The current issue and full text archive of this journal is available at
www.emeraldinsight.com/0264-4401.htm

Stochastic sensitivity analysis using preconditioning approach

R. Chowdhury and S. Adhikari
School of Engineering, Swansea University, Singleton Park, Swansea, UK

Received 1 September 2009
Reviewed 26 December 2009
Revised 5 January 2010
Accepted 14 January 2010

Abstract

Purpose – High-dimensional model representation (HDMR) is a general set of quantitative model assessment and analysis tools for capturing the high-dimensional relationships between sets of input and output model variables. It is an efficient formulation of the system response, if higher-order cooperative effects are weak, allowing the physical model to be captured by the lower-order terms. The paper's aim is to develop a new computational tool for estimating probabilistic sensitivity of structural/mechanical systems subject to random loads, material properties and geometry.

Design/methodology/approach – When first-order HDMR approximation of the original high-dimensional limit state is not adequate to provide the desired accuracy to the sensitivity analysis, this paper presents an enhanced HDMR (eHDMR) method to represent the higher-order terms of HDMR expansion by expressions similar to the lower-order ones with monomial multipliers. The accuracy of the HDMR expansion can be significantly improved using preconditioning with a minimal number of additional input-output samples without directly invoking the determination of second- and higher-order terms. As a part of this effort, the efficacy of HDMR, which is recently applied to uncertainty analysis, is also demonstrated. The method is based on computing eHDMR approximation of system responses and score functions associated with probability distribution of a random input. Surrogate model is constructed using moving least squares interpolation formula. Once the surrogate model form is defined, both the probabilistic response and its sensitivities can be estimated from a single probabilistic analysis, without requiring the gradients of performance functions.

Findings – The results of two numerical examples involving mathematical function and structural/solid-mechanics problems indicate that the sensitivities obtained using eHDMR approximation provide significant accuracy when compared with the conventional Monte Carlo method, while requiring fewer original model simulations.

Originality/value – This is the first time where application of eHDMR concepts is explored in the stochastic sensitivity analysis. The present computational approach is valuable to the practical modelling and design community.

Keywords Sensitivity analysis, Structural engineering, Modelling, Mechanical behaviour of materials

Paper type Research paper

1. Introduction

Sensitivity analysis provides an important insight towards the understanding of the physical mechanisms underpinning the failure of engineering structures. It is also necessary for modifying the design towards mitigating risk. Significant advancements have been made over the past few decades in developing methods such that the sensitivity information is provided as a "by-product" of the analysis. Studies on sensitivities of the random variables are carried out in the literature (McClendon and Rabitz, 1988; Asmussen and Rubinstein, 1993; Helton, 1997; Melchers and Ahammed, 2004; Oakley and O'Hagan, 2004; Gunawan *et al.*, 2005; Bae *et al.*, 2006a, b; Ahammed

The first author acknowledges the support of Royal Society through Newton Internal Fellowship. The second author acknowledges the support of UK Engineering and Physical Sciences Research Council (EPSRC) through the award of an Advanced Research Fellowship and The Leverhulme Trust for the award of the Philip Leverhulme Prize.

Engineering Computations: International Journal for Computer-Aided Engineering and Software
Vol. 27 No. 7, 2010
pp. 841-862

0264-4401
DOI 10.1108/02644401011073683

and Melchers, 2006; Alyanak *et al.*, 2008) in order to establish a systematic framework for identifying relative importance of parameters that merit descriptions through random variables or optimizing a system's performance with an acceptable risk. Generally three classes of approaches are available for estimating the sensitivity of a general probabilistic response, which are named as finite-difference (L'Ecuyer and Perron, 1994; Plyasunov and Arkin, 2007), perturbation (Ho and Cao, 1991; Glasserman, 1991; Adhikari and Friswell, 2007) and score function (Rubinstein and Shapiro, 1993; Rahman, 2009; Millwater, 2009). In finite-difference approach (Plyasunov and Arkin, 2007) repeated probabilistic analyses needs to performed, for nominal and perturbed values of system parameters and thus often expensive. In contrast to finite-difference approach, both perturbation and score function approach require only single stochastic analysis to estimate both performance as well sensitivities. For perturbation analysis (Glasserman, 1991; Ghosh *et al.*, 2004; Adhikari and Friswell, 2007), the probability measure is fixed, and the gradient of a limit state is taken. Score function approach (Rubinstein and Shapiro, 1993; Millwater, 2009) involves continuously varied probability measure. Theoretically both perturbation and score function approach can be employed in conjunction with the direct Monte Carlo simulation (MCS) but the problem lies in the calculation of gradients. The gradient estimation of a stochastic system, embedded in commercial software, using the direct MCS can be impractical. Due to this fact neither perturbation approach nor score function method established their way in stochastic sensitivity analysis of structural/mechanical systems.

The direct differentiation method provides an attractive alternative to the finite-difference method for calculating stochastic sensitivities. In conjunction with classical approaches (Nair and Keane, 2002; Adhikari, 2005) of reliability analysis, Liu and Der Kiureghian (1991) has significantly contributed to the development of such methods for obtaining reliability sensitivities. In contrast, the three sensitivity methods described in the preceding are independent of underlying stochastic analysis. In a recent study (Rahman, 2009), a score function approach is implemented with dimensional decomposition, which is same as high-dimensional model representation (HDMR) although not mentioned explicitly in the paper, to estimate stochastic sensitivities. In this article, enhanced HDMR (eHDMR) is explored in conjunction with score function for calculating stochastic sensitivities of structural/mechanical system with respect to probability distribution parameters. The novelty of the present approach lies in the fact that, it uses constant and first-order terms of HDMR and approximates the higher-order terms by using monomials, instead of directly invoking the higher-order HDMR expansion.

The paper is organized as follows. Section 2 presents a brief overview of statistical moments. Section 3 describes the concept of sensitivity analysis using score function. Section 4 presents a brief overview of HDMR. Section 5 portrays the mathematical formulation of eHDMR. Section 6 presents approximation of the original limit state/performance function using eHDMR. Section 7 presents the estimation of failure probability, statistical moments and sensitivities by MCS using the approximate limit state/performance function generated by HDMR. Numerical examples involving elementary mathematical functions and structural problems are presented in section 8 to illustrate the proposed method.

2. Statistical moments

Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$ denote the uncertainties in loads, material properties and geometry of structural/mechanical system. The probabilistic description of the random variables is completely defined by joint density function $\{f_{\boldsymbol{x}}(\boldsymbol{x}; \boldsymbol{\theta}),\ \boldsymbol{x} \in \Re^N,\ \boldsymbol{\theta} \in \Re^N\}$

that is associated with the probability measure $\{P_{\boldsymbol{\theta}},\ \boldsymbol{\theta} \in \Re^N\}$ (Sudret and Der-Kiureghian, 2000). In this study, it is assumed that the limit state $g(\boldsymbol{x})$ is not an explicit function of $\boldsymbol{\theta}$, although $g(\boldsymbol{x})$ implicitly depends on $\boldsymbol{\theta}$ through the probabilistic description of the random variables considered. The objective of probabilistic sensitivity analysis is to obtain the partial derivatives of a probabilistic characteristic of $g(\boldsymbol{x})$ with respect to a parameter $\theta_i,\ i = 1, 2, \ldots, M$, given a reasonably arbitrary probability law of $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$.

The qth moment of $g(\boldsymbol{x})$ can be defined by:

$$M_q(\boldsymbol{\theta}) = \mathsf{E}_\theta[g^q(x)] = \int_{\Re^N} g^q(x) f_x(x; \boldsymbol{\theta}) dx;\ q = 1, 2, \ldots \tag{1}$$

A similar integral appears in time-invariant reliability analysis, which entails calculating the failure probability:

$$P_F(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}[\boldsymbol{x} \in \Omega F] = \int_{\Re^N} \mathrm{J}_{\Omega_F}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) d\boldsymbol{x} \tag{2}$$

where Ω_F is the failure set for component reliability analysis and,

$$\mathrm{J}_{\Omega_F}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega_F \\ 0, & \boldsymbol{x} \in \Omega \backslash \Omega_F \end{cases};\ \boldsymbol{x} \in \Re^N \tag{3}$$

is an indicator function. Therefore, both the expressions in equations (1) and (2) can be consolidated into a generic probabilistic response:

$$h(\boldsymbol{\theta}) = \mathsf{E}_{\boldsymbol{\theta}}[g(\boldsymbol{x})] = \int_{\Re^N} g(\boldsymbol{x}) f_X(\boldsymbol{x}; \boldsymbol{\theta}) d\boldsymbol{x} \tag{4}$$

where $h(\boldsymbol{\theta})$ and $g(\boldsymbol{x})$ are either $m_q(\boldsymbol{\theta})$ and $g^q(\boldsymbol{x})$, respectively, for statistical moment analysis; $P_F(\boldsymbol{\theta})$ and $\mathrm{J}_{\Omega_F}(\boldsymbol{x})$, respectively, for reliability analysis.

3. Sensitivity analysis using score function

Consider a distribution parameter $\theta_i,\ i = 1, 2, \ldots, M$, and suppose that the gradient of a generic probabilistic response $h(\boldsymbol{\theta})$, which is either statistical moment or reliability of a structural/mechanical system, with respect to θ_i is sought (Rahman, 2009). According to Rubinstein and Shapiro (1993), pioneers of the score function method, few assumptions are necessary for such sensitivity analysis. Details of these assumptions can be found elsewhere in Rubinstein and Shapiro (1993). Taking the partial derivative of both sides of equation (4) with respect to θ_i yields:

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \int_{\Re^N} g(\boldsymbol{x}) f_X(\boldsymbol{x}; \boldsymbol{\theta}) d\boldsymbol{x} \tag{5}$$

By invoking Lebesgue dominated convergence theorem, the differential and integral

operators can be interchanged (Rahman, 2009), which yields:

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta i} = \int_{\Re^N} g(\boldsymbol{x}) \frac{\partial f_X(\boldsymbol{x};\boldsymbol{\theta})}{\partial \theta_i}\, d\boldsymbol{x} = \int_{\Re^N} g(\boldsymbol{x}) \frac{\partial \ln f_X(\boldsymbol{x};\boldsymbol{\theta})}{\partial \theta_i} f_X(\boldsymbol{x};\boldsymbol{\theta}) d\boldsymbol{x} = \mathsf{E}_{\boldsymbol{\theta}}\left[g(\boldsymbol{x}) \frac{\partial \ln f_X(\boldsymbol{x};\boldsymbol{\theta})}{\partial \theta_i} \right]; \quad i = 1, \ldots, M \tag{6}$$

Define:

$$\mathsf{K}_{\theta}^{(1)}(\boldsymbol{x};\boldsymbol{\theta}) = \frac{\partial \ln f_{\boldsymbol{x}}(\boldsymbol{x};\boldsymbol{\theta})}{\partial \theta_i} \tag{7}$$

which is known as the first-order score function for the parameter θ_i (Rahman, 2009; Millwater, 2009). Therefore, the first-order sensitivity of $h(\boldsymbol{\theta})$ can be expressed by (Rahman, 2009):

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = \mathsf{E}_{\boldsymbol{\theta}}[g(\boldsymbol{x}) \mathsf{K}_{\theta}^{(1)}(\boldsymbol{x};\boldsymbol{\theta})];\ i = 1, \ldots, M \tag{8}$$

The score function method requires differentiating only the probability density function. Also in most cases, resulting score functions can be easily determined analytically. In contrast, the infinitesimal perturbation analysis requires derivatives or perturbation of the limit state/performance function, which is always expensive in stochastic mechanics applications (Rahman, 2009). Furthermore, if the performance function is not differentiable, interchangeability of differential and integral operators is violated and the direct differentiation-based approaches will not work. In the score function method, $g(\boldsymbol{x})$ can be discontinuous – for example, the indicator function $\mathrm{J}_{\Omega_F}(\boldsymbol{x})$ arising in the reliability analyses – but the method still allows evaluation of the sensitivity if the density function is differentiable. Due to these facts, in this paper the score function method is chosen as a tool for efficient computation of probabilistic sensitivity (Rahman, 2009).

The score functions $\mathsf{K}_{\theta}^{(1)}(\boldsymbol{x};\boldsymbol{\theta})$ depend only on the joint probability density function of the random input vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$. When the distribution of x_i is either independent or both independent and identical, the expressions of the score functions simplify slightly. Since a major application of sensitivity analysis is the design optimization, where the second-moment properties of random input play the role of design parameters, attention is confined to the score functions associated with the mean and standard deviations of input (Rahman, 2009). In general, the log-derivatives of a marginal probability density function are required in determining the score functions.

4. High-dimensional model representation

The fundamental principle underlying the HDMR (Yaman and Demiralp, 2009; Tunga and Demiralp, 2004, 2005; Sobol, 2003; Alis and Rabitz, 2001; Li *et al.*, 2001a; Rabitz and Alis, 1999; Wang *et al.*, 1999) is that, from the perspective of the output/response, the order of cooperative effects between the independent variables will die off rapidly. This assertion does not eliminate strong variable dependence or even the possibility that all the variables are important. Various sources (Alis and Rabitz, 2001; Rabitz and Alis, 1999; Wang *et al.*, 1999) of information support this point of there being limited high-order correlations. First, the variables in most systems are chosen to enter as independent entities. Second, traditional statistical analyses of system behavior have revealed that a variance and covariance analysis of the output in relation to the input

variables often adequately describes the physics of the problem. These general observations lead to a dramatically reduced computational scaling when one seeks to map input-output relationships of complex systems.

Evaluating the input-output mapping of a system generates an HDMR (Rabitz and Alis, 1999) of that system. This is achieved by expressing system response as a hierarchical, correlated function expansion of a mathematical structure and evaluating each term of the expansion independently. One may show that a general system response $g(\boldsymbol{x}) = g(x_1, x_2, \ldots, x_N)$, which is a function of N input variables, can be expressed as summands of different dimensions:

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1 \le i_1 < i_2 \le N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \ldots + \sum_{1 \le i_1 < \ldots < i_l \le N} g_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l}) + \ldots + g_{12\ldots N}(x_1, x_2, \ldots, x_N) \tag{9}$$

where g_0 is a constant term representing the zeroth-order component function or the mean response of $g(\boldsymbol{x})$. The function $g_i(x_i)$ is a first-order term expressing the effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\boldsymbol{x})$. The function $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is a second-order term which describes the cooperative effects of the variables x_{i_1} and x_{i_2} upon the output $g(\boldsymbol{x})$.

The expansion functions are determined by evaluating the input-output responses of the system relative to the defined reference point $\bar{\boldsymbol{x}} = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N\}$ along associated lines, surfaces, subvolumes, etc. in the input variable space. This process reduces to the following relationship for the component functions in equation (9):

$$g_0 = g(\bar{\boldsymbol{x}}) \tag{10}$$

$$g_i(x_i) = g(x_i, \bar{\boldsymbol{x}}^i) - g_0 \tag{11}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g_{i_1 i_2}(x_{i_1}, x_{i_2}, \bar{\boldsymbol{x}}^{i_1 i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0 \tag{12}$$

where the notation $g(x_i, \bar{\boldsymbol{x}}^i) = g(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N)$ denotes that all the input variables are at their reference point values except x_i. The process of subtracting off the lower-order expansion functions removes their dependence to assure a unique contribution from the new expansion function.

Considering terms up to first-order in equation (9) yields:

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \mathrm{R}_2 \tag{13}$$

Substituting equations (10)-(11) into equation (13) leads to:

$$g(\boldsymbol{x}) = \sum_{i=1}^{N} g(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N) - (N-1) g(\bar{\boldsymbol{x}}) + \mathrm{R}_2 \tag{14}$$

Therefore, first-order HDMR approximation of $g(\boldsymbol{x})$ can be represented as:

$$\tilde{g}(\boldsymbol{x}) \equiv g(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} g(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N) - (N-1)g(\bar{\boldsymbol{x}}) \tag{15}$$

Comparison of equations (14) and (15) indicates that the first-order approximation leads to the residual error $g(\boldsymbol{x}) - \tilde{g}(\boldsymbol{x}) = \mathrm{R}_2$, which includes contributions from terms of two and higher-order component functions. It can be shown that (Li *et al.*, 2001a), the first-order component function $g_i(x_i)$ is the sum of all the Taylor series terms which contain and only contain variable x_i. Furthermore, the approximations contain contributions from all the input variables. Thus, the infinite numbers of terms in the Taylor series are partitioned into finite different groups and each group corresponds to one HDMR component function (Chowdhury *et al.*, 2009). Therefore, any truncated HDMR expansion provides a better approximation and convergent solution of $g(\boldsymbol{x})$ than any truncated Taylor series because the latter only contains a finite number of terms of Taylor series (Li *et al.*, 2001a).

5. Enhanced HDMR

Very often the contribution of higher-order HDMR terms (equation (9)) to the multivariate function approximation are small thereby making low-order HDMR approximations satisfactory for practical purposes. However, in some cases the lower-order (e.g. first-order) HDMR approximation may not provide satisfactory accuracy, and higher-order HDMR approximations might have to be considered (Li *et al.*, 2001b). For HDMR, the higher-order terms demand a polynomially increasing number of finite-element (FE) model runs (Rao and Chowdhury, 2009). If the higher-order component functions of HDMR can be approximately represented in a similar fashion as those for the zeroth and first-order component functions, then higher-order approximation of HDMR can be included without dramatically increasing the number of experiments or model runs as well as reducing computer storage requirements. One way to realize this concept is to represent a high-order HDMR component function as a sum of preconditioned low-order HDMR component functions. The preconditioning may be accomplished by multiplying each low-order HDMR component function with a suitable known function of the remaining input variables.

When $g(\boldsymbol{x})$ is approximated by the lth HDMR at reference point $\bar{\boldsymbol{x}}$, the error of this approximation is given by the residual:

$$r_l(\boldsymbol{x}) = g(\boldsymbol{x}) - g_0 - \sum_{i=1}^{N} g_i(x_i) - \sum_{1 \le i_1 < i_2 \le N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) - \ldots - \sum_{1 \le i_1 < \ldots < i_l \le N} g_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l}) \tag{16}$$

As outlined in section 4, $g_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l})$ is the sum of all terms in Taylor series which only contain $x_{i_1}, x_{i_2}, \ldots, x_{i_l}$ when $g(\boldsymbol{x})$ can be expanded as a convergent Taylor series at reference point $\bar{\boldsymbol{x}}$. Since the collective HDMR component functions $g_{i_1 i_2 \ldots i_s}(x_{i_1}, x_{i_2}, \ldots, x_{i_s})$ $(s = 0, 1, \ldots, l)$ remove all the Taylor series terms of $g(\boldsymbol{x})$ with up to l variables, then $r_l(\boldsymbol{x})$ comprises only of the Taylor series terms containing more than l variables.

In order to estimate the contribution of the next term beyond that contained in the HDMR expansion in equation (16), consider a subset I from the set of the input variables considered, $\{1, 2, \ldots, N\}$, i.e.:

$$\boldsymbol{I} = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, N\}; \quad k = l + 1 \tag{17}$$

and let,

$$\boldsymbol{x_I} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \tag{18}$$

Then $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ (where $\bar{\boldsymbol{x}}^{\boldsymbol{I}}$ is the $\bar{\boldsymbol{x}}$ without elements $\{\bar{x}_{i_1}, \bar{x}_{i_2}, \ldots, \bar{x}_{i_k}\}$) is the residual value with all variables evaluated at $\bar{\boldsymbol{x}}$ except of the elements in $\bar{\boldsymbol{x}}^{\boldsymbol{I}}$. Considering that $r_l(\boldsymbol{x})$ may be viewed as composed of the products of monomial bases $(x_i - \bar{x}_i)$ $(i = 1, 2, \ldots, N)$, therefore $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ only contains the Taylor series terms with the variable in $\boldsymbol{x_I}$. This implies that $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ is a kth order HDMR component function:

$$r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = g_{i_1 i_2 \ldots i_k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \tag{19}$$

The objective is to find an approximation of equation (19). In order to estimate the approximation of equation (19), it is convenient to write $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ as:

$$r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = \varphi(\boldsymbol{x_I}) \frac{r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})}{\varphi(\boldsymbol{x_I})} = \varphi(\boldsymbol{x_I}) h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \tag{20}$$

where $\varphi(\boldsymbol{x_I})$ is some appropriate known function (e.g. the product of monomials). If $h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ can be represented by first-order HDMR approximations about some other suitable center other than $\bar{\boldsymbol{x}}$, then $\varphi(\boldsymbol{x_I})$ $h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ will provide an first-order approximation of $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$. By this way $\varphi(\boldsymbol{x_I})$ can be viewed as a preconditioning function that extracts some characteristics behavior from $g_{i_1 i_2 \ldots i_k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. If $\varphi(\boldsymbol{x_I})$ is the product of monomials, i.e.:

$$\varphi(\boldsymbol{x_I}) = \prod_{s=1}^{k} (x_{i_s} - \bar{x}_{i_s}) \tag{21}$$

then,

$$h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = \frac{r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})}{\prod_{s=1}^{k} (x_{i_s} - \bar{x}_{i_s})} \tag{22}$$

and the process is referred to as monomial based preconditioning.

Now consider approximating $h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ by first-order HDMR at a new reference point:

$$\boldsymbol{b} = \{b_1, b_2, \ldots, b_N\} \tag{23}$$

such that, $b_i \neq \bar{x}_i$ for all i, to avoid the singularity in $h_{l\boldsymbol{I}}(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$, and set:

$$\boldsymbol{b}_{\boldsymbol{I}} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \tag{24}$$

which leads to:

$$h_{l\boldsymbol{I}}(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \approx \frac{r_l(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})}{\prod_{s=1}^{k}(b_{i_s} - \bar{x}_{i_s})} + \sum_{s=1}^{k}\left(\frac{r_l(x_{i_s}, \boldsymbol{b}_{\boldsymbol{I}}^{i_s}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})}{(x_{i_s} - \bar{x}_{i_s})\prod_{r=1,\ i_r \neq i_s}^{k}(b_{i_r} - \bar{x}_{i_r})} - \frac{r_l(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})}{\prod_{r=1}^{k}(b_{i_r} - \bar{x}_{i_r})}\right) \tag{25}$$

The resultant first-order HDMR component functions for $h_{l\boldsymbol{I}}(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ are then multiplied by $\varphi(\boldsymbol{x}_{\boldsymbol{I}}) = \prod_{s=1}^{k}(x_{i_s} - \bar{x}_{i_s})$ in equation (21), which gives an approximation for kth order component function:

$$\begin{aligned} g_{i_1 i_2 \ldots i_k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) &= r_l(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \\ &= \prod_{s=1}^{k}(x_{i_s} - \bar{x}_{i_s}) h_{l\boldsymbol{I}}(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \\ &\approx \prod_{s=1}^{k}\frac{(x_{i_s} - \bar{x}_{i_s})}{(b_{i_s} - \bar{x}_{i_s})} r_l(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) + \sum_{s=1}^{k}\left(\begin{array}{l} \prod_{r=1,\ i_r \neq i_s}^{k}\frac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_l(x_{i_s}, \boldsymbol{b}_{\boldsymbol{I}}^{i_s}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) - \\ \prod_{r=1,\ i_r \neq i_s}^{k}\frac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_l(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \end{array}\right) \end{aligned} \tag{26}$$

where $\boldsymbol{b}_{\boldsymbol{I}}^{i_s}$ is just $\boldsymbol{b}_{\boldsymbol{I}}$ without elements $\boldsymbol{b}_{i_s}$. When all possible choices of $\boldsymbol{I}$ are considered, the collective terms results in an approximation of the $(l+1)$th order component functions of HDMR to $g(\boldsymbol{x})$ without directly evaluating $g_{i_1 i_2 \ldots i_k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$.

With an aim to improve the accuracy in approximating $g(\boldsymbol{x})$ for subsequent reliability and sensitivity analysis application, at same time without increasing exorbitantly the effort required in evaluating the higher-order component functions of HDMR, the present work restricts to an approximation to first-order HDMR terms. Therefore, considering $l = 1$ and $k = 2$, in equations (16)-(27) leads to, respectively:

$$r_1(\boldsymbol{x}) = g(\boldsymbol{x}) - g_0 - \sum_{i=1}^{N} g_i(x_i) \tag{28}$$

$$\boldsymbol{I} = \{i_1, i_2\} \subseteq \{1, 2, \ldots, N\} \tag{29}$$

and,

$$\boldsymbol{x}_{\boldsymbol{I}} = \{x_{i_1}, x_{i_2}\} \tag{30}$$

Considering $k = 2$ in equation (24) results in:

$$\boldsymbol{b}_{\boldsymbol{I}} = \{b_{i_1}, b_{i_2}\} \tag{31}$$

Therefore, the first-order approximation of $r_l(\boldsymbol{x}_{\boldsymbol{I}}, \boldsymbol{c}^{\boldsymbol{I}})$ at reference point $\boldsymbol{b} = \{b_1, b_2, \ldots, b_N\}$ is given by:

$$r_1(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = \wp_0 + \sum_{s=1}^{k} \wp_{i_s} \tag{32}$$

where,

$$\wp_0 = \prod_{s=1}^{k} \frac{(x_{i_s} - \bar{x}_{i_s})}{(b_{i_s} - \bar{x}_{i_s})} r_1(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \tag{33}$$

$$\wp_{i_s} = \prod_{r=1, i_r \neq i_s}^{k} \frac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_1(x_{i_s}, \boldsymbol{b}_{\boldsymbol{I}}^{i_s}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) - \prod_{r=1}^{k} \frac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_1(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \tag{34}$$

and,

$$r_1(\boldsymbol{b}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = g(b_{i_1}, b_{i_2}, \bar{\boldsymbol{x}}^{i_1 i_2}) - g(b_{i_1}, \bar{\boldsymbol{x}}^{i_1}) - g(b_{i_2}, \bar{\boldsymbol{x}}^{i_2}) + g(\bar{\boldsymbol{x}}) \tag{35}$$

$$r_1(x_{i_1}, \boldsymbol{b}_{\boldsymbol{I}}^{i_1}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) = g(x_{i_1}, b_{i_2}, \bar{\boldsymbol{x}}^{i_1 i_2}) - g(x_{i_1}, \bar{\boldsymbol{x}}^{i_1}) - g(x_{i_2}, \bar{\boldsymbol{x}}^{i_2}) + g(\bar{\boldsymbol{x}}) \tag{36}$$

Hence, eHDMR approximation of the function $g(\boldsymbol{x})$ is obtained as follows:

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{\boldsymbol{I}} [r_1(\boldsymbol{x}_{\boldsymbol{I}}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})] \tag{37}$$

which can be rewritten as:

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{\boldsymbol{I}} \left[\wp_0 + \sum_{s=1}^{k} \wp_{i_s} \right] \tag{38}$$

6. Generation of eHDMR approximation

HDMR in equation (9) is exact along any of the cuts, and the approximation of the response function $g(\boldsymbol{x})$ using eHDMR at a point $\boldsymbol{x}$ can be obtained by following the procedure in Steps 1 to 3 below:

(1) *Step 1*. Interpolate each of the low-dimensional HDMR expansion terms with respect to the input values of the point $\boldsymbol{x}$. For example, consider the first-order component function $g(x_i, \bar{\boldsymbol{x}}^i) = g(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N)$. If for $x_i = x_i^j$, n function values:

$$g(x_i^j, \bar{\boldsymbol{x}}^i) = g(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N); \; j = 1, 2, \ldots, n \tag{39}$$

are given at $n(=3,5,7 \text{ or } 9)$ regularly spaced sample points, the function value for arbitrary x_i can be obtained by the moving least square (MLS) interpolation as (Lancaster and Salkauskas, 1986):

$$g(x_i, \bar{\boldsymbol{x}}^j) = \sum_{j=1}^{n} \phi_j(x_i)\, g(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N) \tag{40}$$

where $\phi_j(x_i)$ is the MLS interpolation function.

By using equation (40), $g_i(x_i)$can be generated if n function values are given at corresponding sample points. The same procedure shall be repeated for all the first-order component functions, i.e.:

$$g_i(x_i);\ i = 1, 2, \ldots, N$$

(2) *Step 2*. Sum the interpolated values of HDMR expansion terms calculated in Step 1. This leads to first-order HDMR approximation of the function $g(\boldsymbol{x})$ as:

$$\tilde{\boldsymbol{g}}(\boldsymbol{x}) = \sum_{i=1}^{N} \sum_{j=1}^{n} \phi_j(x_i) g(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N) - (N-1) g(\bar{\boldsymbol{x}}) \tag{41}$$

(3) *Step 3*. Compute first-order approximation of $r_l(\boldsymbol{x_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}})$ using equation (32) and subsequently add to first-order HDMR approximation (calculated in Step 2), which leads to eHDMR approximation of the function $g(\boldsymbol{x})$ as:

$$\begin{aligned} \tilde{\boldsymbol{g}}(\boldsymbol{x}) = & \sum_{i=1}^{N} \sum_{j=1}^{n} \phi_j(x_i) g(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i^j, \bar{x}_{i+1}, \ldots, \bar{x}_N) - (N-1) g(\bar{\boldsymbol{x}}) \\ & + \sum_{\boldsymbol{I}} \left[\begin{array}{l} \prod_{s=1}^{k} \dfrac{(x_{i_s} - \bar{x}_{i_s})}{(b_{i_s} - \bar{x}_{i_s})} r_1(\boldsymbol{b_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) + \prod_{r=1, i_r \neq i_s}^{k} \dfrac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_1(x_{i_s}, \boldsymbol{b}_{\boldsymbol{I}}^{i_s}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \\ - \prod_{r=1}^{k} \dfrac{(x_{i_r} - \bar{x}_{i_r})}{(b_{i_r} - \bar{x}_{i_r})} r_1(\boldsymbol{b_I}, \bar{\boldsymbol{x}}^{\boldsymbol{I}}) \end{array} \right] \end{aligned} \tag{42}$$

6.1 Computational effort

If n is the number of sample points taken along each of the variable axis and s is the order of the component function considered, starting from zeroth-order to k-th order, then the total number of function evaluation for HDMR approximation is given by $\sum_{s=0}^{k} (N!(n-1)^s)/((N-s)!s!)$, which grows polynomially with n and s.

When new component functions of the kth order eHDMR are added to the kth order HDMR, the increase in the number of function evaluation for different k is given by Table I, where:

$$k C_N^k = k \frac{N!}{k!(N-k)!} \tag{43}$$

The significance of this table can be understood by considering a case when $N = 3$. If we add the error function $r_l(\boldsymbol{x}_{\boldsymbol{I}}, \boldsymbol{c}^{\boldsymbol{I}})$ in the first-order HDMR (equation (41)) to generate eHDMR approximation (equation (42)), then there are total $3 + 2C_3^2 = 9$ one variable functions, $g(x_i, \boldsymbol{c}^i)$ for all possible combinations of $\boldsymbol{I}$. These additional component functions composed new lines passing through $N = 3$ dimensional space upon which the function $g(\boldsymbol{x})$ is exactly represented. Thus, the accuracy of eHDMR approximation can be dramatically improved over the first-order HDMR approximation. Now suppose that, each variable is sampled at n sample points. Then the total number of function evaluation for kth order component functions of HDMR is $C_N^k n^k$, but kth order eHDMR approximation component functions only need $C_N^k(kn + C_k^2 n^2)$ function evaluations. The ratio is:

$$R = \frac{\text{Function evaluation of } k\text{-th order eHDMR}}{\text{Function evaluation of } k\text{-th order HDMR}} = \frac{C_N^k(kn + C_k^2 n^2)}{C_N^k n^k} = \frac{k + C_k^2 n}{n^{k-1}} \tag{44}$$

For $k = 2$ and $n = 10$, $R \approx 1$. For $k = 3$ and $n = 10$, $R \approx 1/3$. For $k = 5$ and $n = 10$, $R \approx 1/100$. Therefore, the saving in computational cost is obvious. As a few low-order component functions of eHDMR are used, the sample savings due to eHDMR are significant compared to traditional sampling. Hence the reliability and subsequent sensitivity analysis using eHDMR relies on an accurate reduced model being generated with a small number of model simulations.

7. Estimation of failure probability and sensitivity

Recall that $\tilde{g}(\boldsymbol{x})$ is the approximate function of the original limit state/performance function. Based on this approximation, let $\hat{\Omega}_{FS} = \{\boldsymbol{x} : \tilde{g}(\boldsymbol{x}) < 0\}$ define the approximate failure set in a reliability analysis. Therefore, the Monte Carlo method estimates of the failure probability $P_F(\boldsymbol{\theta})$ and its sensitivity $\partial P_F(\boldsymbol{\theta})/\partial\theta_i$, employing HDMR and eHDMR approximation, are given by:

$$P_F(\boldsymbol{\theta}) \cong \mathsf{E}_\theta[\mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x})] = \lim_{N_s \to \infty} \frac{1}{N_S} \sum_{i=1}^{N_S} \mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i) \tag{45}$$

and,

K	$g(x_{i_s}, \boldsymbol{b}_{\boldsymbol{I}}^i, \boldsymbol{c}^{\boldsymbol{I}})$
2	$2C_N^2$
3	$3C_N^3$
K	kC_N^k

Table I.
Increasing numbers of one-dimensional function evaluation

$$\frac{\partial P_F(\boldsymbol{\theta})}{\partial \theta_i} \cong \mathsf{E}_\theta[\mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x})\mathsf{K}_\theta^{(1)}(\boldsymbol{x};\boldsymbol{\theta})] = \lim_{N_s\to\infty}\frac{1}{N_S}\sum_{i=1}^{N_S}\mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i)\mathsf{K}_\theta^{(1)}(\boldsymbol{x}^i;\boldsymbol{\theta}) \tag{46}$$

respectively. Here $\boldsymbol{x}^i$ is i^{th} realization of $\boldsymbol{x}$, N_S is the sample size, $\mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i)$ is an indicator of fail or safe state such that:

$$\mathrm{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i) = \begin{cases} 1, & \boldsymbol{x}^i \in \hat{\Omega}_{FS} \\ 0, & \boldsymbol{x}^i \in \Omega\backslash\hat{\Omega}_{FS} \end{cases} \tag{47}$$

The Monte Carlo estimate of the moments of HDMR and eHDMR approximation can be found as:

$$m_q(\boldsymbol{\theta}) \cong \mathsf{E}_\theta[\tilde{g}^q(\boldsymbol{x})] = \lim_{N_s\to\infty}\frac{1}{N_S}\sum_{i=1}^{N_S}\tilde{g}^q(\boldsymbol{x}^i) \tag{48}$$

and its sensitivity:

$$\frac{\partial m_q(\boldsymbol{\theta})}{\partial \theta_i} \cong \mathsf{E}_\theta[\tilde{g}^q(\boldsymbol{x})\mathsf{K}_\theta^{(1)}(\boldsymbol{x};\boldsymbol{\theta})] = \lim_{N_s\to\infty}\frac{1}{N_S}\sum_{i=1}^{N_S}\tilde{g}^q(\boldsymbol{x}^i)\mathsf{K}_\theta^{(1)}(\boldsymbol{x}^i;\boldsymbol{\theta}) \tag{49}$$

8. Numerical examples

Two numerical examples are presented to illustrate the proposed approach, for obtaining the sensitivity of the moment and reliability. Whenever possible, the finite-difference method and the direct MCS are employed to evaluate the accuracy and computational efficiency of the present method. The sample sizes for the direct MCS and the MCS in conjunction with HDMR approximation vary from 10^4 to 10^6, depending on the examples, but they are identical for a specific problem. A flow diagram for eHDMR approximation and the uncertainty analysis by MCS is shown in Figure 1. For first-order HDMR, n regularly spaced sample points are deployed along the variable axis through the reference point $\bar{\boldsymbol{x}}$. Sampling scheme for the reduced-order modelling of a function having one variable (x) and two variables (x_1 and x_2) using the first-order HDMR is shown in Figures 2(a) and (b), respectively. For the eHDMR, reference point $\boldsymbol{b}$ is arbitrarily chosen from within the domain of the problem. It is documented (Li *et al.*, 2001b) that, interchanging the $\bar{\boldsymbol{x}}$ and $\boldsymbol{b}$ does not change the tendency of improvement in approximation. A recent study (Rao and Chowdhury, 2009) concludes that the approximation of eHDMR is almost unchanged irrespective of reference point $\boldsymbol{b}$. Due to these facts, reference point $\boldsymbol{b}$ is selected arbitrarily within the problem domain.

8.1 Example 1: cubic function with two variables

Consider a cubic limit state/performance function (Grandhi and Wang, 1999) of the following form:

$$g(\boldsymbol{x}) = 2.2257 - \frac{0.025\sqrt{2}}{27}(x_1 + x_2 - 20)^3 + \frac{33}{140}(x_1 - x_2) \tag{50}$$

with two independent normal variables. The mean and standard deviation of the

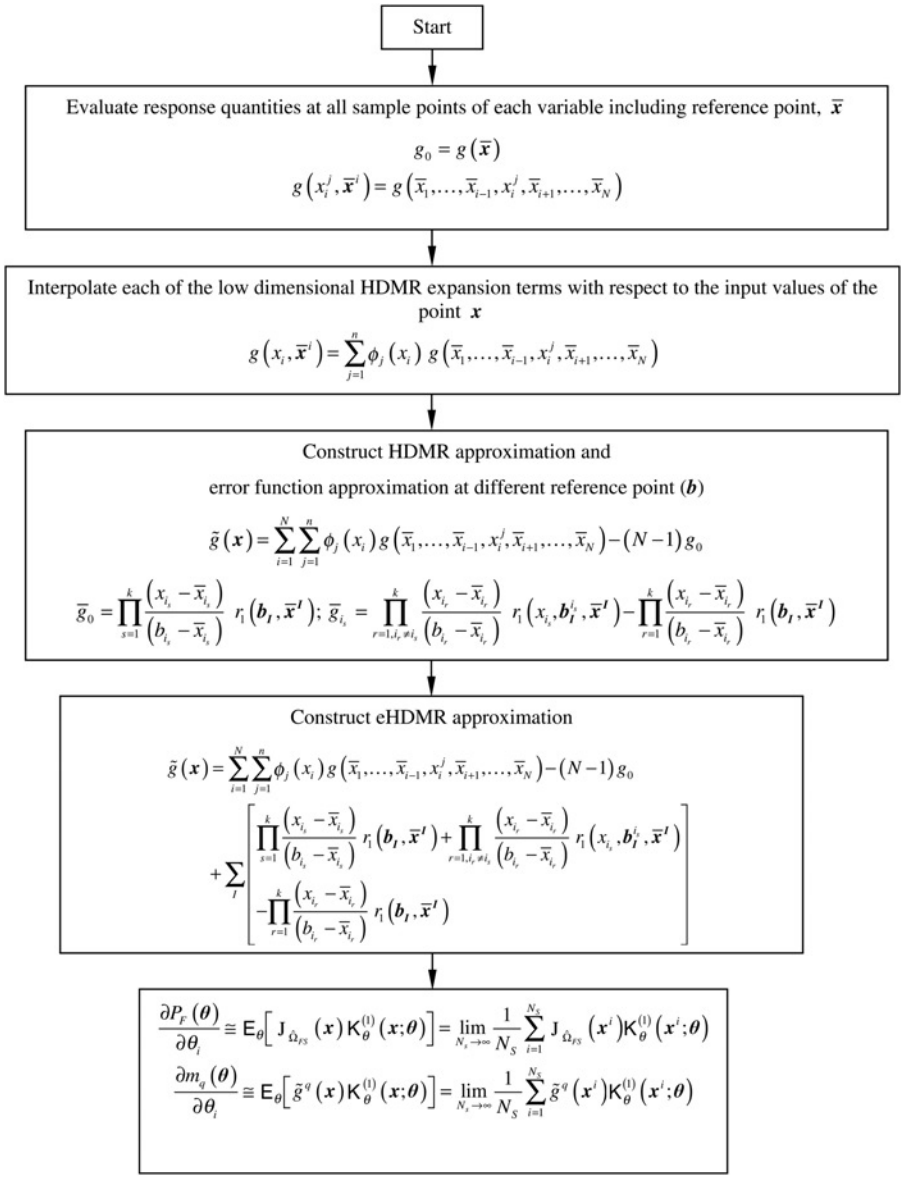
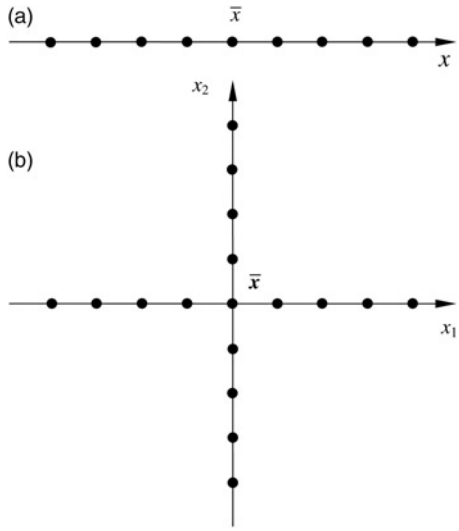


Figure 1.
Flowchart of sensitivity analysis using eHDMR approximation

random variables are 10 and 3, respectively. For evaluating the failure probability P_F and subsequent sensitivity analysis, HDMR/eHDMR approximation is constructed by deploying five equally spaced sample points ($n = 5$) along each of the variable axis. The reference point $\bar{\boldsymbol{x}}$ is chosen as (10, 10) and the reference point $\boldsymbol{b}$ is arbitrarily chosen as (5, 13). Table II compares the results obtained by the present method using HDMR and eHDMR approximation with direct MCS. A sampling size $N_S = 10^6$ is considered in direct MCS to evaluate the failure probability P_F. Table II also contains the computational effort in terms of number of function evaluations, associated with



Notes: (a) For a function having one variable (x); (b) for a function having two variables ($x1$ and $x2$)

Figure 2.
Sampling scheme for first-order HDMR

Method	Failure probability	Number of function evaluation[a]
FORM	0.01302	21
SORM (Adhikari, 2005)	0.01302	204
HORM (Grandhi and Wang, 1999)	0.01818	Not reported
First-order HDMR	0.01583	9[b]
eHDMR	0.01894	35[c]
Direct MCS	0.01907	10^6

Notes: [a]Total number of times the original performance function is calculated; [b]$(n-1)\times N+1 = (5-1)\times 2+1 = 9$;
[c]$\frac{N!}{k!(N-k)!}\left[k\times n+\frac{k!}{2!(k-2)!}n^2\right] = \frac{2!}{2!(2-2)!}\left[2\times 5+\frac{2!}{2!(2-2)!}5^2\right] = 35$

Table II.
Estimation of failure probability for Example 1

each of the methods. Compared with the failure probability obtained using the direct MCS ($P_F = 0.01907$), the first-order HDMR and eHDMR approximation underestimates the failure probability by 16.99 percent ($P_F = 0.01583$) and 0.68 percent ($P_F = 0.01894$), respectively. Table III lists the first-order sensitivities of failure probability with respect to mean and standard deviation of the random variables $\partial P_F/\partial \mu_i$ and $\partial P_F/\partial \sigma_i$ for $i = 1, 2$. Similarly, Table IV presents first three moments m_q and their first-order sensitivities $\partial m_q(\boldsymbol{\theta})/\partial \mu$ and $\partial m_q(\boldsymbol{\theta})/\partial \sigma$ for $q = 1, 2, 3$. The agreement between the results of the proposed approach using simulation and the exact solution is excellent.

8.2 Example 2: 6-DOF dynamical system

This example considers a four-story building excited by a single period sinusoidal pulse of ground motion, studied by Gavin and Yau (2008). Figure 3(a) shows the four-story building with isolation systems and Figure 3(b) presents the acceleration history. The building contains isolated equipment resting on the second floor. The motion of the ground floor is resisted mainly by base isolation bearings (Wen, 1976) and if its displacement exceeds D_c (=0.50 m) then an additional stiffness force contributes to the resistance. Mass, stiffness and damping coefficient m_f, k_f and c_f, respectively, at each floor are assumed to be same.

There are two isolated masses, representing isolated, shock-sensitive equipment resting on the second floor. The larger mass m_1(=500 kg) is connected to the floor by a relatively flexible spring, k_1(=2500 N/m), and a damper, c_1(=350 N/m/s), representing the isolation system. The smaller mass ($m_2 = 100$ kg) is connected to the larger mass by a relatively stiff spring, k_2(=10^5 N/m), and a damper, c_2(=200 N/m/s), representing the equipment itself. All variables are assumed to be lognormal and independent. Probabilistic descriptions of the random variables are listed in Table V. The limit state/performance function is defined by the combination of three failure modes leading to system failure and is the following form:

	First-order HDMR	eHDMR	Direct MCS[a]
$\partial P_F/\partial\mu_1$	-5.56×10^{-3}	-5.06×10^{-3}	-5.07×10^{-3}
$\partial P_F/\partial\mu_2$	4.38×10^{-3}	1.38×10^{-2}	1.36×10^{-2}
$\partial P_F/\partial\sigma_1$	1.19×10^{-2}	2.63×10^{-2}	2.67×10^{-2}
$\partial P_F/\partial\sigma_2$	2.32×10^{-2}	2.45×10^{-2}	2.48×10^{-2}

Note: [a]For sensitivity estimation, finite difference with 1 percent perturbation is used

Table III. Sensitivities of failure probability (Example 1)

	First-order HDMR	eHDMR	Direct MCS[a]
m_1	2.2270	2.2245	2.2245
m_2	6.0002	6.1110	6.1088
m_3	17.9941	18.7497	18.7502
$\partial m_1/\partial\mu_1$	0.2008	0.1630	0.1632
$\partial m_2/\partial\mu_1$	0.8918	0.7214	0.7215
$\partial m_3/\partial\mu_1$	3.5453	2.7451	2.7448
$\partial m_1/\partial\mu_2$	−0.2725	−0.3082	−0.3082
$\partial m_2/\partial\mu_2$	−1.2125	−1.3826	−1.3827
$\partial m_3/\partial\mu_2$	−5.0903	−5.9408	−5.9410
$\partial m_1/\partial\sigma_1$	0.0004	0.0002	0.0002
$\partial m_2/\partial\sigma_1$	0.1713	−0.2911	−0.2912
$\partial m_3/\partial\sigma_1$	1.1388	−1.9266	−1.9267
$\partial m_1/\partial\sigma_2$	-8.3321×10^{-4}	0.0002	0.0002
$\partial m_2/\partial\sigma_2$	0.5792	−0.7211	−0.7210
$\partial m_3/\partial\sigma_2$	3.8612	−4.8141	−4.8140

Note: [a]For sensitivity estimation, finite difference with 1 percent perturbation is used

Table IV. Moments and sensitivities of moments (Example 1)

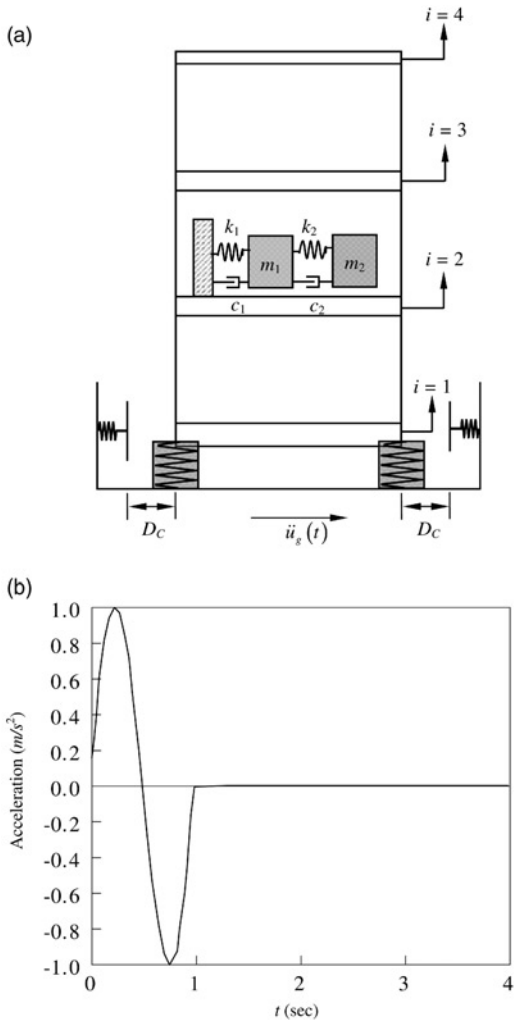


Figure 3.
Problem statement (Example 2)

Notes: (a) Base isolated structure with an equipment isolation system on the second floor; (b) acceleration history

$$g(\boldsymbol{x}) = 12.50(0.04 - \max_t |x_{f_i}(t) - x_{f_{i-1}}(t)|)_{i=2,3,4} + (0.50 - \max_t |\ddot{\boldsymbol{u}}_g(t) + \ddot{\boldsymbol{x}}_{m_2}(t)|) + 2.0(0.25 - \max_t |x_{f_2}(t) - x_{m_1}(t)|) \quad (51)$$

where $x_{f_i}(t)$ refers to the displacement of i^{th} floor and $(x_{f_i}(t) - x_{f_{i-1}}(t))$ is the inter story

drift. $\ddot{u}_g(t)$ is the ground acceleration and $\ddot{x}_{m_2}(t)$ is the acceleration smaller mass block. The displacement $x_{m_1}(t)$ is of the larger mass block, and represents the displacement of the equipment isolation system. The limit state/performance function in equation (51) is the overall representation of three failure modes. The first term describes the damage to the structural system due to excessive deformation. The second term represents the damage to equipment caused by excessive acceleration. The last term represents the damage of the isolation system. The weighing factors, multiplied with each term in equation (51), are mainly to emphasize the equal contribution of the individual failure modes to the overall failure of system. It is desirable that

(1) inter story drift is limited to 0.04 m;

(2) the peak acceleration of the equipment is less than 0.5 m/s^2; and

(3) the displacement across the equipment isolation system is less than 0.25 m.

Equation (51) signifies overall system failure, which does not necessarily occur when above-mentioned one or two failure criteria satisfies. For estimating sensitivity and failure probability P_F, first-order HDMR and eHDMR approximation is constructed by deploying five regularly spaced sample points ($n = 5$) along each of the variable axis. The reference point $\boldsymbol{x}$ is chosen as mean values of variables and the reference point $\boldsymbol{b}$ is arbitrarily chosen as some point within the domain. Table VI compares the results obtained using first-order HDMR and eHDMR approximation with first- and second-order reliability method (FORM/SORM), direct MCS and other existing results. Table VI

Random variable	Units	Description	Mean	COV
m_f	kg	Floor mass	6000	0.10
k_f	N/m	Floor stiffness	3×10^7	0.10
c_f	N/m/s	Floor damping coefficient	6×10^4	0.20
f_y	N	Isolation yield force	2×10^4	0.20
d_y	m	Isolation yield displacement	0.05	0.20
k_c	N/m	Isolation contact stiffness	3×10^7	0.30
T	s	Force period	1.0	0.20
A	m/m/s	Force amplitude	1.0	0.50

Table V.
Properties of the random variables for Example 2

Method	Failure probability	Number of function evaluation[a]
FORM	0.22549	86
SORM (Adhikari, 2005)	0.21410	357
First-order HDMR	0.19968	33[b]
eHDMR	0.19576	980[c]
SRSM (Gavin and Yau, 2008)	0.19355	6561
HO-SRSM (Gavin and Yau, 2008)	0.19643	2106
Direct MCS	0.19599	10^5

Notes: [a]Total number of times the original performance function is calculated; [b]$(n-1) \times N + 1 = (5-1) \times 8 + 1 = 33$;
[c]$\frac{N!}{k!(N-k)!}\left[k \times n + \frac{k!}{2!(k-2)!}n^2\right] = \frac{8!}{2!(8-2)!}\left[2 \times 5 + \frac{2!}{2!(2-2)!}5^2\right] = 980$

Table VI.
Estimation of failure probability for Example 2

also presents the computational effort in terms of number of function evaluations, associated with each of the methods. The benchmark solution of the failure probability is obtained by direct MCS with $N_S = 10^5$. Compared with the benchmark solution ($P_F = 0.19599$), FORM and SORM overestimate the failure probability by around 15.05 percent ($P_F = 0.22549$) and 8.46 percent ($P_F = 0.21410$), respectively. First-order HDMR approximation (Chowdhury *et al.*, 2009) overestimates the failure probability by about 1.88 percent ($P_F = 0.19968$) and it needs only 33 function evaluations. eHDMR approximation underestimates the failure probability by about 0.16 percent ($P_F = 0.19567$) and it needs 980 function evaluations, while FORM, SORM and direct MCS requires 86, 356 and 10^5 number of original function evaluations, respectively. This shows the accuracy and the efficiency (in terms of original function calculations) of the first-order HDMR approximation, over FORM, SORM and direct MCS. Compared to the first-order HDMR, the error in the estimated failure probability reduces from +1.88 to −0.16 percent, and the number of function evaluations increases from 33 to 980, using eHDMR approximation.

Tables VII and VIII present the first-order sensitivities of failure probability with respect to mean and standard deviation of random variables $\partial P_F/\partial \mu_i$ and $\partial P_F/\partial \sigma_i$ for $i = 1, \ldots, 8$, respectively. Table IX lists the first three moments m_q and their first-order sensitivities $\partial m_1(\boldsymbol{\theta})/\partial \mu_i$ and $\partial m_1(\boldsymbol{\theta})/\partial \sigma_i$ for $i = 1, \ldots, 8$. The agreement between the results of the proposed approach and the exact solution demonstrates that

	First-order HDMR	eHDMR	Direct MCS[a]
$\partial P_F/\partial \mu_1$	-7.8376×10^{-6}	-8.1671×10^{-6}	-8.2371×10^{-6}
$\partial P_F/\partial \mu_2$	-5.9548×10^{-10}	-6.1203×10^{-10}	-6.1242×10^{-10}
$\partial P_F/\partial \mu_3$	9.1605×10^{-8}	8.9812×10^{-8}	8.9905×10^{-8}
$\partial P_F/\partial \mu_4$	5.4109	5.4267	5.4309
$\partial P_F/\partial \mu_5$	5.2208×10^{-6}	5.6898×10^{-6}	5.7218×10^{-6}
$\partial P_F/\partial \mu_6$	6.5064×10^{-11}	6.9064×10^{-11}	6.9064×10^{-11}
$\partial P_F/\partial \mu_7$	0.9655	0.9788	0.9834
$\partial P_F/\partial \mu_8$	0.3705	0.4226	0.4215

Note: [a]For sensitivity estimation, finite difference with 1 percent perturbation is used

Table VII.
Sensitivities of failure probability with respect to mean (Example 2)

	First-order HDMR	eHDMR	Direct MCS[a]
$\partial P_F/\partial \sigma_1$	-5.0739×10^{-6}	-5.1684×10^{-6}	-5.1702×10^{-6}
$\partial P_F/\partial \sigma_2$	-1.9825×10^{-10}	-1.9961×10^{-10}	-1.9965×10^{-10}
$\partial P_F/\partial \sigma_3$	1.2518×10^{-7}	1.3109×10^{-7}	1.3117×10^{-7}
$\partial P_F/\partial \sigma_4$	−0.7237	−0.7346	−0.7342
$\partial P_F/\partial \sigma_5$	-4.4779×10^{-6}	-4.4888×10^{-6}	-4.4878×10^{-6}
$\partial P_F/\partial \sigma_6$	1.3931×10^{-10}	1.3941×10^{-10}	1.3942×10^{-10}
$\partial P_F/\partial \sigma_7$	0.4060	0.4110	0.4106
$\partial P_F/\partial \sigma_8$	-6.4636×10^{-3}	-6.4741×10^{-3}	-6.4737×10^{-3}

Note: [a]For sensitivity estimation, finite difference with 1 percent perturbation is used

Table VIII.
Sensitivities of failure probability with respect to standard deviation (Example 2)

	First-order HDMR	eHDMR	Direct MCS[a]
m_1	0.3741	0.3718	0.3722
m_2	0.3358	0.3351	0.3354
m_3	0.2615	0.2615	0.2613
$\partial m_1/\partial \mu_1$	1.3873×10^{-5}	1.4617×10^{-5}	1.4621×10^{-5}
$\partial m_1/\partial \mu_2$	2.9342×10^{-10}	1.3488×10^{-9}	1.3476×10^{-9}
$\partial m_1/\partial \mu_3$	-6.0250×10^{-8}	-1.2108×10^{-7}	-1.2107×10^{-7}
$\partial m_1/\partial \mu_4$	−9.4181	−9.7684	−9.7682
$\partial m_1/\partial \mu_5$	-9.1614×10^{-6}	-8.9295×10^{-6}	-8.9308×10^{-6}
$\partial m_1/\partial \mu_6$	-2.2457×10^{-11}	-1.7345×10^{-10}	-1.7385×10^{-10}
$\partial m_1/\partial \mu_7$	−1.5725	−1.5691	−1.5708
$\partial m_1/\partial \mu_8$	−0.7712	−0.7722	−0.7735
$\partial m_1/\partial \sigma_1$	8.9684×10^{-6}	9.5182×10^{-6}	9.5198×10^{-6}
$\partial m_1/\partial \sigma_2$	-4.9191×10^{-11}	-1.3693×10^{-9}	-1.3716×10^{-9}
$\partial m_1/\partial \sigma_3$	-1.1677×10^{-7}	1.3160×10^{-7}	1.3271×10^{-7}
$\partial m_1/\partial \sigma_4$	2.5100	2.9262	2.9269
$\partial m_1/\partial \sigma_5$	8.1566×10^{-6}	8.8669×10^{-6}	8.8689×10^{-6}
$\partial m_1/\partial \sigma_6$	-1.7508×10^{-10}	-4.1083×10^{-10}	-4.1108×10^{-10}
$\partial m_1/\partial \sigma_7$	0.1116	0.1022	0.1027
$\partial m_1/\partial \sigma_8$	0.2909	0.2968	0.2974

Note: [a]For sensitivity estimation, finite difference with 1 percent perturbation is used

Table IX. Moments and sensitivities of first-moment (Example 2)

the proposed approach can indeed account high nonlinearity and large input uncertainties.

9. Conclusions and outlook

This paper addressed a new computational method for estimating stochastic sensitivities of structural/mechanical systems with respect to probability distribution parameters. The methods are based on eHDMR and score functions associated with the probability distribution of a random input. Following are the concluding remarks of the present approach:

- Both the probabilistic response and its sensitivities can be estimated from a single stochastic analysis, without requiring limit state/performance function gradients. The effort in obtaining probabilistic sensitivities can be viewed as calculating the conditional response at a selected deterministic input, defined by sample points. Therefore, the methods can be easily adapted for solving stochastic problems involving third-party, commercial FE codes.
- First-order and eHDMR approximation are employed to solve two numerical problems, where the performance functions are linear or nonlinear, include Gaussian and/or non-Gaussian random variables, and are described by simple mathematical functions or mechanical responses from FE analysis.
- The results indicate that HDMR approximation, in particular the eHDMR, provide very accurate estimates of sensitivities of statistical moments or reliability.
- EHDMR approximation, which generally outperforms first-order HDMR approximation, demands a higher cost scaling, making it more expensive than

first-order approximation. Nonetheless, both approaches are far less expensive than the finite-difference method or the existing score function method entailing direct MCS.

- In contrast to the direct differentiation method, which can calculate sensitivities with respect to both distribution as well as limit state/performance function parameters, the proposed approach is limited to sensitivity analysis with respect to the distribution parameters only. Therefore, future effort in extending the present approaches to account for the performance function parameters will be undertaken.

The method proposed here is applied to structural/mechanical systems as examples. However, as shown in the paper, the method has the potential for application in other scientific disciples and multiphysics problems. Further, work is, however, needed to realize this potential.

References

Adhikari, S. (2004), "Reliability analysis using parabolic failure surface approximation", *Journal of Engineering Mechanics, ASCE*, Vol. 130 No. 12, pp. 1407-27.

Adhikari, S. (2005), "Asymptotic distribution method for structural reliability analysis in high dimensions", *Proceedings of Royal Society of London, Series – A*, Vol. 461 No. 2062, pp. 3141-58.

Adhikari, S. and Friswell, M.I. (2007), "Random matrix eigenvalue problems in structural dynamics", *International Journal for Numerical Methods in Engineering*, Vol. 69 No. 3, pp. 562-91.

Ahammed, M. and Melchers, R.E. (2006), "Gradient and parameter sensitivity estimation for systems evaluated using Monte Carlo analysis", *Reliability Engineering and System Safety*, Vol. 91 No. 5, pp. 594-601.

Alis, O.F. and Rabitz, H. (2001), "Efficient implementation of high dimensional model representations", *Journal of Mathematical Chemistry*, Vol. 29 No. 2, pp. 127-42.

Alyanak, E., Grandhi, R. and Bae, H. (2008), "Gradient projection for reliability-based design optimization using evidence theory", *Engineering Optimization*, Vol. 40 No. 10, pp. 923-35.

Asmussen, S. and Rubinstein, R.Y. (1993), "Response surface estimation and sensitivity analysis via efficient change of measure", *Stochastic Models*, Vol. 9 No. 3, pp. 313-39.

Bae, H., Grandhi, R.V. and Canfield, R.A. (2006a), "Sensitivity analysis of structural response uncertainty propagation using evidence theory", *Journal of Structural and Multidisciplinary Optimization*, Vol. 31 No. 4, pp. 270-9.

Bae, H., Grandhi, R.V. and Canfield, R.A. (2006b), "Efficient successive reanalysis technique for engineering structures", *AIAA Journal*, Vol. 44 No. 8, pp. 1883-9.

Chowdhury, R., Rao, B.N. and Prasad, A.M. (2009), "High dimensional model representation for structural reliability analysis", *Communications in Numerical Methods in Engineering*, Vol. 25, pp. 301-37.

Gavin, H.P. and Yau, S.C. (2008), "High-order limit state functions in the response surface method for structural reliability analysis", *Structural Safety*, Vol. 30 No. 2, pp. 162-79.

Ghosh, R., Chakraborty, S. and Bhattacharyya, B. (2004), "Stochastic sensitivity analysis of structures using first-order perturbation", *Meccanica*, Vol. 36 No. 3, pp. 291-6.

Glasserman, P. (1991), *Gradient Estimation via Perturbation Analysis*, Kluwer, Norwell, MA.

Grandhi, R. and Wang, L. (1999), "Higher-order failure probability calculation using nonlinear approximations", *Computer Methods in Applied Mechanics and Engineering*, Vol. 168 Nos 1-4, pp. 185-206.

Gunawan, R., Cao, Y., Petzold, L. and Doyle, F.J. (2005), "Sensitivity analysis of discrete stochastic systems", *Biophysical Journal*, Vol. 88 No. 4, pp. 2530-40.

Helton, J.C. (1997), "Uncertainty and sensitivity analysis in the presence of stochastic and subjective uncertainty", *Journal of Statistical Computation and Simulation*, Vol. 57 Nos 1-4, pp. 3-76.

Ho, Y.C. and Cao, X.R. (1991), *Discrete Event Dynamic Systems and Perturbation Analysis*, Kluwer, Norwell, MA.

Lancaster, P. and Salkauskas, K. (1986), *Curve and Surface Fitting: An Introduction*, Academic Press, London.

L'Ecuyer, P. and Perron, G. (1994), "On the convergence rates of IPA and FDC derivative estimators", *Operations Research*, Vol. 42 No. 4, pp. 643-56.

Li, G., Rosenthal, C. and Rabitz, H. (2001a), "High dimensional model representations", *Journal of Physical Chemistry A*, Vol. 105, pp. 7765-77.

Li, G., Wang, S.W. and Rabitz, H. (2001b), "High dimensional model representations generated from low dimensional data samples—I. mp—Cut(HDMR", *Journal of Mathematical Chemistry*, Vol. 30 No. 1, pp. 1-30.

Liu, P.L. and Der Kiureghian, A. (1991), "Finite element reliability of geometrically nonlinear uncertain structures", *Journal of Engineering Mechanics, ASCE*, Vol. 117 No. 8, pp. 1806-25.

McClendon, M. and Rabitz, H. (1988), "Sensitivity analysis in stochastic mechanics", *Physical Review A*, Vol. 37 No. 9, pp. 3493-8.

Melchers, R.E. and Ahammed, M. (2004), "A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability", *Computers & Structures*, Vol. 82 No. 1, pp. 55-61.

Millwater, H. (2009), "Universal properties of kernel functions for probabilistic sensitivity analysis", *Probabilistic Engineering Mechanics*, Vol. 24 No. 1, pp. 89-99.

Nair, P.B. and Keane, A.J. (2002), "Stochastic reduced basis methods", *AIAA Journal*, Vol. 40 No. 8, pp. 1653-64.

Oakley, J.E. and O'Hagan, A. (2004), "Probabilistic sensitivity analysis of complex models: a Bayesian approach", *Journal of the Royal Statistical Society: Series B*, Vol. 66 No. 3, pp. 751-69.

Plyasunov, S. and Arkin, A.P. (2007), "Efficient stochastic sensitivity analysis of discrete event systems", *Journal of Computational Physics*, Vol. 221 No. 8, pp. 724-38.

Rabitz, H. and Alis, O.F. (1999), "General foundations of high dimensional model representations", *Journal of Mathematical Chemistry*, Vol. 25 Nos 2/3, pp. 197-233.

Rahman, S. (2009), "Stochastic sensitivity analysis by dimensional decomposition and score function", *Probabilistic Engineering Mechanics*, Vol. 24, pp. 278-87.

Rao, B.N. and Chowdhury, R. (2009), "Enhanced high dimensional model representation for reliability analysis", *International Journal for Numerical Methods in Engineering*, Vol. 77, pp. 719-50.

Rubinstein, R.Y. and Shapiro, A. (1993), *Discrete Event Systems — Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, John Wiley & Sons, New York, NY.

Sobol, I.M. (2003), "Theorems and examples on high dimensional model representations", *Reliability Engineering and System Safety*, Vol. 79 No. 2, pp. 187-93.

Sudret, B. and Der-Kiureghian, A. (2000), *Stochastic Finite Element Methods and Reliability*, Report No. UCB/SEMM-2000/08, Department of Civil & Environmental Engineering, University of California, Berkeley, CA.

Tunga, M.A. and Demiralp, M. (2004), "A factorized high dimensional model representation on the partitioned random discrete data", *Applied Numerical Analysis and Computational Mathematics*, Vol. 1 No. 1, pp. 231-41.

Tunga, M.A. and Demiralp, M. (2005), "A factorized high dimensional model representation on the nods of a finite hyperprismatic regular grid", *Applied Mathematics and Computation*, Vol. 164, pp. 865-83.

Wang, S.W., Levy II, H., Li, G. and Rabitz, H. (1999), "Fully equivalent operational models for atmospheric chemical kinetics within global chemistry-transport models", *Journal of Geophysical Research*, Vol. 104 No. D23, pp. 30417-26.

Wen, Y.K. (1976), "Method for random vibration of hysteretic systems", *Journal of Engineering Mechanics, ASCE*, Vol. 102 No. EM2, pp. 249-63.

Yaman, I. and Demiralp, M. (2009), "A new rational approximation technique based on transformational high dimensional model representation", *Numerical Algorithms*, Vol. 52 No. 3, pp. 1017-398.

Corresponding author
R. Chowdhury can be contacted at: R.Chowdhury@swansea.ac.uk