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Piezoelectric energy harvesting from broadband random vibrations

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Abstract
Energy harvesting for the purpose of powering low power electronic sensor systems has received explosive attention in the last few years. Most works using deterministic approaches focusing on using the piezoelectric effect to harvest ambient vibration energy have concentrated on cantilever beams at resonance using harmonic excitation. Here, using a stochastic approach, we focus on using a stack configuration and harvesting broadband vibration energy, a more practically available ambient source. It is assumed that the ambient base excitation is stationary Gaussian white noise, which has a constant power-spectral density across the frequency range considered. The mean power acquired from a piezoelectric vibration-based energy harvester subjected to random base excitation is derived using the theory of random vibrations. Two cases, namely the harvesting circuit with and without an inductor, have been considered. Exact closed-form expressions involving non-dimensional parameters of the electromechanical system have been given and illustrated using numerical examples.

(Some figures in this article are in colour only in the electronic version)

Nomenclature

α	dimensionless time constant $\alpha = \omega_n C_p R_l$
β	dimensionless constant $\beta = \omega_n^2 L C_p$
κ	electromechanical coupling coefficient $\kappa = \theta^2/kC_p$
Ω	dimensionless frequency $\Omega = \omega/\omega_n$
ω	frequency
ω_n	natural frequency of the harvester
θ	electromechanical coupling
ζ	damping factor
c	damping of the harvester
C_p	capacitance of the piezoelectric layer
k	equivalent stiffness of the harvester
L	inductance
m	equivalent mass of the harvester
R_l	load resistance
t	time
$V(\omega)$	Fourier transform of voltage $v(t)$
$v(t)$	voltage
$X(\omega)$	Fourier transform of displacement $x(t)$
$x(t)$	displacement of the mass
$X_b(\omega)$	Fourier transform of base excitation $x_b(t)$
$x_b(t)$	base excitation to the harvester
$(\bullet)^*$	complex conjugation
$\det[\bullet]$	determinant of a matrix
$E[\bullet]$	expectation operator
$\Phi_\bullet(\omega)$	spectral density
$R_\bullet$	autocorrelation function

1. Introduction

The harvesting of ambient vibration energy for use in powering low energy electronic devices has formed the focus of much recent research [1–6]. Of the published results that focus on the piezoelectric effect as the transduction method, almost all have focused on harvesting using cantilever beams and on single-frequency ambient energy, i.e. resonance-based energy harvesting. The exceptions are Sodano *et al* [7] who looked at random ambient vibration disturbances via strictly experimental means and Tanner *et al* [8] who developed a piezoceramic stack harvesting device to power a magneto-rheological damper, again only experimentally. The design of an energy harvesting device must be tailored to the ambient energy available. In some applications the ambient excitation

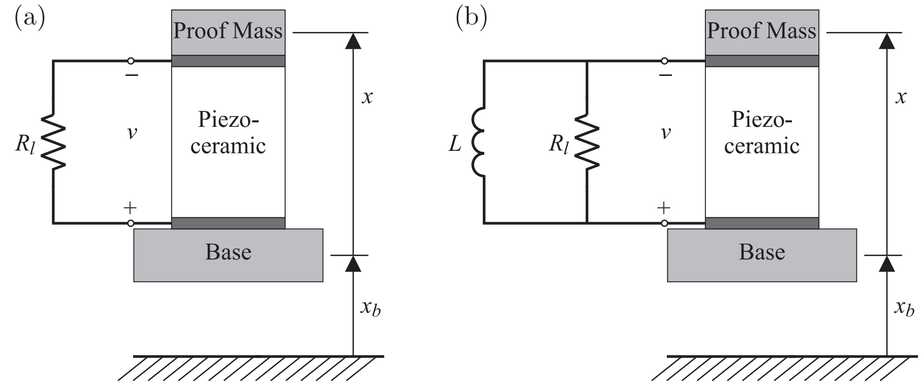

Figure 1. Schematic diagrams of piezoelectric energy harvesters with two different harvesting circuits. (a) Harvesting circuit without an inductor. (b) Harvesting circuit with an inductor.

will be at a single frequency, and most studies have designed resonant harvesting devices based on this. Such devices have to be tuned to the excitation and may not be robust to variations in the excitation frequency. In many applications the ambient energy is random and broadband, and the design of the harvester must account for this form of excitation. This paper examines the use of a stack harvester using the d33 piezoelectric constant, with broadband random ambient base acceleration as the input.

Energy harvesting of ambient vibration has become important and new electronic devices are being developed that require very low power. Completely wireless sensor systems are desirable and this can only be accomplished by using batteries and/or harvested energy. Harvesting is attractive because harvested energy can be used directly or used to recharge batteries or other storage devices, which enhances battery life. Soliman *et al* [9] considered energy harvesting under wide band excitation. Liu *et al* [10] proposed acoustic energy harvesting using an electromechanical resonator. Shu *et al* [11–13] conducted detailed analysis of the power output for piezoelectric energy harvesting systems. Several authors [14–17] have proposed methods to optimize the parameters of the system to maximize the harvested energy.

Most of the works reported above consider that the (base) excitation has some known form. Typically harmonic excitation is considered. However, it is easy to envisage situations where energy harvesting devices are operating under unknown or random excitations. In such situations the ambient vibration should be described using the theory of random processes [18] and the analysis of harvested power should be performed using the framework of probability theory. Lefeuvre *et al* [19] were possibly the first to consider random vibrations in the context of energy harvesting due to random vibrations. However, Halvorsen [20] first used linear random vibration theory to obtain closed-form expressions for the harvested energy. They also derived the Fokker–Planck equation governing the probability density function of the harvested power. In this paper we consider a similar approach. Specifically, we derive expressions for the mean normalized harvested power of a system subjected to Gaussian white noise base acceleration.

The outline of this paper is as follows. A single-degree-of-freedom electromechanical model with and without an inductor under deterministic excitation is discussed in section 2. A brief overview of linear stationary random vibrations is given in section 3. The mean power for a system without an inductor is derived in section 4 and the equivalent expression for the system with an inductor in section 5. Numerical illustrations of the derived expressions are shown in section 6. Based on the study considered in this paper, a set of conclusions are drawn up in section 7.

2. Single-degree-of-freedom electromechanical model

We consider stack-type piezoelectric harvesting circuits as shown in figure 1. We have considered two types of circuits, namely without and with an inductor, as shown in figures 1(a) and 1(b), respectively. Energy is harvested through base excitations and the piezoceramic is operated in the {33} direction. Here we use a simple single-degree-of-freedom model for the mechanical motion of the harvester.

A more detailed model, along with correction factors for a single-degree-of-freedom model that accounts for distributed mass effects, was given by Erturk and Inman [21–24]. This enables the analysis described here to be used in a wide range of practical applications, providing that the broadband base acceleration does not excite high vibration modes of the harvester. The single-degree-of-freedom model could be extended to multi-degree-of-freedom mechanical systems by using a modal decomposition of the response. This paper only considers a linear model of the piezoelectric material along the d33 direction, which allows the application of linear random vibration theory. The relaxation of the linearity assumption would require the use of nonlinear random vibration theory which is not considered in this initial work.

2.1. Circuit without an inductor

duToit and Wardle [25] expressed the coupled electromechanical behaviour by the linear ordinary differential equations:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \theta v(t) = -m\ddot{x}_{\mathrm{b}}(t) \qquad (1)$$

$$\theta \dot{x}(t) + C_{\rm p}\dot{v}(t) + \frac{1}{R_{\rm l}}v(t) = 0. \qquad (2)$$

Equation (1) is simply Newton's equation of motion for a single-degree-of-freedom system, where t is the time, $x(t)$ is the displacement of the mass, m, c and k are, respectively. the mass, damping and stiffness of the harvester and $x_{\rm b}(t)$ is the random base excitation. In this paper we consider the base excitation to be a random process. θ is the electromechanical coupling and the mechanical force is modelled as proportional to the voltage across the piezoceramic, $v(t)$. Equation (2) is obtained from the electrical circuit, where the voltage across the load resistance arises from the mechanical strain through the electromechanical coupling, θ, and the capacitance of the piezoceramic, $C_{\rm p}$. Transforming both the equations into the frequency domain and dividing the first equation by m and the second equation by $C_{\rm p}$ we obtain

$$(-\omega^2 + 2{\rm i}\omega\zeta\omega_{\rm n} + \omega_{\rm n}^2)X(\omega) - \frac{\theta}{m}V(\omega) = \omega^2 X_{\rm b}(\omega) \qquad (3)$$

$${\rm i}\omega\frac{\theta}{C_{\rm p}}X(\omega) + \left({\rm i}\omega + \frac{1}{C_{\rm p}R_{\rm l}}\right)V(\omega) = 0. \qquad (4)$$

Here $X(\omega)$, $V(\omega)$ and $X_{\rm b}(\omega)$ are, respectively, the Fourier transforms of $x(t)$, $v(t)$ and $x_{\rm b}(t)$. The natural frequency of the harvester, $\omega_{\rm n}$, and the damping factor, ζ, are defined as

$$\omega_{\rm n} = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \zeta = \frac{c}{2m\omega_{\rm n}}. \qquad (5)$$

Dividing the preceding equations by $\omega_{\rm n}$ and writing in matrix form one has

$$\begin{bmatrix} (1-\Omega^2) + 2{\rm i}\Omega\zeta & -\frac{\theta}{k} \\ {\rm i}\Omega\frac{\alpha\theta}{C_{\rm p}} & ({\rm i}\Omega\alpha + 1) \end{bmatrix} \begin{Bmatrix} X \\ V \end{Bmatrix} = \begin{Bmatrix} \Omega^2 X_{\rm b} \\ 0 \end{Bmatrix}, \qquad (6)$$

where the dimensionless frequency and dimensionless time constant are defined as

$$\Omega = \frac{\omega}{\omega_{\rm n}} \qquad \text{and} \qquad \alpha = \omega_{\rm n} C_{\rm p} R_{\rm l}. \qquad (7)$$

α is the time constant of the first-order electrical system, non-dimensionalized using the natural frequency of the mechanical system. Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_1}\begin{bmatrix} ({\rm i}\Omega\alpha + 1) & \frac{\theta}{k} \\ -{\rm i}\Omega\frac{\alpha\theta}{C_{\rm p}} & (1-\Omega^2) + 2{\rm i}\Omega\zeta \end{bmatrix}\begin{Bmatrix} \Omega^2 X_{\rm b} \\ 0 \end{Bmatrix} = \begin{Bmatrix} ({\rm i}\Omega\alpha + 1)\Omega^2 X_{\rm b}/\Delta_1 \\ -{\rm i}\Omega^3\frac{\alpha\theta}{C_{\rm p}}X_{\rm b}/\Delta_1 \end{Bmatrix}, \qquad (8)$$

where the determinant of the coefficient matrix is

$$\Delta_1({\rm i}\Omega) = ({\rm i}\Omega)^3\alpha + (2\zeta\alpha + 1)({\rm i}\Omega)^2 + (\alpha + \kappa^2\alpha + 2\zeta)({\rm i}\Omega) + 1 \qquad (9)$$

and the non-dimensional electromechanical coupling coefficient is

$$\kappa^2 = \frac{\theta^2}{kC_{\rm p}}. \qquad (10)$$

2.2. Circuit with an inductor

For this case, following Renno *et al* [17], the electrical equation becomes

$$\theta\ddot{x}(t) + C_{\rm p}\ddot{v}(t) + \frac{1}{R_{\rm l}}\dot{v}(t) + \frac{1}{L}v(t) = 0 \qquad (11)$$

where L is the inductance of the circuit. Transforming equation (11) into the frequency domain and dividing by $C_{\rm p}\omega_{\rm n}^2$ one has

$$-\Omega^2\frac{\theta}{C_{\rm p}}X + \left(-\Omega^2 + {\rm i}\Omega\frac{1}{\alpha} + \frac{1}{\beta}\right)V = 0 \qquad (12)$$

where the second dimensionless constant is defined as

$$\beta = \omega_{\rm n}^2 L C_{\rm p}, \qquad (13)$$

and is the ratio of the mechanical to electrical natural frequencies. Similar to equation (6), this equation can be written in matrix form with the equation of motion of the mechanical system (3) as

$$\begin{bmatrix} (1-\Omega^2) + 2{\rm i}\Omega\zeta & -\frac{\theta}{k} \\ -\Omega^2\frac{\alpha\beta\theta}{C_{\rm p}} & \alpha(1-\beta\Omega^2) + {\rm i}\Omega\beta \end{bmatrix}\begin{Bmatrix} X \\ V \end{Bmatrix} = \begin{Bmatrix} \Omega^2 X_{\rm b} \\ 0 \end{Bmatrix}. \qquad (14)$$

Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_2}\begin{bmatrix} \alpha(1-\beta\Omega^2) + {\rm i}\Omega\beta & \frac{\theta}{k} \\ \Omega^2\frac{\alpha\beta\theta}{C_{\rm p}} & (1-\Omega^2) + 2{\rm i}\Omega\zeta \end{bmatrix} \times \begin{Bmatrix} \Omega^2 X_{\rm b} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \left(\alpha(1-\beta\Omega^2) + {\rm i}\Omega\beta\right)\Omega^2 X_{\rm b}/\Delta_2 \\ \Omega^4\frac{\alpha\beta\theta}{C_{\rm p}}X_{\rm b}/\Delta_2 \end{Bmatrix} \qquad (15)$$

where the determinant of the coefficient matrix is

$$\begin{aligned}\Delta_2({\rm i}\Omega) = {} & ({\rm i}\Omega)^4\beta\alpha + (2\zeta\beta\alpha + \beta)({\rm i}\Omega)^3 \\ & + (\beta\alpha + \alpha + 2\zeta\beta + \kappa^2\beta\alpha)({\rm i}\Omega)^2 \\ & + (\beta + 2\zeta\alpha)({\rm i}\Omega) + \alpha. \end{aligned} \qquad (16)$$

3. A brief overview of stationary random vibration

We consider that the base excitation $x_{\rm b}(t)$ is a random process. It is assumed that $x_{\rm b}(t)$ is a weakly stationary, Gaussian, broadband random process. Mechanical systems driven by this type of excitation have been discussed by Lin [26], Nigam [27], Bolotin [28], Roberts and Spanos [29] and Newland [30] within the scope of random vibration theory. To obtain the samples of the random response quantities such as the displacement of the mass $x(t)$ and the voltage $v(t)$, one needs to solve the coupled stochastic differential equations (1) and (2) or (1) and (11). However, analytical results developed within the theory of random vibration allow us to bypass numerical solutions because we are interested in the average values of the output random processes. Here we extend the available results to the energy harvester. Since $x_{\rm b}(t)$ is a weakly stationary random process, its autocorrelation function depends only on the difference in the time instants, and thus

$$E[x_{\rm b}(\tau_1)x_{\rm b}(\tau_2)] = R_{x_{\rm b}x_{\rm b}}(\tau_1 - \tau_2). \qquad (17)$$

This autocorrelation function can be expressed as the inverse Fourier transform of the spectral density $\Phi_{x_b x_b}(\omega)$ as

$$R_{x_b x_b}(\tau_1-\tau_2)=\int_{-\infty}^{\infty}\Phi_{x_b x_b}(\omega)\exp[\mathrm{i}\omega(\tau_1-\tau_2)]\,\mathrm{d}\omega. \quad (18)$$

In this paper we are interested in the average harvested power given by

$$E[P(t)]=E\left[\frac{v^2(t)}{R_l}\right]=\frac{E\left[v^2(t)\right]}{R_l}. \quad (19)$$

For a damped linear system of the form $V(\omega)=H(\omega)X_b(\omega)$, it can be shown that [26, 27] the spectral density of V is related to the spectral density of X_b by

$$\Phi_{VV}(\omega)=|H(\omega)|^2\Phi_{x_b x_b}(\omega). \quad (20)$$

Thus, for large t, we obtain

$$E\left[v^2(t)\right]=R_{vv}(0)=\int_{-\infty}^{\infty}|H(\omega)|^2\Phi_{x_b x_b}(\omega)\,\mathrm{d}\omega. \quad (21)$$

This expression will be used to obtain the average power for the two cases considered. We assume that the base acceleration $\ddot{x}_b(t)$ is Gaussian white noise so that its spectral density is constant with respect to frequency.

The calculation of the integral on the right-hand side of equation (21) in general requires the calculation of integrals of the following form:

$$I_n=\int_{-\infty}^{\infty}\frac{\Xi_n(\omega)\,\mathrm{d}\omega}{\Lambda_n(\omega)\Lambda_n^*(\omega)} \quad (22)$$

where the polynomials have the form

$$\Xi_n(\omega)=b_{n-1}\omega^{2n-2}+b_{n-2}\omega^{2n-4}+\cdots+b_0 \quad (23)$$

$$\Lambda_n(\omega)=a_n(\mathrm{i}\omega)^n+a_{n-1}(\mathrm{i}\omega)^{n-1}+\cdots+a_0. \quad (24)$$

Following Roberts and Spanos [29] this integral can be evaluated as

$$I_n=\frac{\pi}{a_n}\frac{\det[\mathbf{D}_n]}{\det[\mathbf{N}_n]}. \quad (25)$$

Here the $m\times m$ matrices are defined as

$$\mathbf{D}_n=\begin{bmatrix} b_{n-1} & b_{n-2} & & \cdots & & & b_0\\ -a_n & a_{n-2} & -a_{n-4} & a_{n-6} & \cdots & 0 & \cdots\\ 0 & -a_{n-1} & a_{n-3} & -a_{n-5} & \cdots & 0 & \cdots\\ 0 & a_n & -a_{n-2} & a_{n-4} & \cdots & 0 & \cdots\\ 0 & \cdots & & & \cdots & 0 & \cdots\\ 0 & 0 & & & \cdots & -a_2 & a_0 \end{bmatrix} \quad (26)$$

and

$$\mathbf{N}_n=\begin{bmatrix} a_{n-1} & -a_{n-3} & a_{n-5} & -a_{n-7} & & & \\ -a_n & a_{n-2} & -a_{n-4} & a_{n-6} & \cdots & 0 & \cdots\\ 0 & -a_{n-1} & a_{n-3} & -a_{n-5} & \cdots & 0 & \cdots\\ 0 & a_n & -a_{n-2} & a_{n-4} & \cdots & 0 & \cdots\\ 0 & \cdots & & & \cdots & 0 & \cdots\\ 0 & 0 & & & \cdots & -a_2 & a_0 \end{bmatrix}. \quad (27)$$

4. Mean power for systems without an inductor

From equation (8) we obtain the voltage in the frequency domain as

$$V=\frac{-\mathrm{i}\Omega^3\frac{\alpha\theta}{C_p}}{\Delta_1(\mathrm{i}\Omega)}X_b. \quad (28)$$

Following duToit and Wardle [25] we are interested in the mean of the normalized harvested power when the base acceleration is Gaussian white noise, that is $|V|^2/(R_l\omega^4\Phi_{x_b x_b})$. Note that $\omega^4\Phi_{x_b x_b}$ is the spectral density of the acceleration and is assumed to be constant. After some algebra, from equation (28), the normalized power is

$$\tilde{P}=\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}=\frac{k\alpha\kappa^2}{\omega_n^3}\frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)}. \quad (29)$$

Using equation (21), the average normalized power can be obtained as

$$E[\tilde{P}]=E\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}\right]=\frac{k\alpha\kappa^2}{\omega_n^3}\int_{-\infty}^{\infty}\frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)}\,\mathrm{d}\omega. \quad (30)$$

From equation (9) we observe that $\Delta_1(\mathrm{i}\Omega)$ is a third-order polynomial in $(\mathrm{i}\Omega)$. Noting that $\mathrm{d}\omega=\omega_n\,\mathrm{d}\Omega$ and from equation (9), the average harvested power can be obtained from equation (30) as

$$E[\tilde{P}]=E\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}\right]=m\alpha\kappa^2 I^{(1)} \quad (31)$$

where

$$I^{(1)}=\int_{-\infty}^{\infty}\frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)}\,\mathrm{d}\Omega. \quad (32)$$

Comparing $I^{(1)}$ with the general integral in equation (22) we have

$$n=3,\quad b_2=0,\quad b_1=1,\quad b_0=0,$$
$$a_3=\alpha,\quad a_2=(2\zeta\alpha+1),\quad a_1=(\alpha+\kappa^2\alpha+2\zeta),$$
$$a_0=1. \quad (33)$$

Now using equation (25), the integral can be evaluated as

$$I^{(1)}=\frac{\pi}{\alpha}\frac{\det\begin{bmatrix}0 & 1 & 0\\ -\alpha & \alpha+\kappa^2\alpha+2\zeta & 0\\ 0 & -2\zeta\alpha-1 & 1\end{bmatrix}}{\det\begin{bmatrix}2\zeta\alpha+1 & -1 & 0\\ -\alpha & \alpha+\kappa^2\alpha+2\zeta & 0\\ 0 & -2\zeta\alpha-1 & 1\end{bmatrix}}. \quad (34)$$

Combining this with equation (31) we finally obtain the average harvested power due to white noise base acceleration as

$$E[\tilde{P}]=E\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}\right]=\frac{\pi m\alpha\kappa^2}{(2\zeta\alpha^2+\alpha)\kappa^2+4\zeta^2\alpha+(2\alpha^2+2)\zeta}. \quad (35)$$

Since α and κ^2 are positive the average harvested power is monotonically decreasing with damping ratio ζ. Thus the mechanical damping in the harvester should be minimized. For fixed α and ζ the average harvested power is monotonically increasing with the coupling coefficient κ^2, and hence the electromechanical coupling should be as large as possible. Maximizing the average power with respect to α gives the condition

$$\alpha^2(1+\kappa^2)=1 \tag{36}$$

or in terms of physical quantities

$$R_1^2 C_p(kC_p+\theta^2)=m. \tag{37}$$

5. Mean power for systems with an inductor

From equation (15) we obtain the voltage in the frequency domain as

$$V=\frac{\Omega^4\frac{\alpha\beta\theta}{C_p}}{\Delta_2(\mathrm{i}\Omega)}X_b. \tag{38}$$

Following Renno *et al* [17] the average normalized harvested power can be obtained as

$$E[\tilde{P}]=E\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}\right]=m\alpha\beta^2\kappa^2 I^{(2)} \tag{39}$$

where

$$I^{(2)}=\int_{-\infty}^{\infty}\frac{\Omega^4}{\Delta_2(\mathrm{i}\Omega)\Delta_2^*(\mathrm{i}\Omega)}\,\mathrm{d}\Omega. \tag{40}$$

Using the expression of $\Delta_2(\mathrm{i}\Omega)$ in equation (16) and comparing $I^{(2)}$ with the general integral in equation (22) we have

$$\begin{gathered} n=4,\quad b_3=0,\quad b_2=1,\quad b_1=0,\\ b_0=0,\quad a_4=\beta\alpha,\quad a_3=(2\zeta\beta\alpha+\beta),\\ a_2=(\beta\alpha+\alpha+2\zeta\beta+\kappa^2\beta\alpha),\quad a_1=(\beta+2\zeta\alpha),\\ a_0=\alpha. \end{gathered} \tag{41}$$

Now using equation (25), the integral can be evaluated as

$$I^{(2)}=\frac{\pi}{\beta\alpha}\times\det\begin{bmatrix}0 & 1 & 0 & 0\\ -\beta\alpha & \beta\alpha+\alpha+2\zeta\beta+\kappa^2\beta\alpha & -\alpha & 0\\ 0 & -2\zeta\beta\alpha-\beta & \beta+2\zeta\alpha & 0\\ 0 & -\beta\alpha & \beta\alpha+\alpha+2\zeta\beta+\kappa^2\beta\alpha & \alpha\end{bmatrix}\Bigg/\det\begin{bmatrix}2\zeta\beta\alpha+\beta & -\beta-2\zeta\alpha & 0 & 0\\ -\beta\alpha & \beta\alpha+\alpha+2\zeta\beta+\kappa^2\beta\alpha & -\alpha & 0\\ 0 & -2\zeta\beta\alpha-\beta & \beta+2\zeta\alpha & 0\\ 0 & -\beta\alpha & \beta\alpha+\alpha+2\zeta\beta+\kappa^2\beta\alpha & \alpha\end{bmatrix}. \tag{42}$$

Combining this with equation (31) we finally obtain the average normalized harvested power as

$$\begin{aligned} E[\tilde{P}]=E\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_b x_b})}\right]&=m\alpha\beta\kappa^2\pi(\beta+2\alpha\zeta)/\\ &[(4\beta\alpha^3\zeta^2+2\beta\alpha^2(\beta+1)\zeta+\beta^2\alpha)\kappa^2+8\beta\alpha^2\zeta^3\\ &+4\beta\alpha(\beta+1)\zeta^2+2(\beta^2\alpha^2+\beta^2-2\beta\alpha^2+\alpha^2)\zeta]\\ &=\frac{m\alpha\beta\kappa^2\pi(\beta+2\alpha\zeta)}{\beta(\beta+2\alpha\zeta)(1+2\alpha\zeta)(\alpha\kappa^2+2\zeta)+2\alpha^2\zeta(\beta-1)^2}. \end{aligned} \tag{43}$$

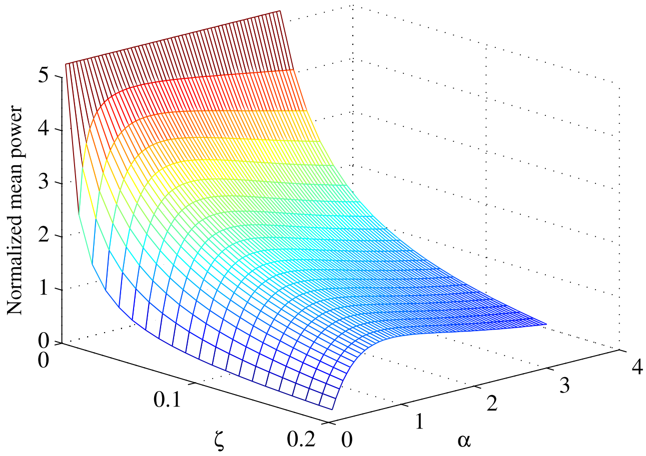

Figure 2. The normalized mean power of a harvester without an inductor as a function of α and ζ, $\kappa=0.6$.

This is the complete closed-form expression of the normalized harvested power under Gaussian white noise base acceleration.

Since α, β and κ^2 are positive the average harvested power is monotonically decreasing with damping ratio ζ. Thus the mechanical damping in the harvester should be minimized. For fixed α, β and ζ the average harvested power is monotonically increasing with the coupling coefficient κ^2, and hence the electromechanical coupling should be as large as possible. These are the same conclusions as for the case without an inductor, although only slightly more difficult to prove.

We can also determine optimum values for α and β. Dividing both the numerator and denominator of the last expression in equation (43) by $\beta(\beta+2\alpha\zeta)$ shows that the optimum value of β for all values of the other parameters is $\beta=1$. This value of β implies that $\omega_n^2 LC_p=1$, and thus the mechanical and electrical natural frequencies are equal. With $\beta=1$ the average normalized harvested power is

$$E[\tilde{P}]=\frac{m\alpha\kappa^2\pi}{(1+2\alpha\zeta)(\alpha\kappa^2+2\zeta)}. \tag{44}$$

If κ and ζ are fixed then the maximum power with respect to α is obtained when $\alpha=1/\kappa$.

6. Numerical illustrations

The expressions of sections 4 and 5 are now illustrated numerically for a system with unit mass. In figure 2 the normalized mean power of a harvester without an inductor, as given by equation (35), is shown as a function of α and ζ. For illustration, the value of the coupling coefficient κ is kept fixed at 0.6. The increased harvested energy as the damping ratio ζ decreases is clearly seen. Also there is a maximum in the harvested energy for $\alpha=0.86$, corresponding to the optimum predicted by equation (36).

In figure 3 the normalized mean power of a harvester with an inductor, as given by equation (43), is shown as a function of α and β. For illustration, the value of the coupling coefficient κ is again kept fixed at 0.6, while the value of damping factor ζ is kept fixed at 0.1. There is clearly a well-defined maximum

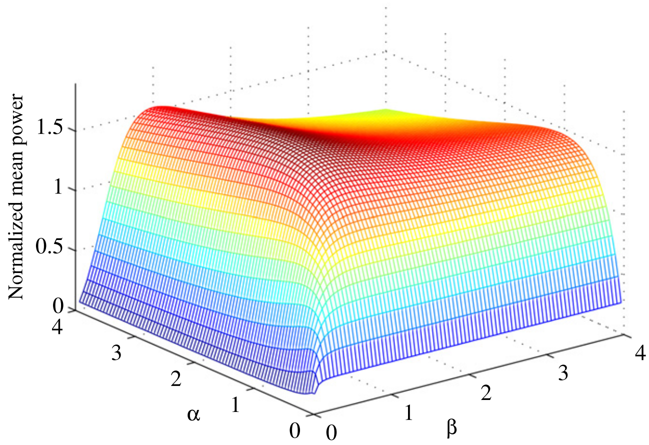

Figure 3. The normalized mean power of a harvester with an inductor as a function of α and β, with $\zeta = 0.1$ and $\kappa = 0.6$.

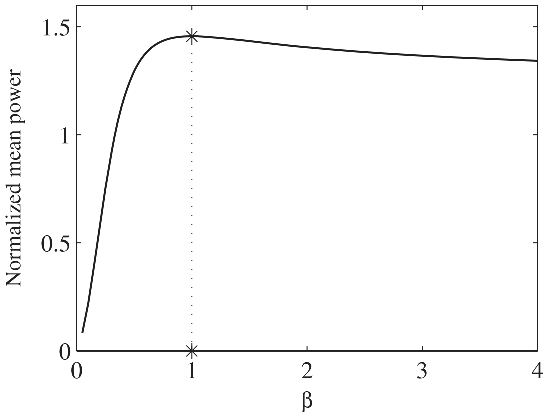

Figure 4. The normalized mean power of a harvester with an inductor as a function of β for $\alpha = 0.6$, $\zeta = 0.1$ and $\kappa = 0.6$. The $*$ corresponds to the optimal value of $\beta(=1)$ for the maximum mean harvested power.

harvested energy at $\beta = 1$ and $\alpha = 1/\kappa = 1.667$, as predicted in section 5, although this is better illustrated by taking sections through this 3D surface.

The mean harvested power as a function of β is shown in figure 4, with the other parameters fixed at $\alpha = 0.6$, $\zeta = 0.1$ and $\kappa = 0.6$. The optimum value occurs at $\beta = 1$, which is shown by the star in figure 4. It was highlighted in section 5 that this value of β is optimum for all values of α. The mean harvested power as a function of α is shown in figure 5 for $\beta = 1$ and clearly shows the maximum at $\alpha = 1.667$.

7. Conclusions

Vibration-energy-based piezoelectric energy harvesters are expected to operate under a wide range of ambient environments. This paper considers energy harvesting of such systems under broadband random excitations. Specifically, analytical expressions of the normalized mean harvested power due to stationary Gaussian white noise base excitation has been derived. The resulting two-dimensional stochastic differential equations are solved using the theory of linear random vibrations. Two cases, namely the harvesting circuit

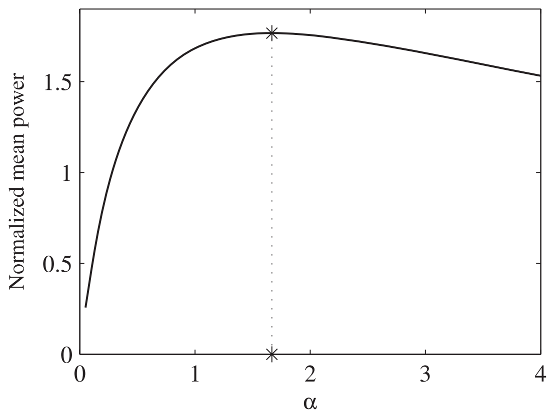

Figure 5. The normalized mean power of a harvester with an inductor as a function of α for $\beta = 1$, $\zeta = 0.1$ and $\kappa = 0.6$. The $*$ corresponds to the optimal value of $\alpha(=1.667)$ for the maximum mean harvested power.

with and without an inductor, have been considered. For both cases exact closed-form expressions of the harvested power involving the non-dimensional time constants, the non-dimensional electromechanical coupling coefficient and the mechanical viscous damping factor have been derived. Optimal values of the parameter for which the harvested power is maximum have been discussed. It was shown that, in order to maximize the mean of the harvested power, (a) the mechanical damping in the harvester should be minimized and (b) the electromechanical coupling should be as large as possible. The electrical circuits may also be optimized to obtain the maximum mean power, and the expressions for these optima have been given. For the circuit with an inductor the maximum mean power occurs when the natural frequency of the electrical circuit is equal to that of the mechanical system.

The expressions derived in this paper are useful in quantifying the harvested power under random vibration. The analysis presented here, equation (43) in particular, can be used to (a) design energy harvesters subject to random excitation and (b) to provide insight into the physical nature of harvesting when subject to random ambient energy. To date, researchers have only provided expressions for the energy harvested due to deterministic ambient energy at a single frequency. The importance of the result presented here is to provide a key to designing harvesters in the more practical case when the ambient vibration is random. The approach described in this paper can be extended to filtered white noise and non-Gaussian excitation that may be described as a rational fraction polynomial in the frequency domain. Such excitation will simulate more realistic excitation spectra compared to the pure Gaussian white noise base acceleration considered in this paper, although the derived expressions will be complicated. Further work is also needed to obtain the higher-order moments of the harvested power, such as the standard deviation, in addition to the mean power derived here.

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