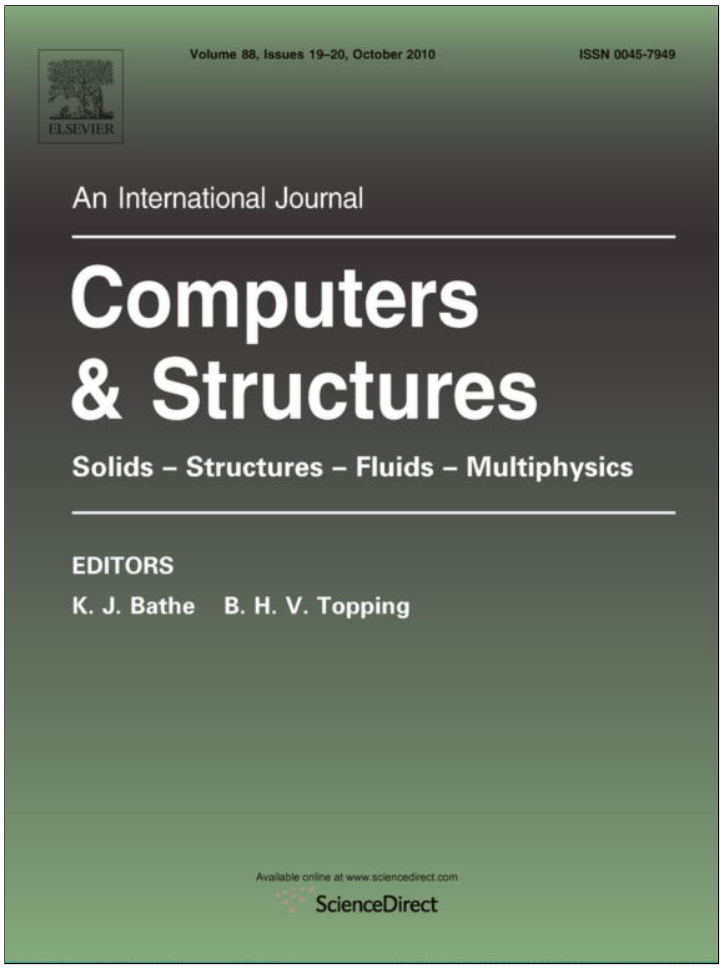

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A joint diagonalisation approach for linear stochastic systems

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ABSTRACT

In stochastic finite element problems the solution of a system of coupled linear random algebraic equations is needed. This problem in turn requires the calculation of the inverse of a random matrix. Over the past four decades several approximate analytical methods and simulation methods have been proposed for the solution of this problem in the context of probabilistic structural mechanics. In this paper we present a new solution method for stochastic linear equations. The proposed method is based on Neumann expansion and the recently developed joint diagonalisation solution strategy. Unlike the classical Neumann expansion, here only the inversion of a diagonal matrix is needed. Numerical examples are given to illustrate the use of the expressions derived in the paper.

1. Introduction

The quantification of uncertainty in the response of a mechanical system plays a crucial role in establishing the credibility of the underlying numerical model. There are two complimentary approaches to quantify uncertainties in a model. The first is the *parametric approach* and the second is the *non-parametric approach*. In the parametric approach the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element method (see for example [1–23]). This type of approach is suitable to quantify aleatoric uncertainties. Epistemic uncertainty on the other hand do not explicitly depend on the systems parameters. For example, there can be unquantified errors associated with the equation of motion (linear or non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods (e.g., discretisation of displacement fields, truncation and roundoff errors, tolerances in the optimization and iterative algorithms, step-sizes in the time-integration methods). The parametric approach is not suitable to quantify this type of uncertainties and a non-parametric approach (see for example [24–27]) is needed for this purpose.

In many stochastic mechanics problems, whether a parametric or a non-parametric method is used, one finally needs to solve a system of linear stochastic equations

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \tag{1}$$

Here $\mathbf{K} \in \mathbb{R}^{n\times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^n$ is a n-dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^n$ is a n-dimensional real uncertain output vector which we want to determine. It is obvious that the inverse of a random matrix is necessary to solve the set of linear equation (1). Equations of this type typically arise due to the discretisation of stochastic partial differential equations. In the context of linear structural mechanics, $\mathbf{K}$ is known as the stiffness matrix, $\mathbf{f}$ is the forcing vector and $\mathbf{u}$ is the vector of structural displacements. The central aim of a stochastic structural analysis is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of $\mathbf{u}$. It is often difficult to obtain the probably density function (pdf) of the response. As a consequence, engineers often restrict themselves to obtain only the first few moments of the response quantity.

In this paper we propose a new solution method for stochastic linear systems. The proposed method is based on Neumann expansion and the recently developed joint diagonalisation solution strategy [23]. The outline of the paper is as follows. In Section 2 discretisation methods for random material properties are discussed. In Section 3 some current methods to obtain the statistics of $\mathbf{u}$ in Eq. (1) are briefly reviewed to put the proposed method into a proper perspective. The proposed joint diagonalisation approach is discussed in Section 4 and a step-by-step algorithm to implement the proposed approach is summarized in Section 5. The

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expressions derived in the paper are illustrated by numerical examples in Section 6.

2. Discretisation of stochastic material parameters

For heterogeneous materials such as rocks, soils and composite materials with random inclusions, their material properties vary irregularly through the medium and they are often termed random media in engineering. In a practical problem with the presence of heterogeneous materials, the random material properties are usually modeled by stochastic fields. In most cases, the stochastic material parameters are initially defined by lower-order statistical moments, e.g. expectation and covariance functions. However, in a stochastic finite element formulation, an explicit discretisation of the stochastic material parameters is required.

Over the past few decades, different numerical techniques have been developed for this task and they include the middle point method [28,15], the local averaging method [29], the shape function method [16–18], the least-squares discretisation method [30] and the trigonometric series approximation method [31,32]. However, some of these are either inefficient (measured by the total number of random variables required) or inaccurate (measured by the error between the interpolated stochastic field and the real one) for an advanced SFEM formulation. To date, the most significant step forward for the discretisation of stochastic fields of material parameters is the application of Karhunen–Loève (K–L) expansions [33], which was first introduced into SFEM research by Spanos and Ghanem [2], and since then has been widely used in various SFEM formulations to discretise random material properties.

For stationary random media, the classic K–L expansion method using a background mesh has been recently extended by Li et al. [34,35] and a Fourier–Karhunen–Loéve (F–K–L) representation method is proposed. Briefly speaking, a stationary stochastic field $H(\mathbf{x},\omega)$ representing a material parameter (e.g. Young's modulus or Poisson's ratio) can be explicitly expressed as

$$H(\mathbf{x},\omega) = H_0(\mathbf{x}) + \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\xi_j(\omega)\phi_j(\mathbf{x}) \tag{2}$$

where $H_0(\mathbf{x})$ is the expectation function of $H(\mathbf{x},\omega)$, $\xi_j(\omega)$ are uncorrelated random variables and λ_j and $\phi_j(\mathbf{x})$ are the eigen-values and eigen-functions of the stochastic field. In particular, due to the stationarity of the stochastic field, the eigen-pairs λ_j and $\phi_j(\mathbf{x})$ in the F–K–L expansion scheme can be explicitly obtained via fast Fourier transform (FFT) without solving any equation. Numerical examples have shown that compared to the K–L expansion method using a background mesh, both the eigen-values and the eigen-functions can be obtained to a very high accuracy in the F–K–L expansion scheme. However, this advantage is compromised by the restricted use of the latter scheme, i.e. only applicable to stationary stochastic fields, while the former one is applicable to any stochastic field with finite second-order statistics. An important issue in the discretisation of (2) is how to determine the probability distribution of random variables $\xi_j(\omega)$. For Gaussian stochastic fields, these random variables are orthonormal Gaussian random variables, but for non-Gaussian stochastic fields, the determination of these random variables usually requires information of higher-order statistical moments.

3. Current methods for response-statistics calculation

After explicitly representing the random material parameters with discretised stochastic fields and discretising the unknown functions (e.g. the displacement field $\mathbf{u}$) with a finite element mesh, the stochastic linear system (1) can be readily constructed following a standard procedure of finite element formulation [2,21]. The solution of the random linear algebraic system (1) is fundamental to stochastic mechanics problems. As a result, a number of papers have been written on this topic over the past four decades. The purpose of this section is to give a brief overview on some of the existing methods in order to put the proposed method into a proper perspective. Readers are referred to the review papers [4,36,19,37,20,21] for further details. In a SFEM formulation, the stiffness matrix can always be represented as

$$\mathbf{K} = \mathbf{K}^0 + \Delta\mathbf{K} \tag{3}$$

where $\mathbf{K}^0 \in \mathbb{R}^{n\times n}$ is the deterministic part and $\Delta\mathbf{K} \in \mathbb{R}^{n\times n}$ is the random part. The random part is often expressed as

$$\Delta\mathbf{K} = \sum_{j=1}^{m} \xi_j \mathbf{K}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{j} \xi_j\xi_l \mathbf{K}_{jl}^{II} + \cdots \tag{4}$$

where m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n\times n}$, $\forall j,l$ are deterministic matrices and ξ_j, $\forall j$ are real random variables. Eq. (4) can be viewed as the Taylor expansion of the random stiffness matrix with respect to the random variables at the origin. These random variables may be Gaussian and orthonormalized in many problems. Under these settings, the following approaches have been employed to obtain the probabilistic descriptions of the response vector $\mathbf{u}$.

3.1. Perturbation based approach

The perturbation method can be applied in various forms. Here we consider the expansion of the response vector as

$$\mathbf{u} = \mathbf{u}^0 + \sum_{j=1}^{m} \xi_j \mathbf{u}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{j} \xi_j\xi_l \mathbf{u}_{jl}^{II} + \cdots \tag{5}$$

The vectors $\mathbf{u}^0, \mathbf{u}_j^I, \mathbf{u}_{jl}^{II} \in \mathbb{R}^n$ need to be determined. In an alternative formulation, $\mathbf{u}$ can be viewed as a function of the vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)^T$ and can be expanded in a Taylor series about the mean of $\boldsymbol{\xi}$ or some other suitable point. The mathematical details to be outlined will be similar for both approaches. Substituting $\mathbf{K}$ and $\mathbf{u}$ from Eqs. (3) and (5) into the governing equation (1) and equating the corresponding coefficients associated with the random variables we have

$$\mathbf{K}^0\mathbf{u}^0 = \mathbf{f} \tag{6}$$

$$\mathbf{K}_j^I\mathbf{u}^0 + \mathbf{K}^0\mathbf{u}_j^I = \mathbf{0} \tag{7}$$

$$\text{and} \quad \mathbf{K}_{jl}^{II}\mathbf{u}^0 + \mathbf{K}_j^I\mathbf{u}_l^I + \mathbf{K}^0\mathbf{u}_{jl}^{II} = \mathbf{0} \tag{8}$$

Solving these equations one has

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}}\mathbf{f} \tag{9}$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}}\mathbf{K}_j^I\mathbf{u}^0, \quad \forall j \tag{10}$$

$$\text{and} \quad \mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}}[\mathbf{K}_{jl}^{II}\mathbf{u}^0 + \mathbf{K}_j^I\mathbf{u}_l^I + \mathbf{K}_l^I\mathbf{u}_j^I], \quad \forall j,l. \tag{11}$$

The above equations completely define the unknown vectors appearing in the perturbation expansion (5).

Another variant of the perturbation type approach is the so called Neumann expansion method [15]. Provided $\|\mathbf{K}^{0^{-1}}\Delta\mathbf{K}\|_F < 1$, the inverse of the random matrix can be expanded in a binomial type of series as

$$\begin{aligned}\mathbf{K}^{-1} &= \left[\mathbf{K}^0(\mathbf{I}_n + \mathbf{K}^{0^{-1}}\Delta\mathbf{K})\right]^{-1} \\ &= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}}\Delta\mathbf{K}\mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}}\Delta\mathbf{K}\mathbf{K}^{0^{-1}}\Delta\mathbf{K}\mathbf{K}^{0^{-1}} + \cdots\end{aligned} \tag{12}$$

From this expansion one has

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f} = \mathbf{u}^0 - \mathbf{T}\mathbf{u}_0 + \mathbf{T}^2\mathbf{u}^0 + \cdots \tag{13}$$

where $\mathbf{u}^0$ is as defined in Eq. (11) and $\mathbf{T} = \mathbf{K}^{0^{-1}}\Delta\mathbf{K} \in \mathbb{R}^{n\times n}$ is a random matrix. An improved version of the perturbation method was proposed by Elishakoff et al. [38]. Later Adhikari and Manohar [7] extended the Neumann expansion method to complex symmetric random matrices arising in dynamic problems.

The first- and second-order statistics of $\mathbf{u}$ can be calculated from Eq. (5) or (13). The following general points may be noted for all perturbation based approaches

- If the random variations are large, the higher-order terms may not be negligibly small.
- The calculation of response statistics become difficult if the elements of $\Delta\mathbf{K}$ are non-Gaussian random variables.
- Even if the elements of $\Delta\mathbf{K}$ are Gaussian random variables, the inclusion of higher-order terms (more than second-order) results in very messy calculations.

For the above reasons, other methods have been proposed in the literature.

3.2. Projection methods

The perturbation based solutions can be viewed as a local approximation around the deterministic solution. The subspace projection schemes for stochastic finite element analysis on the other hand achieve global representation. The basic idea is simple, powerful and is based on solid theoretical foundations [20,21]. Here one 'projects' the solution vector onto a complete stochastic basis. Depending on how the basis is selected, several methods are proposed. Using the Polynomial Chaos (PC) projection scheme [2] one can express the solution vector as

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{14}$$

where $\mathbf{u}_j \in \mathbb{R}^n,\ \forall j$ are unknown vectors and the polynomial chaoses $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials with respect to $\boldsymbol{\xi}$. Instead of choosing Hermite polynomials, Xiu and Karniadakis [22,39] used a generalized orthogonal basis which utilizes functions from the Askey family of hypergeometric polynomials. In the context of dynamic problems, Adhikari and Manohar [7] proposed the random eigenfunction expansion method where the random basis is selected to be the complete set of eigen-functions of the undamped dynamic stiffness matrix. The number of terms in expansion (14) is given by [2,23]

$$P = \frac{(m+r)!}{m!r!} \tag{15}$$

where r is the order of the PC expansion. Substituting $\mathbf{u}$ from Eq. (14) into Eq. (1) and imposing the Galerkin condition, the unknown $\mathbf{u}_j$ vectors can be obtained by solving a set of nP dimensional deterministic linear algebraic equations. To reduce the computational effort, Nair and Keane [11] proposed the stochastic reduced basis method. Here the solution vector is represented using basis vectors spanning the preconditioned stochastic Krylov subspace and consequently the application of the Galerkin scheme leads to a reduced-order deterministic system of equations. A comparison of different PC projection schemes for stochastic finite element analysis is given by Sachdeva et al. [13]. Recently Sarkar et al. [40,41] have proposed a domain decomposition approach to obtain $\mathbf{u}_j$ which utilizes PC expansion and can be implemented in parallel to solve large scale problems. Once all $\mathbf{u}_j$ are known, the statistics of $\mathbf{u}$ can be obtained from (14) in a relatively straight-forward manner.

3.3. Monte Carlo simulation and other methods

Once the statistical properties of the random matrix $\mathbf{K}$ is known, the samples can be generated using Monte Carlo simulation and the moments and pdf of $\mathbf{u}$ can be obtained by standard statistical methods. Several authors have also proposed other methods to obtain statistical properties of $\mathbf{u}$ and a partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems is given in Table 1.

4. Joint diagonalisation approach

For the solution of the stochastic linear system (1), a novel solution strategy, namely the joint diagonalisation method, was recently proposed by Li et al. [23]. In this approach, Eq. (1) is first reorganized as follows

$$(\mathbf{K}_0 + \xi_1(\omega)\mathbf{K}_1 + \xi_2(\omega)\mathbf{K}_2 + \cdots + \xi_m(\omega)\mathbf{K}_m)\mathbf{u} = \mathbf{f} \tag{16}$$

where $\mathbf{K}_j,\ \forall j$ are real symmetric deterministic matrices and $\xi_j(\omega),\ \forall j$ are real random variables. The above stochastic linear system is commonly obtained in a SFEM formulation after discretising with the K–L (or F–K–L) expansion method the random material parameters which appear in linear terms inside the stiffness matrix.

It is shown by Li et al. [23] that by using a sequence of Givens transformations, stiffness matrices $\mathbf{K}_j$ in Eq. (16) can be simultaneously diagonalised such that

$$\mathbf{Q}^{-1}\mathbf{K}_j\mathbf{Q} = \Lambda_j + \Delta_j \approx \Lambda_j, \quad j = 1, 2, \ldots, m \tag{17}$$

where $\Lambda_j,\ \forall j$ are diagonal matrices and $\Delta_j,\ \forall j$ are matrices with zero diagonal entries and small magnitude off-diagonal entries. The transform matrix $\mathbf{Q}$ is explicitly obtained as the product of the Givens rotation matrices

$$\mathbf{Q} = \mathbf{G}_1^T\mathbf{G}_2^T\cdots\mathbf{G}_k^T \tag{18}$$

where k is the total number of Givens transformations and the Givens rotation matrices $\mathbf{G}_j,\ \forall j$ take the following form

$$\mathbf{G} = \mathbf{G}(p,q,\theta) \triangleq \begin{array}{c} \\ 1 \\ \\ p \\ \\ q \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} p & & & & q & & \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \cos\theta & & \sin\theta & \\ & & & & 1 & & \\ & & & -\sin\theta & & \cos\theta & \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \end{array} \tag{19}$$

The joint diagonalisation works in an iterative manner as shown in Eq. (18) and the algorithm is implemented to scan in turn all the off-diagonal entries (p,q), $p \neq q$ in the matrix system. In each iteration, a set of matrix elements $(\mathbf{K}_j)_{pq}, j = 1,\ldots,m$ are selected and to minimize the corresponding off-diagonal entries the Givens rotation matrix (19) is constructed. As matrices $\mathbf{K}_j,\ \forall j$ are real symmetric, the optimal Givens rotation angle θ_{opt} that in the current iteration maximizes the diagonal entries and minimizes the off-diagonal entries can be accurately obtained by solving the following characteristic equation

$$(\cos 2\theta_{opt} \ \ \sin 2\theta_{opt})^T\mathbf{J} = \lambda_J(\cos 2\theta_{opt} \ \ \sin 2\theta_{opt})^T \tag{20}$$

Table 1
A partial summary of the solution techniques for coupled linear algebraic equations arising in stochastic mechanics problems.

	Methods	Sub-methods
1.	Perturbation based methods	First and second-order perturbation [1,3,16] Neumann expansion [15,7] Improved perturbation method [38]
2.	Projection methods	Polynomial chaos expansion [2] Random eigenfunction expansion [7] Stochastic reduced basis method [11,13,14] Wiener–Askey chaos expansion [22,39,47] Domain decomposition method [40,41]
3.	Monte Carlo simulation and other methods	Simulation methods [48,49] Analytical method in references [50–54] Exact solutions for beams [55,12]

where $\mathbf{J}$ is a two by two matrix given by

$$\mathbf{J}=\sum_{j=1}^{m}\begin{pmatrix} 2(\mathbf{K}_j)_{pq}^2 & (\mathbf{K}_j)_{pq}\left((\mathbf{K}_j)_{qq}-(\mathbf{K}_j)_{pp}\right) \\ (\mathbf{K}_j)_{pq}\left((\mathbf{K}_j)_{qq}-(\mathbf{K}_j)_{pp}\right) & \frac{1}{2}\left((\mathbf{K}_j)_{qq}-(\mathbf{K}_j)_{pp}\right)^2 \end{pmatrix} \tag{21}$$

and λ_J is the smallest eigen-value of $\mathbf{J}$.

In the original joint diagonalisation solution [23], the off-diagonal entries $\mathbf{\Delta}_j$, $\forall j$ in Eq. (17) are completely ignored and this significantly simplifies the random equation system which in turn leads to an explicit solution

$$\mathbf{u}\approx\mathbf{Q}(\mathbf{\Lambda}_0+\xi_1(\omega)\mathbf{\Lambda}_1+\cdots+\xi_m(\omega)\mathbf{\Lambda}_m)^{-1}\mathbf{Q}^{-1}\mathbf{f} \tag{22}$$

The above solution relies on the assumption that the off-diagonal matrices $\mathbf{\Delta}_j$, $\forall j$ are negligible comparing to the diagonal matrices $\mathbf{\Lambda}_j$, $\forall j$. However, for larger matrices $\mathbf{K}_j$, $\forall j$, to reduce matrices $\mathbf{\Delta}_j$, $\forall j$ to a negligible level is extremely time consuming and with limited computational power available, the accuracy of the solution (22) decreases as the dimension of matrices increases. In order to reduce the computational load of joint diagonalisation and overcome the problem of inefficiency, the contribution from the off-diagonal matrices $\mathbf{\Delta}_j$, $\forall j$ must be taken into account in the solution. Substituting Eq. (17) into Eq. (16) yields

$$\mathbf{Q}\left[\left(\mathbf{\Lambda}_0+\sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Lambda}_j\right)+\left(\mathbf{\Delta}_0+\sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Delta}_j\right)\right]\mathbf{Q}^{-1}\mathbf{u}=\mathbf{f} \tag{23}$$

Let

$$\begin{aligned}\mathbf{V}&=\mathbf{\Lambda}_0+\sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Lambda}_j\\ \mathbf{A}&=\mathbf{\Delta}_0+\sum_{j=1}^{m}\xi_j(\omega)\mathbf{\Delta}_i\end{aligned} \tag{24}$$

then the solution of (23) can be expressed as

$$\mathbf{u}=\mathbf{Q}\left[\mathbf{V}(\mathbf{I}_n+\mathbf{V}^{-1}\mathbf{A})\right]^{-1}\mathbf{Q}^{-1}\mathbf{f} \tag{25}$$

Noting that matrix $\mathbf{V}$ is a diagonal matrix whose inverse can be explicitly obtained, the above expression can be further simplified by using the Neumann expansion as

$$\mathbf{u}=\mathbf{Q}[\mathbf{V}^{-1}-(\mathbf{V}^{-1}\mathbf{A})\mathbf{V}^{-1}+(\mathbf{V}^{-1}\mathbf{A})^2\mathbf{V}^{-1}-\cdots]\mathbf{Q}^{-1}\mathbf{f} \tag{26}$$

Eq. (26) provides an exact explicit solution of the stochastic linear system (16) in a series form and the expression converges when the spectral radius of matrix $\mathbf{V}^{-1}\mathbf{A}$ is less than 1. Although we do not have a rigorous estimate for the spectral radius $\rho(\mathbf{V}^{-1}\mathbf{A})$, it is verified in numerical examples that this condition can be quickly satisfied after a few joint diagonalisation iterations. Compared to the original joint diagonalisation solution (22), the above solution can be obtained more efficiently and without compromising the accuracy.

5. Summary of the proposed approach

The joint diagonalisation approach is applicable to any real-symmetric matrices. The response statistics for static (or steady-state) stochastic systems can be obtained by the following steps:

(1) Discretise the random material parameters, which are implicitly defined by their lower-order statistical moments, by using the F–K–L expansion scheme [34,35] such that an explicit representation is obtained in the form of Eq. (2).
(2) In space dimension, discretise the unknown field (e.g. displacement in statics problems or temperature in heat conduction problems) with a finite element mesh.

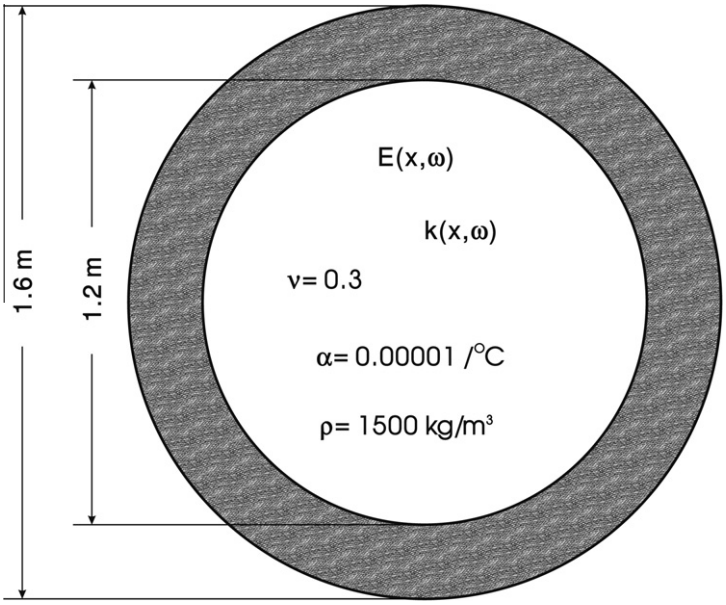

Fig. 1. Cross-section of the wastewater pipe.

(3) Following a standard finite element formulation procedure and taking into consideration the F–K–L expansion (2) of random material parameters, construct the stochastic linear system in the form of Eq. (16).
(4) Approximate joint diagonalisation of all matrices in Eq. (16):
- Check in turn all the off-diagonal entries in matrices $\mathbf{K}_j$ and find an entry (p,q), $p \neq q$ such that $\sum_{j=1}^{m}(\mathbf{K}_j)_{pq}^2 \neq 0$.
- For every entry (p,q) satisfying the above condition, compute the optimal Givens rotation angle θ_{opt} according to Eqs. (20) and (21), and form the corresponding Givens rotation matrix as shown in (19).
- Apply Givens similarity transformation to all the matrices $\mathbf{K}_j$, $\forall j$ and update these matrices such that $\mathbf{K}_j = \mathbf{G}(p,q,\theta_{opt})\mathbf{K}_j\mathbf{G}^T(p,q,\theta_{opt})$.
- Repeat the above procedure until the process converges below the given threshold from which the transform matrix $\mathbf{Q}$ and the jointly diagonalised matrices $\mathbf{\Lambda}_j + \mathbf{\Delta}_j$, $\forall j$ are obtained as (18) and (17) respectively.

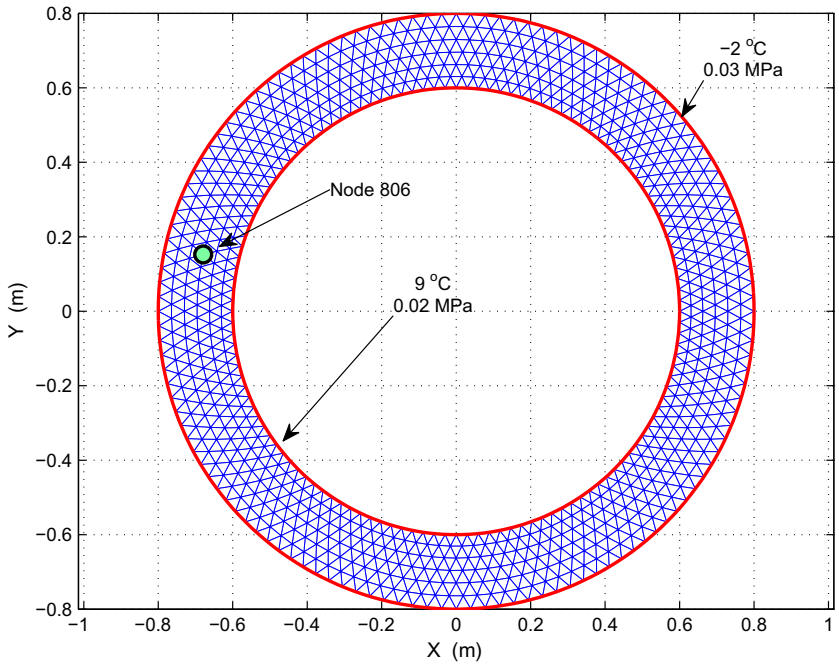

Fig. 2. Finite element mesh and boundary conditions.

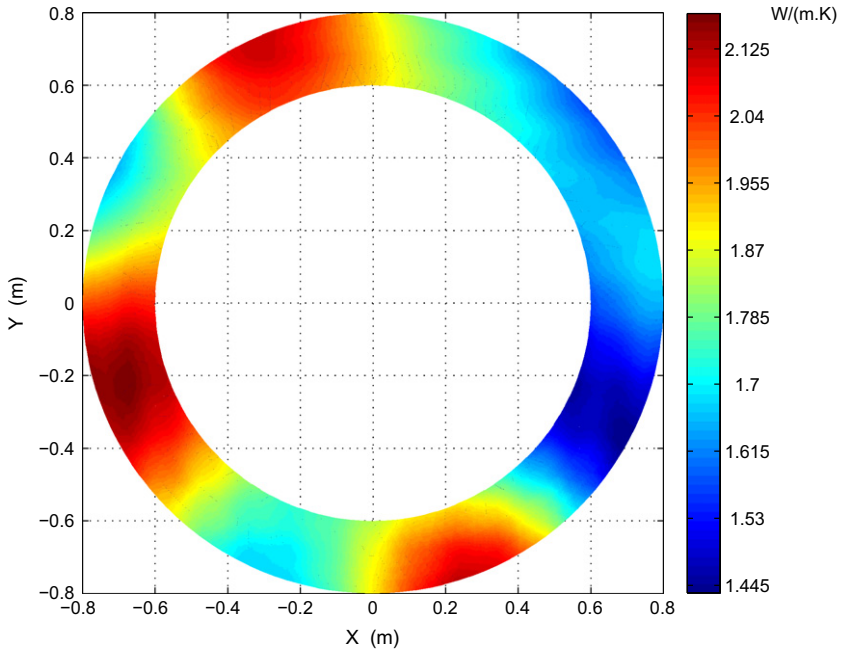

Fig. 3. Thermal conductivity reconstructed from the F–K–L expansion.

(5) For a specific realization of random variables $\xi_j(\omega)$, $\forall j$ and by using Neumann expansion, the response vector **u** is obtained as shown in (26).

6. Numerical examples

A concrete wastewater pipe buried in ground is analysed with the proposed stochastic approach. The cross-section of the pipe is shown in Fig. 1 and its geometry is represented by a finite element mesh with 896 nodes as shown in Fig. 2.

The pipe is subjected to both thermal and pressure loads from the inner wastewater flow and the surrounding soil. The temperature at the inner and outer surfaces are 9 °C and −2 °C, respectively; and the pressure on the inner and outer surfaces are 0.02 MPa and 0.03 MPa, respectively. The concrete material is assumed to be isotropic and its major material properties including

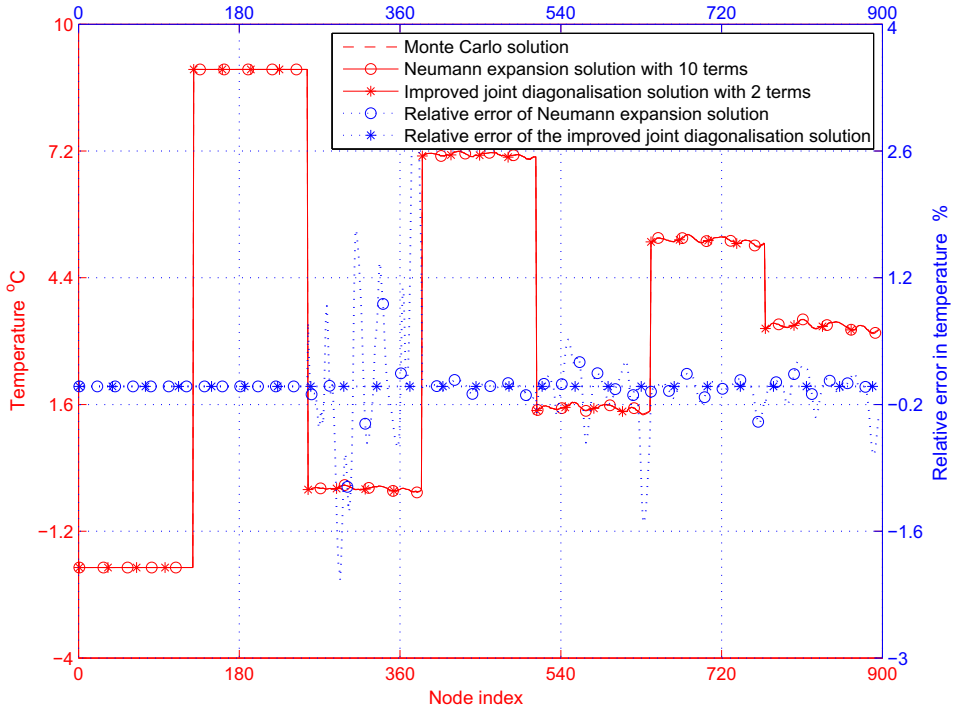

Fig. 4. Comparison of the improved joint diagonalisation solution, the Neumann expansion solution and the Monte Carlo solution.

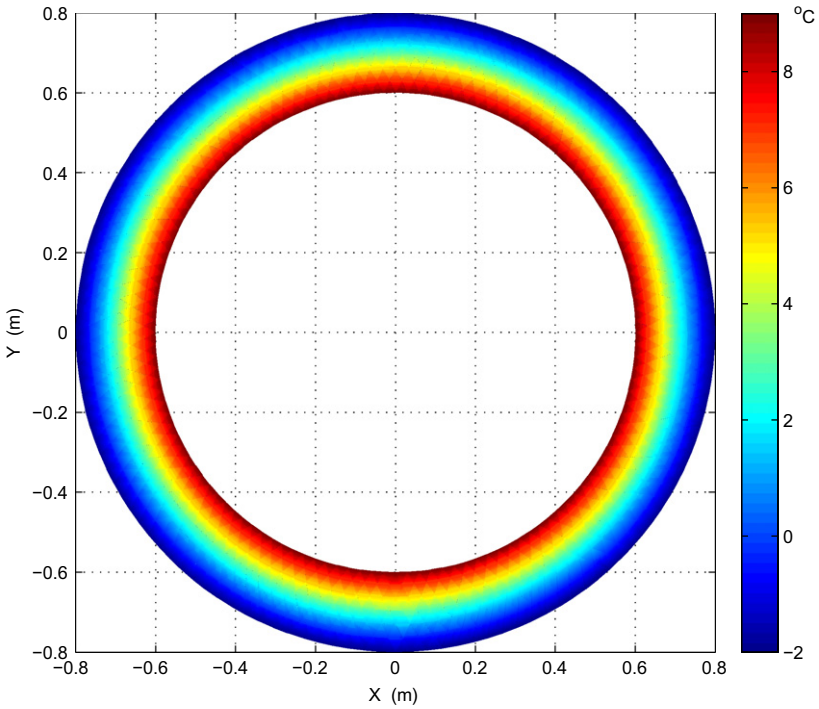

Fig. 5. A sample solution of temperature distribution.

thermal conductivity $k = k(x,\omega)$ and Young's modulus $E = E(x,\omega)$ are approximately modeled as independent stationary Gaussian stochastic fields, while density $\rho = 1500\ \mathrm{kg/m^3}$, coefficient of thermal expansion $\alpha = 10^{-5}\ (^\circ\mathrm{C})^{-1}$ and Poisson's ratio $\nu = 0.2$ are treated as deterministic constants. The mean values of thermal conductivity and Young's modulus are 1.7 W/(m K) and 60 GPa, respectively; and their corresponding covariance functions are $0.116e^{-\left(\frac{(x_1-x_2)^2+(y_1-y_2)^2}{0.2}\right)^{2/3}}\ \mathrm{W^2/(m\,K)^2}$ and $144e^{-\left(\frac{(x_1-x_2)^2+(y_1-y_2)^2}{0.2}\right)^{2/3}}\ \mathrm{GPa^2}$, respectively. In the first example (Section 6.1), a heat conduction analysis is performed and the randomly distributed temperature obtained is then put into a mechanical analysis as addressed in the second example (Section 6.2), where the stress distribution of the concrete pipe under thermal and pressure loads is finally solved.

Remarks. Although the random material parameters (both thermal conductivity $k(x,\omega)$ and Young's modulus $E(x,\omega)$) are approximately modeled as Gaussian stochastic fields in the following examples, it should be noted that the proposed joint diagonalisation approach is not restricted to any specific probability distribution of random variables. This can be clearly seen from Eqs. (24)–(26) as well as the original joint diagonalisation approach [23]. For a non-Gaussian random material model, higher-order statistical moments are generally required to determine the probability distribution of random variables in the K–L (or F–K–L) expansion. It is noted that, although without direct experimental verification, a log-Gaussian stochastic field model is also frequently used in recent literature [42–46].

6.1. Temperature distribution of the concrete pipe due to temperature difference on the inner and outer surfaces

As outlined in Section 5, the F–K–L expansion [34,35] is first employed to obtain an explicit representation of the random thermal conductivity. In the F–K–L expansion, the approximation error due to truncation is controlled within 10% in terms of the difference of variance, which after discretising the space dimension with a finite element mesh leads to a stochastic linear system (16) consisting of 22 random variables. To give an intuitive picture of the random material model, a specific realization of random thermal conductivity is reconstructed from the truncated F–K–L expansion and shown in Fig. 3.

For the heat conduction analysis, there are 23 matrices in Eq. (16) including the mean matrix $\mathbf{K}_0$ and the dimension of each matrix is 896. A total number of 8960 sample solutions of the stochastic linear system are computed using the proposed joint diagonalisation method. To check the accuracy, these sample solutions are compared with Monte Carlo solutions and Neumann expansion solutions. For a specific sample solution, it is shown in Fig. 4 that by using only two terms in Eq. (26), the improved joint diagonalisation method agrees well with the Monte Carlo method

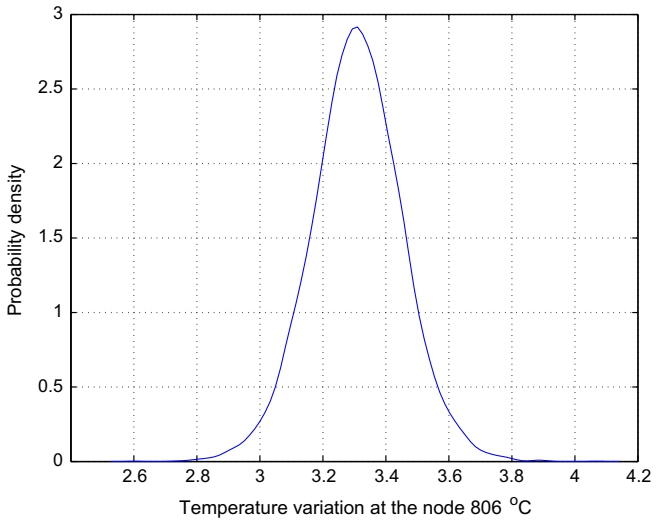

Fig. 7. Probability distribution of temperature at node 806.

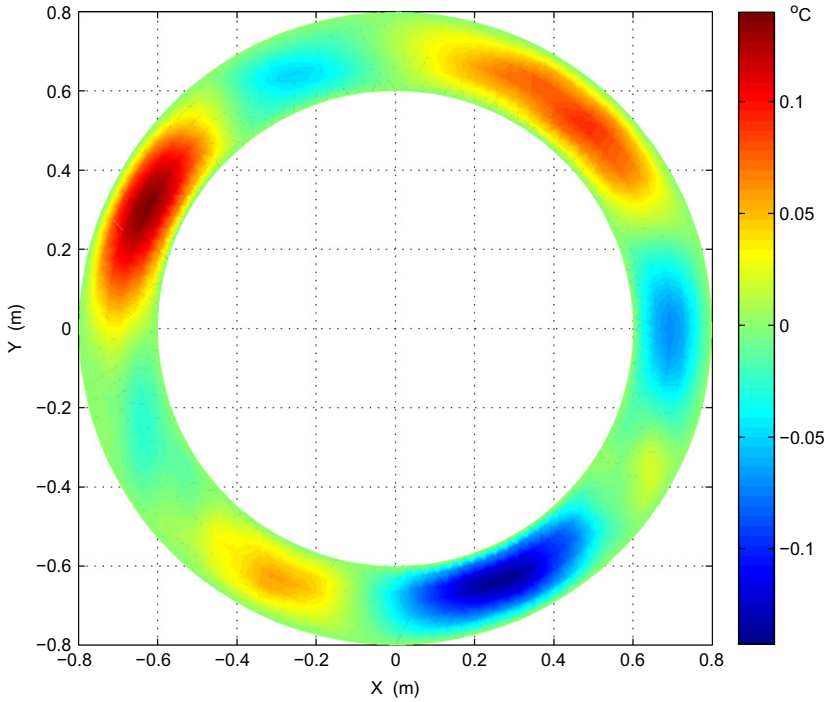

Fig. 6. Difference between the random temperature distribution and the deterministic mean-value temperature distribution.

such that the relative error is virtually 0%. The basic Neumann expansion method is however converging much slower and by using 10 terms in Eq. (13), it converges to a solution with a relative error of about 2%. It is also observed in numerical experiments that a few Neumann expansion solutions do not converge as the largest eigen-value of the corresponding matrix **T** in (13) is greater than 1. For calculating 8960 sample solutions, the CPU time costs for the proposed joint diagonalisation method, the standard Neumann expansion method and the Monte Carlo method are 7768 s, 24,855 s, and 22,373 s respectively. These calculations are performed on a standard PC with a 2.4 GHz Intel Core 2 Duo processor and 2 GB of RAM. For the Monte Carlo method, a direct solver based on LU factorization is used; for the joint diagonalisation method and the Neumann expansion method, the relative error is set as 2% in comparison to the Monte Carlo method. The original joint diagonalisation method [23] only provides approximate solutions to the stochastic linear system. This example demonstrates that by using Neumann expansion (26), those off-diagonal random entries ignored in [23] can be effectively taken into account to give accurate solutions. The improved joint diagonalisation method is superior to the basic Neumann expansion method in terms of convergence rate. Implementation of the perturbation method (Sec-

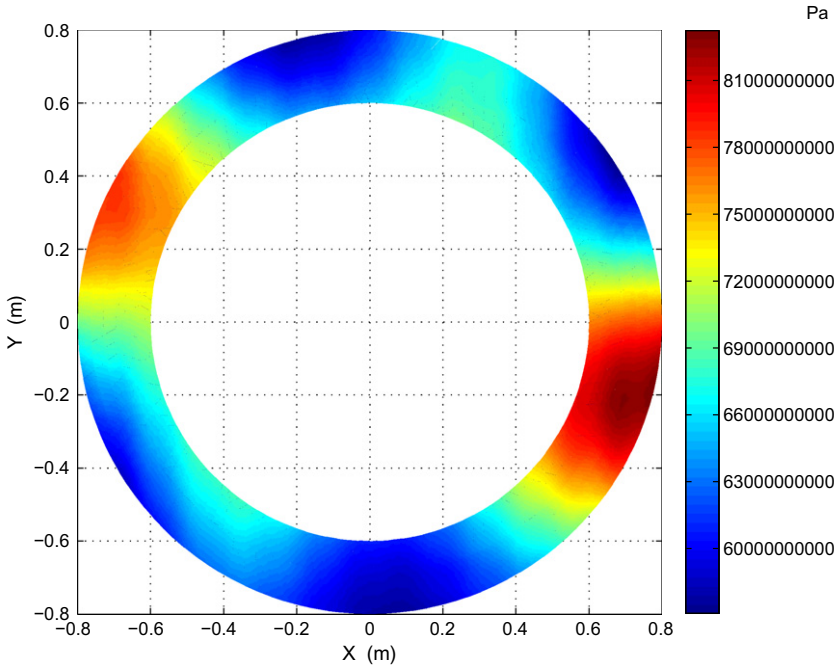

Fig. 8. Young's modulus reconstructed from the F–K–L expansion.

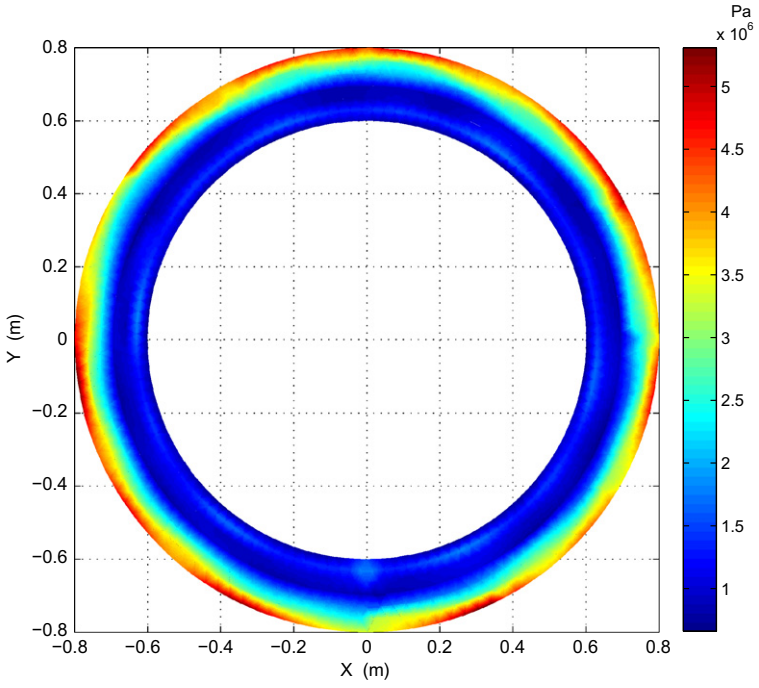

Fig. 9. A sample solution of the first principal stress.

tion 3.1) and projection methods (Section 3.2) becomes increasingly more complex as the number of random variables grows, and as a result these two solution strategies are not particularly favored in solving stochastic systems containing a relatively large number of random variables. Hence, in this work, the perturbation method and the projection method are not implemented for comparison.

A specific sample solution of temperature distribution of the concrete pipe is plotted in Fig. 5. Although the variation of thermal conductivity is quite significant (shown in Fig. 3), the random variation of temperature distribution in Fig. 5 is quite small and the result is rather similar to the deterministic case due to the strong fixed-temperature boundary conditions applied on both the inner- and the outer-pipe surfaces. Shown in Fig. 6 is the difference between the random solution in Fig. 5 and the deterministic solution in which the thermal conductivity is set as the mean value 1.7 W/(m K). One of the major advantages of the stochastic heat conduction analysis is that it can provide the probability distribution of temperature at any point of interest. For example, from a simple statistical counting of the sample solutions, the probability distri-

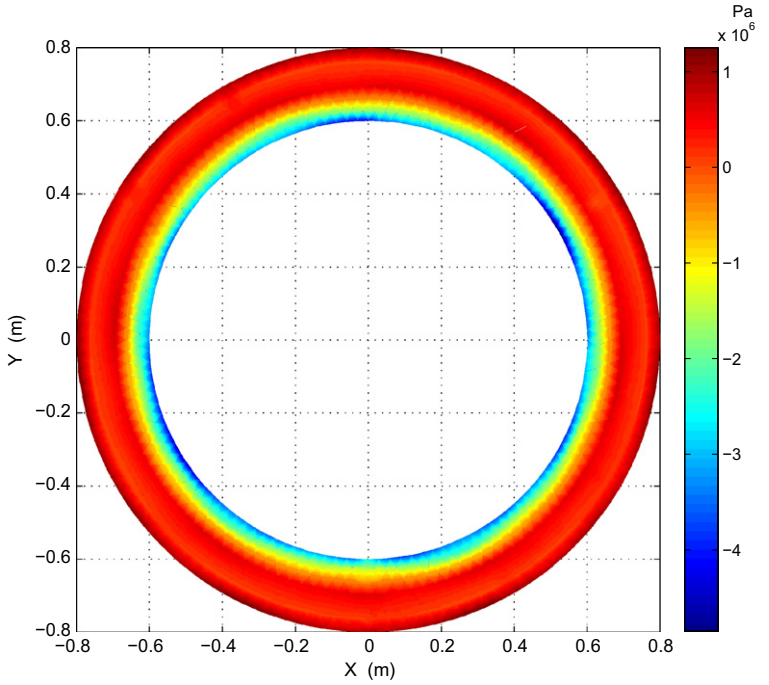

Fig. 10. A sample solution of the second principal stress.

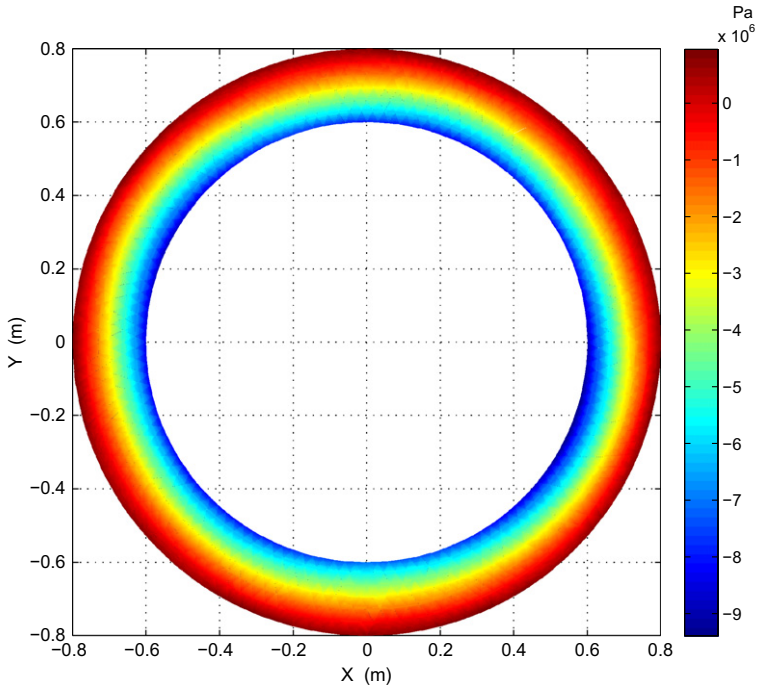

Fig. 11. A sample solution of the third principal stress.

bution of temperature at node 806 (see Fig. 2) is given in Fig. 7. Such information in pdf can provide a better technical reference to design engineers. Compared to conventional design based on a blind safety factor, the new design is potentially more accurate, more clear and more flexible.

6.2. Stress distribution of the concrete pipe under both thermal and pressure loads on the inner and outer surfaces

This example computes the random stress distribution of the concrete pipe. As shown in Fig. 2, the concrete pipe buried in ground has both temperature and pressure boundary conditions, which introduce respectively thermal and pressure loads into the system. The reference temperature of the concrete pipe is assumed as −2 °C, the same as the soil temperature. The thermal load due to the nonuniform change of temperature can be readily computed according to the temperature distribution obtained in example (Section 6.1). As shown in Fig. 1, there are two random material parameters, i.e. the thermal conductivity and the Young's modulus. Other material properties including the Poisson's ratio, the coefficient of thermal expansion and the density are assumed as deterministic constants.

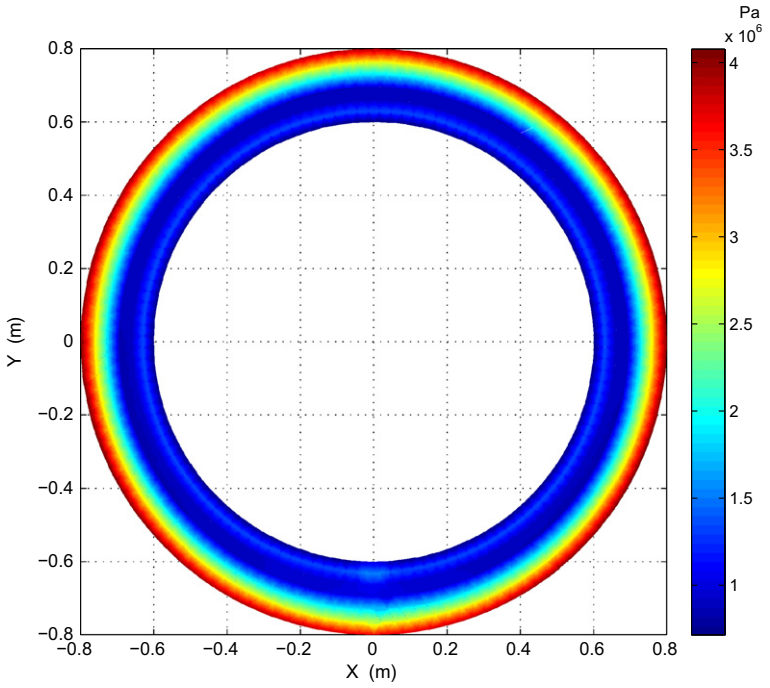

Fig. 12. The mean solution of the first principal stress.

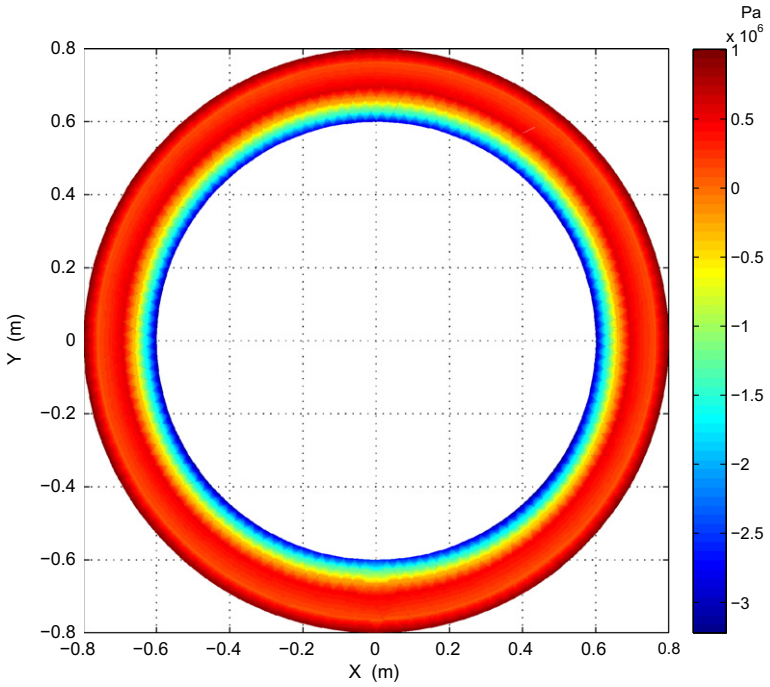

Fig. 13. The mean solution of the second principal stress.

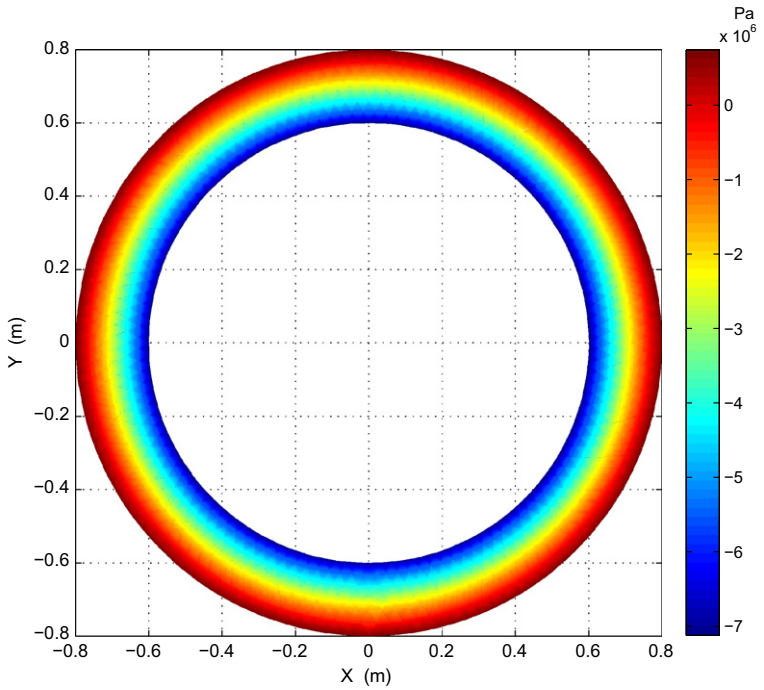

Fig. 14. The mean solution of the third principal stress.

Following the stochastic finite modeling algorithm as described in Section 5, the random Young's modulus and thermal conductivity are discretised by using the F–K–L representation scheme, where the error tolerance is set as 10%. Similar to the contour plot of the sample thermal conductivity shown in Fig. 3, a sample Young's modulus is plotted in Fig. 8. After discretising the unknown displacement field with a finite element mesh and following a plane strain stochastic finite element formulation, a stochastic linear system containing 23 real-symmetric matrices (1792 by 1792 in dimension) is constructed. Here, in the resulting stochastic linear system, not only the stiffness matrix is random, but also the load vector is random. This is because the thermal load contribution is random due to the random thermal conductivity.

The stochastic linear system is solved by using the proposed joint diagonalisation approach and a total number of 17,920 sample solutions are computed. A set of sample solutions of principal stresses are shown in Figs. 9–11. The mean solutions of principal stresses are shown in Figs. 12–14. It can be seen that the random variations through the medium differ for different principal stress components. For the node 806 as shown in Fig. 2, the probability density functions for all three principal stresses are plotted in Fig. 15. It can be seen that, for different principal stress components, the probability distributions differ not only in the variation range, but also in the type of distribution. This information of spatial and random variation of principal stresses can not be obtained using conventional deterministic analysis. In particular, the variation difference among different stress components is not reflected in the traditional design code based on deterministic analysis and a blind safety factor. Thus, it is clear that the SFEM technique does have the potential advantage of providing more meaningful information for the design of structures with random properties.

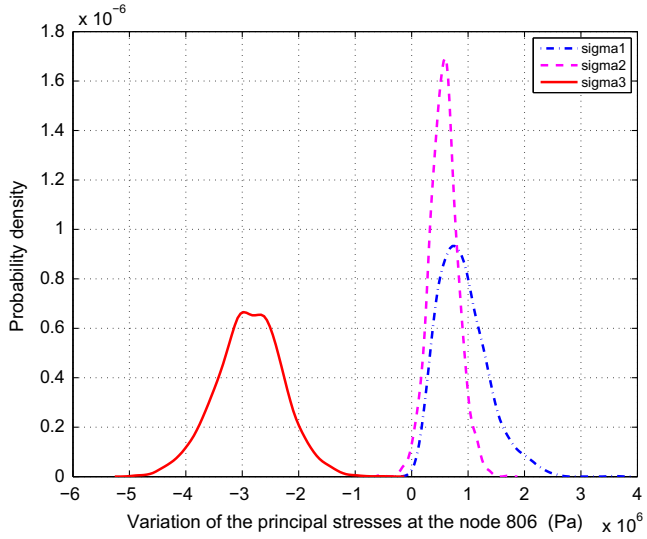

Fig. 15. Probability distribution of principal stresses at node 806.

7. Conclusion

For the solution of static and steady-state problems of random media, this paper presents an improved joint diagonalisation solution framework. Firstly, the random material properties are discretised by using the Fourier–Karhunen–Loève expansion scheme. Then, after discretising the unknowns with finite element meshes and following a similar formulation procedure as that in the standard finite element method, the stochastic linear system consisting of uncorrelated random variables and deterministic real-symmetric matrices is constructed. Finally, the random solution of the stochastic linear system is obtained by using the improved joint diagonalisation approach. The recently developed Jacobi-like joint diagonalisation algorithm is employed to simultaneously and approximately diagonalise all the real-symmetric matrices arising in the stochastic equation. The Neumann expansion is employed to improve the accuracy of the random solution by taking into account the influence from the off-diagonal entries in the approximately diagonalised matrix system. Numerical examples of heat conduction analysis and elastostatics analysis are presented to investigate the performance of the proposed stochastic approach. It is shown that with moderate additional computational cost,

the modified joint diagonalisation approach has a clear advantage, in terms of accuracy and efficiency, over the traditional Neumann expansion approach and the original joint diagonalisation approach. The numerical examples also demonstrate how the proposed stochastic approach, as a complete solution scheme, can be applied in practical random engineering structures to provide meaningful reference information to design engineers.

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