A Reduced Second-Order Approach for Linear Viscoelastic Oscillators

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This paper proposes a new approach for the reduction in the model-order of linear multiple-degree-of-freedom viscoelastic systems via equivalent second-order systems. The assumed viscoelastic forces depend on the past history of motion via convolution integrals over kernel functions. Current methods to solve this type of problem normally use the state-space approach involving additional internal variables. Such approaches often increase the order of the eigenvalue problem to be solved and can become computationally expensive for large systems. Here, an approximate reduced second-order approach is proposed for this type of problems. The proposed approximation utilizes the idea of generalized proportional damping and expressions of approximate eigenvalues of the system. A closed-form expression of the equivalent second-order system has been derived. The new expression is obtained by elementary operations involving the mass, stiffness, and the kernel function matrix only. This enables one to approximately calculate the dynamical response of complex viscoelastic systems using the standard tools for conventional second-order systems. Representative numerical examples are given to verify the accuracy of the derived expressions. [DOI: 10.1115/1.4000913]

1 Introduction

The characterization of energy dissipation in complex vibrating structures such as aircrafts and helicopters is of fundamental importance. Noise and vibration are not only uncomfortable to the users of these complex dynamical systems, but also may lead to fatigue, fracture, and even failure of such systems. The increasing use of composite structural materials, active control, and damage tolerant systems in the aerospace and automotive industries have lead to renewed demand for energy absorbing and high damping materials. Effective applications of such materials in complex engineering dynamical systems require robust and efficient analytical and numerical methods. Due to the superior damping characteristics, the dynamics of viscoelastic materials and structures have received significant attention over the past 2 decades. This paper is aimed at developing a computationally efficient numerical method by approximating a higher-order viscoelastic system with an equivalent reduced second-order system. Here, the word "system" is used in a general sense to represent oscillatory mechanical structures. Depending on the application, a system may imply a complete structure, e.g., a car body-in-white, or it can also imply a small substructure, e.g., a beam or a plate.

The key feature of a viscoelastic system is the incorporation of the time history of the state-variables in the equation of motion. The equation of motion of an N-degree-of-freedom linear viscoelastic system can be expressed by coupled integro-differential equations as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{u}}(\tau)\mathrm{d}\tau + \mathbf{K}_e\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

Here, $\mathbf{u}(t) \in \mathbb{R}^N$ is the displacement vector, $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector, $\mathbf{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K}_e \in \mathbb{R}^{N\times N}$ is the elastic stiffness matrix, and $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is the matrix of viscoelastic kernel functions. Here, $\mathbb{R}^N$ denotes the space of N-dimensional real vectors and $\mathbb{R}^{N\times N}$ denotes the space of $N\times N$ real matrices. The kernel functions $\boldsymbol{\mathcal{G}}(t)$ are known as retardation functions, heredity functions, after-effect functions, or relaxation functions in the context of different subjects. In the limit when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, Eq. (1) reduces to the case of elastic system with viscous damping. A wide variety of mathematical expressions could be used for the kernel functions $\boldsymbol{\mathcal{G}}(t)$, as long as the rate of energy dissipation is nonnegative. Some of the kernel functions used in the literature are shown in Table 1.

Taking the Laplace transform of Eq. (1), the equation of motion can be expressed as

$$\mathbf{D}(s)\bar{\mathbf{u}}(s) = \bar{\mathbf{f}}(s) \tag{2}$$

where $\bar{\mathbf{u}}(s)$ and $\bar{\mathbf{f}}(s)$ are, respectively, the Laplace transforms of $\mathbf{u}(t)$ and $\mathbf{f}(t)$, and the dynamic stiffness matrix $\mathbf{D}(s)$ is given by

$$\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}_e \in \mathbb{C}^{N\times N} \tag{3}$$

Here, $\mathbb{C}^{N\times N}$ denotes the space of $N\times N$ complex matrices.

Current methods for dynamic analysis of viscoelastic systems are dominated by state-space based approaches. Bagley and Torvik [1] used an extended state-space approach for linear systems with fractional derivative damping models. They have expressed the extended state-vector in terms of various fractional order differentials of the displacement vector. Golla and Hughes [2] and McTavish and Hughes [3] used an internal variables based approach (Golla-Hughes-McTavish (GHM) method) in the context of viscoelastic structures. Another approach, known as the anelastic displacement field (ADF) method, was developed by Lesieutre and co-workers [4,5]. Like GHM, the ADF method is also an internal variable based viscoelastic model, but distinguished from GHM in that it is first-order in time, not second-order. Muravyov and co-worker [6,7] proposed an extended state-space method for systems with exponential viscoelastic kernels associated with the stiffness operator. Wagner and Adhikari [8–10] proposed a symmetric state-space approach for linear systems with the Biot model. Muscolino et al. [11] and Palmeri et al. [12,13] used state-space based approach time-domain analysis viscoelastic systems subject to random excitations.

Methods, which are not based on state-space approach, are less common. The main reasons for seeking an alternative to the state-

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Table 1 Some viscoelastic functions in the Laplace domain

Model number	Viscoelastic function	Author and year of publication
1	$G(s)=\Sigma_{k=1}^{n}\frac{a_k}{s+b_k}$	Biot [34] - 1955
2	$G(s)=\frac{E_1 s^{\alpha}-E_0 b s^{\beta}}{1+b s^{\beta}}\ (0<\alpha,\beta<1)$	Bagley and Torvik [1] - 1983
3	$G(s)=G^{\infty}\left[\Sigma_k \alpha_k \frac{s+2\hat{\xi}_k\hat{\omega}_k}{s^2+2\hat{\xi}_k\hat{\omega}_k s+\hat{\omega}_k^2}\right]$	Golla and Hughes [2] – 1985 and McTavish and Hughes [3] – 1993
4	$G(s)=1+\Sigma_{k=1}^{n}\frac{\Delta_k s}{s+\beta_k}$	Lesieutre and Mingori [4] – 1990
5	$G(s)=c\frac{1-e^{-st_0}}{st_0}$	Adhikari and Woodhouse [35] – 1998
6	$G(s)=\frac{c}{st_0}\frac{1+2(st_0/\pi)^2-e^{-st_0}}{1+2(st_0/\pi)^2}$	Adhikari and Woodhouse [35] – 1998
7	$G(s)=ce^{s^2/4\mu}\left[1-\mathrm{erf}\left(\frac{s}{2\sqrt{\mu}}\right)\right]$	Adhikari and Woodhouse [36] – 2001

space approach are: (a) although exact in nature, the state-space approach for linear viscoelastic systems is relatively computationally intensive for real-life multiple degrees of freedom (MDOF) systems, due to the huge number of internal variables, and (b) the well-understood mathematical insights offered by methods in the original space (e.g., the modal analysis) is lost in a state-space based approach. For example, a 30 degree-of-freedom oscillatory system with a four-term Biot model (see Table 1) will, in general, result into a state-space matrix of dimension $30\times(2+4)=180$. Compared with this, a nonproportionally viscously damped system will result into a state-space matrix of dimension $30\times 2=60$. This is a key motivation to seek for reduced nonstate-space based approaches.

It should be emphasized that some kind of approximation needs to be employed for nonstate-space approaches. Woodhouse [14] and Adhikari [15,16] proposed approximate methods based on a small damping assumption. Based on a variational principle, Qian and Hansen [17] derived a substructure synthesis method, where the viscoelastic system eigensolution is obtained from the undamped system eigensolution. Several authors [18,19] discussed the nature of eigensolutions of viscoelastic systems to develop an understanding for approximate analyses. Daya and Potier-Ferry [20] proposed an asymptotic numerical method for the calculation of natural frequencies and loss-factors of viscoelastic systems. Using the small viscoelasticity assumption, Segalman [21] presented an approach to obtain the damping and stiffness matrices for viscoelastic systems. Bilbao et al. [22] considered proportional damping approximation for structures with viscoelastic dampers. Friswell et al. [23,24] proposed reduced-order models for viscoelastic systems with the GHM model. Adhikari and Pascual [25] proposed approximation methods based on the Taylor series expansion of the kernel function $\mathbf{G}(s)$ in the complex plane.

In this paper, we aim to obtain an equivalent second-order system for a general viscoelastic system. This is effectively a strategy for the reduction in the model-order. The main motivation behind seeking this approximation are:

- the physical understandings are very well developed for second-order systems, compared with viscoelastic systems, which are, in general, higher-order systems;
- wide ranging resources such as computational tools, books, and software are available for second-order systems;
- the computational cost for the analysis of a second-order system is significantly less, compared with a viscoelastic system, as there is no need to employ additional dissipation coordinates.

The approximation proposed here utilized the generalized proportional damping [26,27] and approximate eigenvalues [16,25] of viscoelastic systems. First, proportionally damped systems are considered in Sec. 2, and later, the results are extended to the general case in Sec. 3. The newly derived results are applied to a linear array of viscoelastic spring-mass system with 25dof in Sec. 4 for numerical illustration.

2 Proportionally Damped Systems

2.1 Approximation of the Eigenvalues. We consider the simplest case when the kernel function matrix takes the form

$$\mathbf{G}(s)=G(s)\mathbf{C} \tag{4}$$

and the coefficient matrix $\mathbf{C}$ can be simultaneously diagonalized with the mass and stiffness matrices using the undamped eigenvector corresponding to the underlying elastic system. The condition for the simultaneous diagonalization can be expressed [28,29] as $\mathbf{M}\mathbf{C}^{-1}\mathbf{K}=\mathbf{K}\mathbf{C}^{-1}\mathbf{M}$. The undamped elastic eigenvalue problem is given by

$$\mathbf{K}_e\boldsymbol{\phi}_j=\omega_j^2\mathbf{M}\boldsymbol{\phi}_j,\quad j=1,2,\ldots,N \tag{5}$$

where ω_j^2 and $\boldsymbol{\phi}_j$ are, respectively, the eigenvalues and mass-normalized eigenvectors of the system. We define the matrices

$$\boldsymbol{\Omega}=\mathrm{diag}[\omega_1,\omega_2,\ldots,\omega_N]\quad\text{and}\quad\boldsymbol{\Phi}=[\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\ldots,\boldsymbol{\phi}_N] \tag{6}$$

so that

$$\boldsymbol{\Phi}^T\mathbf{K}_e\boldsymbol{\Phi}=\boldsymbol{\Omega}^2\quad\text{and}\quad\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi}=\mathbf{I}_N \tag{7}$$

where $\mathbf{I}_N$ is an N-dimensional identity matrix. Using these, Eq. (2) can be transformed into the modal coordinates as

$$[s^2\mathbf{I}_N + s\mathbf{G}'(s) + \mathbf{\Omega}^2]\bar{\mathbf{u}}' = \bar{\mathbf{f}}' \tag{8}$$

where $(\bullet)'$ denotes the quantities in the modal coordinates

$$\mathbf{G}'(s) = \mathbf{\Phi}^T\mathbf{G}(s)\mathbf{\Phi} = G(s)\mathbf{C}', \quad \bar{\mathbf{u}} = \mathbf{\Phi}\bar{\mathbf{u}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{9}$$

Here, $\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$ is a diagonal matrix. Denoting the eigenvalues of the system as s_j, the jth characteristic equation corresponding to Eq. (8) can be obtained as

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 = 0 \tag{10}$$

$$\text{or} \quad d_j(s_j) = 0, \quad \text{where} \quad d_j(s) = s^2 + sG'_{jj}(s) + \omega_j^2 \tag{11}$$

Using the small viscoelasticity assumption [21], the solution of this equation can be approximated [14,16,25]. Adhikari and Woodhouse [30] gave a detailed discussion on the quantification of viscoelasticity. In the context of this paper, in simple mathematical terms, small viscoelasticity implies less variation in the function $\mathbf{G}(s)$ with respect to s. Here, based on Ref. [25], we outline an approach suitable for the derivation of the equivalent second-order system. We define the modal damping factor ζ_j as

$$\zeta_j = \frac{\lim_{s\to 0} G'_{jj}(s)}{2\omega_j} = C'_{jj}/2\omega_j \tag{12}$$

If Eq. (8) was a truly second-order viscously damped elastic system, its eigenvalues [31] would have been given by

$$s_{0_j} = -\zeta_j\omega_j \pm i\omega_j\sqrt{1-\zeta_j^2}, \quad \approx -\zeta_j\omega_j \pm i\omega_j \tag{13}$$

Since, in general, this is not the case, the difference between the elastic solution and the true solution of the characteristic equation (10) is essentially arising due to the "varying" nature of the function $G(s)$. The approximate solutions obtained here are based on keeping this fact in mind.

The central idea is that the actual solution of the characteristic equation (10) can be obtained by expanding the solution in a Taylor series around s_{0_j}. The error arising in the resulting solution would then depend on the "degree of variability" of the function $G(s)$. We assume that the true solution of Eq. (10) can be expressed as

$$s_j = s_{0_j} + \delta_j \tag{14}$$

where δ_j is a small quantity. Substituting this into the characteristic equation (11), we have

$$d_j(s_{0_j} + \delta_j) = 0 \tag{15}$$

Expanding $d_j(s_{0_j}+\delta_j)$ in a Taylor series in δ_j around s_{0_j}, one has

$$d_j(s_{0_j}) + \delta_j\frac{\partial d_j(s_{0_j})}{\partial s} + \frac{1}{2}\delta_j^2\frac{\partial^2 d_j(s_{0_j})}{\partial s^2} + \dots = 0 \tag{16}$$

Keeping only the first-order terms in δ_j, we have

$$\delta_j \approx -\left[\frac{\partial d_j(s_{0_j})}{\partial s}\right]^{-1} d_j(s_{0_j}) \tag{17}$$

where

$$\frac{\partial d_j(s_{0_j})}{\partial s} = s_{0_j}\left(2 + \frac{\partial G_{jj}(s_{0_j})}{\partial s}\right) + G_{jj}(s_{0_j}) \tag{18}$$

The δ_j in Eq. (17) is a first-order approximation. Higher-order approximations can be obtained by retaining higher-order terms in δ_j in Eq. (16). Thus, the complete approximate solution can be expressed as

$$s_j \approx \tilde{s}_j = -\zeta_j\omega_j + i\omega_j - \delta_j, \quad -\zeta_j\omega_j - i\omega_j - \delta_j^* \tag{19}$$

The expression of the approximate eigenvalue derived here shows that they can be obtained by postprocessing of the undamped eigenvalue ω_n and equivalent viscous damping factor ζ_n. This approximation of the complex conjugate eigenvalues is valid for any kernel function. It could therefore be used with any viscoelastic models given by Table 1.

Once the approximate eigenvalues of the system are known, the key idea proposed here is that $\mathbf{G}'(s)$ in Eq. (8) can be approximated by $\mathbf{G}'(\tilde{s}_j)$, that is

$$\mathbf{G}'_{\text{eqv}} = \mathbf{G}'(\tilde{s}_j) \quad \forall\, j \tag{20}$$

This approximation is based on the physical understanding that damping has a significant effect on the dynamic response only around the natural frequencies. Using this approximation, we have

$$[s^2\mathbf{I}_N + s\mathbf{G}'(s) + \mathbf{\Omega}^2] \approx [s^2\mathbf{I}_N + s\mathbf{G}'_{\text{eqv}} + \mathbf{\Omega}^2] \tag{21}$$

Because the matrix $\mathbf{G}'(\tilde{s}_j)$ is a diagonal matrix for all j, the dynamic response of the system can be obtained using the usual modal analysis approach. This approximation can be expressed in a more elegant way using the generalized proportional damping approach.

2.2 Generalized Proportional Damping. The concept of generalized proportional damping was introduced by Adhikari [26]. The generalized proportional damping expresses the damping matrix in terms of a smooth continuous function involving specially arranged mass and stiffness matrices so that the system still posses classical normal modes. The main result is below.

Theorem. *Viscously damped linear systems will have classical normal modes if the damping matrix can be represented by* $\mathbf{C} = \mathbf{M}f(\mathbf{M}^{-1}\mathbf{K})$*, where* $f(\bullet)$ *is a smooth analytic function in the neighborhood of all the undamped eigenvalues.*

This enables one to model the variations in the modal damping factors with respect to the frequency in a simplified manner. We use this observation to obtain an equivalent second-order system. Using $\mathbf{C} = \mathbf{M}f(\mathbf{M}^{-1}\mathbf{K})$, it can be shown [26,27] that

$$\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = f(\mathbf{\Omega}^2) \quad \text{or} \quad 2\zeta_j\omega_j = f(\omega_j^2), \quad \forall\, j \tag{22}$$

The reconstruction of a damping matrix using the generalized proportional damping requires the functional variation to be expresses as a function of ω_j^2. Therefore, it is convenient to write the approximate eigenvalues in Eq. (19) as

$$\tilde{s}_j = h(\omega_j^2) \tag{23}$$

where $h(\bullet)$ is a smooth function. Applying Eq. (22) to the $\mathbf{G}'(\tilde{s}_j)$ and using the expression of $\mathbf{G}(s)$ in Eq. (4), one can write

$$G'_{jj}(\tilde{s}_j) = C'_{jj}G(\tilde{s}_j) = C'_{jj}G(h(\omega_j^2)), \quad \forall\, j \tag{24}$$

From this equation, the generalized proportional damping can be identified [26,27] as

$$\mathbf{G}_{\text{eqv}} = \mathbf{C}G(h(\mathbf{M}^{-1}\mathbf{K})) \tag{25}$$

Observe that $G(h(\mathbf{M}^{-1}\mathbf{K}))$ is now a function of a matrix argument. The scalar functions described in Table 1 need to be adjusted to be used with matrix arguments. For example, for the Biot model, $G(s) = \sum_{k=1}^n a_k/(s+b_k)$ needs to be expressed as $G(\mathbf{S}) = \sum_{k=1}^n a_k[\mathbf{S} + b_k\mathbf{I}_N]^{-1}$, where $\mathbf{S}$ is a $N \times N$ complex matrix.

The equivalence between Eqs. (24) and (25) is a key result for the developments proposed in this paper. To prove this, using Eq. (7), we have

$$\mathbf{M} = \mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1}, \quad \mathbf{K} = \mathbf{\Phi}^{-T}\mathbf{\Omega}^2\mathbf{\Phi}^{-1} \quad \text{and} \quad \mathbf{M}^{-1}\mathbf{K} = \mathbf{\Phi}\mathbf{\Omega}^2\mathbf{\Phi}^{-1} \tag{26}$$

Because the function $f(\bullet)$ is assumed to be analytic in the neighborhood of all the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, it can be expressed in polynomial forms using the Taylor series expansion. Following Bellman [32, Chapter 6], we can obtain

$$f(\mathbf{M}^{-1}\mathbf{K}) = \mathbf{\Phi}f(\mathbf{\Omega}^2)\mathbf{\Phi}^{-1} \tag{27}$$

Premultiplying and postmultiplying Eq. (25) by $\mathbf{\Phi}^T$ and $\mathbf{\Phi}$, respectively, one obtains

$$\mathbf{G}'_{\mathrm{eqv}} = \mathbf{\Phi}^T\mathbf{G}_{\mathrm{eqv}}\mathbf{\Phi} = \mathbf{\Phi}^T\mathbf{C}G(h(\mathbf{M}^{-1}\mathbf{K}))\mathbf{\Phi} \tag{28}$$

Using the relationship in Eq. (27) and considering $f(\bullet)=G(h(\bullet))$, this equation can be expressed as

$$\begin{aligned}\mathbf{G}'_{\mathrm{eqv}} &= \mathbf{\Phi}^T\mathbf{C}[\mathbf{\Phi}G(h(\mathbf{\Omega}^2))\mathbf{\Phi}^{-1}]\mathbf{\Phi} = [\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}]G(h(\mathbf{\Omega}^2))\\ &= \mathbf{C}'G(h(\mathbf{\Omega}^2))\end{aligned} \tag{29}$$

The diagonal of the preceding equation is Eq. (24), which completes the proof. One fact immediately obvious from this proof is that, on the contrary to what was assumed before, matrix $\mathbf{C}$ does not have to be simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$ for the equivalence between Eqs. (24) and (25). The only consequence will be the fact that the eigenvalue Eq. (10) will become approximate for a nonproportional $\mathbf{C}$ matrix. Based on this insight, we generalize this approach to a nonproportionally damped system with a general kernel function in Sec. 3.

3 The General Case

In this section, we consider a general form of the $\mathbf{G}(s)$ matrix. For example, generalizing the Biot model to the matrix case, one has

$$\mathbf{G}(s) = \mathbf{K}_{v_0} + \sum_{k=1}^{n}\frac{a_k}{s+b_k}\mathbf{K}_{v_k} \tag{30}$$

where $\mathbf{K}_{v_j}, j=0,1,2,\ldots$ are $N\times N$ real symmetric matrices. For a proportionally damped system, the validity of the equivalent second-order system depends on the accuracy of two approximations, namely, (a) approximate eigenvalues of the system $\tilde{s}_j$ given by Eq. (19) and (b) $\mathbf{G}(s)$ can be "replaced" by $\mathbf{G}(\tilde{s}_j)$. This approximation may likely to be valid for lightly viscoelastic systems. For systems with general nonproportional $\mathbf{G}(s)$, it is further assumed that the system is lightly nonproportionally damped [33] for the purpose of the calculation of the approximate eigenvalues. This implies that $\mathbf{G}'(s)$ is a diagonally dominant matrix. Therefore, when the equation for approximate eigenvalues (19) is applied to nonproportionally damped systems, it contains two approximations, namely, (a) the system is lightly viscoelastic and (b) the system is lightly nonproportional. Under these assumptions, we can extend the approximations derived in Sec. 2 of the general case and have

$$\mathbf{G}_{\mathrm{eqv}} = \mathbf{G}(h(\mathbf{M}^{-1}\mathbf{K})) \tag{31}$$

This is the main result of the paper. Using this equivalent damping matrix, the equation of motion of a viscoelastic system can be approximately expressed in the second-order form. Now we assume $\tilde{\mathbf{S}}=h(\mathbf{M}^{-1}\mathbf{K})$ for notational convenience and simplify the term $h(\mathbf{M}^{-1}\mathbf{K})$ for an easy practical implementation.

From Eq. (12), note that

$$\omega_j\zeta_j = \lim_{s\to 0} G'_{jj}(s)/2 = G_{0_{jj}}/2 \tag{32}$$

where

$$\mathbf{G}_0 = \lim_{s\to 0}\mathbf{G}(s) \tag{33}$$

As an example, for the Biot model in Eq. (30), one has $\mathbf{G}_0=\mathbf{K}_{v_0}+\sum_{k=1}^{n}\frac{a_k}{b_k}\mathbf{K}_{v_k}$. Rewriting Eq. (32) in the matrix form, one has

$$\mathbf{\Omega}\boldsymbol{\zeta} = \frac{1}{2}\mathbf{M}^{-1}\mathbf{G}_0 \tag{34}$$

where $\boldsymbol{\zeta}$ is the diagonal matrix containing the damping factors ζ_j

$$\boldsymbol{\zeta} = \mathrm{diag}[\zeta_1,\zeta_2,\ldots,\zeta_N] \tag{35}$$

Using Eq. (34) and taking the positive sign for illustration, the eigenvalue matrix can be expressed from Eq. (13) as

$$\mathbf{S}_0 = -\mathbf{M}^{-1}\mathbf{G}_0/2 + \mathrm{i}\sqrt{\mathbf{M}^{-1}\mathbf{K}} \tag{36}$$

From Eq. (17) and using the idea of generalized proportional damping, the correction term δ_j can be expressed in a matrix form as

$$\mathbf{\Delta} = -[\mathbf{D}_p(\mathbf{S}_0)]^{-1}\mathbf{D}(\mathbf{S}_0) \tag{37}$$

where

$$\mathbf{D}(\mathbf{S}_0) = \mathbf{S}_0^2\mathbf{M} + \mathbf{S}_0\mathbf{G}(\mathbf{S}_0) + \mathbf{K}_e \tag{38}$$

and

$$\mathbf{D}_p(\mathbf{S}_0) = \left.\frac{\partial\mathbf{D}(s)}{\partial s}\right|_{s=\mathbf{S}_0} = \mathbf{S}_0\left(2\mathbf{M} + \left.\frac{\partial\mathbf{G}(s)}{\partial s}\right|_{s=\mathbf{S}_0}\right) + \mathbf{G}(\mathbf{S}_0) \tag{39}$$

As an example, for the Biot model in Eq. (30), we can obtain

$$\frac{\partial\mathbf{G}(s)}{\partial s} = -\sum_{k=1}^{n}\frac{a_k}{(s+b_k)^2}\mathbf{K}_{v_k} \tag{40}$$

so that

$$\mathbf{G}(\mathbf{S}_0) = \mathbf{K}_{v_0} + \sum_{k=1}^{n}a_k\mathbf{K}_{v_k}(\mathbf{S}_0 + b_k\mathbf{I}_N)^{-1} \tag{41}$$

and

$$\left.\frac{\partial\mathbf{G}(s)}{\partial s}\right|_{s=\mathbf{S}_0} = -\sum_{k=1}^{n}a_k\mathbf{K}_{v_k}(\mathbf{S}_0 + b_k\mathbf{I}_N)^{-2} \tag{42}$$

Using these expressions, from Eq. (14), we have

$$\tilde{\mathbf{S}} = \mathbf{S}_0 + \mathbf{\Delta} \tag{43}$$

and

$$\mathbf{G}_{\mathrm{eqv}} = \mathbf{G}(\tilde{\mathbf{S}}) \in \mathbb{C}^{N\times N} \tag{44}$$

The results obtained in this paper can be conveniently summarized in the following conjecture.

Conjecture. *A viscoelastic system with a kernel function matrix* $\mathbf{G}(s)\in\mathbb{C}^{N\times N}$ *in the Laplace domain can be approximated by a second-order system with an equivalent damping matrix obtained by simply replacing the argument s with a complex* $N\times N$ *matrix* $\tilde{\mathbf{S}}$*, that is,* $\mathbf{G}_{\mathrm{eqv}}=\mathbf{G}(\tilde{\mathbf{S}})\in\mathbb{C}^{N\times N}$*, where*

$$\begin{aligned}\tilde{\mathbf{S}} &= \mathbf{S}_0 + \mathbf{\Delta} \in \mathbb{C}^{N\times N}\\ \mathbf{S}_0 &= -\mathbf{M}^{-1}\mathbf{G}_0/2 + \mathrm{i}\sqrt{\mathbf{M}^{-1}\mathbf{K}} \in \mathbb{C}^{N\times N}\\ \mathbf{G}_0 &= \lim_{s\to 0}\mathbf{G}(s) \in \mathbb{R}^{N\times N}\\ \mathbf{\Delta} &= -[\mathbf{D}_p(\mathbf{S}_0)]^{-1}[\mathbf{S}_0^2\mathbf{M} + \mathbf{S}_0\mathbf{G}(\mathbf{S}_0) + \mathbf{K}_e] \in \mathbb{C}^{N\times N}\\ \mathbf{D}_p(\mathbf{S}_0) &= \mathbf{S}_0\left(2\mathbf{M} + \left.\frac{\partial\mathbf{G}(s)}{\partial s}\right|_{s=\mathbf{S}_0}\right) + \mathbf{G}(\mathbf{S}_0) \in \mathbb{C}^{N\times N}\end{aligned} \tag{45}$$

This conjecture completely defines the proposed reduced-order approximation and it is valid for a general $\mathbf{G}(s)$, as shown in Table 1. The approximation scheme will remain the same; the only changes will be due to the different functional forms of $\mathbf{G}(s)$. One only needs to solve the undamped eigenvalue problem to obtain the equivalent second-order system to use this approximation. As a result of this conjecture, the equation of motion in the Laplace domain can be expressed by

$$\tilde{\mathbf{D}}(s)\bar{\mathbf{u}}(s) = \bar{\mathbf{f}}(s) \tag{46}$$

where the equivalent second-order dynamic stiffness matrix $\tilde{\mathbf{D}}(s)$ is given by

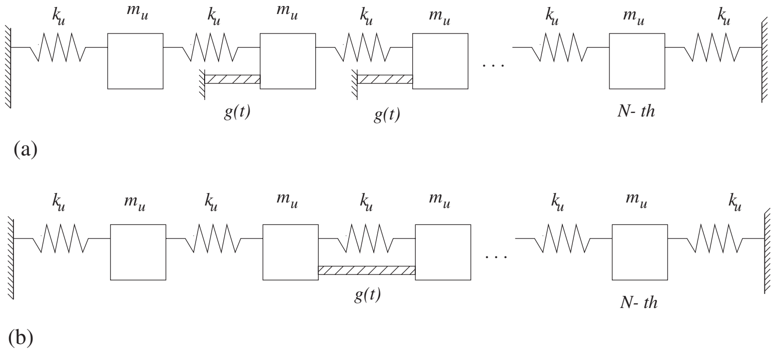

Fig. 1 Linear array of N spring-mass oscillators N=25, m_u=1 kg, and k_u=4.0×10^5 N/m. Viscoelastic dampers are attached between the eighth and 17th masses with c_a=40.0 N s/m and c_b=120.0 N s/m for the two cases are considered.

$$\tilde{\mathbf{D}}(s) = s^2\mathbf{M} + s\mathbf{G}_{\text{eqv}} + \mathbf{K}_e \in \mathbb{C}^{N\times N} \tag{47}$$

The matrix $\mathbf{G}_{\text{eqv}}$ is, in general, nonproportional and complex. All conventional approaches for nonproportionally damped systems can be used for Eq. (46). This approximation eliminates the need to employ additional dissipation coordinates and significantly reduces the computational cost. The suggested approximation is simple and involves only elementary matrix operations involving the system matrices. In Sec. 4, we verify the accuracy of the proposed approximation.

4 Numerical Examples

A partly damped linear array of the spring-mass oscillator is considered to illustrate the application of the proposed approximation. The system, together with the numerical values assumed for different parameters, are shown in Fig. 1.

The mass matrix of the system can be given by $\mathbf{M}=m_u\mathbf{I}$, where $\mathbf{I}$ is the $N\times N$ identity matrix. The stiffness matrix of the system is given by

$$\mathbf{K}_e = k_u\mathbf{J}, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix} \tag{48}$$

The following two cases are considered for the viscoelastic kernel.

- For the system shown in Fig. 1(*a*), certain of the masses have dissipative elements connecting them to the ground. In this case, the viscoelastic damping force depends only on the absolute motion of the individual masses. For this case the kernel function matrix has the form of Eq. (4), that is

$$\mathbf{G}(s) = G(s)\mathbf{C} \tag{49}$$

  where the coefficient matrix $\mathbf{C}=c_a\tilde{\mathbf{I}}$. The matrix $\tilde{\mathbf{I}}$ has a similar form of an $N\times N$ identity matrix, except that it has nonzero diagonal entries corresponding to the masses attached with the dampers only.
- For the system shown in Fig. 1(*b*), by contrast, dissipative elements are connected between certain adjacent pairs of masses. In this case, the viscoelastic damping force depends on the relative motion of the two adjacent masses. The kernel function matrix has the form of Eq. (49) with the coefficient matrix $\mathbf{C}=c_b\tilde{\mathbf{J}}$. The matrix $\tilde{\mathbf{J}}$ has a similar form to the matrix $\mathbf{J}$ in Eq. (48), except that it has nonzero entries corresponding to the masses attached with the viscoelastic dampers only.

With the $\mathbf{C}$ matrices considered here, it is easy to verify that, for both cases, the system is actually nonproportionally damped. For numerical calculations, we have considered a 25-degree-of-

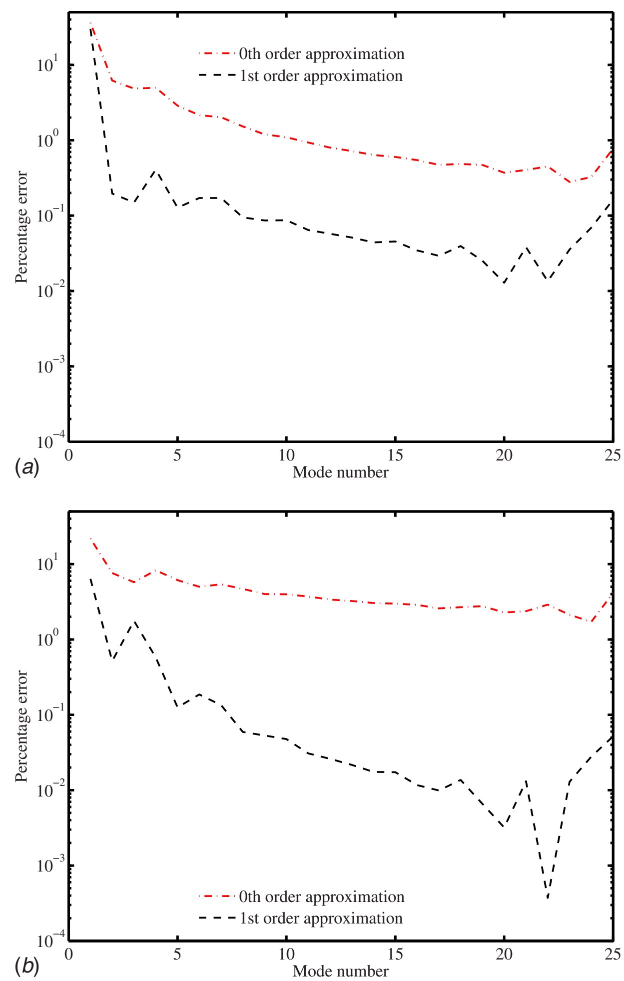

Fig. 2 Percentage error in the eigenvalues obtained using the approximate expressions for case (a). (*a*) Error in the real parts. (*b*) Error in the imaginary parts.

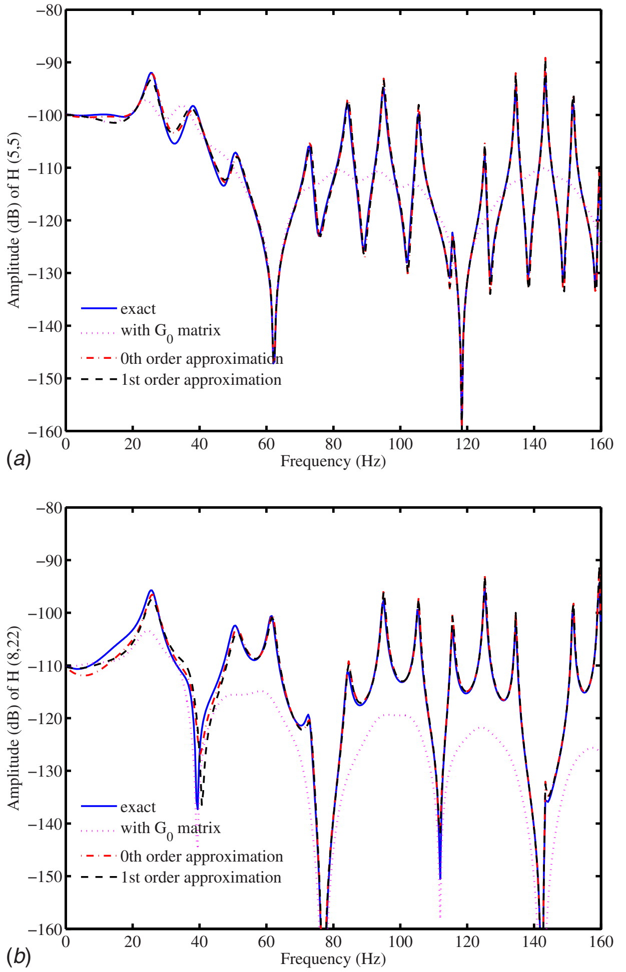

Fig. 3 FRF for case (a). (*a*) Driving point FRF at node 5. (*b*) Cross FRF between nodes 8 and 22.

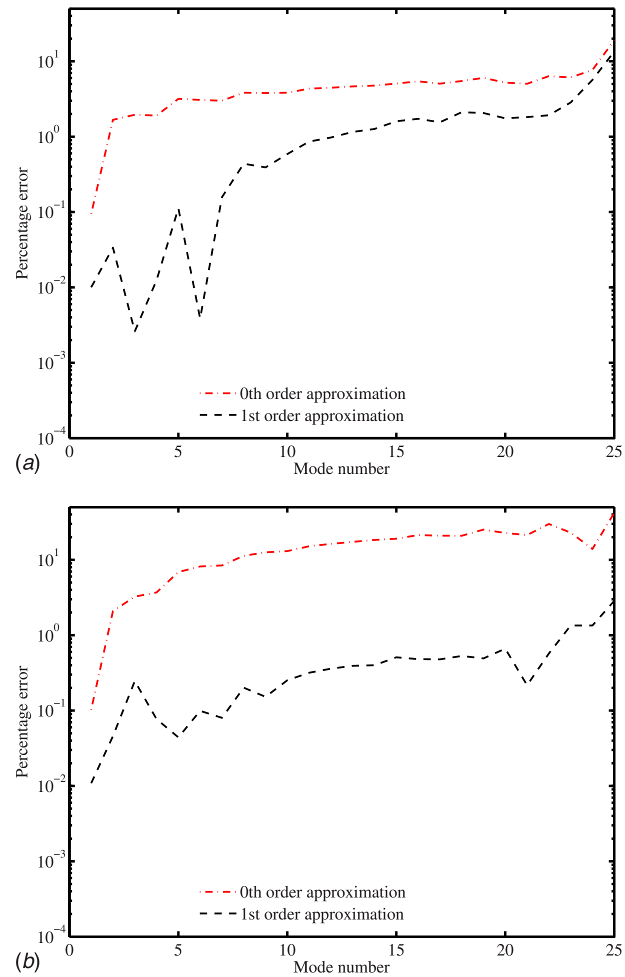

Fig. 4 Percentage error in the eigenvalues obtained using the approximate expressions for case (b). (*a*) Error in the real parts. (*b*) Error in the imaginary parts.

freedom system so that $N=25$. Values of the mass and stiffness associated with each unit are assumed to be the same with numerical values of $m=1$ kg and $k=4.00\times 10^5$ N/m. The resulting undamped natural frequencies then range from approximately 20 Hz to 200 Hz. The value $c_a=40.0$ N s/m and $c_b=120.0$ N s/m has been used for cases (a) and (b), respectively. The function $G(s)$ is assumed to be the Biot model with four terms as

$$G(s)=\sum_{k=1}^{4}\frac{b_k}{s+b_k} \tag{50}$$

with $b_k=\{20.4703, 188.1373, 243.7953, 300.2269\}$ s^{-1}. The accuracy of the approximate second-order system proposed here depends on the accuracy of the approximate eigenvalues. As a result, we first look at how the eigenvalues are approximated using the two approximate expressions derived in Sec. 2.1.

In Fig. 2, the percentage error in the real sand imaginary parts of the eigenvalues obtained using the two approximate expressions for case (a) have been shown. For the calculation of the percentage error, the eigenvalues obtained from the state-space approach [8,9], involving additional dissipation coordinates, were used. For this 25 degrees-of-freedom system, the order of the state-space problem turns out to be 90. This demonstrates the computational need for the model-order reduction in viscoelastic systems and highlights the justification behind the approximation proposed in the paper.

The zeroth order approximation shown in Fig. 2 corresponds to the eigenvalues obtained using Eq. (13). The first-order approximation is obtained using Eq. (19). As expected, the first-order approximation is more accurate than the zeroth order approximation. A driving point and a cross frequency response function (FRF) of the system corresponding to case (a) are shown in Fig. 3. With the exact FRF, three more approximations are shown in this diagram.

The FRF with $\mathbf{G}_0$ is obtained from a second-order system with a damping matrix given by Eq. (33). This is perhaps the simplest possible approximation. However, as can be seen from the plots, this approximation introduces a significant error and clearly not suitable for this problem. The other two approximations, namely, zeroth and first-order approximation, produce fairly similar results. The zeroth order approximation can be obtained from Eq. (45) by substituting $\mathbf{\Delta}=\mathbf{O}$. This is particularly a very simple approximation for which $\mathbf{G}_{\mathrm{eqv}}=\mathbf{G}(-\mathbf{M}^{-1}\mathbf{G}_0/2+\mathrm{i}\sqrt{\mathbf{M}^{-1}\mathbf{K}})\in\mathbb{C}^{N\times N}$. The first-order approximation can be obtained by using all the equations in Eq. (45). The computational time is slightly more than the zeroth order approximation, as one has to calculate a matrix inversion to obtain the correction term $\mathbf{\Delta}$. From Fig. 3, it can be observed that both of these approximation turn out to be accurate for this problem. The approximations are less accurate in the lower frequency range, which collaborate with the observation in Fig. 2 that errors in the lower modes are more than that in the higher modes.

Percentage errors in the real sand imaginary parts of the eigenvalues obtained using the two approximate expressions for case (b) are shown in Fig. 4.

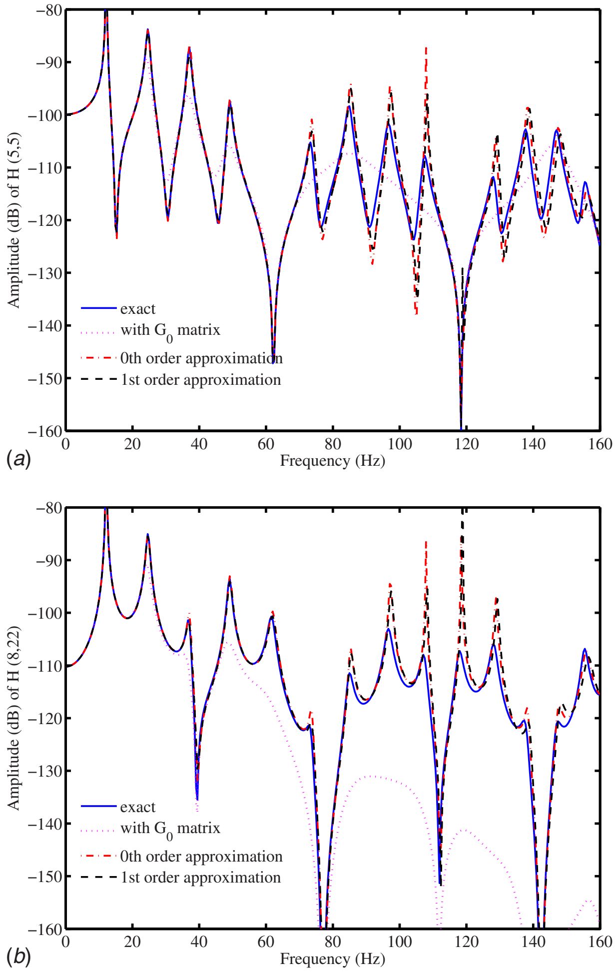

Fig. 5 FRF for case (b). (*a*) Driving point FRF at node 5. (*b*) Cross FRF between nodes 8 and 22.

As expected, the first-order approximation is more accurate that the zeroth order approximation. The trend of errors in these diagrams is opposite to the previous case shown in Fig. 2. Here, errors increase with increasing frequency. As a consequence, the error in using the proposed second-order approximation might be more in the high frequency regions, compared with the low frequency regions. This can be confirmed from Fig. 5, where the exact and approximate FRFs are plotted for case (b).

Like the previous case, we observe that the zeroth and first-order approximations are very close and significantly better than the approximation with matrix $\mathbf{G}_0$ only. These numerical results show that the simple approximation developed here can be used to significantly reduce the order of linear multiple-degree-of-freedom viscoelastic systems.

5 Conclusions

The order of the state-space model of a multiple-degree-of-freedom linear viscoelastic system can be very large due to the additional dissipation coordinates. To reduce the computational cost, the model-order reduction in viscoelastic systems has been considered. In particular, the possibility of using an equivalent second-order system has been discussed. The main contribution of the paper is that an equivalent second-order model can be obtained by simply treating the viscoelastic kernel function as a matrix function of a suitable argument matrix. This particular matrix can, in turn, be obtained using elementary matrix operations of the known system matrices. The idea behind this approximation arisen from the concept of generalized proportional damping, where any frequency depended damping can be incorporated within the scope of the proportional damping approximation. Here, generalized proportional damping is used together with the expressions of approximate eigenvalues of the viscoelastic system. Like any approximate method, this approach has regions of acceptable accuracy. The error arising in the eigenvalue approximation depends on the validity of two assumptions, namely, (a) small viscoelasticity and (b) small nonproportionality. As a result, the accuracy of the proposed second-order model depends on these two conditions. The proposed approach is valid for any general viscoelastic kernel function (smooth functions of frequency) such as the GHM, ADF, fractional derivative, and Biot model.

In the numerical examples, a system with order 90 in the state-space is reduced to a second-order system with dimension 25 using the proposed method. Acceptable agreements between the full system and the reduced system were obtained for the frequency response functions. Further reduction from the second-order system may be possible, for example, using the Guyan type reduction. The results derived in this paper may be a good starting point for future research in this direction.

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