

MODAL ANALYSIS OF LINEAR ASYMMETRIC NONCONSERVATIVE SYSTEMS

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ABSTRACT: In this work, classical modal analysis has been extended to treat lumped parameter asymmetric linear dynamic systems. In the presence of general nonconservative forces, the damping matrix is not simultaneously diagonalizable with the mass and stiffness matrices. The proposed method utilizes left and right eigenvectors of the second-order system and does not require conversion of the equations of motion into the first-order form. Left and right eigenvectors of the nonconservative system are derived in terms of the left and right eigenvectors of the corresponding conservative system using a Galerkin error minimization approach in conjunction with a Neumann expansion method. Transfer functions for the asymmetric nonconservative system are derived in terms of the left and right eigenvectors of the nonconservative system. Suitable numerical examples are given to illustrate the proposed method.

INTRODUCTION

Modal analysis plays a central role in the vibrational studies of linear engineering structures. Since the publication of Rayleigh's classic monograph (1945, originally 1897), extensive work has been done in this area over the last 11 decades. The method was originally proposed for undamped structures whose inertia and stiffness properties can be represented by symmetric matrices or self-adjoint differential operators. However, Rayleigh himself noted that the real-life structural systems are not undamped, but that they possess some kind of energy dissipation mechanism or damping. While there are excellent theories in classical mechanics about why inertia and stiffness properties can be represented by symmetric matrices or self-joint operators when the system executes a small oscillation around a stable equilibrium, no such theory is available for energy dissipation. To solve this problem, in analogy with "potential energy" and "kinetic energy," Rayleigh assumed a "dissipation function," which is a nonnegative definite functional of instantaneous generalized velocities. This kind of damping model is known as viscous damping. So that the modal analysis of undamped systems is applicable to damped systems, Rayleigh made one more assumption by taking the viscous damping forces to be proportional to the inertia and stiffness forces. Since its introduction, this model has been used extensively and is known as "classical damping" or "proportional damping."

Rayleigh's argument behind proportional damping was intuitive rather than theoretical, and it was formulated for mathematical convenience only. Until Caughey and O'Kelly (1965) gave necessary and sufficient conditions for a damped system to have classical normal modes, the nature of proportional damping was not very clear. This topic is well understood now, but unfortunately, there is no mathematical theory or sufficient experimental evidence to show why a physical system should obey Caughey and O'Kelly's criterion. In fact, practical experiences show that most of the real-life structures do not satisfy this criterion as they possess complex modes instead of real normal modes. This leads to the notion of nonproportional or nonclassical damping. Extending modal analysis to nonproportionally damped systems is still an active area of research.

Currently, there have been efforts to extend the classical

modal analysis procedure to systems whose inertia, stiffness, and damping properties cannot be represented by symmetric matrices or self-adjoint differential operators. These kinds of problems arise in the dynamics of actively controlled structures and in many general nonconservative dynamic systems, for example, a moving vehicle on the road, a missile on its trajectory, a ship's motion in seawater, or the study of aircraft flutter. The asymmetry of damping and stiffness terms are often addressed in the context of gyroscopic and follower forces and asymmetry in mass (Soom and Kim 1983). Many authors have considered this kind of general nonconservative linear system. Fawzy and Bishop (1976) presented several relationships satisfied by eigenvectors and eigenrows of the second-order system and they also presented a method to normalize them. Inman (1983) considered a class of asymmetric systems that can be transformed into symmetric systems using a linear transformation. Conditions for the existence of classical normal modes in this kind of asymmetric systems were given by Ahmadian and Inman (1984). Later, Caughey and Ma (1993) gave conditions under which a general asymmetric system can be decoupled. In a subsequent paper, Ma and Caughey (1995) used equivalence transform to analyze asymmetric systems. However, their work was restricted to conservative systems or to special kinds of nonconservative systems that can be decoupled by an equivalence transformation.

In this paper general asymmetric discrete nonconservative dynamic systems are considered. The method presented does not require conversion of the equations of motion to the first-order form. In Section 2, left and right eigenvectors of the nonconservative system are derived in terms of left and right eigenvectors of the corresponding conservative system. A Galerkin error minimization approach in conjunction with a Neumann expansion method have been used for this purpose. Transfer functions for the asymmetric system are derived in Section 3. The derivation method uses the state-space representation of equations of motion at the intermediate steps, but finally relates the transfer functions to the left and right eigenvectors of the second-order system so that the first-order eigen-solutions are not required. Applications of the proposed method and the related numerical issues are discussed in Section 4 using a three-degree-of-freedom asymmetric system. Finally, Section 5 summarizes the main results of the study reported in this paper.

EIGENVALUES AND EIGENVECTORS OF NONCONSERVATIVE DYNAMIC SYSTEMS

Background

The equations of motion of a linear damped discrete system with N degrees of freedom can be written as

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Note. Associate Editor: Apostolos Papageorgiou. Discussion open until May 1, 2000. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on May 26, 1999. This paper is part of the *Journal of Engineering Mechanics*, Vol. 125, No. 12, December, 1999. ©ASCE, ISSN 0733-9399/99/0012-1372-1379/\$8.00 + \$.50 per page. Paper No. 21056.

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \hat{\mathbf{f}}(t) \quad (1)$$

where \mathbf{M} , \mathbf{C} , and $\mathbf{K} \in \mathbb{R}^{N \times N}$ are mass, damping, and stiffness matrices, respectively. $\mathbf{u}(t) \in \mathbb{R}^N$ is the response vector, $t \in \mathbb{R}^+$ denotes time, and $\hat{\mathbf{f}}(t) \in \mathbb{R}^N$ is the forcing vector. Traditional restrictions of symmetry and positive definiteness are not imposed on \mathbf{M} , \mathbf{C} , and \mathbf{K} ; however, it is assumed that \mathbf{M}^{-1} exist, that is the system is not defective or degenerate. It is well known that for any linear system, if the forcing function is harmonic, that is $\hat{\mathbf{f}}(t) = \mathbf{f} \exp[st]$ with $s = i\omega$ and amplitude vector $\mathbf{f} \in \mathbb{R}^N$, in steady state the response will also be harmonic at frequency $\omega \in \mathbb{R}^+$. Therefore, we seek a solution of the form $\mathbf{u}(t) = \tilde{\mathbf{u}} \exp[st]$, where $\tilde{\mathbf{u}} \in \mathbb{C}^N$ is the response vector in the frequency domain. Substitution of $\mathbf{u}(t)$ and $\hat{\mathbf{f}}(t)$ in (1) results in

$$s^2 \tilde{\mathbf{M}}\tilde{\mathbf{u}} + s\tilde{\mathbf{C}}\tilde{\mathbf{u}} + \tilde{\mathbf{K}}\tilde{\mathbf{u}} = \mathbf{f} \quad (2)$$

The right eigenvalue problem associated with the above equation can be represented by the λ -matrix problem (Lancaster 1966)

$$s_i^2 \mathbf{M}\mathbf{u}_i + s_i \mathbf{C}\mathbf{u}_i + \mathbf{K}\mathbf{u}_i = 0, \quad \forall i = 1, \dots, N \quad (3)$$

where $s_i \in \mathbb{C}$ is the i th latent root (eigenvalue); and $\mathbf{u}_i \in \mathbb{C}^N$ is the i th right latent vector (right eigenvector). Similarly, the left eigenvalue problem can be represented by

$$s_i^2 \mathbf{v}_i^T \mathbf{M} + s_i \mathbf{v}_i^T \mathbf{C} + \mathbf{v}_i^T \mathbf{K} = 0, \quad \forall i = 1, \dots, N \quad (4)$$

where $\mathbf{v}_i \in \mathbb{C}^N$ is the i th left latent vector (left eigenvector); and $(\cdot)^T$ denotes the matrix transpose.

When \mathbf{M} , \mathbf{C} , and \mathbf{K} are general asymmetric matrices, the left and right eigenvectors can easily be obtained from the first-order formulations, for example, state-space method (Newland 1989), Duncan forms (Meirovitch 1980), etc. Although exact in nature, the first-order methods require significant numerical efforts for obtaining the eigensolutions as the size of the problem doubles. Moreover, these methods also lack some of the intuitive simplicity of the analysis based on N -space. For these reasons the determination of eigenvalues and eigenvectors in N -space for asymmetric nonconservative systems is very desirable. Ma and Caughey (1995, Theorem 3) have shown that in the special case, when $\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{M}^{-1}\mathbf{K}$ commute in product, the linear asymmetric nonconservative system (1) can be decoupled by an equivalence transformation, and hence, the N -space method can be used. But in general, linear nonconservative systems do not satisfy this condition and some kind of approximate methods must be used for further analysis. Meirovitch and Ryland (1985) and Malone et al. (1997) have used a perturbation method to determine the eigensolutions of gyroscopic systems. The difficulty with these kinds of perturbation methods is that any k th order ($k > 2$) term for eigenvalues or eigenvectors requires the determination of all $(k - 1)$ terms of eigenvalues and eigenvectors, and they are correlated among themselves. It may be noted that eigenvalue determination is essentially a numerical method and leads to the solution of

$$\det(s_i^2 \mathbf{M} + s_i \mathbf{C} + \mathbf{K}) = 0 \quad (5)$$

The above equation is a polynomial of order $2N$ and yields $2N$ values of s_i , which appear in complex conjugate pairs. In this paper it is assumed that all of the eigenvalues are distinct. Several efficient numerical methods are available to solve (5), and so for eigenvalue determination perturbation methods are most likely not necessary.

Determination of Eigenvectors

Unlike the eigenvalues, the left and right eigenvectors for general asymmetric nonconservative systems cannot be deter-

mined by such simple procedures. In this paper we try to determine these quantities in terms of left and right eigenvectors of the associated asymmetric conservative system. The eigenproblem of asymmetric conservative systems has been well studied in the literature (Huseyin 1978; Ma and Caughey 1995). Here we briefly outline the main features for further reference.

Consider the undamped right eigenvalue problem

$$\mathbf{K}\mathbf{x}_{u_i} = \omega_i^2 \mathbf{M}\mathbf{x}_{u_i}, \quad \forall i = 1, \dots, N \quad (6)$$

where $\omega_i \in \mathbb{R}$ is the i th natural frequency and $\mathbf{x}_{u_i} \in \mathbb{R}^N$ is the i th undamped right eigenvector (mode shape). Similarly, the undamped left eigenvalue problem can be defined by

$$\mathbf{x}_{v_i}^T \mathbf{K} = \omega_i^2 \mathbf{x}_{v_i}^T \mathbf{M}, \quad \forall i = 1, \dots, N \quad (7)$$

where $\mathbf{x}_{v_i} \in \mathbb{R}^N$ is the i th undamped left eigenvector. For distinct eigenvalues it is easy to show that the left and right eigenvectors satisfy the biorthogonality relationship with respect to \mathbf{M} and \mathbf{K} (Huseyin 1978). We also normalize the eigenvectors such that

$$\mathbf{x}_{v_j}^T \mathbf{M}\mathbf{x}_{u_i} = \delta_{ji}, \quad \mathbf{x}_{v_j}^T \mathbf{K}\mathbf{x}_{u_i} = \omega_i^2 \delta_{ji}, \quad \forall j, i = 1, \dots, N \quad (8)$$

where δ_{ji} is the Kronecker's delta function. Because all of the undamped eigenvalues (ω_i^2) are assumed to be distinct, \mathbf{x}_{u_i} , $\forall i = 1, \dots, N$ form a complete set of vectors, so that \mathbf{u}_i can be expanded as a complex combination of \mathbf{x}_{u_i} . Similarly, \mathbf{v}_i can also be expanded in terms of \mathbf{x}_{v_i} . Thus, an expansion of the form

$$\mathbf{u}_i = \sum_{j=1}^N \alpha_j^{(i)} \mathbf{x}_{u_j}; \quad \mathbf{v}_i = \sum_{j=1}^N \beta_j^{(i)} \mathbf{x}_{v_j} \quad (9a,b)$$

may be considered. Now, without any loss of generality, we can assume $\alpha_i^{(i)} = 1$ and $\beta_i^{(i)} = 1$ (normalization), which leaves us to determine $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\forall j \neq i$. A Galerkin type of error minimization approach combined with a complex Neumann expansion method is adopted for this purpose.

Substituting the expansion of \mathbf{u}_i in (3) and using the usual definition of natural frequencies in structural dynamics, i.e., $\lambda_i = s_i/i$, the error vector for the i th mode can be expressed as

$$\Delta^{(i)} = \sum_{j=1}^N -\lambda_i^2 \alpha_j^{(i)} \mathbf{M}\mathbf{x}_{u_j} + i\lambda_i \alpha_j^{(i)} \mathbf{C}\mathbf{x}_{u_j} + \alpha_j^{(i)} \mathbf{K}\mathbf{x}_{u_j} \in \mathbb{C}^N \quad (10)$$

Consider the undamped left eigenvectors \mathbf{x}_{v_k} , $\forall k = 1, \dots, N$ as "weighting functions," and following the Galerkin method we have $(\mathbf{x}_{v_k}, \Delta^{(i)}) = 0$ or $\mathbf{x}_{v_k}^T \Delta^{(i)} = 0$, where (\cdot, \cdot) denotes the standard inner product norm in \mathbb{C}^N . Using the biorthogonality property of the undamped left and right eigenvectors described in (8), one obtains

$$-\lambda_i^2 \alpha_k^{(i)} + i\lambda_i \sum_{j=1}^N \alpha_j^{(i)} C'_{kj} + \omega_k^2 \alpha_k^{(i)} = 0, \quad \forall k = 1, \dots, N \quad (11)$$

where $C'_{kj} = \mathbf{x}_{v_k}^T \mathbf{C}\mathbf{x}_{u_j}$. The i th equation of this set obtained by setting $k = i$ is a trivial case because $\alpha_i^{(i)} = 1$ has already been assumed. From the above set of equations, excluding this trivial case, one has

$$-\lambda_i^2 \alpha_k^{(i)} + i\lambda_i \left(C'_{ki} + \alpha_k^{(i)} C'_{kk} + \sum_{j \neq k \neq i}^N \alpha_j^{(i)} C'_{kj} \right) + \omega_k^2 \alpha_k^{(i)} = 0, \quad \forall k = 1, \dots, N \neq i \quad (12)$$

which can be written further in matrix form as

$$[\mathbf{P}^{(i)} - \mathbf{Q}^{(i)}] \hat{\mathbf{a}}^{(i)} = \mathbf{g}_i^{(i)} \quad (13)$$

In the above equation

$$\mathbf{P}^{(i)} = \text{diag} \left[\frac{\omega_1^2 - \lambda_i^2 + i\lambda_i C'_{11}}{-i\lambda_i}, \dots, \{\text{ith term deleted}\}, \dots, \frac{\omega_N^2 - \lambda_i^2 + i\lambda_i C'_{NN}}{-i\lambda_i} \right] \in \mathbb{C}^{(N-1) \times (N-1)} \quad (14)$$

and the traceless matrix

$$\mathbf{Q}^{(i)} = \begin{bmatrix} 0 & C'_{21} & \cdots & \{\text{ith term deleted}\} & \cdots & C'_{1N} \\ C'_{21} & 0 & \vdots & \vdots & \vdots & C'_{2N} \\ \vdots & \vdots & \vdots & \{\text{ith term deleted}\} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C'_{N1} & C'_{N2} & \cdots & \{\text{ith term deleted}\} & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(N-1) \times (N-1)} \quad (15)$$

$$\mathbf{g}_u^{(i)} = \{C'_{1i}, C'_{2i}, \dots, \{\text{ith term deleted}\}, \dots, C'_{Ni}\}^T \in \mathbb{R}^{(N-1)} \quad (16)$$

and

$$\hat{\mathbf{a}}^{(i)} = \{\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \{\text{ith term deleted}\}, \dots, \alpha_N^{(i)}\}^T \in \mathbb{C}^{(N-1)} \quad (17)$$

is the vector of unknown $\alpha_k^{(i)}$, $\forall k = 1, \dots, N, k \neq i$. From (13), $\hat{\mathbf{a}}^{(i)}$ must be determined by performing the associated matrix inversion, and this is achieved by using the Neumann expansion method. A similar procedure was used by Adhikari and Manohar (1999) in the context of inversion of the dynamic stiffness matrix of structures with stochastic properties. Now, using the Neumann expansion we have

$$\begin{aligned} \hat{\mathbf{a}}^{(i)} &= [\mathbf{I}_{N-1} - \mathbf{P}^{(i-1)} \mathbf{Q}^{(i)}]^{-1} \{\mathbf{P}^{(i-1)} \mathbf{g}_u^{(i)}\} \\ &= [\mathbf{I}_{N-1} + \mathbf{R}_u^{(i)} + \mathbf{R}_u^{(i)2} + \mathbf{R}_u^{(i)3} + \cdots] \mathbf{a}_0^{(i)} \end{aligned} \quad (18)$$

where \mathbf{I}_{N-1} is $(N-1) \times (N-1)$ identity matrix

$$\mathbf{R}_u^{(i)} = \mathbf{P}^{(i-1)} \mathbf{Q}^{(i)} \in \mathbb{C}^{(N-1) \times (N-1)}; \quad \mathbf{a}_0^{(i)} = \mathbf{P}^{(i-1)} \mathbf{g}_u^{(i)} \in \mathbb{C}^{(N-1)} \quad (19a,b)$$

Because $\mathbf{P}^{(i)}$ is a diagonal matrix, its inversion can be carried out analytically, and subsequently, the closed-form expressions of $\mathbf{R}_u^{(i)}$ and $\mathbf{a}_0^{(i)}$ can be obtained (see Section 4). This makes further calculations involving these quantities simpler. From (18), $\hat{\mathbf{a}}^{(i)}$ can be calculated in an efficient way, as one can write

$$\hat{\mathbf{a}}^{(i)} = \mathbf{a}_0^{(i)} + \mathbf{a}_1^{(i)} + \mathbf{a}_2^{(i)} + \cdots + \mathbf{a}_k^{(i)} + \cdots \quad (20)$$

where

$$\mathbf{a}_1^{(i)} = \mathbf{R}_u^{(i)} \mathbf{a}_0^{(i)}, \mathbf{a}_2^{(i)} = \mathbf{R}_u^{(i)} \mathbf{a}_1^{(i)}, \dots, \mathbf{a}_k^{(i)} = \mathbf{R}_u^{(i)} \mathbf{a}_{k-1}^{(i)} \quad (21)$$

This implies that all of the $\mathbf{a}_k^{(i)}$ can be obtained using successive matrix-vector multiplications only. Now noting that $\hat{\mathbf{a}}^{(i)}$ is the vector of $\alpha_k^{(i)}$, $\forall k = 1, \dots, N, k \neq i$, the substitution of it in (9a) will give the right eigenvectors associated with the nonconservative system (1). It is easy to see that by taking more terms in the series represented by (20), one can obtain the right eigenvectors to any arbitrary precision, provided the complex matrix power series $\mathbf{I}_{N-1} + \mathbf{R}_u^{(i)} + \mathbf{R}_u^{(i)2} + \mathbf{R}_u^{(i)3} + \cdots$ is convergent. The convergence issue of this series is addressed in the next subsection.

Similarly, the left eigenvectors can be obtained by substituting the expansion of \mathbf{v}_i in (4), and letting \mathbf{x}_u as weighting functions while applying the Galerkin method. Following the procedure employed for the right eigenvectors, one can write

$$[\mathbf{P}^{(i)} - \mathbf{Q}^{(i)T}] \hat{\mathbf{b}}^{(i)} = \mathbf{g}_v^{(i)} \quad (22)$$

where

$$\mathbf{g}_v^{(i)} = \{C'_{i1}, C'_{i2}, \dots, \{\text{ith term deleted}\}, \dots, C'_{iN}\}^T \in \mathbb{R}^{(N-1)} \quad (23)$$

and

$$\hat{\mathbf{b}}^{(i)} = \{\beta_1^{(i)}, \beta_2^{(i)}, \dots, \{\text{ith term deleted}\}, \dots, \beta_N^{(i)}\}^T \in \mathbb{C}^{(N-1)}$$

is the vector of unknown $\beta_k^{(i)}$, $\forall k = 1, \dots, N, k \neq i$. Now, using the Neumann expansion method and defining

$$\mathbf{R}_v^{(i)} = \mathbf{P}^{(i-1)} \mathbf{Q}^{(i)T} \in \mathbb{C}^{(N-1) \times (N-1)}; \quad \mathbf{b}_0^{(i)} = \mathbf{P}^{(i-1)} \mathbf{g}_v^{(i)} \in \mathbb{C}^{(N-1)} \quad (24a,b)$$

from (22) one obtains $\hat{\mathbf{b}}^{(i)}$ as a series

$$\begin{aligned} \hat{\mathbf{b}}^{(i)} &= [\mathbf{I}_{N-1} + \mathbf{R}_v^{(i)} + \mathbf{R}_v^{(i)2} + \mathbf{R}_v^{(i)3} + \cdots] \mathbf{b}_0^{(i)} \\ &= \mathbf{b}_0^{(i)} + \mathbf{b}_1^{(i)} + \mathbf{b}_2^{(i)} + \cdots + \mathbf{b}_k^{(i)} + \cdots \end{aligned} \quad (25)$$

Here

$$\mathbf{b}_1^{(i)} = \mathbf{R}_v^{(i)} \mathbf{b}_0^{(i)}, \mathbf{b}_2^{(i)} = \mathbf{R}_v^{(i)} \mathbf{b}_1^{(i)}, \dots, \mathbf{b}_k^{(i)} = \mathbf{R}_v^{(i)} \mathbf{b}_{k-1}^{(i)} \quad (26)$$

The i th eigenvector of the nonconservative system \mathbf{v}_i can be obtained by substituting $\hat{\mathbf{b}}^{(i)}$ in the second equation of (9). This method does not require much computational time as closed-form expressions for $\mathbf{R}_v^{(i)}$ and $\mathbf{b}_0^{(i)}$ are available (see Section 4). It may be noted that by taking more terms in the series (25), one can obtain \mathbf{v}_i to arbitrary precision if the complex matrix power series $\mathbf{I}_{N-1} + \mathbf{R}_v^{(i)} + \mathbf{R}_v^{(i)2} + \mathbf{R}_v^{(i)3} + \cdots$ is convergent.

Convergence

A necessary condition for validity of the series expression of $\hat{\mathbf{a}}^{(i)}$ and $\hat{\mathbf{b}}^{(i)}$ in (20) and (25), required for the determination of the right and left eigenvectors of the nonconservative system (1), is that the complex matrix power series

$$\mathbf{S}_u = \mathbf{I}_{N-1} + \mathbf{R}_u^{(i)} + \mathbf{R}_u^{(i)2} + \mathbf{R}_u^{(i)3} + \cdots \quad (27)$$

and

$$\mathbf{S}_v = \mathbf{I}_{N-1} + \mathbf{R}_v^{(i)} + \mathbf{R}_v^{(i)2} + \mathbf{R}_v^{(i)3} + \cdots \quad (28)$$

are convergent. Looking at the expression of $\mathbf{R}_u^{(i)}$ and $\mathbf{R}_v^{(i)}$ in (19) and (24), it may be revealed that they are quite similar, and it is sufficient to study the convergence property of any one of the series. Here the series in (27) will be studied.

Condition 1

The complex matrix power series \mathbf{S}_u converges if, and only if, for all of the eigenvalues $\sigma_j^{(i)}$ of the matrix $\mathbf{R}_u^{(i)}$, the inequality $|\sigma_j^{(i)}| < 1$ holds.

Proof of this directly follows from Wilkinson (1965). Although the above condition is both necessary and sufficient, checking convergence for all $i = 1, \dots, N$ is often not feasible. Therefore, we look for a sufficient condition that is relatively easy to check and that ensures convergence for all $i = 1, \dots, N$.

Condition 2

The complex matrix power series \mathbf{S}_u converges for any λ_i , ω_i if \mathbf{C}' is a diagonally dominant matrix.

Proof

Because a matrix norm is always greater than or equal to its maximum eigenvalue, it follows from Condition 1 that, convergence of the series is guaranteed if $\|\mathbf{R}_u^{(i)}\| < 1$. Writing the sum of absolute values of entries of $\mathbf{R}_u^{(i)}$ (see Section 4) results in the following inequality as the required sufficient condition for convergence:

$$\sum_{k=1}^N \sum_{j=1, j \neq i}^N \left| \frac{\lambda_i C'_{kj}}{\omega_k^2 - \lambda_i^2 + i\lambda_i C'_{kk}} \right| (1 - \delta_{ik}) < 1 \quad (29)$$

Dividing the numerator and denominator by λ_i , the above inequality can be written as

$$\sum_{k=1}^N \sum_{j=1, j \neq i \neq k}^N \frac{|C'_{kj}|}{|1/\lambda_i(\omega_k^2 - \lambda_i^2) + iC'_{kk}|} < 1 \quad (30)$$

Taking the maximum for all $k \neq i$, this condition can further be represented as

$$\max_{k \neq i} \frac{\sum_{j=1, j \neq i, k}^N |C'_{kj}|}{\left[\frac{1}{|\lambda_i|^2} (|\lambda_i|^2 - \omega_k^2)^2 + C'_{kk}{}^2 \right]^{1/2}} < 1 \quad (31)$$

It is clear that (31) always holds if

$$\sum_{j=1, j \neq i \neq k}^N |C'_{kj}| < |C'_{kk}|, \quad \forall k \neq i \quad (32)$$

which, in turn, implies that for all $i = 1, \dots, N$, the inequality $\|\mathbf{R}_u^{(i)}\| < 1$ holds if \mathbf{C}' is diagonally dominant. It is important to note that the diagonal dominance of \mathbf{C}' is only a sufficient condition, and that the lack of it does not necessarily preclude the convergence of \mathbf{S}_u .

Discussion

Following the procedure outlined in the previous subsection one can obtain right and left eigenvectors of the nonconservative systems up to any desired level of accuracy without using first-order formalisms. It may be noted that the expressions developed here are somewhat different from perturbation solutions [for example, Cronin (1990) for symmetric systems], because higher-order terms of eigenvectors do not depend on the higher-order terms of the eigenvalues. The eigenvalues are determined exactly by solving the polynomial equation (5), and subsequently, the right and left eigenvectors of the nonconservative system can be obtained from the series expression (9). The vector of complex constants $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ appearing in (9) are further obtained from (20) and (25), respectively. For many engineering problems it is often observed that entries in the \mathbf{C} matrix are not very "big," and that by retaining only a few terms in the series, (20) and (25) will result in an acceptable accuracy. Closed-form approximate expressions of right and left eigenvectors by retaining one and two terms in these series are given in Appendix I. These expressions might be useful whenever we find that the entries of the \mathbf{C} matrix are small compared with that of \mathbf{M} and \mathbf{K} .

TRANSFER FUNCTIONS

Transfer functions of a system completely define its input-output relationship. Suppose the forcing vector \mathbf{f} in (2) is zero for all of the entries except the m th, which has an entry p . Due to this force, if the response at some n th degree of freedom is \tilde{u}_n , then the transfer function $H_{nm}(\omega)$ can be defined by

$$H_{nm}(\omega) = \frac{\tilde{u}_n}{p} \quad (33)$$

In principle, one can carry out the associated matrix inversion with (2) and subsequently calculate $\tilde{\mathbf{u}}$ to obtain \tilde{u}_n . However, this would be a somewhat expensive numerical exercise and may not offer much physical insight to the analyst. Instead, we seek a solution analogous to the classical modal series solution of the undamped or proportionally damped symmetric system (Meirovitch 1967). At this stage it turns out

to be useful to perform the calculations in state-space and then relate the result to the left and right eigenvectors of the second-order system determined in the last section.

Eq. (1) can be transformed into first-order (state-space) form as

$$\dot{\mathfrak{z}}(t) = \mathbf{A}\mathfrak{z}(t) + \mathcal{P}(t) \quad (34)$$

where $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$ = system matrix; $\mathcal{P}(t) \in \mathbb{R}^{2N}$ = forcing vector; and $\mathfrak{z}(t) \in \mathbb{R}^{2N}$ = response vector in the state-space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_N \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \quad \mathfrak{z}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} \quad (35a,b)$$

$$\mathcal{P}(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{Bmatrix} \quad (35c)$$

In the above equation $\mathbf{0}$ is the $N \times N$ null matrix and \mathbf{I}_N is the $N \times N$ identity matrix. For periodic forcing $\mathcal{P}(t) = \mathbf{P} \exp[st]$, assume periodic solution of the form $\mathfrak{z}(t) = \tilde{\mathbf{z}} \exp[st]$, and substituting it in (34) we obtain

$$s\tilde{\mathbf{z}} = \mathbf{A}\tilde{\mathbf{z}} + \mathbf{P} \quad (36)$$

In the above equation $\tilde{\mathbf{z}} \in \mathbb{C}^{2N}$ is the frequency domain response vector in state-space and can be related to the corresponding vector of the second-order system as

$$\tilde{\mathbf{z}} = \begin{Bmatrix} \tilde{\mathbf{u}} \\ s\tilde{\mathbf{u}} \end{Bmatrix} \quad (37)$$

and the amplitude of the forcing vector is $\mathbf{P} = \{\mathbf{0}, \mathbf{M}^{-1}\mathbf{f}\}^T \in \mathbb{R}^{2N}$. The right eigenvalue problem associated with (36) can be expressed as

$$\mathbf{A}\mathbf{z}_i = s_i\mathbf{z}_i, \quad \forall i = 1, \dots, 2N \quad (38)$$

where $s_i \in \mathbb{C}$ is the i th eigenvalue and $\mathbf{z}_i \in \mathbb{C}^{2N}$ is the i th right eigenvector which is related to the i th right eigenvector of the second-order system as

$$\mathbf{z}_i = \begin{Bmatrix} \mathbf{u}_i \\ s_i\mathbf{u}_i \end{Bmatrix} \quad (39)$$

The left eigenvector $\mathbf{y}_i \in \mathbb{C}^{2N}$ associated with s_i is defined by the equation

$$\mathbf{y}_i^T \mathbf{A} = s_i \mathbf{y}_i^T \quad (40)$$

For distinct eigenvalues it is easy to show that the right and left eigenvectors satisfy an orthogonality relationship, that is

$$\mathbf{y}_i^T \mathbf{z}_j = 0, \quad \forall j \neq i \quad (41)$$

and we also normalize the eigenvectors so that

$$\mathbf{y}_i^T \mathbf{z}_i = 1 \quad (42)$$

The above two equations imply that the dynamic system defined by (34) possesses a set of biorthogonal eigenvectors with respect to the system matrix \mathbf{A} .

Because it has been already assumed that \mathbf{A} has distinct eigenvalues, the right eigenvectors $\mathbf{z}_i, \forall i = 1, \dots, 2N$, form a complete set of vectors. Thus, the solution of (36) can be expanded as

$$\tilde{\mathbf{z}} = \sum_{j=1}^{2N} e_j \mathbf{z}_j \quad (43)$$

where $e_j \in \mathbb{C}, \forall j = 1, \dots, 2N$ are set of constants to be determined. Note that the above equation is a vector equation with $2N$ rows: The first N rows are a solution of the second-order system (2), which we aim to obtain. Substituting $\tilde{\mathbf{z}}$ in (36), and premultiplying by the left eigenvector \mathbf{y}_k^T , one has

$$\sum_{j=1}^{2N} \mathbf{y}_k^T (s - \mathbf{A}) \mathbf{e}_j \mathbf{z}_j = \mathbf{y}_k^T \mathbf{P} \quad (44)$$

Using the biorthogonality relationship of the left and right eigenvectors of \mathbf{A} , the above equation results in

$$e_k = \frac{\mathbf{y}_k^T \mathbf{P}}{s - s_k}, \quad \forall k = 1, \dots, 2N \quad (45)$$

The e_k expressed above is not very useful because it is in terms of left and right eigenvectors of the first-order system. To obtain a relationship with the eigenvectors of the second-order system, assume

$$\mathbf{y}_i = \begin{Bmatrix} \mathbf{y}_{1i} \\ \mathbf{y}_{2i} \end{Bmatrix} \quad (46)$$

where $\mathbf{y}_{1i}, \mathbf{y}_{2i} \in \mathbb{C}^N$. Substituting \mathbf{y}_i in (40) and taking the transpose, one obtains

$$s_i \mathbf{y}_{1i} = -\mathbf{K}^T \mathbf{M}^{-1T} \mathbf{y}_{2i} \quad (47a)$$

$$s_i \mathbf{y}_{2i} = \mathbf{y}_{1i} - \mathbf{C}^T \mathbf{M}^{-1T} \mathbf{y}_{2i} \quad \text{or} \quad \mathbf{y}_{1i} = [s_i \mathbf{I}_N + \mathbf{C}^T \mathbf{M}^{-1T}] \mathbf{y}_{2i} \quad (47b)$$

Elimination of \mathbf{y}_{1i} from the above two equations yield

$$s_i (s_i \mathbf{y}_{2i} + \mathbf{C}^T \mathbf{M}^{-1T} \mathbf{y}_{2i}) = -\mathbf{K}^T \mathbf{M}^{-1T} \mathbf{y}_{2i} \quad \text{or}$$

$$(\mathbf{y}_{2i}^T \mathbf{M}^{-1}) [s_i^2 \mathbf{M} + s_i \mathbf{C} + \mathbf{K}] = 0 \quad (48)$$

By comparing this equation with (4), it can be observed that the vector $\mathbf{y}_{2i}^T \mathbf{M}^{-1}$ is parallel to \mathbf{v}_i^T , that is, there exists a non-zero $\gamma_i \in \mathbb{C}$, such that

$$\mathbf{y}_{2i}^T \mathbf{M}^{-1} = \gamma_i \mathbf{v}_i^T \quad \text{or} \quad \mathbf{y}_{2i} = \gamma_i \mathbf{M}^T \mathbf{v}_i \quad (49)$$

Substituting $\mathbf{y}_{1i}, \mathbf{y}_{2i}$, and also \mathbf{z}_i from (39) into the normalization condition (42), we have the scalar equation

$$\mathbf{y}_{1i}^T \mathbf{u}_i + s_i \mathbf{y}_{2i}^T \mathbf{u}_i = 1 \quad \text{or} \quad \gamma_i \mathbf{v}_i^T [\mathbf{C} + s_i \mathbf{M}] \mathbf{u}_i + \gamma_i s_i \mathbf{v}_i^T \mathbf{M} \mathbf{u}_i = 1 \quad (50)$$

From the above equation, the scalar constant γ_i can be obtained as

$$\gamma_i = \frac{1}{\mathbf{v}_i^T [2s_i \mathbf{M} + \mathbf{C}] \mathbf{u}_i} \quad (51)$$

Now taking the first N rows of (43) and using (49), one obtains

$$\tilde{\mathbf{u}} = \sum_{j=1}^{2N} \frac{\mathbf{y}_{2j}^T \mathbf{M}^{-1} \mathbf{f}}{s - s_j} \mathbf{u}_j = \sum_{j=1}^{2N} \frac{p \gamma_j \mathbf{v}_m^{(j)}}{s - s_j} \mathbf{u}_j \quad (52)$$

where $\mathbf{v}_m^{(j)}$ represents the m th element of the j th left eigenvector of the second-order system \mathbf{v}_j .

It is well known that for any system with real coefficient matrices, the eigenvalues and eigenvectors appear in complex conjugate pairs. This implies that with the usual definition of natural frequencies, $s_j = i\lambda_j$ and $s_j^* = -i\lambda_j^*$ occurs with $\mathbf{u}_j, \mathbf{v}_j$, and $\mathbf{u}_j^*, \mathbf{v}_j^*$, respectively. Here $(\cdot)^*$ denotes complex conjugation. Now from the definition of the transfer function in (33) and from (52), writing the terms corresponding to s_i and s_i^* separately, we finally obtain

$$H_{mm}(\omega) = \sum_{j=1}^N \left\{ -\frac{\mu_j \mathbf{v}_m^{(j)} \mathbf{u}_n^{(j)}}{\omega - \lambda_j} + \frac{\mu_j^* \mathbf{v}_m^{(j)*} \mathbf{u}_n^{(j)*}}{\omega + \lambda_j^*} \right\} \quad (53)$$

where $\mu_j = i\gamma_j = 1/(\mathbf{v}_j^T [2\lambda_j \mathbf{M} - i\mathbf{C}] \mathbf{u}_j)$. This is a generalization of the known expression of the transfer function for symmetric conservative systems to asymmetric nonconservative systems. Transfer functions for several interesting special cases may be obtained from (53):

1. Symmetric conservative system (Rayleigh 1897): In this

case $\mathbf{M} = \mathbf{M}^T$, $\mathbf{K} = \mathbf{K}^T$, and $\mathbf{C} = 0$ results in $\mathbf{v}_j = \mathbf{u}_j = \mathbf{x}_{u_j}$, and $\mu_j = 1/2\omega_j$, which reduces expression (53) to

$$H_{mm}(\omega) = \sum_{j=1}^N \frac{\chi_{u_m}^{(j)} \chi_{u_n}^{(j)}}{\omega_j^2 - \omega^2} \quad (54)$$

2. Asymmetric conservative system (Huseyin 1978): In this case $\mathbf{C} = 0$ results in $\mathbf{u}_j = \mathbf{x}_{u_j}$, $\mathbf{v}_j = \mathbf{x}_{v_j}$, and $\mu_j = 1/2\omega_j$, which reduces expression (53) to

$$\mathbf{H}_{mm}(\omega) = \sum_{j=1}^N \frac{\chi_{v_m}^{(j)} \chi_{u_n}^{(j)}}{\omega_j^2 - \omega^2} \quad (55)$$

3. Symmetric nonconservative system (Vigneron 1986): In this case $\mathbf{M} = \mathbf{M}^T$, $\mathbf{K} = \mathbf{K}^T$, and $\mathbf{C} = \mathbf{C}^T$ results in $\mathbf{v}_j = \mathbf{u}_j$ and $\mu_j = 1/(\mathbf{u}_j^T [2\lambda_j \mathbf{M} - i\mathbf{C}] \mathbf{u}_j)$, which reduces expression (53) to

$$H_{mm}(\omega) = \sum_{j=1}^N \left\{ -\frac{\mu_j \mathbf{u}_m^{(j)} \mathbf{u}_n^{(j)}}{\omega - \lambda_j} + \frac{\mu_j^* \mathbf{u}_m^{(j)*} \mathbf{u}_n^{(j)*}}{\omega + \lambda_j^*} \right\} \quad (56)$$

From the transfer function expression [(53)], the steady-state response due to harmonic load or response due to broadband random excitation can be obtained directly. However, response due to transient loads or that due to initial conditions can also be obtained by familiar methods using convolution integrals in the time domain after obtaining the impulse response function by taking the Fourier transform of (53) (Meirovitch 1967, section 7.6).

APPLICATIONS AND EXAMPLES

Following the procedure outlined before the frequency domain response of an asymmetric nonconservative system can be obtained using the right and left eigenvectors. The method presented here is very similar to that of the classical modal analysis because it appears that only the undamped modes and frequencies have to be replaced appropriately by right and left eigenvectors and complex eigenvalues of the asymmetric nonconservative system. The left and right eigenvectors can be obtained from that of the corresponding conservative system using a series expansion in N -space and the eigenvalues can be obtained directly from the characteristic equation. Transfer functions are further expressed in terms of these left and right eigenvectors and complex eigenvalues. Here we briefly summarize the steps to be followed:

1. Obtain the eigenvalues ω_i , right eigenvectors \mathbf{x}_{u_i} , and left eigenvectors \mathbf{x}_{v_i} of the conservative system from $\mathbf{K}\mathbf{x}_{u_i} = \omega_i^2 \mathbf{M}\mathbf{x}_{u_i}$ and $\mathbf{x}_{v_i}^T \mathbf{K} = \omega_i^2 \mathbf{x}_{v_i}^T \mathbf{M}$ for all $i = 1, \dots, N$. Normalize \mathbf{x}_{u_i} and \mathbf{x}_{v_i} so that $\mathbf{x}_{v_i}^T \mathbf{M}\mathbf{x}_{u_i} = \delta_{ij}$.
2. Determine the eigenvalues λ_i of the nonconservative system by solving the polynomial equation $\det(s_i^2 \mathbf{M} + is_i \mathbf{C} + \mathbf{K}) = 0$. Obtain the complex natural frequencies $\lambda_i = s_i/i$.
3. Set up the \mathbf{C}' matrix using $C'_{kj} = \mathbf{x}_{v_k}^T \mathbf{C}\mathbf{x}_{u_j}$, $\forall j, k = 1, \dots, N$. Calculate the matrices $\mathbf{R}_u^{(i)}$ and $\mathbf{R}_v^{(i)}$ [obtained by simplifying (19) and (24), respectively], using

$$R_{u_{k,j_i}}^{(i)} = \frac{-i\lambda_i C'_{kj} (1 - \delta_{k,j_i})}{\omega_k^2 - \lambda_i^2 + i\lambda_i C'_{kk}}$$

and

$$R_{v_{k,j_i}}^{(i)} = \frac{-i\lambda_i C'_{jk} (1 - \delta_{k,j_i})}{\omega_k^2 - \lambda_i^2 + i\lambda_i C'_{kk}}, \quad \forall j, k = 1, \dots, N, \neq i$$

For keeping the dimension of $\mathbf{R}_u^{(i)}$ and $\mathbf{R}_v^{(i)}$ to $(N-1) \times (N-1)$, express k_1 , and consequently, j_1 as $k_1 = k - \mathcal{U}_{(k,i)}$ and $j_1 = j - \mathcal{U}_{(j,i)}$. Here the function $\mathcal{U}_{(k,i)}$ (and similarly $\mathcal{U}_{(j,i)}$) is defined as

$$u_{(k,i)} = \begin{cases} 0, & \text{if } k < i \\ 1, & \text{if } k > i \\ \text{not defined} & \text{if } k = i \end{cases} \quad (57)$$

Also obtain the vectors $\mathbf{a}_0^{(i)}$ and $\mathbf{b}_0^{(i)}$, using

$$a_{0_i}^{(i)} = \frac{-i\lambda_i C'_{ji}}{\omega_j^2 - \lambda_i^2 + i\lambda_i C'_{jj}}$$

and

$$b_{0_i}^{(i)} = \frac{-i\lambda_i C'_{ij}}{\omega_j^2 - \lambda_i^2 + i\lambda_i C'_{jj}}, \quad \forall j = 1, \dots, N, \neq i$$

4. Select the number of terms, say r , to be retained in the expansion of the eigenvectors [see (20) and (25) for reference]. Calculate $\mathbf{a}_k^{(i)} = \mathbf{R}_u^{(i)} \mathbf{a}_{k-1}^{(i)}$ and $\mathbf{b}_k^{(i)} = \mathbf{R}_v^{(i)} \mathbf{b}_{k-1}^{(i)}$ for all $k = 1, \dots, r$, and subsequently obtain $\hat{\mathbf{a}}^{(i)} = \sum_{k=1}^r \mathbf{a}_{k-1}^{(i)}$ and $\hat{\mathbf{b}}^{(i)} = \sum_{k=1}^r \mathbf{b}_{k-1}^{(i)}$.
5. Obtain the right eigenvectors of the nonconservative system as

$$\mathbf{u}_i = \mathbf{x}_{u_i} + \sum_{\substack{j=1 \\ j \neq i}}^N \hat{a}_{j_i}^{(i)} \mathbf{x}_{u_j}$$

and left eigenvectors as

$$\mathbf{v}_i = \mathbf{x}_{v_i} + \sum_{\substack{j=1 \\ j \neq i}}^N \hat{b}_{j_i}^{(i)} \mathbf{x}_{v_j}$$

for all $i = 1, \dots, N$.

6. Finally, calculate the transfer function matrix using these left and right eigenvectors

$$\mathbf{H}(\omega) = \sum_{j=1}^N \left\{ -\frac{\mu_j \mathbf{u}_j \mathbf{v}_j^T}{\omega - \lambda_j} + \frac{\mu_j^* \mathbf{u}_j^* \mathbf{v}_j^{*T}}{\omega + \lambda_j^*} \right\},$$

$$\text{with } \mu_j = \frac{1}{\mathbf{v}_j^T [2\lambda_j \mathbf{M} - i\mathbf{C}] \mathbf{u}_j} \quad (58)$$

or alternatively, in the time domain, obtain the impulse response function matrix

$$\mathbf{h}(t) = \sum_{j=1}^N \left\{ -\mu_j \mathbf{v}_j \mathbf{u}_j^T e^{i\lambda_j t} + \mu_j^* \mathbf{v}_j^* \mathbf{u}_j^{*T} e^{-i\lambda_j^* t} \right\} \quad (59)$$

Compare to the state-space approach, the procedure outlined above offers a significant reduction in computational effort because two eigenvalue problems of size $N \times N$ have to be solved (Step 1) instead of solving two eigenvalue problems of size $2N \times 2N$. Moreover, this method provides a better physical insight because only the familiar N -space eigenvectors are used. Various efficient numerical algorithms are available (Press et al. 1992) to solve the characteristic polynomial equation (Step 2). Many of these algorithms are iterative in nature and to increase the efficiency of the solution procedure, the eigenvalues obtained from a first-order perturbation method given by (76) in Appendix I can be used as an initial guess. The procedure to be followed later for obtaining the eigenvectors of the nonconservative system, i.e., Step 3 to Step 5, is quite systematic and straightforward and does not involve intensive computation.

To illustrate the proposed method, a numerical example of a three-degree-of-freedom system is considered. The \mathbf{M} , \mathbf{C} , and \mathbf{K} matrices appearing in (1) are defined by

$$\mathbf{M} = \begin{bmatrix} 0.5740 & 1.3858 & 1.3858 \\ 0.7070 & 0.7070 & -0.7070 \\ 0.4620 & -0.1914 & -0.1914 \end{bmatrix} \quad (60)$$

$$\mathbf{C} = \begin{bmatrix} 2.6710 & 2.9592 & 2.9651 \\ 0.4843 & 1.2606 & -0.6119 \\ 0.2875 & -0.5808 & 1.2272 \end{bmatrix} \quad (61)$$

$$\mathbf{K} = \begin{bmatrix} 1.3748 & 10.9440 & 25.2975 \\ 1.2625 & 2.8770 & -17.4195 \\ 0.7455 & -4.1244 & 0.8625 \end{bmatrix} \quad (62)$$

Numerical values for the entries of \mathbf{M} and \mathbf{K} matrices are taken from Ma and Caughey (1995). It may be easily verified that all of the above matrices are neither symmetric nor positive definite. The problem of determining the left and right eigenvectors will be solved by following the steps described earlier.

Step 1: Solution of the right and left eigenvalue problem of the associated conservative system given by (6) and (7) and normalization according to (8) yields the natural frequencies $\omega_1 = 1.3506$, $\omega_2 = 3.0913$, $\omega_3 = 4.8527$, and the eigenvectors

$$\mathbf{X}_u = [\mathbf{x}_{u_1}, \mathbf{x}_{u_2}, \mathbf{x}_{u_3}] = \begin{bmatrix} 1.0741 & -0.6240 & 0.4421 \\ -0.0292 & 0.9635 & -0.5428 \\ -0.0047 & -0.0290 & 0.8731 \end{bmatrix} \quad (63)$$

$$\mathbf{X}_v = [\mathbf{x}_{v_1}, \mathbf{x}_{v_2}, \mathbf{x}_{v_3}] = \begin{bmatrix} 0.3082 & 0.4876 & 0.5167 \\ 0.5025 & 0.3000 & -0.7973 \\ 0.8983 & -0.9954 & 0.5915 \end{bmatrix} \quad (64)$$

Step 2: By solving the characteristic determinant, $\det(s_1^2 \mathbf{M} + i s_1 \mathbf{C} + \mathbf{K}) = 0$, the eigenvalues of the nonconservative system are obtained; $s_1 = -0.7725 + 1.1965i$, $s_2 = -0.7251 + 3.0560i$, $s_3 = -1.3949 + 4.3092i$, together with the corresponding complex conjugate values. The complex natural frequencies are obtained by $\lambda_i = s_i/i$: $\lambda_1 = 1.1965 + 0.7725i$, $\lambda_2 = 3.0560 + 0.7251i$, $\lambda_3 = 4.3092 + 1.3949i$.

Step 3: The \mathbf{C}' matrix calculated from $\mathbf{C}' = \mathbf{X}_v^T \mathbf{C} \mathbf{X}_u$ as

$$\mathbf{C}' = \begin{bmatrix} 1.3850 & 0.1101 & 1.5220 \\ 1.1774 & 1.5858 & -0.7533 \\ 1.2321 & -0.6323 & 2.8142 \end{bmatrix} \quad (65)$$

From the \mathbf{C}' matrix we can further obtain $\mathbf{R}_u^{(i)}$, $i = 1, 2, 3$, using the closed-form expression described earlier

$$\mathbf{R}_u^{(1)} = \begin{bmatrix} 0 & -0.0768 + 0.1207i \\ -0.0209 + 0.0384i & 0 \end{bmatrix} \quad (66a)$$

$$\mathbf{R}_u^{(2)} = \begin{bmatrix} 0 & -0.1235 + 0.5850i \\ -0.0244 - 0.2886i & 0 \end{bmatrix} \quad (66b)$$

$$\mathbf{R}_u^{(3)} = \begin{bmatrix} 0 & 0.0010 + 0.0280i \\ 0.0981 + 0.4920i & 0 \end{bmatrix} \quad (66c)$$

Similarly $\mathbf{R}_v^{(i)}$, $i = 1, 2, 3$, can also be calculated

$$\mathbf{R}_v^{(1)} = \begin{bmatrix} 0 & -0.0645 + 0.1013i \\ -0.0250 + 0.0457i & 0 \end{bmatrix} \quad (67a)$$

$$\mathbf{R}_v^{(2)} = \begin{bmatrix} 0 & -0.1000 + 0.4735i \\ -0.0301 - 0.3565i & 0 \end{bmatrix} \quad (67b)$$

$$\mathbf{R}_v^{(3)} = \begin{bmatrix} 0 & 0.0102 + 0.2996i \\ 0.0092 + 0.0460i & 0 \end{bmatrix} \quad (67c)$$

Now obtain $\mathbf{a}_0^{(i)}$, $\mathbf{b}_0^{(i)}$, $i = 1, 2, 3$ using the closed-form expression described earlier

$$\mathbf{a}_0^{(1)} = \begin{Bmatrix} 0.1201 - 0.1887i \\ 0.0408 - 0.0748i \end{Bmatrix} \quad (68a)$$

$$\mathbf{a}_0^{(2)} = \begin{Bmatrix} -0.0089 + 0.0423i \\ 0.0125 + 0.1481i \end{Bmatrix} \quad (68b)$$

$$\mathbf{a}_0^{(3)} = \begin{Bmatrix} 0.0132 + 0.3872i \\ -0.0627 - 0.3148i \end{Bmatrix} \quad (68c)$$

$$\mathbf{b}_0^{(1)} = \begin{Bmatrix} 0.0112 - 0.0176i \\ 0.0504 - 0.0924i \end{Bmatrix} \quad (69a)$$

$$\mathbf{b}_0^{(2)} = \begin{Bmatrix} -0.0955 + 0.4525i \\ 0.0149 + 0.1764i \end{Bmatrix} \quad (69b)$$

$$\mathbf{b}_0^{(3)} = \begin{Bmatrix} 0.0107 + 0.3135i \\ -0.0527 - 0.2642i \end{Bmatrix} \quad (69c)$$

Step 4: Assume the number of terms to be retained $r = 7$, in the series expansion (20) and (25). Computing each of the terms using (21) and (26) and summing them results

$$\hat{\mathbf{a}}^{(1)} = \begin{Bmatrix} 0.1247 - 0.1782i \\ 0.0450 - 0.0663i \end{Bmatrix} \quad (70a)$$

$$\hat{\mathbf{a}}^{(2)} = \begin{Bmatrix} -0.1181 + 0.0348i \\ 0.0255 + 0.1812i \end{Bmatrix} \quad (70b)$$

$$\hat{\mathbf{a}}^{(3)} = \begin{Bmatrix} 0.0205 + 0.3800i \\ -0.2477 - 0.2674i \end{Bmatrix} \quad (70c)$$

$$\hat{\mathbf{b}}^{(1)} = \begin{Bmatrix} 0.0173 - 0.0067i \\ 0.0503 - 0.0914i \end{Bmatrix} \quad (71a)$$

$$\hat{\mathbf{b}}^{(2)} = \begin{Bmatrix} -0.2312 + 0.5273i \\ 0.2099 + 0.2429i \end{Bmatrix} \quad (71b)$$

$$\hat{\mathbf{b}}^{(3)} = \begin{Bmatrix} 0.0872 + 0.2913i \\ -0.0653 - 0.2575i \end{Bmatrix} \quad (71c)$$

Step 5: Using the assumed expansion of \mathbf{u}_i and \mathbf{v}_i in (9), and also noting that $\hat{\mathbf{a}}^{(i)}$ and $\hat{\mathbf{b}}^{(i)}$ are vectors of $\alpha_j^{(i)}$ and $\beta_j^{(i)}$, we finally obtain the right and left eigenvectors of nonconservative system

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$$

$$= \begin{bmatrix} 1.0162 + 0.0819i & -0.7395 + 0.1175i & 0.6187 + 0.5750i \\ 0.0665 - 0.1357i & 0.9531 - 0.0994i & -0.7821 - 0.2688i \\ 0.0310 - 0.0527i & -0.0062 + 0.1580i & 0.8802 + 0.0060i \end{bmatrix} \quad (72)$$

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

$$= \begin{bmatrix} 0.3426 - 0.0505i & 0.5248 + 0.2880i & 0.5117 - 0.0358i \\ 0.4676 + 0.0709i & 0.0164 + 0.0714i & -0.7731 + 0.0691i \\ 0.9109 - 0.0475i & -1.0789 + 0.6174i & 0.7347 + 0.5180i \end{bmatrix} \quad (73)$$

It is useful to check the accuracy of these quantities against the exact ones obtained from the state-space method. The exact right and left eigenvectors of nonconservative system are obtained as

$$\mathbf{U}_{\text{exact}}$$

$$= \begin{bmatrix} 1.0162 + 0.0819i & -0.7395 + 0.1175i & 0.6187 + 0.5750i \\ 0.0665 - 0.1357i & 0.9530 - 0.0994i & -0.7821 - 0.2688i \\ 0.0310 - 0.0527i & -0.0063 + 0.1582i & 0.8802 + 0.0060i \end{bmatrix} \quad (74)$$

$$\mathbf{V}_{\text{exact}}$$

$$= \begin{bmatrix} 0.3426 - 0.0505i & 0.5248 + 0.2880i & 0.5117 - 0.0358i \\ 0.4676 + 0.0709i & 0.0162 + 0.0713i & -0.7731 + 0.0691i \\ 0.9109 - 0.0475i & -1.0789 + 0.6182i & 0.7347 + 0.5180i \end{bmatrix} \quad (75)$$

For comparison, the exact eigenvectors are normalized to

have the same numerical value in the first element. It is clear that the results obtained from the method outlined hereby match the exact solutions to an excellent accuracy. An interesting point to be noted is that the matrix \mathbf{C}' is not diagonally dominant as $|C'_{11}| < |C'_{13}|$. However, all of the eigenvalues $\sigma_j^{(i)}$ of the matrix $\mathbf{R}_u^{(i)}$ satisfy $|\sigma_j^{(i)}| < 1$, as we have obtained $\max(\sigma^{(1)}) = 0.0791$, $\max(\sigma^{(2)}) = 0.4161$, and $\max(\sigma^{(3)}) = 0.1186$. This demonstrates that the condition established in (32) is a sufficient condition, and there may be cases when one can obtain the eigenvectors of the nonconservative system even if this condition is not satisfied.

CONCLUSIONS

The problem of dynamic analysis of nonconservative linear multiple degrees-of-freedom systems has been considered. The central theme of the approach adopted in this paper is to utilize the left and right eigenvectors of the second-order system so that conversion of the equations of motion to the first-order form can be avoided. The analysis is restricted to systems with nonrepetitive eigenvalues and with a nonsingular mass matrix. It has been assumed that, in general, the mass, damping, and stiffness matrices are neither symmetric nor positive definite and cannot be simultaneously diagonalized by any linear transformation. Complex eigenvalues of the system are obtained from the characteristic equation. The left and right eigenvectors of the nonconservative system are expressed as a complex combination of the left and right eigenvectors of the corresponding conservative system. The vectors of these complex constants for both eigenvectors are further determined from a series obtained by the Neumann expansion method in conjunction with a Galerkin-type error minimization. A useful sufficient condition for convergence of this series is established. Transfer functions of the system considered are derived as a series involving the left and right eigenvectors and complex natural frequencies of the nonconservative system. The method described is quite similar to the classical modal analysis where undamped natural frequencies and modes must be appropriately replaced by complex natural frequencies and left and right eigenvectors of the nonconservative system. Compared with the state-space approach, the procedure outlined herein offers a reduction in computational effort and provides more physical insight. The expressions developed for the eigenvectors and transfer functions in this paper are very general in nature, and most of the familiar linear dynamic systems, for example, undamped/classically damped symmetric systems, nonclassically damped symmetric systems, undamped asymmetric systems, damped/undamped gyroscopic systems, etc., can be treated as special cases.

APPENDIX I. APPROXIMATE EXPRESSIONS OF RIGHT AND LEFT EIGENVECTORS

The expressions of the right and left eigenvectors obtained by taking one term in the series (20) and (25) produce similar expressions as those obtained from the first-order perturbation analysis. The validity of this analysis relies on the fact that the entries of the \mathbf{C} matrix are not very large. Considering the i th set of (11), and neglecting the second-order terms involving $\alpha_j^{(i)}$ and C'_{ij} , $\forall j \neq i$, and also noting that $a_i^{(i)} = 1$, one obtains

$$-\lambda_i^2 + i\lambda_i C'_{ii} + \omega_i^2 = 0, \quad \text{or} \quad \lambda_i \approx \pm\omega_i + iC'_{ii}/2 \quad (76)$$

This is the first-order approximate expression for the complex natural frequencies of system (1). To obtain an approximate expression of the right and left eigenvectors, one simply considers only the first term of the series (20) and (25) and substitutes $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$ in (9) to obtain

$$\mathbf{u}_i \approx \mathbf{x}_{u_i} + \sum_{k=1, k \neq i}^N \frac{i\lambda_i C'_{ki} \mathbf{x}_{u_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)} \quad (77a)$$

$$\mathbf{v}_i \approx \mathbf{x}_{v_i} + \sum_{k=1, k \neq i}^N \frac{i\lambda_i C'_{ik} \mathbf{x}_{v_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)} \quad (77b)$$

In deriving the above expression, using an approximate expression of the complex frequencies in (77), we have factorized the denominator as $\omega_k^2 - \lambda_i^2 + i\lambda_i C'_{ik} \approx -(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)$. Using this factorization and retaining the first two terms of the series expression (20) and (25), and substituting $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$ in (9), one obtains

$$\mathbf{u}_i \approx \mathbf{x}_{u_i} + \sum_{k=1, k \neq i}^N \frac{i\lambda_i C'_{ki} \mathbf{x}_{u_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)} - \sum_{k=1, k \neq i}^N \sum_{j=1, j \neq i \neq k}^N \frac{\lambda_i^2 C'_{kj} C'_{ji} \mathbf{x}_{u_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)(\lambda_i - \lambda_j)(\lambda_i + \lambda_j^*)} \quad (78a)$$

$$\mathbf{v}_i \approx \mathbf{x}_{v_i} + \sum_{k=1, k \neq i}^N \frac{i\lambda_i C'_{ik} \mathbf{x}_{v_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)} - \sum_{k=1, k \neq i}^N \sum_{j=1, j \neq i \neq k}^N \frac{\lambda_i^2 C'_{jk} C'_{ij} \mathbf{x}_{v_k}}{(\lambda_i - \lambda_k)(\lambda_i + \lambda_k^*)(\lambda_i - \lambda_j)(\lambda_i + \lambda_j^*)} \quad (78b)$$

The above are second-order approximate expressions for the right and left eigenvectors of the nonconservative system (1).

ACKNOWLEDGMENTS

The writer is indebted to Dr. Atanas Popov for his careful reading of the manuscript and valuable comments on this problem.

APPENDIX II. REFERENCES

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