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A General Derivation of Dynamic Response of Viscoelastic Structures

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Abstract

The calculation of dynamic response of multiple-degree-of-freedom viscoelastic linear systems is considered. Viscoelastic forces depend on the past history of motion via convolution integrals over exponentially decaying kernel functions. Exact closed-form expressions for the dynamic response due to general forces and initial conditions are derived in terms of the eigensolutions of the system in the original space. Eigensolutions of the viscoelastic system in turn are obtained approximately as functions of the elastic eigensolutions. This enables one to approximately calculate the dynamic response of complex viscoelastic systems by simple post-processing of the elastic (undamped) eigensolutions. Suitable examples are given to illustrate the derived results.

Key words: viscoelasticity, structural dynamics, modal analysis, frequency response, eigensolutions

1. INTRODUCTION

The characterization of energy dissipation in complex vibrating structures such as aircrafts and helicopters is of fundamental importance. Noise and vibration are not only uncomfortable to the users of these complex dynamical systems, but also may lead to fatigue, fracture and even failure of such systems. Increasing use of composite structural materials, active control and damage tolerant systems in the aerospace and automotive industries has lead to renewed demand for energy absorbing and high damping materials. Effective applications of such materials in complex engineering dynamical systems require robust and efficient analytical and numerical methods. Due to the superior damping characteristics, the dynamics of viscoelastic materials and structures have received significant attention over the past two decades. This paper is aimed at developing computationally efficient and physically insightful approximate numerical methods for linear dynamical systems with viscoelastic materials.

A key feature of viscoelastic systems is the incorporation of the time history of the state-variables in the equation of motion. Here we use the Biot model [10] which allows one to incorporate wide range of functions in the frequency domain by means of summation of simple 'pole residue forms'. Several authors have considered this model due to its simplicity and generality (see for example [17, 18,21]). The equations of motion of a N-degree-of-freedom linear system with such material can be expressed by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \mathcal{G}(t-\tau)\dot{\mathbf{u}}(\tau)d\tau + \mathbf{K}_e\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

together with the initial conditions

$$\mathbf{u}(t=0 = \mathbf{u}_0 \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_0 \in \mathbb{R}^N. \tag{2}$$

Here $\mathbf{u}(t) \in \mathbb{R}^N$ is the displacement vector, $\mathrm{f}(t) \in \mathbb{R}^N$ is the forcing vector, $\mathbf{M} \in \mathbb{R}^n$ is the mass matrix, $\mathbf{K}_e \in \mathbb{R}^n$ is the elastic stiffness matrix and $\mathcal{G}(t-\tau)$ is the matrix of viscoelastic stiffness kernel functions. The kernel functions $\mathcal{G}(t-\tau)$, or functions similar to them, are known by many names such as retardation functions, heredity functions, after-effect functions or relaxation functions in the context of different subjects. Equation (1) is very general and for any engineering applications some specific

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form of $\mathcal{G}(t)$ has to be assumed. A wide variety of mathematical expressions could be used for the kernel functions $\mathcal{G}(t)$. Here we will use a viscoelastic material model for which the kernel function matrix has the special form

$$\mathcal{G}(t) = \mathbf{K}_{v0} + \sum_{k=1}^{n} a_k e^{-b_k t} \mathbf{K}_{vk} \tag{3}$$

or in the Laplace domain

$$\mathbf{G}(s) = \mathbf{K}_{v_0} + \sum_{k=1}^{n} \frac{a_k}{s + b_k} \mathbf{K}_{v_k} . \tag{4}$$

Constants a_k, $b_k \in \mathbb{R}^+$ are viscous parameters depending on the viscoelastic material used, n denotes the number of perturbing terms and $\mathbf{K}_{v0}$, $\mathbf{K}_{vk}$ are the viscous stiffness matrices. Some other viscoelastic modeling approaches, such as the GHM (Golla-Hughes-McTavish) approach [12,15] and ADF (Anelastic Displacement Field) approach [13,14], although physically different, can be mathematically represented by a pole-residue form similar to equation (4).

Equation (1) together with the kernel in equation (3) represent a set of coupled integro-differential equations. Several authors have proposed [7,9,12–15,17–19] a state-space approach based on the internal variables for this type of equations. The main reasons for an alternative to the state-space approach for structural dynamics include, but are not limited to:

1. although exact in nature, the state-space approach usually needed for this type of damped systems is computationally very intensive for real-life systems;
2. the physical insights offered by methods in the original space (eg, the modal analysis) is lost in a state-space based approach

Regarding the first point, Woodhouse [20] and Adhikari [3] proposed approximate methods in the space of the original problem. These methods are applied to frequency dependent damping when the damping is small and they neglect the overdamped modes, the equation arising from that problem is almost identical to the one faced here. A direct time-domain approach to obtain the solution of equation (1) was proposed by Adhikari and Wagner [8]. This method is computationally efficient and accurate but does not provide much physical insight. Regarding the second point, in general, a linear system with viscoelastic model is expected to have complex modes [2,4]. The physical justification of complex modes obtained directly from the state-space analysis is still not clear. In this paper we aim to address these two issues.

In section 2, the dynamic response of a general MDOF viscoelastic system is obtained exactly in closed-form using the eigensolutions of the system in the original space. The calculation of the eigensolutions by solving the non-linear eigenvalue problem corresponding to the equation of motion (1) is a key challenge and the main topic of the rest of the paper. We have derived closed-form approximate expressions of the eigenvalues and eigenvectors of the system for four mathematically different cases based on the values of (a) number of degrees of freedom, and (b) number of kernel functions. The approximations utilize Taylor series expansion in the complex domain and are based on certain simplifying assumptions. The validity of the assumptions and the accuracy of the results are verified by numerical calculations.

2. DYNAMIC RESPONSE FOR THE GENERAL CASE

Taking the Laplace transform of equation (1) and considering the initial conditions in equation (2) we have

$$\begin{aligned} & s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q}_0 - \mathbf{M}\dot{\mathbf{q}}_0 + s\mathbf{G}(s)\bar{\mathbf{q}} - \mathbf{G}(s)\mathbf{q}_0 + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \\ \text{or} \quad & \mathbf{D}(s)\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_0 + [s\mathbf{M} + \mathbf{G}(s)]\mathbf{q}_0 \end{aligned} \tag{5}$$

Here the *dynamic stiffness matrix* is defined as

$$\mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}_e \in \mathbb{C}^{n} . \tag{6}$$

The inverse of the dynamics stiffness matrix, known as the transfer function matrix, is given by

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) \in \mathbb{C}^n \tag{7}$$

Using the residue-calculus the transfer function matrix can be expressed like a viscously damped system as

$$\mathbf{H}(s) = \sum_{j=1}^{m} \frac{\boldsymbol{R}_j}{s - s_j} \tag{8}$$

where m is the number of non-zero eigenvalues (order) of the system. Following Adhikari [1], the residue matrices can be expressed as

$$\mathbf{R}_j = \frac{\mathbf{z}_j \mathbf{z}_j^T}{\mathbf{z}_j^T \frac{\partial D(s_j)}{\partial s_i} \mathbf{z}_j}. \tag{9}$$

Here s_j and $\mathbf{z}_j$ are respectively the eigenvalues and eigenvectors of the system, which are solutions of the non-linear eigenvalue problem

$$\mathbf{D}(s_j)\mathbf{z}_j = \mathbf{0}, \text{ for } j = 1, \cdots, m \tag{10}$$

where

$$\mathbf{D}(s_j) = s_j^2 \mathbf{M} + s_j \mathbf{G}(s_j) + \mathbf{K}_e. \tag{11}$$

This expression of $\mathbf{H}(s)$ in equation (8) allows the response to be expressed as modal summation as

$$\overline{\mathbf{q}}(s) = \sum_{j=1}^{m} y_j \frac{\mathbf{z}_j^T \overline{f}(s) + \mathbf{z}_j^T M\, q_0 + s\mathbf{z}_j^T M\, q_0 + \mathbf{z}_j^T \mathbf{G}(s) q_0(s)}{s - s_j} z_j \tag{12}$$

where

$$y_j = \frac{1}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j}. \tag{13}$$

From equations (8) and (9) it can be immediately seen that conventional modal analysis can be extended to system with viscoelastic materials in a familiar manner provided the nonlinear eigenvalue problem in equation (10) can be solved efficiently. The rest of paper is devoted to address this challenging problem.

3. NON-LINEAR EIGENVALUE PROBLEM FOR VISCOELASTIC SYSTEMS

The eigenvalue problem associated with a linear system with viscoelastic material model can be expressed from equation (10) as

$$\left[s_j^2 \mathbf{M} + s_j \left(\mathbf{K}_{v0} + \sum_{k=1}^{n} \frac{a_k}{s_j + b_k} \mathbf{K}_{vk} \right) + \mathbf{K}_e \right] \mathbf{z}_j = \mathbf{0}, \quad \text{for } j = 1, \ldots, m. \tag{14}$$

For systems with only elastic stiffness matrix the order of the characteristic polynomial $m = 2N$. For viscoelastic systems in general m is more than $2N$, that is $m = 2N + p$; $p > 0$. Using the state-space approach [19] it was shown that $p = \Sigma_{k=1}^{n}$ rank $(\mathbf{K}_v)$ Therefore, if all matrices $\mathbf{K}_{vk}$ are of full rank, then $p = nN$. This shows that although the system has N degrees-of-freedom, the number of eigenvalues is more than $2N$. This is a major difference between a viscoelastic system and an elastic system where the number of eigenvalues is exactly $2N$, including any multiplicities. When s_j appear in complex conjugate pairs, $\mathbf{z}_j$ also appear in complex conjugate pairs, and when s_j is real $\mathbf{z}_j$ is also real. Corresponding to the $2N$ complex conjugate pairs of eigenvalues, the N eigenvectors together with their complex conjugates are called *elastic modes* or *vibration modes*. These modes are related to the N modes of vibration of the structural system. Physically, the assumption of '$2N$ complex conjugate pairs of eigenvalues' implies that all the elastic modes are oscillatory in nature, that is, they are sub-critically damped. The modes

corresponding to the 'additional' p eigenvalues are called *viscous modes* or *overdamped modes*. For stable passive systems the non-viscous modes are over-critically damped (i.e., negative real eigenvalues) and not oscillatory in nature. Non-viscous modes, or similar to these, are known by different names in the literature of different subjects, for example, 'wet modes' in the context of ship dynamics [11] and 'damping modes' in the context of viscoelastic structures [15]. In this paper both the complex conjugate modes and the damping modes will be derived.

For the convenience of analytical development, the following four cases are considered:

- single-degree-of-freedom system with single exponential kernel *(*$N = 1$**,** $n = 1$)
- single-degree-of-freedom system with multiple exponential kernels *(*$N = 1$**,** $n > 1$)
- multiple-degree-of-freedom system with single exponential kernel *(*$N > 1$**,** $n = 1$)
- multiple-degree-of-freedom system with multiple exponential kernels *(*$N > 1$**,** $n > 1$)

In the following sections closed-form approximate expressions of eigenvalues and eigenvectors are derived for the elastic modes and non-viscous modes.

4. SINGLE-DEGREE-OF-FREEDOM SYSTEM WITH SINGLE EXPONENTIAL KERNEL

Computational cost and other relevant issues identified before in the paper do not strictly affect the eigenvalue problem of a single-degree-of-freedom (SDOF) viscoelastic system. The main reason for considering an SDOF system is that in many cases the underlying approximation method can be extended to MDOF systems in a relatively straight-forward manner. For this case when $N = 1$, $n = 1$ the eigenvalue equation can be simplified from equation (14) as

$$s^2 m_u + s g(s) + k_{eu} = 0 \quad \text{where } g(s) = k_{v0} + \frac{a}{s+b} k_{vu}. \tag{15}$$

Here we have omitted the subscripts j and k for notational convenience and the matrices appearing in equation (14) have been replaced by corresponding scalars. Equation (15) is a third-order polynomial in s and it can be solved exactly in closed-form. A more detailed study on the properties of the exact solutions have been carried out in references [5, 6]. In the next two sections we derive the approximate solutions with the vision that they can be generalized to MDOF systems with minor modifications.

4.1. Complex-conjugate Solution

The main motivation of the approximations is that the approximate solution can be 'constructed' from the solution of the equivalent viscously damped system. The solution of the equivalent viscously damped system can in turn be expressed in terms of the undamped eigensolutions. Combining these together, one can therefore obtain the eigensolutions of viscoelastic systems by simple 'post-processing' of the eigensolutions of the equivalent viscously damped system only. The eigenvalues (appearing in a complex conjugate pair) of the equivalent viscously damped system are given by [16]

$$s_0 = -\zeta_n \omega_n \pm \mathrm{i}\omega_n \sqrt{1-\zeta_n^2} \approx -\zeta_n \omega_n \pm \mathrm{i}\omega_n \tag{16}$$

where the undamped natural frequency $\omega_n = \sqrt{k_{eu}/m_u}$ and the viscous damping factor $\zeta_n = (k_{v0} + a k_v/b)$ / $2\sqrt{k_{vu} m_u}$. A viscous damped system is a special case of equation (15) where the function $g(s)$ is replaced by $g(s \to 0)$. For that case the solution given by equation (16) would have been the exact solution of the characteristic equation (15). Since in general this is not the case, the difference between the viscous solution and the true solution of the characteristic equation (15) is essentially arising due to the 'varying' nature of the function $g(s)$. The approximate solutions obtained here are based on keeping this fact in mind.

The central idea here is that the actual solution of the characteristic equation (15) can be obtained by expanding the solution in a Taylor series around s_0. The error arising in the resulting solution would then depend on the 'degree of variability' of the function $g(s)$. We assume that the true solution of equation (15) can be expressed as

$$s = s_0 + \delta \tag{17}$$

where δ is a small quantity. Substituting this into the characteristic equation we have

$$(s_0+\delta)^2 m_u + (s_0+\delta)g(s_0+\delta) + k_{e_u} = 0. \tag{18}$$

Expanding $g(s_0+\delta)$ in a Taylor series in δ around s_0 and keeping only the first-order terms in δ we have

$$\delta^{(1)} = -\frac{s_0(s_0 m_u + g(s_0)) + k_{e_u}}{s_0(2m_u + g'(s_0)) + g(s_0)} \tag{19}$$

where

$$g(s_0) = k_{v_0} + \frac{a}{s_0+b}k_{v_u} \quad \text{and} \quad g'(s_0) = -\frac{a}{(s_0+b)^2}k_{v_u}. \tag{20}$$

Here the superscript 1 is used to denote that this is a first-order approximation. Further simplifying, the final solution can be expressed as

$$s = s_0 + \delta^{(1)} = s_0 - \frac{s_0(s_0 m_u + k_{v_0} + k_{v_u}\frac{a}{b+s_0}) + k_{e_u}}{s_0(2m_u - \frac{a}{(b+s_0)^2}k_{v_u}) + k_{v_0} + \frac{a}{b+s_0}k_{v_u}}. \tag{21}$$

One can improve the accuracy by retaining higher-order terms in δ. Retaining up to second-order terms in δ in the Taylor expansion of equation (18) we have

$$\delta^{(2)} = \frac{-B-\sqrt{B^2-4AC}}{2A} \tag{22}$$

where

$$A = (m_u + \frac{g''(s_0)}{2!}s_0 + g'(s_0)) \tag{23}$$

$$B = (2m_u s_0 + s_0 g'(s_0) + g(s_0)) \tag{24}$$

$$\text{and} \quad C = (s_0^2 m_u + s_0 g(s_0) + k_{e_u}). \tag{25}$$

In the above expressions $g'(s_0)$ and $g''(s_0)$ are respectively the first and second order derivative of $g(s)$ evaluated at $s = s_0$. Our numerical works show that retaining terms higher than the second order results in considerably complex expressions and the accuracy gained is not very significant. As a result we have not perused this approach in the rest of the paper. Based on our numerical works we recommend the second-order expression in equation (22) as it gives excellent accuracy and additional computation cost is just marginally higher compared to the first-order approximation. The expressions of the approximate eigenvalue derived here show that the complex-conjugate solutions of a general viscoelastic system can be obtained by post-processing of the undamped eigenvalue ω_n and equivalent viscous damping factor ζ_n.

4.2. Real Solution

While the complex-conjugate solution can be expected to be close to the solution of the equivalent viscously damped system, no such analogy can be made for the real solution as the equivalent viscously damped system doesn't have one. In order to proceed, we first multiply the characteristic equation (15) by $(s+b)$ and rewrite as

$$(s^2 m_u + s k_{v_0} + k_{e_u})(b+s) + sak_{v_u} = 0. \tag{26}$$

To obtain the initial guess we consider that the damping is small so that $sak_{v_u} \approx 0$. Since $(s^2 m_u + s k_{v_0} + k_{e_u}) \neq 0$ as we are considering the real solution only, the first guess is obtained as

$$b + s_0 = 0 \quad \text{or} \quad s_0 = -b. \tag{27}$$

We take the first approximation of the real root as

$$s = s_0 + \Delta = -b + \Delta \tag{28}$$

and substitute it into the characteristic equation to obtain

$$((-b+\Delta)^2 m_u + (-b+\Delta)k_{v_0} + k_{e_u})\Delta + (-b+\Delta)ak_{v_u} = 0. \tag{29}$$

After neglecting all the terms associated with Δ^n for $n > 1$ we have

$$\Delta \approx \frac{abk_{v_u}}{b^2 m_u - bk_{v_0} + k_{eu} + ak_{v_u}}. \tag{30}$$

Therefore, the real solution is given by

$$s = s_0 + \Delta \approx -b + \frac{abk_{v_u}}{b^2 m_u - bk_{v_0} + k_{e_u} + ak_{v_u}}. \tag{31}$$

5. SINGLE-DEGREE-OF-FREEDOM SYSTEM WITH MULTIPLE EXPONENTIAL KERNELS

For this case the kernel function $g(s)$ takes the form

$$g(s) = k_{v_o} + \sum_{k=1}^{n} \frac{a_k}{s+b_k} k_{v_k}, \quad n > 1. \tag{32}$$

Using this series, a wide range of viscoelastic functions can be modelled in the frequency domain. The characteristic function with this type of kernel function can be expressed as

$$s^2 m_u + s\left(k_{v_o} + \sum_{k=1}^{n} \frac{ak}{s+b_k} k_{v_k}\right) + k_{e_u} = 0. \tag{33}$$

This is a polynomial in s of order $(n + 2)$ and therefore, this equation has $(n + 2)$ roots. In this section we derive approximate solutions with the view of generalizing them to more realistic MDOF systems.

5.1. Complex-conjugate Solution

If the system is vibrating, then equation (33) must have a pair of complex conjugate root. The complex conjugate solutions derived in 4.1 are for a general function $g(s)$ and therefore are valid for this case also. The solutions are therefore given by equations (17) and (19), where $g(s)$ appearing in these equations needs to be replaced by the series in equation (32). The necessary details are obvious and consequently omitted here.

5.2. Real Solutions

There are n number of pure real roots corresponding to n terms in the series in equation (32). Multiplying the characteristic equation (33) by the product $\prod_j^n (s + b_j)$ we have

$$(s^2 m_u + sk_{v_0} + k_{e_u}) \prod_{j=1}^{n} (s+b_j) + s \sum_{k=1}^{n} (a_k k_{v_k}) \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j + s) = 0. \tag{34}$$

Like the previous case, we use the approximation

$$s_k = -b_k + \Delta_k. \tag{35}$$

Substituting this into the characteristic equation and retaining only the first-order terms in Δ_k, after some simplifications we have

$$\Delta_k \approx \frac{b_k a_k k_{vk} p_1}{\left[b_k^2 m_u - b_k k_{v_0} + k_{e_u}\right] p_1 + [-b_k (p_2 + p_3) + a_k k_{v_0} p_1]} \tag{36}$$

$$p_1 = \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j - b_k), \quad p_2 = a_k k_{v_k} \sum_{\substack{j=1 \\ j \neq k}}^{n} \prod_{\substack{r=1 \\ r \neq j \\ r \neq k}}^{n} (b_r - b_k) \tag{37}$$

$$\text{and } p_3 = \sum_{\substack{j=1 \\ j\neq k}}^{n} a_j k_{vj} \prod_{\substack{r=1 \\ r\neq j,k}}^{n} (b_r - b_k).$$

5.3. Numerical Results

We consider a single-degree-of-freedom system with eight exponential kernels to investigate the accuracy of the approximate solutions derived in this section. For numerical calculation we assume m_u = 1 kg, k_{eu} = 2 N/m, k_{v_0} = 0 kg/s. It is assumed that all $a_k k_{v_k}/b_k$ are of same value so that $a_k k_{v_k} = 2\zeta\omega_n$,

Table 1. Exact and approximate eigenvalues of the SDOF system.

μ	Exact solution (state-space)	Proposed approximate solution	Percentage error
1.9442	–1.4649	–1.5135	3.3169
1.5231	–1.5136	–1.5185	0.3237
1.9317	–1.7123	–1.7579	2.6643
1.7657	–1.7576	–1.7613	0.2101
1.7454	–1.8954	–1.9253	1.5767
1.9558	–1.9380	–1.9375	0.0282
2.0677	–1.9517	–1.9527	0.0516
1.4973	–2.0560	–2.0592	0.1559
Complex	–0.0619	–0.0619	0.0003
Conjugate	±1.4718i	±1.4718i	±0i
solution			

$\forall k$ = 1, 2.., 8. The values of b_k for k = 1, 2.., 8 are selected as 1.9442, 1.5231, 1.9317, 1.7657, 1.7454, 1.9558, 2.0677, 1.4973. The values are generated randomly as $b_k = 0.75(1 + r_k)\omega_n$, where r_k are uniform random numbers between 0–1. The approximate eigenvalues obtained using the proposed method is compared with the results obtained from exact state-space solution in Table 1. The complex conjugate eigenvalues are obtained very accurately using the proposed approximation. The real eigenvalues are

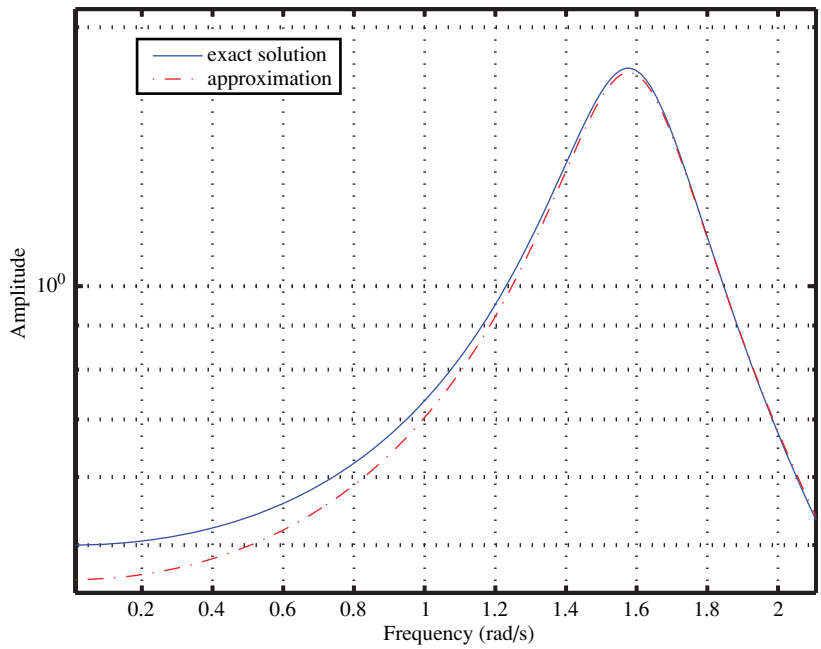

Figure 1. Frequency response function of the SDOF system obtained using exact and approximate eigenvalues

not as accurate as the complex conjugate eigenvalues. However, recall that the motion corresponding to the real eigenvalues are purely dissipative in nature and therefore do not significantly affect the dynamic response of the system. To show this, we have plotted the frequency response function of the system in Figure 1. Because the complex conjugate eigenvalues are approximated very well, the frequency response function of the system obtained using the exact and approximate eigenvalues match very well. Again recall that all of the approximate eigenvalues are obtained by post-processing of the undamped eigenvalues only.

6. MULTIPLE-DEGREE-OF-FREEDOM SYSTEM

Based on the results for the SDOF system, we derive the complex eigenvalues and eigenvectors for a general $\mathbf{G}(s)$. Complex eigensolutions are also known as elastic eigensolutions as they appear in underdamped elastic systems. For the real eigenvalues two cases, namely, when $n = 1$ and when $n > 1$, are considered for analytical convenience.

6.1. Complex-conjugate Solutions

The aim of this section is to obtain the complex conjugate eigensolutions using the elastic eigensolutions. The undamped eigenvalue problem of a MDOF system is given by

$$K_e x_j = \omega_j^2 M x_j \tag{38}$$

where ω_j^2 and $\mathbf{x}_j$ are the eigenvalues and eigenvectors of the system. The eigenvectors are mass-normalized so that

$$\mathbf{x}_l^T \mathbf{M} \mathbf{x}_j = \delta_{lj} \tag{39}$$

$$\text{and } \mathbf{x}_l^T \mathbf{K}_e \mathbf{x}_j = \omega_j^2 \delta_{lj}, \forall l, j = 1,\ldots,N \tag{40}$$

where δ_{lj} is the Kroneker delta function. For distinct undamped eigenvalues (ω_j^2), $\mathbf{x}_l$, $\forall l = 1,\ldots,N$, form a *complete* set of vectors. For this reason, $\mathbf{z}_j$ can be expanded as a complex linear combination of $\mathbf{x}_l$. Thus, an expansion of the form

$$\mathbf{z}_j = \sum_{l=1}^{N} \alpha_l^{(j)} \mathbf{x}_l \tag{41}$$

may be considered. Now, without any loss of generality, we can assume that $\alpha_j^{(j)} = 1$ (normalization) which leaves us to determine $\alpha_l^{(j)}$, $\forall l \neq j$. Substituting the expansion of $\mathbf{z}_j$, from equation (10) one obtains

$$\sum_{l=1}^{N} s_j^2 \alpha_l^{(j)} \mathbf{M} \mathbf{x}_l + s_j \alpha_l^{(j)} \mathbf{G}(s_j) \mathbf{x}_l + \alpha_l^{(j)} \mathbf{K} \mathbf{x}_l = \mathbf{0}. \tag{42}$$

Premultiplying the above equation by $\mathbf{x}_k^T$ and using the mass-orthogonality property of the undamped eigenvectors one obtains

$$s_j^2 \alpha_k^{(j)} + s_j \sum_{l=1}^{N} \alpha_l^{(j)} G'_{kl}(s_j) + \omega_k^2 \alpha_k^{(j)} = 0, \forall k = 1,\ldots,N \tag{43}$$

where $G'_{kl}(s_j) = \mathbf{x}_k^T \mathbf{G}(s_j) \mathbf{x}_l$. We consider that the off-diagonal entries of the viscous stiffness function matrix are small compared to the diagonal entries, that is $G'_{kl}(s_j) \leq G'_{kk}(s_j)$, $\forall k \neq l$. Considering the j-th set of equation (43) and neglecting the second-order terms involving $\alpha_k^{(j)}$ and $G'_{kl}(s_j)$, $\forall k \neq l$, and also noting that $\alpha_k^{(j)} = 1$, one obtains

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 \approx 0. \tag{44}$$

This equation is similar to (15) and can be solved in exactly the same way.

To obtain the eigenvectors we rewrite Eq. (43) for $j \neq k$ as

$$\begin{aligned} &s_j^2 \alpha_k^{(j)} + s_j \left(G'_{kj}(s_j) + \alpha_k^{(j)} G'_{kk}(s_j) + \sum_{l \neq k \neq j}^{N} \alpha_l^{(j)} G'_{kl}(s_j) \right) + . \\ &\omega_k^2 \alpha_k^{(j)} = 0, \forall k = 1,\ldots,N; \neq j. \end{aligned} \tag{45}$$

Keeping only the first-order terms in G'_{kj}, and following [1], we obtain

$$\mathbf{z}_j \approx \mathbf{x}_j - \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)}. \tag{46}$$

Retaining the second-order terms in G'_{kj} a more accurate expression is obtained

$$\begin{aligned}\mathbf{z}_j \approx \mathbf{x}_j &- \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j^2 G'_{kl}(s_j)\mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_{k}(s_j)} \\ &+ \sum_{\substack{k=1 \\ k\neq j}}^{N} \sum_{\substack{l=1 \\ l\neq j\neq k}}^{N} \frac{s_j^2 G'_{kl}(s_j) G'_{lj}(s_j)\mathbf{x}_k}{\left(\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)\right)\left(\omega_l^2 + s_j^2 + s_j G'_{ll}(s_j)\right)}.\end{aligned} \tag{47}$$

The above equation is the second-order approximate expression of the complex eigenvectors of the system.

6.2. Real Solutions

The real solutions are obtained using an approach similar to SDOF systems. After neglecting the off-diagonal terms of the viscoelastic stiffness matrices, the governing characteristic equation for every mode can be expressed by equation (44).

This equation can be solved for the real eigenvalues. For systems with single exponential kernel, following equations (28)-(31), the real eigenvalues can be expressed as

$$s_j = -b + \Delta_t \tag{48}$$

where

$$\Delta_t \approx \frac{abK_{v_{ll}}}{b^2 + \omega_l^2 + \alpha K_{v_{ll}} - bK_{0_{ll}}}; \quad \forall l = 1,2,...,N. \tag{49}$$

Assuming all coefficient matrices are of full rank, for systems with n kernels there are in general nN number of purely real eigenvalues. The approximate eigenvalues can be obtained using equations (35) and (36) as

$$s_{lk} = -b_k + \Delta_{lk} \tag{50}$$

where

$$\Delta_k = \frac{b_k a_k K_{v_{kll}} p_1}{\left[b_k^2 - b_k K_{v_{0ll}} + \omega_l^2\right] p_1 + \left[-b_k(p_2 + p_3) + a_k k_{v_{0ll}} p_1\right]} \tag{51}$$

with

$$\begin{aligned} p_1 &= \prod_{\substack{j=1 \\ j\neq k}}^{n} (b_j - bk), \;\; p_2 = a_k K_{v_{kll}} \sum_{\substack{j=1 \\ j\neq k}}^{n} \prod_{\substack{r=1 \\ r\neq j \\ r\neq k}}^{n} (b_r - b_k) \\ p_3 &= \sum_{\substack{j=1 \\ j\neq k}}^{n} a_j K_{v_{jll}} \prod_{\substack{j=1 \\ r\neq j,k}}^{n} (b_r - b_k) \end{aligned} \tag{52}$$

and

$$K_{v_{kll}} = \mathbf{x}_l^T \mathbf{K}_{v_{kll}} \mathbf{x}_l, \quad K_{v_{0ll}} = \mathbf{x}_l^T \mathbf{K}_{v_{0ll}} \mathbf{x}_l. \tag{53}$$

If viscoelastic stiffness matrices of the system are not full rank, equations to approximate the real eigenvalues can still be applied if a dimension reduction process is applied to equation (14). An eigenvector matrix $\mathbf{Y}_l$ is calculated for each viscous matrix $\mathbf{K}_{v_l}$, mathematically,

$$\mathbf{Y}_l^T \mathbf{K}_{v_l} \mathbf{Y}_l = \Lambda_l. \tag{54}$$

Where Λ_l are diagonal matrices of dimension rank ($\mathbf{K}_{vl}$), and matrices $\mathbf{Y}_l$ are rectangular matrices of dimensions $N \times$ rank ($\mathbf{K}_{vl}$) . Now, from the equilibrium equation (14) we have

$$\mathbf{Y}_l^T \left[s_j^2 \mathbf{M} + s_j \left(\mathbf{K}_{v_0} + \sum_{k=1}^{n} \frac{ak}{s_j + b_k} \mathbf{K}_{v_k} \right) + \mathbf{K}_e \right] \mathbf{Y}_l \mathbf{y}_{lj} = \mathbf{0}. \tag{55}$$

Using the diagonalisation in Eq. (54) this can be simplified to

$$\left[s_j^2 \hat{\mathbf{M}}_l + s_j \left(\hat{\mathbf{K}}_{v_{0l}} + \sum_{\substack{k=1 \\ k \neq l}}^{n} \frac{a_k}{s_j + b_k} \hat{\mathbf{K}}_{vkl} + \frac{a_l}{s_j + b_l} \Lambda_l \right) + \hat{\mathbf{K}}_{et} \right] y_{lj} = \mathbf{0}. \tag{56}$$

Where

$$\begin{aligned} &\mathbf{Y}_l \mathbf{y}_{lj} = \mathbf{z}_j, \hat{\mathbf{M}}_l = \mathbf{Y}_l^T \mathbf{M}_l \mathbf{Y}_l, \hat{\mathbf{K}}_{v0l} \mathbf{Y}_l \\ &\hat{\mathbf{K}}_{v_{kl}} = \mathbf{Y}_l^T \mathbf{K}_{v_{kl}} \mathbf{Y}_l \text{ and } \hat{\mathbf{K}}_{e_l} = \mathbf{Y}_l^T \mathbf{K}_{el} \mathbf{Y}_l \end{aligned} \tag{57}$$

Eigenvalues are then obtained using the derived equations (48) and (49) to the reduced system, as it has been explained for the full rank case. We note here that the number of real solutions obtained has to be $p = \Sigma_{k=1}^{n}$ rank ($\mathbf{K}_{vk}$). Here, superficially, the number of roots we can obtain is $n\ \Sigma_{k=1}^{n}$ rank ($\mathbf{K}_v$). That is, n roots for each nth order polynomial equation multiplied by $\Sigma_{k=1}^{n}$ rank ($\mathbf{K}_v$) number of equations. However, recall that, each preconditioning matrix $\mathbf{Y}_l$ is specific to a matrix $\mathbf{K}_{v_l}$, related to a particular value b_l. Therefore, from each nth order polynomial equation only one eigenvalue is kept as the correct one. This is the eigenvalue whose first approximation is $s_0 = -b_l$. In this way, the number of eigenvalues obtained with this method matches the true number of real eigenvalues $p = \Sigma_{k=1}^{n}$ rank ($\mathbf{K}_v$).

The calculation of eigenvectors in the reduced system follows equation (46). After calculating $\mathbf{y}_{lj}$, the original system corresponding eigenvector is recovered with $\mathbf{z}_j = \mathbf{Y}_l \mathbf{y}_{lj}$. A set of approximated real eigenvalues and eigenvectors is then obtained. After calculating the complex conjugate eigenvalues and eigenvectors obtained from the original system following equations (17) and (19), an approximation to the exact eigenvalues and eigenvectors can be obtained for all modes.

6.3. Numerical Results

We consider a three degrees-of-freedom system to illustrate the proposed method. The mass, elastic stiffness and the viscous stiffness matrices in the Laplace domain for the problem are considered as:

$$\boldsymbol{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \mathbf{K}_e = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{58}$$

$$\mathbf{G}(s) = \sum_{k=1}^{2} \frac{a_k}{s + b_k} K_{v_k} \tag{59}$$

For the numerical value we consider m_u = 3 kg, k_u = 2 and the values of b_1 = 1 and b_2 = 5. The approximate eigenvalues obtained using the proposed method are compared with the results obtained

Table 2. Exact and approximate eigenvalues of the Three-DOF system.

Exact solution (state-space)	Proposed approximate solution	Percentage error
Real solutions		
–0.8649	–0.8873	2.60
–0.9324	–0.9357	0.35
–4.8744	–4.9116	0.76
Complex Conjugate solutions		
–0.0559± 0.6628i	–0.0581 ± 0.6613i	0.20 ± 0.04i
–0.0402 ± 1.1838i	–0.0375 ± 1.1837i	0.02 + 0.23i
–0.0680 ± 1.5569i	–0.0682 ± 1.5577i	–0.05 ± 0.01i

Table 3. Exact and approximate eigenvectors of the Three-DOF system.

z_j	Exact solution	Approximation	Percentage error
1^{st}	$1.4329 \pm 0.0383i$	$1.4329 \pm 0.0387i$	$-0.00 \pm 0.03i$
	$1.0593 \pm 0.1342i$	$1.0577 \pm 0.1331i$	$0.16 \pm 0.08i$
2^{nd}	$0.0762 \pm 0.0100i$	$0.0731 \pm 0.0145i$	$3.26 \pm 6.35i$
	$-0.9582 \pm 0.1168i$	$-0.9621 \pm 0.1124i$	$-0.34 \pm 0.51i$
3^{rd}	$-1.4139 \pm 0.1757i$	$-1.4202 \pm 0.1686i$	$-0.38 \pm 0.55i$
	$0.9235 \pm 0.1443i$	$0.9240 \pm 0.1382i$	$0.05 \pm 0.66i$
4^{th}	1.2021	1.0000	16.82
	0.3567	0	100.00
5^{th}	−0.8359	−0.9468	−13.27
	−0.2320	0	100.00
6^{th}	0.0263	0	100
	−1.0096	−1.0000	0.95

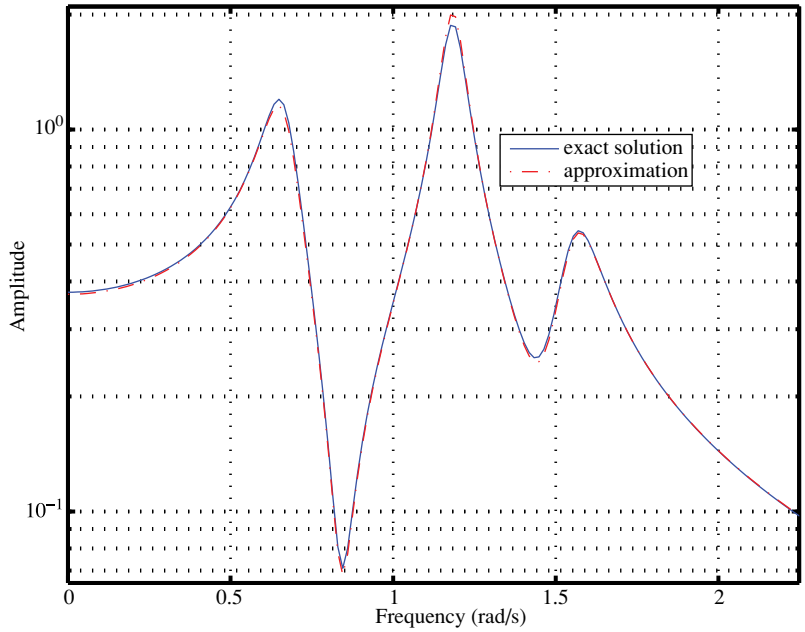

Figure 2. Frequency response function of the MDOF system obtained using exact and approximate eigenvalues

from exact state-space solution in Table 2. The complex conjugate eigenvalues are obtained very accurately using the proposed approximation. The real eigenvalues are not as accurate as the complex conjugate eigenvalues. However, recall that the motion corresponding to the real eigenvalues are purely dissipative in nature and therefore do not significantly affect the dynamic response of the system.

A comparison of eigenvectors is available at Table 3. Each exact and approximated eigenvector is divided by one of the terms of the eigenvector, here, the first term. If the first term of one of the eigenvectors is zero, the normalization is done with the second term (this only happened for the sixth eigenvector). Only the two different terms of each eigenvector are given. The accuracy of the approximation is represented in Figure 2

7. CONCLUSIONS

Multiple degrees-of-freedom viscoelastic linear systems are considered. It has been assumed that, in general, the mass and elastic stiffness matrices as well as the matrix of the kernel functions cannot be simultaneously diagonalized by any linear transformation. The analysis is restricted to systems with non-repetitive eigenvalues. The transfer function matrix of the system was derived in terms of the

eigenvalues and eigenvectors of the second-order system. Exact closed-form expressions of the response due to arbitrary forcing functions and initial conditions were obtained.

The calculation of the eigensolutions of viscoelastic systems requires the solution of a non-linear eigenvalue problem. In this paper new methods are developed for such eigenvalue problems. Approximate expressions are derived for the complex and real eigenvalues of the SDOF system with single and multiple exponential kernels. These results are then extended to MDOF systems. These approximations allow one to obtain the dynamic response of general viscoelastic systems by simple post-processing of undamped eigensolutions. The accuracy of the proposed approximations were verified using numerical examples. The complex conjugate eigensolutions turn out to be more accurate compared to the real eigensolutions. This is particularly encouraging because complex eigensolutions dominate the dynamic response of linear systems. The method presented offers a reduction in computational effort because neither the state-space formalisms nor the additional dissipation coordinates are employed. This approach might provide further physical insight as only familiar undamped natural frequencies and mode shapes are utilized to obtain the eigensolutions and dynamic response of the system.

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