Iterative Methods for Eigenvalues of Viscoelastic Systems

Sondipon Adhikari[1]
Chair of Aerospace Engineering
e-mail: s.adhikari@swansea.ac.uk

Blanca Pascual

School of Engineering,
Swansea University,
Singleton Park,
Swansea SA2 8PP, UK

This paper proposes a new iterative approach for the calculation of eigenvalues of single and multiple degree-of-freedom viscoelastic systems. The Biot model of viscoelasticity is assumed. With this model, the viscoelastic forces depend on the past history of motion via convolution integrals over exponentially decaying kernel functions. Current methods to solve this type of problem normally use the state-space approach involving additional internal variables. Such approaches often increase the order of the eigenvalue problem to be solved and can become computationally expensive for large systems. The method proposed in this paper is aimed to address this issue. In total, five iterative algorithms for the real and complex eigenvalues of single and multiple degree-of-freedom systems have been proposed. The results are obtained in terms of explicit closed-form expressions. This enables one to approximately calculate the eigenvalues of complex viscoelastic systems using the eigenvalues of the underlying elastic systems. Representative numerical examples are given to verify the accuracy of the derived expressions.
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1 Introduction

Due to the superior damping characteristics, the dynamics of viscoelastic materials and structures have received significant attention over the last two decades. The key feature of a viscoelastic system is the incorporation of the time history of the state-variables in the equation of motion. The equation of motion of an N degree-of-freedom linear viscoelastic system can be expressed by coupled integrodifferential equations as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{u}}(\tau)\mathrm{d}\tau + \mathbf{K}_e\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

Here, $\mathbf{u}(t)$ is the displacement vector, $\mathbf{f}(t)$ is the forcing vector, $\mathbf{M}$ is the mass matrix, $\mathbf{K}_e$ is the elastic stiffness matrix, and $\boldsymbol{\mathcal{G}}(t)$ is the matrix of viscoelastic kernel functions. The kernel functions $\boldsymbol{\mathcal{G}}(t)$ are known as retardation functions, heredity functions, after-effect functions, or relaxation functions in the context of different subjects. In the limit when $\boldsymbol{\mathcal{G}}(t-\tau)=\mathbf{C}\delta(t-\tau)$ where $\delta(t)$ is the dirac-delta function, Eq. (1) reduces to the case of elastic system with viscous damping.

A wide variety of mathematical approaches could be used for the determination of kernel functions $\boldsymbol{\mathcal{G}}(t)$ as long as the rate of energy dissipation is non-negative. Here, we will use a viscoelastic material model proposed by Biot [1] for which the kernel function matrix has the special form

$$\boldsymbol{\mathcal{G}}(t) = \mathbf{K}_{v_0} + \sum_{k=1}^{n} a_k e^{-b_k t}\mathbf{K}_{v_k} \tag{2}$$

or in the Laplace domain

$$\mathbf{G}(s) = \mathbf{K}_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s+b_k}\mathbf{K}_{v_k} \tag{3}$$

Constants a_k and b_k are the viscous parameters depending on the viscoelastic material used, n denotes the number of perturbing terms, and $\mathbf{K}_{v_0}$ and $\mathbf{K}_{v_k}$ are the coefficient matrices. Some other viscoelastic modeling approaches, such as the Golla–Hughes–McTavish approach [2,3] and the anelastic displacement field approach [4,5], are although physically different, can be mathematically represented by pole-residue form similar to Eq. (3). Taking the Laplace transform of Eq. (1), the equation of motion can be expressed as

$$\mathbf{D}(s)\bar{\mathbf{u}}(s) = \bar{\mathbf{f}}(s) \tag{4}$$

where $\bar{\mathbf{u}}(s)$ and $\bar{\mathbf{f}}(s)$ are, respectively, the Laplace transforms of $\mathbf{u}(t)$ and $\mathbf{f}(t)$ and the dynamic stiffness matrix $\mathbf{D}(s)$ is given by

$$\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}_e \tag{5}$$

Current methods for dynamic analysis of viscoelastic systems are dominated by state-space based approaches [2–11]. The methods to obtain eigenvalues from the state-space matrices are generally based on Taylor's series expansion and Newton's eigenvalue iteration method [12] or bisection procedure together with a calculation of the number of eigenvalues lying between zero and a given value [13]. In this paper, we aim to propose an iterative method, which is not based on state-space approach [14–20]. Ruge [21] emphasized the advantages for nonstate-space based approaches. The main reasons for seeking an alternative to the state-space approach are (a) although exact in nature, the state-space approach for viscoelastic systems is computationally very intensive for real-life multiple degree-of-freedom (MDOF) systems due to the huge number of internal variables and (b) the physical insights offered by methods in the original space (e.g., the modal analysis) is lost in a state-space based approach. However, it should be emphasized that some kind of approximation needs to be employed for nonstate-space approaches. Recently, Adhikari and Pascual [22] proposed approximation methods based on the Taylor series expansion of the kernel function $\mathbf{G}(s)$ in the complex plane.

In this paper, we propose a new iterative approach to obtain eigenvalues of multiple degrees-of-freedom viscoelastic systems. This approach extends the earlier work by Adhikari and Pascual [22], which is valid for systems with "small viscoelasticity." The work reported in this paper is aimed to relax this limitation. In particular, the previously obtained results are used as the "starting guess" of the proposed iterative approach. The calculation of the eigenvalues by solving the nonlinear eigenvalue problem corresponding to the equation of motion (Eq. (1)) is the main topic of this paper. First, single degree-of-freedom (SDOF) systems are

[1]Corresponding author.

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considered and then, the results are extended to MDOF systems. The validity of the assumptions and the accuracy of the results are verified by numerical calculations.

2 SDOF Systems

The equation of motion can be expressed in the Laplace domain as

$$s^2 m_u \bar{u}(s) + sG(s)\bar{u}(s) + k_e \bar{u}(s) = \bar{f}(s) \quad \text{or} \quad d(s)\bar{u}(s) = \bar{f}(s) \tag{6}$$

Here, the dynamic stiffness

$$d(s) = s^2 m_u + sG(s) + k_e \tag{7}$$

and $\bar{u}(s)$, $\bar{f}(s)$, and $G(s)$ are, respectively, the Laplace transforms of displacement, forcing, and viscoelastic kernel functions. The eigenvalues of the system can be obtained by solving the characteristic equation

$$\begin{gathered} s_j^2 m_u + s_j G(s_j) + k_e = 0 \\ s_j^2 m_u + s_j \left(k_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s_j + b_k} k_{v_k} \right) + k_e = 0, \quad j = 1, \ldots, m \end{gathered} \tag{8}$$

where m is the order of the characteristic polynomial. From Eq. (8), one can observe that in general, $m=2+n$. There are two conjugated complex eigenvalues, which correspond to the oscillatory motions of the system. The rest n eigenvalues are real and negative, as expected for stable, overly damped structures. These overdamped modes are not oscillatory in nature. Here, both the complex conjugate modes and the damping modes will be derived. In the following sections, iterative methods are developed for the elastic modes and damping modes.

2.1 Complex Conjugate Eigenvalues. The eigenvalues of the underlying damped elastic system are complex conjugate in nature. These eigenvalues can be expressed in terms of the undamped eigenvalues. Therefore, if the approximate eigenvalues of the viscoelastic system can be obtained from the eigenvalues of the underlying elastic system, one can effectively obtain them by simple post-processing of the eigenvalues of the underlying undamped systems only. This will reduce computational cost because the undamped eigenvalues can be obtained comparatively efficiently. This is the main motivation behind the iterative approaches proposed in this paper. The eigenvalues of the underlying damped elastic system [23] are given by

$$s^{(0)} = -\zeta_n \omega_n \pm i\omega_n \sqrt{1-\zeta_n^2} \approx -\zeta_n \omega_n \pm i\omega_n \tag{9}$$

where the undamped natural frequency

$$\omega_n = \sqrt{k_e / m_u} \tag{10}$$

and ζ_n is the viscous damping factor. The eigenvalues in Eq. (9) appear in a complex conjugate pair.

Viscously damped elastic system can be considered as a special case of Eq. (6) when the function $G(s)$ is replaced by $G(s \to 0)$. Therefore, for the purpose of numerical approximations, we can obtain an equivalent viscous damping coefficient

$$c = \lim_{s \to 0} G(s) = k_{v_0} + \sum_{k=1}^{n} a_k k_{v_k} / b_k \tag{11}$$

From this expression, the viscous damping factor ζ_n can be obtained as

$$\zeta_n = c / 2\sqrt{k_e m_u} = \left(k_{v_0} + \sum_{k=1}^{n} a_k k_{v_k} / b_k \right) / 2\sqrt{k_e m_u} \tag{12}$$

For a truly damped elastic system, the solution given by Eq. (9) would have been the exact solution of the characteristic Eq. (8). Since in general, this is not the case, the difference between the elastic solution and the true solution of the characteristic Eq. (8) is essentially arising due to the "varying" nature of the function $G(s)$. The actual solution of the characteristic Eq. (8) can be obtained by expanding the solution in a Taylor series around $s^{(0)}$. The error arising in the resulting solution would then depend on the "degree of variability" of the function $G(s)$. We assume that the true solution of Eq. (8) can be expressed as

$$s = s^{(0)} + \delta(s^{(0)}) \tag{13}$$

where δ is a function of $s^{(0)}$ and is expected to be a small quantity. Substituting this into the characteristic equation, following Ref. [22], we have

$$\delta(s^{(0)}) = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{14}$$

where

$$\begin{gathered} A = m_u + \frac{1}{2} \frac{\partial^2 G(s^{(0)})}{\partial s^2} s^{(0)} + \frac{\partial G(s^{(0)})}{\partial s}, \\ B = 2m_u s_0 + s^{(0)} \frac{\partial G(s^{(0)})}{\partial s} + G(s^{(0)}) \end{gathered} \tag{15}$$

and

$$C = s^{(0)^2} m_u + s^{(0)} G(s^{(0)}) + k_e \tag{16}$$

In the above expressions $\partial G(s^{(0)})/\partial s$ and $\partial^2 G(s^{(0)})/\partial s^2$ are, respectively, the first and second-order derivatives of $G(s)$ evaluated at $s=s^{(0)}$. For the exponential kernel function, we have

$$\begin{gathered} \frac{\partial G(s^{(0)})}{\partial s} = -\sum_{k=1}^{n} \frac{a_k}{(s^{(0)} + b_k)^2} k_{v_k} \quad \text{and} \\ \frac{\partial^2 G(s^{(0)})}{\partial s^2} = \sum_{k=1}^{n} \frac{2a_k}{(s^{(0)} + b_k)^3} k_{v_k} \end{gathered} \tag{17}$$

The approximation given by Eqs. (13) and (14) is generally very accurate if the system is not strongly viscoelastic [24]. We propose an iterative approach based on Eq. (13) to relax this restriction. We start with the initial guess $s^{(0)}$ to obtain an updated value of s, say, $s^{(i)}$. Then, this value of s is used in the right-hand side of Eq. (13) to obtain a new estimate of s, say, $s^{(i+1)}$. This process is then repeated until the difference between $s^{(i)}$ and $s^{(i+1)}$ is sufficiently small. The procedure can be implemented by following these simple steps.

Algorithm 1

1. obtain an initial guess $s^{(0)} = -\zeta_n \omega_n \pm i\omega_n \sqrt{1-\zeta_n^2}$, initialize $i=0$ and select a small tolerance ϵ
2. calculate $s^{(i+1)} = s^{(i)} + \delta(s^{(i)})$ where $\delta(s^{(i)})$ can be obtained using Eq. (14)
3. if $|s^{(i+1)} - s^{(i)}| \geq \epsilon$ then set $i=i+1$ and go to step 2, else leave the iteration loop with the current solution as the solution.

Using this iterative approach, the complex conjugate eigenvalues of a general viscoelastic system can be obtained by post-processing of the undamped eigenvalue ω_n and the equivalent viscous damping factor ζ_n. This iterative approach for the complex conjugate eigenvalues is valid for any kernel function. On the contrary, the real eigenvalues are specific to a particular kernel function. In the next section, an iterative approach for the real eigenvalues is derived for the Biot model.

2.2 Real Eigenvalues. The complex conjugate eigenvalues are expected to be close to the eigenvalues of the equivalent elas-

tic damped system. Similarly, the real eigenvalues are expected to be close to the relaxation parameter [22] when the Biot model is used. For the convenience of analytical developments, we consider two cases, namely systems with (a) single exponential kernel and (b) multiple exponential kernels.

2.2.1 Systems With Single Exponential Kernel. For this case eigenvalue equation can be simplified from Eq. (8) as

$$s^2 m_u + sG(s) + k_e = 0 \quad \text{where} \quad G(s) = k_{v_0} + \frac{a_1}{s+b_1} k_{v_1} \tag{18}$$

Equation (18) is a third-order polynomial in s and it can be solved exactly in closed-form. A more detailed study on the properties of the exact solutions have been carried out in Refs. [25–27]. An approximate solution [22] with the motivation of extending to more complex cases can be given by

$$s \approx s^{(0)} + \Delta = -b_1 + \frac{a_1 b_1 k_{v_1}}{b_1^2 m_u - b_1 k_{v_0} + k_e + a_1 k_{v_1}} \tag{19}$$

This approximation is derived using an initial guess of the real solution $s^{(0)} = -b_1$. For the iterative approach proposed in this paper, we take the ith approximation of the real root as

$$s^{(i+1)} = -b_1 + \Delta^{(i)} \tag{20}$$

Substituting this in the equation of motion (18), we have

$$(s^{(i)^2} m_u + s^{(i)} k_{v_0} + k_e)\Delta^{(i)} + (-b_1 + \Delta^{(i)}) a_1 k_{v_1} = 0 \tag{21}$$

$$\text{or} \quad \Delta^{(i)} = \frac{a_1 b_1 k_{v_1}}{s^{(i)^2} m_u + s^{(i)} k_{v_0} + k_e + a_1 k_{v_1}} \tag{22}$$

Equations (20) and (21) can be combined to obtain the following iterative algorithm:

Algorithm 2

1. obtain an initial guess $s^{(0)} = -b_1 + \frac{a_1 b_1 k_{v_1}}{b_1^2 m_u - b_1 k_{v_0} + k_e + a_1 k_{v_1}}$, initialize $i = 0$ and select a small tolerance ϵ
2. calculate $s^{(i+1)} = -b_1 + \frac{b_1 a_1 k_{v_1}}{s^{(i)^2} m_u + s^{(i)} k_{v_0} + k_e + a_1 k_{v_1}}$
3. if $|s^{(i+1)} - s^{(i)}| \geq \epsilon$ then set $i = i+1$ and go to step 2, else leave the iteration loop with the current solution as the solution.

2.2.2 The General Case: Systems With Multiple Exponential Kernels. For this case, the equation of motion is given by Eq. (6). The characteristic Eq. (8) is a polynomial in s of order $(n+2)$ and therefore, has $(n+2)$ roots. In this section, we derive an iterative solution with the view of generalizing it to MDOF systems. There are n number of pure real roots corresponding to n terms in the series in Eq. (8). An approximate solution of the real eigenvalues can be obtained as [22]

$$s_k \approx -b_k + \Delta_k, \quad k = 1,2,\ldots,n \tag{23}$$

where Δ_k is given by

$$\Delta_k = \frac{b_k a_k k_{v_k} p_1}{[b_k^2 m_u - b_k k_{v_0} + k_e] p_1 + [-b_k (p_2 + p_3) + a_k k_{v_k} p_1]} \tag{24}$$

with

$$p_1 = \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j - b_k), \quad p_2 = a_k k_{v_k} \sum_{\substack{j=1 \\ j \neq k}}^{n} \prod_{\substack{r=1 \\ r \neq j \\ r \neq k}}^{n} (b_r - b_k), \quad \text{and}$$
$$p_3 = \sum_{\substack{j=1 \\ j \neq k}}^{n} a_j k_{v_j} \prod_{\substack{r=1 \\ r \neq j,k}}^{n} (b_r - b_k) \tag{25}$$

Using this approximation as the starting guess, we propose an iterative approach. From Eq. (8), the characteristic equation can be expressed as

$$(s^2 m_u + s k_{v_0} + k_e) \prod_{j=1}^{n} (s + b_j) + s \sum_{l=1}^{n} \left(a_l k_{v_l} \prod_{\substack{j=1 \\ j \neq l}}^{n} (b_j + s) \right) = 0 \tag{26}$$

Like the single kernel case, we assume the $(i+1)$th iteration can be related to the previous iteration as

$$s_k^{(i+1)} = -b_k + \Delta_k^{(i)}, \quad k = 1,2,\ldots,n \tag{27}$$

Substituting this into the characteristic Eq. (26), we have

$$(s_k^{(i)^2} m_u + s_k^{(i)} k_{v_0} + k_e) \Delta_k^{(i)} \prod_{\substack{j=1 \\ j \neq k}}^{n} (s_k^{(i)} + b_j) + (-b_k + \Delta_k^{(i)}) \left(a_k k_{v_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j + s_k^{(i)}) \right) + s_k^{(i)} \Delta_{k_i} \sum_{\substack{l=1 \\ l \neq k}}^{n} \left(a_l k_{v_l} \prod_{\substack{j=1 \\ j \neq l \neq k}}^{n} (b_j + s_k^{(i)}) \right) = 0 \tag{28}$$

Simplifying this equation, we obtain

$$\Delta_k^{(i)} = \frac{b_k \left(a_k k_{v_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j + s_k^{(i)}) \right)}{((s_k^{(i)^2} m_u + s_k^{(i)} k_{v_0} + k_e) + a_k k_{v_k}) \prod_{\substack{j=1 \\ j \neq k}}^{n} (s_k^{(i)} + b_j) + s_k^{(i)} \sum_{\substack{l=1 \\ l \neq k}}^{n} a_l k_{v_l} \prod_{\substack{j=1 \\ j \neq l \neq k}}^{n} (b_j + s_k^{(i)})} \tag{29}$$

Equations (27) and (29) can be combined to obtain the following iterative algorithm:

Table 1 The exact and approximate eigenvalues of an SDOF system with a single kernel

b_1	Exact solution (state-space)	Iterative solution (two iterations)	Error (%)
1.5	-1.4500	-1.4501	0.004
Complex conjugate solution	$-0.0250 \pm 1.4382i$	$-0.0250 \pm 1.4382i$	0.0000 ± 0.0000

Table 2 The exact and approximate eigenvalues of an SDOF system with eight kernels

b_k, $k=1,2,\ldots,8$	Exact solution (state-space)	Iterative solution (three iterations)	Error (%)
1.4973	-1.3025	-1.4848	13.9940
1.5231	-1.5105	-1.4963	0.9406
1.7454	-1.6586	-1.6575	0.0687
1.7657	-1.7560	-1.7318	1.3825
1.9317	-1.8668	-1.9146	2.5620
1.9442	-1.9375	-1.9308	0.3451
1.9558	-1.9513	-1.9492	0.1056
2.0677	-2.0511	-2.0509	0.0060
Complex conjugate solution	$-0.1984 \pm 1.5920i$	$-0.1984 \pm 1.5915i$	0.0000 ± 0.0000

Algorithm 3

1. obtain an initial guess $s_k^{(0)}=-b_k+\Delta_k$, $k=1,2,\ldots,n$ where Δ_k is given by Eq. (24), initialize $i=0$ and select a small tolerance ϵ
2. calculate $s_k^{(i+1)}=-b_k+\Delta_k^{(i)}$ where $\Delta_k^{(i)}$ is given by Eq. (29) in terms of $s_k^{(i)}$
3. if $|s_k^{(i+1)}-s_k^{(i)}| \geq \epsilon$ then set $i=i+1$ and go to step 2, else leave the iteration loop with the current solution as the solution.

This iteration need to be carried out for every value of $k=1,2,\ldots,n$ to obtain all the n real solutions.

3 Numerical Examples

To understand the accuracy provided by the iterative approaches derived in the preceding sections, we consider two representative numerical examples covering the two cases. The results are compared with the exact solutions obtained from the state-space method.

3.1 An SDOF System With a Single Kernel. A single degree-of-freedom system with one kernel is studied to investigate the accuracy of the approximate eigenvalues derived in Secs. 2.1 and 2.2.1. For the numerical calculations, we consider $m_u=1$ kg, $k_e=2$ N/m, and $k_{v_0}=0$. It is considered that $a_1 k_{v_1}/b_1=2\zeta_n\omega_n$, $b_1=1.5$, and the damping factor constant $\zeta_n=0.1$. The accuracy of the proposed iterative solution is shown in Table 1.

The complex conjugate eigenvalues turn out to be almost exact while there is a small error in the real eigenvalue. A third iteration gives an approximation to the exact solution with an error 1.3496×10^{-6} for this example.

3.2 The General Case: An SDOF System With Multiple Kernels. We consider a single degree-of-freedom system with eight kernels to investigate the accuracy of the iterative methods derived in Secs. 2.1 and 2.2.2. For the numerical calculations, we consider $m_u=1$ kg, $k_e=2$ N/m, and $k_{v_0}=0$. It is assumed that all $a_k k_{v_k}/b_k$ are of the same value, so that $a_k k_{v_k}/b_k=2\zeta_n\omega_n, \quad \forall k=1,2,\ldots,8$. The damping factor constant $\zeta_n=0.1$ and the values of b_k for $k=1,2,\ldots,8$ are selected as 1.4973, 1.5231, 1.7454, 1.7657, 1.9317, 1.9442, 1.9558, and 2.0677. The eigenvalues obtained using the proposed iterative method is compared with the results obtained from the exact state-space solution in Table 2. The results shown here are obtained using three iterations.

The complex conjugate eigenvalues are obtained very accurately using the proposed iterative approach. The real eigenvalues are not as accurate as the complex conjugate eigenvalues. The error in the real eigenvalues may have some effect in active control applications. However, for passive systems, recall that the motion corresponding to the real eigenvalues is purely dissipative in nature and therefore, do not significantly affect the dynamic response of the system.

4 MDOF System

In this section, the results obtained for the SDOF system are extended to general MDOF systems. We assume that the order of the characteristic polynomial is m, so that there are m eigenvalues of the system. The eigenvalues of the system s_j can be obtained by solving the characteristic polynomial

$$\det[\mathbf{D}(s_j)]=0, \quad j=1,2,\ldots,m \tag{30}$$

where $\mathbf{D}(s)$ is given by Eq. (5). In this work, we consider that all the eigenvalues are distinct. For undamped elastic systems or elastic systems with viscous damping, the order of the characteristic polynomial $m=2N$. For viscoelastic systems in general, m is more than $2N$, that is $m=2N+p$, $p\geq0$. If all of the $\mathbf{K}_{v_k}$ matrices are of full rank, then $p=nN$. This shows that although the system has N degrees-of-freedom, the number of eigenvalues is more than $2N$. This is a major difference between a viscoelastic system and an elastic system where the number of eigenvalues is exactly $2N$ including any multiplicities. Like the SDOF case, N complex conjugate pairs of eigenvalues correspond to *elastic modes* or *vibration modes*. These modes are related to the N modes of vibration of the structural system. Physically, the assumption of "$2N$ complex conjugate pairs of eigenvalues" implies that all the elastic modes are oscillatory in nature, that is, they are subcritically damped. The additional nN eigenvalues correspond to *nonviscous modes* or *overdamped modes*. Here, the complex eigenvalues are derived for a matrix with general kernel functions. For the real eigenvalues, the Biot model with two cases namely, when $n=1$ and when $n>1$ are considered for analytical convenience.

4.1 Complex Conjugate Eigenvalues. The aim here is to obtain the complex conjugate eigenvalues using the undamped elastic eigenvalues. The undamped elastic eigenvalue problem of a MDOF system is given by

$$\mathbf{K}_e\mathbf{x}_j=\omega_j^2\mathbf{M}\mathbf{x}_j, \quad j=1,2,\ldots,N \tag{31}$$

where ω_j^2 and $\mathbf{x}_j$ are the eigenvalues and mass-normalized eigenvectors of the system. We define the matrices

$$\mathbf{\Omega}=\mathrm{diag}[\omega_1,\omega_2,\ldots,\omega_N] \tag{32}$$

$$\text{and} \quad \mathbf{X}=[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N] \tag{33}$$

Using these, the characteristic Eq. (30) can be transformed into the modal coordinates as

$$\det[s_j^2\mathbf{I}_N+s_j\mathbf{G}'(s_j)+\mathbf{\Omega}^2]=0, \quad j=1,2,\ldots,m \tag{34}$$

where $\mathbf{I}_N$ is an N dimensional identity matrix. The viscoelastic kernel function matrix in the modal coordinates $\mathbf{G}'(s)$ is defined as

$$\mathbf{G}'(s) = \mathbf{X}^T\mathbf{G}(s)\mathbf{X} = (\mathbf{X}^T\mathbf{K}_{v_0}\mathbf{X}) + \sum_{k=1}^{n} \frac{a_k}{s+b_k}(\mathbf{X}^T\mathbf{K}_{v_k}\mathbf{X})$$

$$= \mathbf{K}'_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s+b_k}\mathbf{K}'_{v_k} \tag{35}$$

where the matrices

$$\mathbf{K}'_{v_0} = \mathbf{X}^T\mathbf{K}_{v_0}\mathbf{X} \quad \text{and} \quad \mathbf{K}'_{v_k} = \mathbf{X}^T\mathbf{K}_{v_k}\mathbf{X}, \quad k = 1,2,\ldots,n \tag{36}$$

We consider that the system has small nonproportionality, so that the off-diagonal entries of the $\mathbf{G}'$ matrix are small compared with the diagonal entries, that is $G'_{kl}(s_j) \leq G'_{kk}(s_j), \ \forall s_j, \ k \neq l$. This approximation is often employed in the dynamic analysis of damped systems [28–31]. Considering the jth set of Eq. (34), one obtains

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 \approx 0 \tag{37}$$

This equation is similar to Eq. (8) and can be solved in a similar way. The modal damping factor ζ_j can be defined as

$$\zeta_j = \frac{\lim_{s\to 0} G'_{jj}(s)}{2\omega_j} = \left(K'_{v_{0_{jj}}} + \sum_{k=1}^{n} \frac{a_k}{b_k} K'_{v_{k_{jj}}} \right) / 2\omega_j \tag{38}$$

where the matrices $\mathbf{K}'_{v_0}$ and $\mathbf{K}'_{v_k}$ are defined in Eq. (36). Following the results obtained for the SDOF case, the complex conjugate eigenvalues can be obtained using the following iterative algorithm:

Algorithm 4

1. obtain an initial guess $s_j^{(0)} = -\zeta_j\omega_j \pm i\omega_j\sqrt{1-\zeta_j^2}$ for $j = 1,2,\ldots,N$, initialize $i=0$ and select a small tolerance ϵ
2. calculate $s_j^{(i+1)} = s_j^{(i)} + \delta_j^{(i)}$ where

$$\delta_j^{(i)} = \frac{-B_j - \sqrt{B_j^2 - 4A_jC_j}}{2A_j}, \tag{39}$$

$$A_j = 1 + \frac{1}{2}\frac{\partial^2 G'_{jj}(s_j^{(i)})}{\partial s^2} s_j^{(i)} + \frac{\partial G'_{jj}(s_j^{(i)})}{\partial s}, \tag{40}$$

$$B_j = 2s_j^{(i)} + s_j^{(i)}\frac{\partial G'_{jj}(s_j^{(i)})}{\partial s} + G'_{jj}(s_j^{(i)}) \tag{41}$$

$$\text{and} \quad C_j = s_{0_j}^2 + s_j^{(i)} G'_{jj}(s_j^{(i)}) + \omega_j^2 \tag{42}$$

3. if $|s_j^{(i+1)} - s_j^{(i)}| \geq \epsilon$ then set $i = i+1$ and go to step 2, else leave the iteration loop with the current solution as the solution.

In the above expressions, $\partial G'_{jj}(s_j^{(i)})/\partial s$ and $(\partial^2 G'_{jj}(s_j^{(i)})/\partial s^2)$ are, respectively, the first and second-order derivative of $G'_{jj}(s)$ evaluated at $s = s_j^{(i)}$ and can be obtained following Eq. (17). This iteration needs to be carried out for every value of $j = 1,2,\ldots,N$ to obtain all the N complex conjugate solutions.

4.2 Real Eigenvalues. The real eigenvalues are obtained using an approach similar to the SDOF system. After neglecting the off-diagonal terms of the $\mathbf{G}'$ matrix, the governing characteristic equation for every mode can be expressed by Eq. (37). This equation can be solved for the real eigenvalues. For systems with single exponential kernel, there are in total $3N$ number of eigenvalue of which N are real. The method to obtain them is very similar to the SDOF case and therefore, we omit the details.

For systems with n kernels, there are in general, nN number of purely real eigenvalues. The initial guess can be obtained by extending Eqs. (23) and (24) as

$$s_{jk} = -b_k + \Delta_{jk}, \quad \forall\, j = 1,\ldots,N, \quad k = 1,\ldots,n \tag{43}$$

where

$$\Delta_{jk} = \frac{b_k a_k K'_{v_{k_{jj}}} p_1}{[b_k^2 - b_k K'_{v_{0_{jj}}} + \omega_j^2]p_1 + [-b_k(p_2+p_3) + a_k K'_{v_{0_{jj}}} p_1]} \tag{44}$$

with

$$p_1 = \prod_{\substack{l=1 \\ l\neq k}}^{n} (b_l - b_k), \quad p_2 = a_k K'_{v_{k_{jj}}} \sum_{\substack{l=1 \\ l\neq k}}^{n} \prod_{\substack{r=1 \\ r\neq l \\ r\neq k}}^{n} (b_r - b_k), \quad \text{and}$$

$$p_3 = \sum_{\substack{m=1 \\ m\neq k}}^{n} a_m K'_{v_{m_{jj}}} \prod_{\substack{r=1 \\ r\neq m,k}}^{n} (b_r - b_k) \tag{45}$$

From Eq. (43), note that there are N real solutions around each b_k. Equations (43)–(45) completely define the initial guesses for all the nN number of real eigenvalues. Using these, the iterative approach can be obtained by following the procedure similar to the SDOF case as below.

Algorithm 5

1. obtain an initial guess $s_{jk}^{(0)} = -b_k + \Delta_{jk}$, $j = 1,2,\ldots,N$; $k = 1,2,\ldots,n$ where Δ_{jk} is given by Eq. (44), initialize $i=0$ and select a small tolerance ϵ
2. calculate $s_{jk}^{(i+1)} = -b_k + \Delta_{jk}^{(i)}$ where

$$\Delta_{jk}^{(i)} = \frac{b_k\left(a_k K'_{v_{k_{jj}}} \prod_{j=1, j\neq k}^{n} (b_j + s_{jk}^{(i)})\right)}{((s_{jk}^{(i)})^2 m_u + sK'_{v_{0_{jj}}} + k_e) + a_k K'_{v_{k_{jj}}}) \prod_{j=1, j\neq k}^{n} (s_{jk}^{(i)} + b_j) + s_{jk}^{(i)} \sum_{l=1, l\neq k}^{n} a_l K'_{v_{k_{ll}}} \prod_{j=1, j\neq l\neq k}^{n} (b_j + s_{jk}^{(i)})}$$

3. if $|s_{jk}^{(i+1)} - s_{jk}^{(i)}| \geq \epsilon$ then set $i = i+1$ and go to step 2, else leave the iteration loop with the current solution as the solution.

This iteration needs to be carried out for every value of $k = 1,2,\ldots,n$ and $j = 1,2,\ldots,N$ to obtain all the nN real solutions.

Table 3 Exact and approximate eigenvalues of the three-DOF system with lower viscoelasticity

Exact solution (state-space)	Proposed approximate solution (three iterations)	Percentage error
Real eigenvalues		
−0.1979	−0.2146	8.4609
−0.2173	−0.2143	1.4088
−0.2210	−0.2213	0.1382
−0.2566	−0.3005	17.1372
−0.2630	−0.3284	24.8428
−0.2655	−0.3424	28.9518
−0.2934	−0.2909	0.8566
−0.2936	−0.2903	1.1409
−0.2937	−0.2658	9.5091
−0.3162	−0.3019	4.5206
−0.3178	−0.3115	1.9613
−0.3184	−0.3167	0.5512
−0.3420	−0.3476	1.6562
−0.3428	−0.3318	3.2097
−0.3432	−0.3520	2.5484
−0.3565	−0.3537	0.7785
−0.3565	−0.3564	0.0469
−0.3566	−0.3558	0.2050
Complex conjugate eigenvalues		
$-0.0511 \pm 0.9296i$	$-0.0512 \pm 0.9299i$	$0.2885 \pm 0.0302i$
$-0.0337 \pm 1.5942i$	$-0.0337 \pm 1.5942i$	$0.0000 \pm 0.0000i$
$-0.0380 \pm 2.1190i$	$-0.0379 \pm 2.1189i$	$0.1004 \pm 0.0066i$

Table 4 Exact and approximate eigenvalues of the three-DOF system with higher viscoelasticity

Exact solution (state-space)	Proposed approximate solution (three iterations)	Percentage error
Real eigenvalues		
−0.1113	−0.1115	0.2461
−0.1151	−0.1150	0.0783
−0.1157	−0.1157	0.0009
−0.1333	−0.1318	1.1375
−0.1377	−0.1366	0.8388
−0.1387	−0.1384	0.2375
−0.1469	−0.1450	1.3114
−0.1471	−0.1469	0.1339
−0.1472	−0.1470	0.1298
−0.1594	−0.1589	0.3092
−0.1606	−0.1605	0.0933
−0.1610	−0.1609	0.0177
−0.1717	−0.1741	1.4109
−0.1725	−0.1722	0.2042
−0.1728	−0.1711	0.9813
−0.1783	−0.1775	0.4284
−0.1784	−0.1783	0.0204
−0.1784	−0.1782	0.0806
Complex conjugate eigenvalues		
$-0.0159 \pm 0.8563i$	$-0.0159 \pm 0.8564i$	$0.1842 \pm 0.0108i$
$-0.0096 \pm 1.5090i$	$-0.0096 \pm 1.5090i$	$0.0000 \pm 0.0000i$
$-0.0109 \pm 1.9897i$	$-0.0109 \pm 1.9897i$	$0.0534 \pm 0.0021i$

5 Numerical Examples and Analysis

We consider two three degree-of-freedom systems to illustrate the proposed method and verify the accuracy of the approximate expressions. The mass, stiffness, and viscoelastic matrices in the Laplace domain for the problem are considered as

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \quad \mathbf{K}_e = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{46}$$

and

$$\mathbf{G}(s) = \mathbf{K}_v \sum_{k=1}^{6} \frac{a_k}{s + b_k}, \quad \text{where} \quad \mathbf{K}_v = \begin{bmatrix} 0.35 & -0.15 & 0.05 \\ -0.15 & 0.35 & -0.15 \\ 0.05 & -0.15 & 0.35 \end{bmatrix} \tag{47}$$

For the numerical value, we consider $m_u=1$ kg, $k_u=1$ N/m, and $a_k=b_k$. It is assumed that $\mathbf{K}_{v_0}=\underline{0}$, all the $\mathbf{K}_{v_k}$ matrices are the same, and the values of b_k for $k=1,2,\ldots,6$ are selected as 0.3531, 0.3590, 0.2374, 0.2980, 0.2891, and 0.3293 for the first example and one half of these values for the second example, that is $b_k=0.1766$, 0.1795, 0.1187, 0.1490, 0.1446, and 0.1646. The degree of viscoelasticity is, therefore, more for the second examples because in average, the values of the b_k are small and everything else is identical. These two examples are selected to understand the effect of viscoelasticity on the accuracy of the proposed iterative method.

The approximate eigenvalues obtained using the proposed method for the first set of b_k are compared with the results obtained from the exact state-space solution in Table 3. Results for the second set of b_k are given in Table 4. Three iterations are used for both the examples. The accuracy of the real and complex conjugate eigenvalues show contrasting trend. When the system is relatively less viscoelastic (refer to Table 3), the complex conjugate eigenvalues are comparatively more accurate. This is because the approximation of the complex conjugate eigenvalues is derived using an equivalent viscously damped system assumption in Sec. 2.1. When the system is relatively more viscoelastic (refer to Table 4), the real eigenvalues are comparatively more accurate. The reason for this can be traced back to the nature of the proposed approximation in Sec. 2.2 where it is assumed that the system is strongly viscoelastic. Overall, however, the complex conjugate eigenvalues are obtained more accurately.

The computational complexity to solve algebraic eigenvalue problems scale in a cubic manner [32]. Therefore, an estimation of the order of calculations needed to solve the eigenvalue problem of a matrix of size N is $O(N^3)$ for large N. This is the computational time for an undamped system of dimension N. For a general nonproportional viscously damped system, the size of the state-space matrix is $2N$. Therefore, the computational time is in the order $O((2N)^3)=O(8N^3)$. As discussed in Sec. 4, for an N dimensional MDOF system comprising n viscoelastic kernels with full rank coefficient matrices, the dimension of the state-space matrix is $N(2+n)$. The computational time using the direct state-space method is, therefore, in the order $O((N(2+n))^3)=O(N^3(2+n)^3)$. The method proposed to approximate the eigenvalues first requires the solution of the undamped eigenvalue problem and then, the iteration procedure is undertaken for each of the $N(2+n)$ eigenvalues separately. Adding these two, the order of calculations needed to approximate the eigenvalues with the proposed method is $O(N^3+aN(2+n))$, where a is a constant related to the number of iterations and the number of calculations needed for each iteration. The value of a will be higher if more number of iterations are used. It is important to note that the order of computation scales linearly with $N(2+n)$ as opposed to cubically in the case of direct state-space method. Therefore, when N and n become large as expected for practical problems, the proposed method has a clear advantage as $O(N^3+aN(2+n)) \ll O(N^3(2+n)^3)$.

6 Conclusions

The calculation of eigenvalues of viscoelastic systems with Biot model requires the solution of a nonlinear eigenvalue problem. Generally, state-space based methods involving additional internal

variables have been used for this type of problems. In this paper, new iterative approaches have been proposed for such nonlinear eigenvalue problems. It is assumed that all the eigenvalues are distinct. For the MDOF system, it has been additionally assumed that the system can be approximately diagonalized using the undamped elastic modes. Iterative methods have been proposed for both complex and real eigenvalues with single and multiple exponential kernels. In total, five algorithms have been given in this paper. The results are obtained in terms of explicit closed-form expressions. The complex eigenvalues are close to the eigenvalues of the underlying damped elastic system while the real eigenvalues are close to the relaxation parameters. The iterative method for the complex eigenvalues is applicable to any general viscoelastic kernel while the iterative method for the real eigenvalues is specific to the Biot model only. The accuracy of the proposed approximations were verified numerically against the exact state-space eigenvalues for few example problems. The iterative methods presented offer a reduction in computational effort because neither the state-space formalisms nor the additional internal variables are employed. Using the methods developed in this paper, it is possible to obtain the eigenvalues of viscoelastic systems by simple post-processing of the undamped elastic eigenvalues, which in turn, can be obtained using a general purpose finite element software.

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References

[1] Biot, M. A., 1955, "Variational Principles in Irreversible Thermodynamics With Application to Viscoelasticity," Phys. Rev., **97**(6), pp. 1463–1469.
[2] Golla, D. F., and Hughes, P. C., 1985, "Dynamics of Viscoelastic Structures—A Time Domain Finite Element Formulation," ASME J. Appl. Mech., **52**, pp. 897–906.
[3] McTavish, D. J., and Hughes, P. C., 1993, "Modeling of Linear Viscoelastic Space Structures," ASME J. Vibr. Acoust., **115**, pp. 103–110.
[4] Lesieutre, G. A., and Mingori, D. L., 1990, "Finite Element Modeling of Frequency-Dependent Material Properties Using Augmented Thermodynamic Fields," AIAA J., **13**, pp. 1040–1050.
[5] Lesieutre, G. A., and Bianchini, E., 1995, "Time-Domain Modeling of Linear Viscoelasticy Using Anelastic Displacement Fields," ASME J. Vibr. Acoust., **117**(4), pp. 424–430.
[6] Bagley, R. L., and Torvik, P. J., 1983, "Fractional Calculus—A Different Approach to the Analysis of Viscoelastically Damped Structures," AIAA J., **21**(5), pp. 741–748.
[7] Adhikari, S., and Wagner, N., 2003, "Analysis of Asymmetric Non-Viscously Damped Linear Dynamic Systems," ASME J. Appl. Mech., **70**(6), pp. 885–893.
[8] Adhikari, S., and Wagner, N., 2004, "Direct Time-Domain Approach for Exponentially Damped Systems," Comput. Struct., **82**(29–30), pp. 2453–2461.
[9] Muravyov, A., and Hutton, S. G., 1997, "Closed-Form Solutions and the Eigenvalue Problem for Vibration of Discrete Viscoelastic Systems," ASME J. Appl. Mech., **64**, pp. 684–691.
[10] Muravyov, A., 1998, "Forced Vibration Responses of a Viscoelastic Structure," J. Sound Vib., **218**(5), pp. 892–907.
[11] Wagner, N., and Adhikari, S., 2003, "Symmetric State-Space Formulation for a Class of Non-Viscously Damped Systems," AIAA J., **41**(5), pp. 951–956.
[12] Singh, K., and Ram, Y., 2002, "Transcendental Eigenvalue Problem and Its Applications," AIAA J., **40**(7), pp. 1402–1407.
[13] Williams, F., and Kennedy, D., 1988, "Reliable Use of Determinants to Solve Non-Linear Structural Eigenvalue Problems Efficiently," Int. J. Numer. Methods Eng., **26**(8), pp. 1825–1841.
[14] Adhikari, S., 1999, "Modal Analysis of Linear Asymmetric Non-Conservative Systems," J. Eng. Mech., **125**(12), pp. 1372–1379.
[15] Adhikari, S., 2002, "Dynamics of Non-Viscously Damped Linear Systems," J. Eng. Mech., **128**(3), pp. 328–339.
[16] Daya, E. M., and Potier-Ferry, M., 2001, "A Numerical Method for Nonlinear Eigenvalue Problems and Application to Vibrations of Viscoelastic Structures," Comput. Struct., **79**, pp. 533–541.
[17] Segalman, D. J., 1987, "Calculation of Damping Matrices for Linearly Viscoelastic Structures," ASME J. Appl. Mech., **54**, pp. 585–588.
[18] Friswell, M. I., Inman, D. J., and Lam, M. J., 1997, "On the Realisation of GHM Models in Viscoelasticity," J. Intell. Mater. Syst. Struct., **8**(11), pp. 986–993.
[19] Friswell, M. I., and Inman, D. J., 1999, "Reduced-Order Models of Structures With Viscoelastic Components," AIAA J., **37**(10), pp. 1318–1325.
[20] Adhikari, S., 2010, "A Reduced Second-Order Approach for Linear Viscoelastic Oscillators," ASME J. Appl. Mech., **77**(4), p. 041003.
[21] Ruge, P., 1998, "Eigenvalues of Damped Structures: Vectoriteration in the Original Space of DOF," Comput. Mech., **22**(2), pp. 167–173.
[22] Adhikari, S., and Pascual, B., 2009, "Eigenvalues of Linear Viscoelastic Systems," J. Sound Vib., **325**(4–5), pp. 1000–1011.
[23] Meirovitch, L., 1967, *Analytical Methods in Vibrations*, Macmillan, New York.
[24] Adhikari, S., and Woodhouse, J., 2003, "Quantification of Non-Viscous Damping in Discrete Linear Systems," J. Sound Vib., **260**(3), pp. 499–518.
[25] Adhikari, S., 2005, "Qualitative Dynamic Characteristics of a Non-Viscously Damped Oscillator," Proc. R. Soc. London, Ser. A, **461**(2059), pp. 2269–2288.
[26] Adhikari, S., 2008, "Dynamic Response Characteristics of a Non-Viscously Damped Oscillator," ASME J. Appl. Mech., **75**(1), p. 011003.
[27] Muller, P., 2005, "Are the Eigensolutions of a 1-D.O.F. System With Viscoelastic Damping Oscillatory or Not?," J. Sound Vib., **285**(1–2), pp. 501–509.
[28] Caughey, T. K., and O'Kelly, M. E. J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," ASME J. Appl. Mech., **32**, pp. 583–588.
[29] Shahruz, S. M., and Ma, F., 1988, "Approximate Decoupling of the Equations of Motion of Linear Underdamped System," ASME J. Appl. Mech., **55**, pp. 716–720.
[30] Shahruz, S. M., 1990, "Approximate Decoupling of the Equations of Motion of Damped Linear Systems," J. Sound Vib., **136**(1), pp. 51–64.
[31] Adhiakri, S., 2001, "Classical Normal Modes in Non-Viscously Damped Linear Systems," AIAA J., **39**(5), pp. 978–980.
[32] Wilkinson, J. H., 1988, *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, UK.