# MODAL ANALYSIS OF LINEAR ASYMMETRIC NONCONSERVATIVE SYSTEMS[a]

## Discussion by Donald L. Cronin[2]

In this paper Adhikari describes interesting and possibly useful work. There are several matters discussed by him, however, that I believe deserve comment.

The method for eigenvector determination the author describes is approximate. This should have been clearly stated. The author might respond that the use of a Galerkin approach implies an approximation, but this is not adequate. The language employed, moreover, suggests that any accuracy desired for the analysis can be achieved by including additional terms in the associated Neumann series. This is not true, since adding terms to the series will not improve the fundamental approximation. A discussion of the approximation used in the author's analysis and a characterization of its associated error deserve treatment.

Of interest at present, however, is the author's observation that the perturbation-based analysis of Malone et al. (1997) for the eigenanalysis of asymmetric nonconservative systems is approximate. This approach for eigenanalysis—also discussed in Cronin (1988, 1990) and Peres-da-Silva et al. (1995)—is exact. It is exact because all of the terms in the series description for the eigenvalues and eigenvectors were given. Here I use the term ''perturbation-based'' rather than ''perturbation'' to characterize the above work, because the perturbation quantity is eventually set to one in Peres-da-Silva et al. and Malone et al. Thus, the resulting series are not power series in the perturbation quantity, and series convergence was not simply assumed for sufficiently small values of the perturbation quantity. Rather, the convergence of the resulting series was rigorously studied in these papers, and convergence criteria were derived that are suitable for a priori estimation of how much numerical work is required to achieve a particular accuracy for the eigenanalysis of a given dynamic system.

The author's observation that a perturbation-based approach is less suitable for determining system eigenvalues than ''several efficient numerical methods'' is unsupported. The suggested approach—solving a high-order characteristic polynomial to determine eigenvalues—is not regarded as a practical solution method for these problems. Compared with generally formulated numerical analysis schemes, a computational approach based on a specifically tailored analysis will usually be the better choice (the perturbation-based analysis discussed in the above papers is one such). It should be especially useful in view of the associated eigenpair-by-eigenpair a priori convergence estimation capability. The advantage of this analysis should be particularly good if eigenvectors are also a desired output.

Finally, symmetry of the damping matrix was not assumed in Peres-da-Silva et al. One of the examples worked in that paper was a gyroscopic system, and the authors claimed, as has Adhikari, that nothing in their analysis excludes its application to a wide and general class of dynamic systems.

The discusser appreciates the opportunity to comment on the author's paper and to review the record. It is the discusser's belief that this work and the work of others—several are mentioned here, and the nature and extent of their contributions are, I hope, clarified—have produced worthwhile progress in the development of eigenanalysis techniques for nonclassically damped dynamic systems.

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Peres-da-Silva, S. S., Cronin, D. L., and Randolph, T. W. (1995). ''Computation of eigenvalues and eigenvectors on nonclassically damped systems.'' *Comp. and Struct.* 57, 883–891.


## Closure by Sondipon Adhikari,[3] Member, ASCE

The writer wishes to thank the discusser, D. L. Cronin, for his interest and useful comments on the paper. The following is the writer's response to his comments.

The writer disagrees with the discusser's main argument, that the method for eigenvector determination described in the paper is approximate. Motivated by his comment, it was observed that (11) of the paper, from which the rest of the eigenanalysis follows, is in fact exact. To show this, first consider the right eigenvalue problem. Substitute the assumed expansion of the right eigenvectors ($\mathbf{u}_i$) in (9*a*) into the eigenvalue (3). Premultiplying the resulting expression by $\mathbf{x}_{\mathbf{v}_k}^T$ (*k*th undamped left eigenvector) and using the biorthogonality relationship of the undamped right and left eigenvectors, one obtains (11). This approach for deriving (11) is clearly exact, although in the original paper it was derived using a Galerkin approach. A similar argument holds for left eigenvectors. Thus, in contrast to the discusser's opinion, the method for determination of right and left eigenvectors of asymmetric nonconservative systems developed in the paper is exact to the extent of number of terms used in the Neumann series.

Perturbation-based methods for eigenanalysis, as cited by the discusser, were termed approximate in the spirit of comparing them with the exact state-space solutions, and that infinite series assumed for the eigenvalues and eigenvectors needs to be truncated for practical purposes. The writer, however, agrees with the discusser that eigensolutions obtained by the use of the series expansion provided by him and his coworkers [e.g., (25) and (27) in Peres-Da-Silva et al. 1995] will approach the exact solutions if sufficient terms are retained. Notwithstanding, it should be noted that these expressions tend more toward ''numerical'' in nature as opposed to those provided by the writer [e.g., (77) and (78) of the paper], which are more ''physical'' in nature. From these expressions one easily observes that ''complexity'' of a given mode is a function of coupling with the other modes through the off-diagonal entries of the modal damping matrix and separation of the complex natural frequencies. In fact, the matrix appears in the Neumann series ($\mathbf{R}_u^{(i)}$) can be regarded as a measure of modal coupling—if $\|\mathbf{R}_u^{(i)}\|$ is bigger, i.e., if modal coupling is higher, more terms are required for convergence of the series and vice versa. This is essentially equivalent to the convergence criteria mentioned by the discusser, which gives an a priori estimate of the required number of terms to be used in the series expressions.

The writer agrees with the discusser that a specifically tailored method will probably perform better than a generally formulated numerical method for solving the characteristic

[a]December 1999, Vol. 125, No. 12, by Sondipon Adhikari (Paper 21056).


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polynomial. Thinking along this line, the writer feels that ''search algorithms'' used by the polynominal solvers can be made quite efficient with a good initial guess, such as the one obtained from a first-order perturbation method [(76) of the paper]. Observe that the effect of the off-diagonal terms of the modal damping matrix ($\Gamma_1$ in discusser's citations) is of second-order and higher while determining the eigenvalues. Thus, again, based on the norm of $\Gamma_1$ one could come up with a convergence criteria for the iterative schemes that the search algorithms normally use. The (infinite) series expansion approach for eigenvalue determination, cited by the discusser, has the following potential disadvantage. Determination of any *i*th term in this series [see (25) in Peres-Da-Silva et al. 1995] requires the information of *all* (*i* 1) terms of the corresponding series for *eigenvectors* as well as eigenvalues. For large systems, and if higher-order terms are required, this approach not only requires more memory but also requires additional computation time to calculate the terms from the corresponding eigenvector series. Thus, if one's sole interest is in eigenvalues, which approach is numerically most efficient is questionable—a detailed calculation of ''flops'' may be useful. From this discussion what follows is that providing closed-form expressions of general terms in the infinite series for eigenvalues and eigenvectors does not naturally lead the approach to be numerically efficient.

Turning to the discusser's final remark, the writer was aware of the works by Peres-Da-Silva et al. (1995) and Malone et al. (1997) on damped gyroscopic systems where the damping matrix no longer remains symmetric. However, the claim that this approach can be extended to a more general class of asymmetric nonconservative systems is not obvious. One example is when $\mathbf{M}^{-1}\mathbf{K}$ is not symmetrizable [see Inman (1983) for the definition of symmetrizability]. In this case ''undamped'' eigenvalues and eigenvectors would be complex. In the paper under discussion it was implicitly assumed that $\mathbf{M}^{-1}\mathbf{K}$ is symmetrizable. Later it was noted (Adhikari, unpublished, 2000) that this restriction is not necessary, and with little modifications the approach can be extended to this general case. Successful extension of the works cited by the discusser to nonsymmetrizable asymmetric nonconservative systems requires further research and is worth pursuing.

I again thank the discusser for giving me an opportunity to clarify some of the important issues regarding eigenanalysis of asymmetric nonconservative systems, which otherwise wouldn't have been brought out.

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# NONLINEAR BUCKLING AND POSTBUCKLING OF CABLE-STIFFENED PRESTRESSED DOMES[a]


## Discussion by Robert Schmidt,[3] Member, ASCE


It is interesting that the rather unusual problem of the hinged-clamped circular arch with the central angle $\theta = 215°$—employed by the authors for the purpose of comparing their calculated results with those obtained by Simo and Vu-Quoc (1986), and thus validating their own discrete method—has been the subject of several other nonlinear analyses. A very accurate analysis (DaDeppo and Schmidt 1975; Schmidt and DaDeppo 1977), described in greater detail by DaDeppo and Schmidt (1974), confirms the authors' results, where possible. For the central angle $\theta = 2\alpha = 215°$, Table 1 in the 1975 reference presents the following critical values: $PR^2/EI = 8.97$, $u/R = 0.6116$, $v/R = 1.1372$, and $\beta = 0.0498$, where $P$ is the downward point load at the crown, $u$ the horizontal deflection, $v$ the vertical deflection, and $\beta$ the rotation of the crown of the arch (the critical load is the so-called upper-limit load here). The authors' values, as scaled off the not-very-accurate graph in Fig. 4, are $PR^2/EI = 9.17$, $u/R = 0.62$, and $v/R = 1.15$.

The same arch problem has also been analyzed by means of the finite-element method by Wood and Zienkiewicz (1976), and their results have also been presented by Zienkiewicz (1977); but these results are less accurate than those obtained by DaDeppo and Schmidt (1975). However, Zienkiewicz's (1977) Fig. 19.8 seems to coincide, at least qualitatively, with the drawing of the arch in the authors' Fig. 4 (observe the inaccuracy at the clamped end).

The discusser surmises that the stability problems of hinged-clamped arches with the values of the central angle $\theta$ in the range between $\theta = 205°$ and the authors' $\theta = 215°$ have been of interest because of the apparent change of the buckling mode at $\theta \approx 210°$ (DaDeppo and Schmidt 1975)—and the mode shape is of the greatest importance in the intended application.

The significance of the mode changes in geometrically or physically asymmetrical arches with large prebuckling deflections has not been thoroughly investigated yet, even though the first (experimental) investigation by Deutsch took place as early as 1940. Deutsch (1940) induced asymmetry by subjecting two-hinged and hingeless parabolic arches to asymmetrically stepped distributed loads. According to Austin (1971), the exact nature of the buckling phenomenon was not clear.

In the case of the hingeless (geometrically symmetric clamped) arches, the buckling mode shapes are easy to visualize and describe: the snap-through mode changes to the sidesway mode at about $\theta = 270°$ for the central angle (Schmidt and DaDeppo 1972). However, in the case of the asymmetric supports chosen by the authors, the buckling mode shapes are not easily deduced without extensive calculations. Do both modes represent different categories of sidesway?

## APPENDIX. REFERENCES


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[a]October 1999, Vol. 125, No. 10, by Vinasithamby Ragavan and Amde M. Amde (Paper 18726).


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## Closure by Vinasithamby Ragavan,[4] Member, ASCE, and Amde M. Amde,[5] Fellow, ASCE

The writers would like to thank the discusser for his interest in this paper. The writers' intention in using the result of Simo and Vu-Quoc (1986) was simply to validate their algorithm implemented in a nonlinear program, which has been developed alongside this research. In response to the discussion comments, the writers have made additional analyses of the steep circular arches with hinged-clamped support condition. The subtending angles used for two different arch geometries are 205° and 210°, and the corresponding load-deformation curves are shown in Figs. 11 and 12, respectively. The analyses show that the two arch geometries, and the 215° arch shown in Fig. 4, buckle in the same mode. The buckling loads of the arches with subtending angles 205°, 210°, and 215° agree with the discusser's results (DaDeppo et al. 1975).

### APPENDIX. REFERENCE


DaDeppo, D. A., and Schmidt, R. (1975). "Instability of clamped-hinged circular arches subjected to a point load." *J. Appl. Mech.*, 42(4), 894–896.


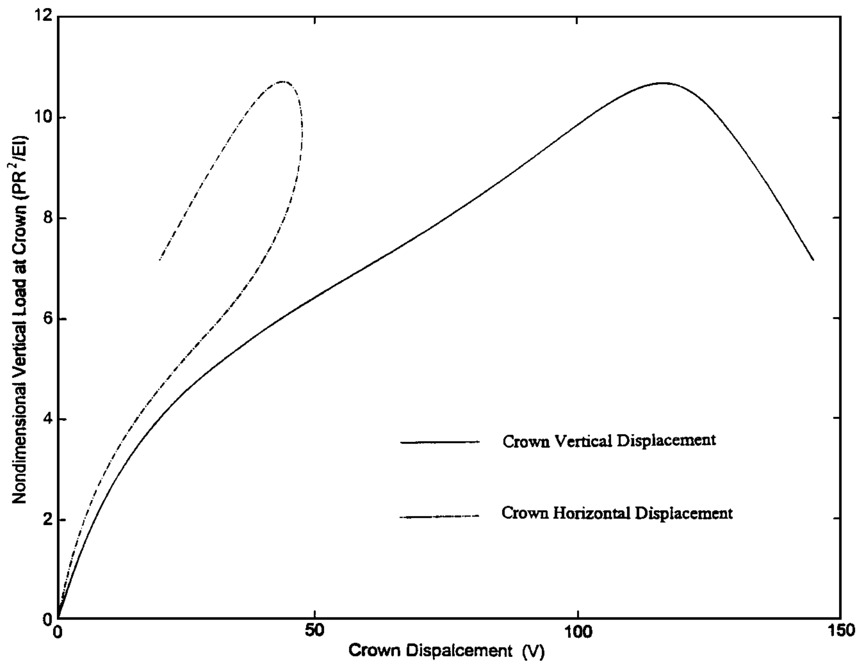


**FIG. 11. Load Deflection Curves for Deep Circular Arch with Subtending Angle 205°**

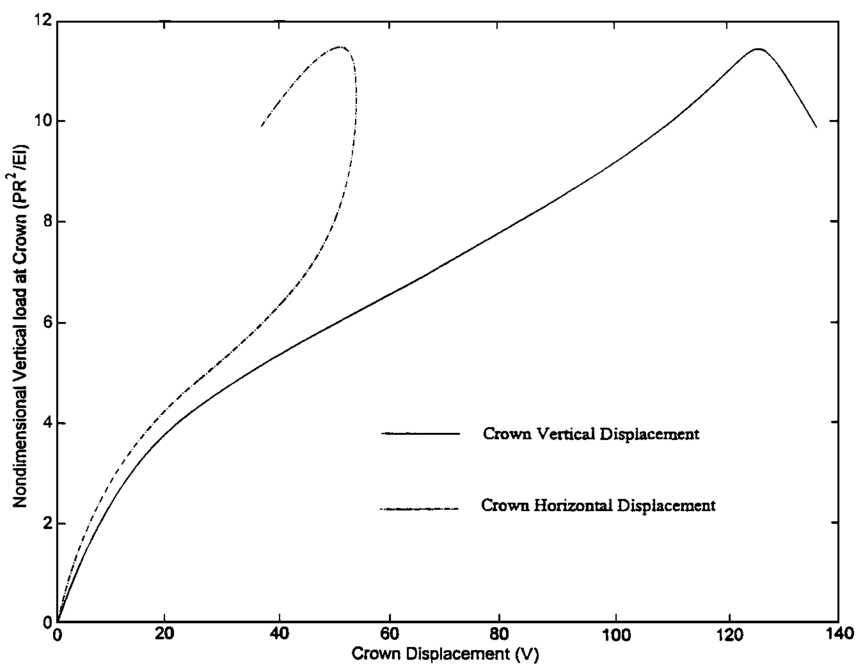


**FIG. 12. Load Deflection Curves for Deep Circular Arch with Subtending Angle 210°**


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