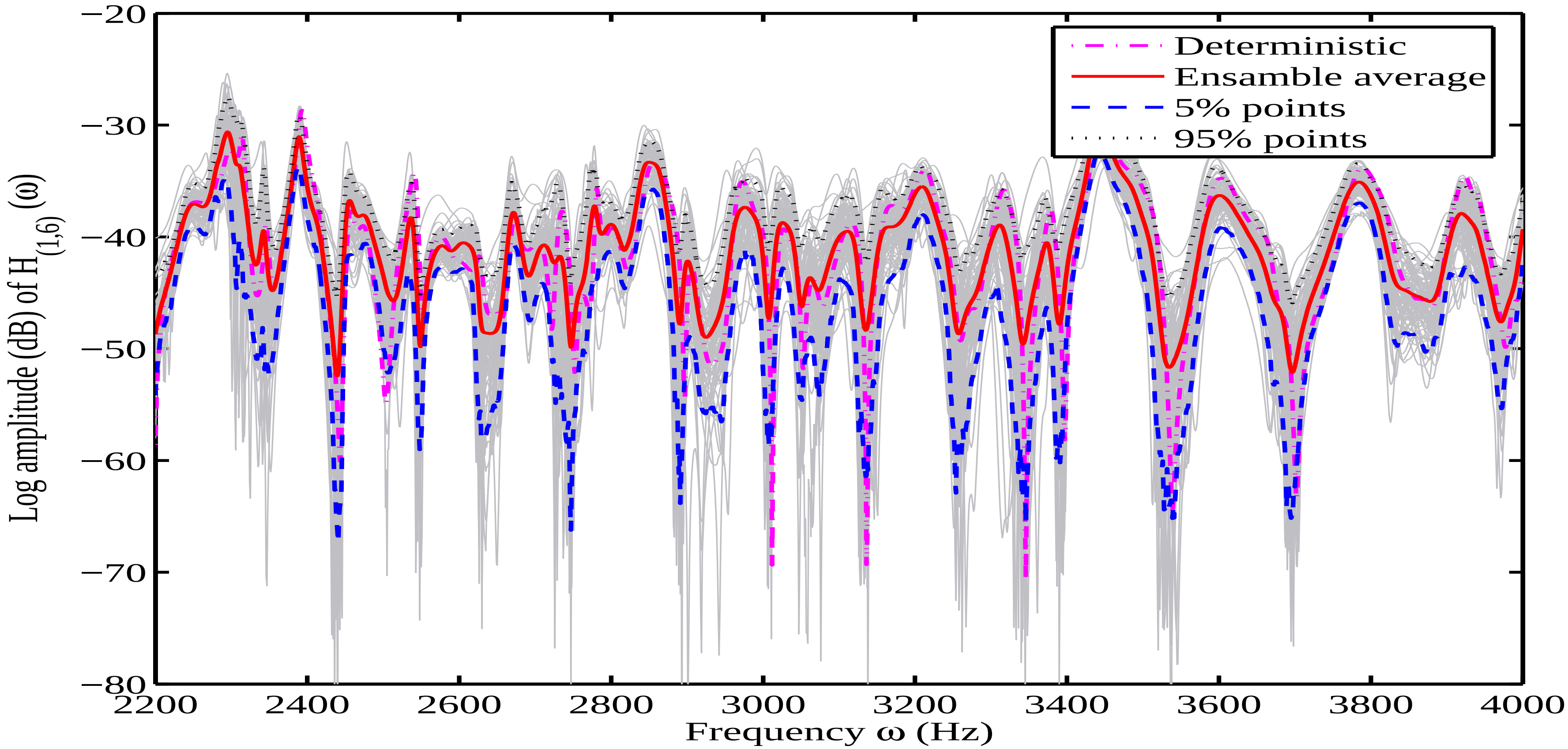
Variability in the amplitude of the FRF.

FRF Variability: Mid Freq



Variability in the amplitude of the FRF.

FRF Variability: High Freq



Variability in the amplitude of the FRF.

Summary

- Using a Matrix Factorization Approach (MFA) it was shown Wishart matrices may be used as the model for the random system matrices in structural dynamics.
- The parameters of the distribution were obtained in closed-form by solving an optimisation problem.
- Only the mean matrix and normalized standard deviation is required to model the system.
- Numerical results show that SFEM and RMT results match well in the mid and high frequency region.

Open problems (OP) - 1

- How to incorporate a given covariance tensor of $\mathbf{G}$ (e.g., obtained using the SFEM)?
  - Possibility: Use non-central Wishart distribution.
- What is the consequence of the zeros in $\mathbf{G}$ are not being preserved?
  - Possibility: Use SVD to preserve the 'structure' of the random matrix realizations and check the results.
- Are we taking model uncertainties ('unknown unknowns') into account? How can we verify it?

OP - RMT

- Pdf of complex linear combinations of real random matrices
- Joint Pdf of the eigenvalues of real random matrices.
- Inverse of complex symmetric random matrices.