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Rates of Change of Eigenvalues and Eigenvectors in Damped Dynamic System

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Rates of change of eigenvalues and eigenvectors of a damped linear discrete dynamic system with respect to the system parameters are presented. A non-proportional viscous damping model is assumed. Due to the non-proportional nature of the damping the mode shapes and natural frequencies become complex, and as a consequence the sensitivities of eigenvalues and eigenvectors are also complex. The results are presented in terms of the complex modes and frequencies of the second order system and the use of rather undesirable state-space representation is avoided. The usefulness of the derived expressions is demonstrated by considering an example of a non-proportionally damped two degree-of-freedom system.

Introduction

Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest. Motivation for this kind of study arises, on one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualise the changes in the dynamic response with respect to system parameters. Besides, this kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods. Rates of change of eigenvalues and eigenvectors are useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration. Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems. Possibly, the earliest work on the sensitivity of the eigenvalues was carried out by Rayleigh[1]. In his classic monograph he derived the changes in natural frequencies due to small changes in system parameters. Fox and Kapoor[2] have given exact expressions for rates of change of eigenvalues and eigenvectors with respect to any design variables. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure in its current state, and have been used extensively in a wide range of application areas of structural dynamics. Nelson[3]. proposed an efficient method to calculate eigenvector derivative which requires only the eigenvalue and eigenvector under consideration. A comprehensive review of research on this kind of sensitivity analysis can be obtained in Adelman and Haftka[4].

The above-mentioned analytical methods are based on the *undamped* free vibration of the system. For damped systems, it is well known that unless the damping matrix of the structure is proportional to the inertia and/or stiffness matrices (proportional damping) or can be represented in the series form derived by Caughey[5], the mode shapes of the system will not coincide with the undamped mode shapes. In the presence of general non-proportional viscous damping, the equations of motion in the modal coordinates will be coupled through the off-diagonal terms of the modal damping matrix, and the mode shapes and natural frequencies of the structure will in general be complex. The solution procedures for such non-proportionally damped systems follow mainly two routes: the state space method and approximate methods in 'N-space'. The state-space method (see Newland[6]) although exact in nature requires significant numerical effort for obtaining the eigensolutions as the size of the problem doubles. Moreover, this method also lacks some of the intuitive simplicity of traditional modal analysis. For these reasons there has been considerable research effort to analyse non-proportionally damped structures in N-space. Most of these methods either seek an optimal decoupling of the equations of motion or simply neglect the off-diagonal terms of the modal damping matrix. It may be noted that following such methodologies the mode shapes of the structure will still be real. The accuracy of these methods, other than the light damping assumption, depends upon various factors, for example, frequency separation between the modes, driving frequency, etc. (see Park *et. al.*[7], Gawronski and Sawicki[8] and the references therein for discussions on these topics). A convenient way to avoid the problems which arise due to the use of real normal modes is to incorporate complex modes in the analysis. Apart from the mathematical consistency, conducting experimental modal analysis also one often identifies complex modes: as Sestieri and Ibrahim[9] have put it ' ... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality ! ' But surprisingly in most of the current application areas of structural dynamics which utilise the eigensolution derivatives, *e.g.* modal updating, damage detection, design optimisation and stochastic finite element methods, do not use complex modes in the analysis but rely on the real undamped modes only. This is partly because of the problem of considering appropriate damping model in the structure and partly because of the unavailability of complex eigensolution sensitivities. Although, there has been considerable research efforts towards damping models, sensitivity of complex eigenvalues and eigenvectors with respect to system parameters appear to have received very little attention in the existing literature.

In this paper we determine the rates of change of complex natural frequencies and mode shapes with respect to some set of design variables in non-proportionally damped discrete linear systems. It is assumed that the system does not posses repeated eigenvalues. In section , we briefly discuss the requisite mathematical background on linear multiple-degree-of-freedom discrete systems needed for further derivations. Sensitivity of complex eigenvalues is derived in section in terms of complex modes, natural frequencies and changes in the system property matrices. The approach taken here avoids the use of state-space formulation. In section , sensitivity of complex eigenvectors is derived. The derivation method uses state-space representation of equations of motion for intermediate calculations and then relates the eigenvector sensitivities to the complex eigenvectors of the second order system and to the changes in the system property matrices. In section , a 2 degree-of-freedom system which shows the 'curve-veering' phenomenon has been considered to illustrate the application of the expression for rates of changes of complex eigenvalues and eigenvectors. The results are carefully analysed and compared with presently available sensitivity expressions of undamped real modes.

Background of Analytical Methods

The equations of motion for free vibration of a linear damped discrete system with N degrees of freedom can be written as

$$\mathbf{M}\ddot{\mathbf{u}}(\mathbf{t}) + \mathbf{C}\dot{\mathbf{u}}(\mathbf{t}) + \mathbf{K}\mathbf{u}(t) = 0; \qquad t \geq 0 \qquad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are mass, damping and stiffness matrices, $\mathbf{u}(t) \in \mathbb{R}^N$ is the vector of the generalised coordinates and $t \in \mathbb{R}^+$ denotes time. We seek a harmonic solution of the form

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$\mathfrak{u}(t) = \mathbf{u}\exp[st]$, where $s = \mathrm{i}\omega$ with $i = \sqrt{-1}$ and ω denotes frequency. Substitution of $\mathfrak{u}(t)$ in equation (1) results

$$s^2\mathbf{M}\mathbf{u} + s\mathbf{C}\mathbf{u} + \mathbf{K}\mathbf{u} = 0. \tag{2}$$

This equation is satisfied by the i-th latent root, s_i, and i-th latent vector, $\mathbf{u_i}$, of the $\lambda-$matrix problem (see Lancaster[10]), so that

$$s_i^2\mathbf{M}\mathbf{u}_i + s_i\mathbf{C}\mathbf{u}_i + \mathbf{K}\mathbf{u}_i = 0, \quad \forall\, i = 1\cdots N. \tag{3}$$

In the context of structural dynamics the $\mathbf{u}_i$ are called mode shapes and the natural frequencies λ_i are defined by $s_i = \mathrm{i}\lambda_i$. Unless system (1) is proportionally damped, *i.e.* $\mathbf{C}$ is simultaneously diagonalisable with $\mathbf{M}$ and $\mathbf{K}$ (conditions were derived by Caughey and O'Kelly,[11]), in general $\lambda_i \in \mathbb{C}$ and $\mathbf{u}_i \in \mathbb{C}^N$. Several authors have proposed methods to obtain complex modes and natural frequencies in N-space. Rayleigh[1] considered approximate methods to determine λ_i and $\mathbf{u}_i$ by assuming the elements of $\mathbf{C}$ are small but otherwise general. Using perturbation analysis, Cronin[12] has given a power series expression of eigenvalues and eigenvectors. Recently Woodhouse[13] has extended Rayleigh's analysis to the case of more general linear damping models described by convolution integrals of the generalised coordinates over the damping kernel functions. Bhaskar[14] developed a procedure to exactly obtain λ_i and $\mathbf{u}_i$ by an iterative method. All of these methods calculate the complex modes and frequencies with varying degree of accuracy depending on various factors: for example amount of damping, separation between the modes and number of terms retained in perturbation expansion, etc.

However, complex modes and frequencies can be exactly obtained by the state space (first order) formalisms. Transforming equation (1) into state space form we obtain

$$\dot{\mathfrak{z}}(t) = \mathbf{A}\mathfrak{z}(t) \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{2N\times 2N}$ is the system matrix and $\mathfrak{z}(t) \in \mathbb{R}^{2N}$ response vector in the state space given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \qquad \mathfrak{z}(t) = \begin{Bmatrix} \mathfrak{u}(t) \\ \dot{\mathfrak{u}}(t) \end{Bmatrix}. \tag{5}$$

In the above equation $\mathbf{0} \in \mathbb{R}^{N\times N}$ is the null matrix and $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix. The eigenvalue problem associated with the above equation is now in term of an asymmetric matrix and can be expressed as

$$\mathbf{A}\mathbf{z}_i = s_i\mathbf{z}_i, \qquad \forall i = 1,\cdots,2N \tag{6}$$

where s_i is the $i-$th eigenvalue and $\mathbf{z}_i \in \mathbb{C}^{2N}$ is the $i-$th *right* eigenvector which is related to the eigenvector of the second order system as

$$\mathbf{z}_i = \begin{Bmatrix} \mathbf{u}_i \\ s_i\mathbf{u}_i \end{Bmatrix}. \tag{7}$$

The *left* eigenvector $\mathbf{y}_i \in \mathbb{C}^{2N}$ associated with s_i is defined by the equation

$$\mathbf{y}_i^T\mathbf{A} = s_i\mathbf{y}_i^T \tag{8}$$

where $(\bullet)^T$ denotes matrix transpose. For distinct eigenvalues it is easy to show that the right and left eigenvectors satisfy an orthogonality relationship, that is

$$\mathbf{y}_j^T\mathbf{z}_i = 0; \qquad \forall j \neq i \tag{9}$$

and we may also normalise the eigenvectors so that

$$\mathbf{y}_i^T\mathbf{z}_i = 1. \tag{10}$$

The above two equations imply that the dynamic system defined by equation (4) posses a set of *biorthonormal* eigenvectors. As a special case, when all eigenvalues are distinct, this set forms a *complete* set. Henceforth in our discussion it will be assumed that all the system eigenvalues are distinct.

Rates of Change of Eigenvalues

Suppose the structural system defined in (1) can be described by a set of m parameters (design variables), $\mathbf{g} = \{g_1, g_2, \cdots g_m\}^T \in \mathbb{R}^m$, so that the mass, damping and stiffness matrices become functions of $\mathbf{g}$, that is $\mathbf{M}, \mathbf{C}$ and $\mathbf{K} : \mathbf{g} \rightarrow \mathbb{R}^{N\times N}$. Assume further that the design variables undergo a small change of the form $\mathbf{\Delta g} = \{\Delta g_1, \Delta g_2, \cdots \Delta g_m\}^T \in \mathbb{R}^m$. For this small change, neglecting higher order terms in the Taylor series, the i-th complex eigenvalue can be expressed as

$$\lambda_i^{(c)} \approx \lambda_i + \mathbf{\Delta g}^T\nabla\lambda_i \tag{11}$$

where $\lambda_i^{(c)} \in \mathbb{C}$ denotes the changed complex eigenvalue and $\nabla\lambda_i = \{\lambda_{i,1}, \lambda_{i,2}, \cdots \lambda_{i,m}\}^T \in \mathbb{C}^m$. Here $\lambda_{i,j} = \frac{\partial\lambda_i}{\partial g_j}$ is the rate of change of i-th eigenvalue with respect to g_j, which is to be found. It may be noted that recently Bhaskar[15] has derived an expression for $\lambda_{i,j}$ by converting equation (3) to the state-space from where the eigenvalue problem takes the Duncan form. Here we try to derive an expression of $\lambda_{i,j}$ without going into the state space.

For i-th set, substituting $s_i = \mathrm{i}\lambda_i$, equation (3) can be rewritten as

$$\mathbf{F}_i\mathbf{u}_i = 0 \tag{12}$$

where the regular matrix pencil

$$\mathbf{F}_i \equiv \mathbf{F}(\lambda_i, \mathbf{g}) = -\lambda_i^2\mathbf{M} + \mathrm{i}\lambda_i\mathbf{C} + \mathbf{K}. \tag{13}$$

Premultiplication of equation (12) by $\mathbf{u}_i^T$ yields

$$\mathbf{u}_i^T\mathbf{F}_i\mathbf{u}_i = 0. \tag{14}$$

Differentiating the above equation with respect to g_j one obtains

$$\mathbf{u}_{i,j}^T\mathbf{F}_i\mathbf{u}_i + \mathbf{u}_i^T\mathbf{F}_{i,j}\mathbf{u}_i + \mathbf{u}_i^T\mathbf{F}_i\mathbf{u}_{i,j} = 0 \tag{15}$$

where $\mathbf{F}_{i,j}$ stands for $\frac{\partial\mathbf{F}_i}{\partial g_j}$, and can be obtained by differentiating equation (13) as

$$\mathbf{F}_{i,j} = \left[\lambda_{i,j}\left(\mathrm{i}\mathbf{C} - 2\lambda_i\mathbf{M}\right) - \lambda_i^2\mathbf{M}_{,j} + \mathrm{i}\lambda_i\mathbf{C}_{,j} + \mathbf{K}_{,j}\right]. \tag{16}$$

Now taking the transpose of equation (12) and using the symmetry property of $\mathbf{F}_i$ it can shown that the first and third terms of the equation (15) are zero. Therefore we have

$$\mathbf{u}_i^T\mathbf{F}_{i,j}\mathbf{u}_i = 0 \tag{17}$$

Substituting $\mathbf{F}_{i,j}$ from equation (16) into the above equation one writes

$$-\lambda_{i,j}\,\mathbf{u}_i^T\left(\mathrm{i}\mathbf{C} - 2\lambda_i\mathbf{M}\right)\mathbf{u}_i = \mathbf{u}_i^T\left[-\lambda_i^2\mathbf{M}_{,j} + \mathrm{i}\lambda_i\mathbf{C}_{,j} + \mathbf{K}_{,j}\right]\mathbf{u}_i \tag{18}$$

and again we note that the scalar term

$$\mathbf{u}_i^T\left(\mathrm{i}\mathbf{C} - 2\lambda_i\mathbf{M}\right)\mathbf{u}_i = -\frac{1}{\lambda_i}\left[\mathbf{u}_i^T\mathbf{F}_i\mathbf{u}_i - \mathbf{u}_i^T(\lambda_i^2\mathbf{M} + \mathbf{K})\mathbf{u}_i\right]. \tag{19}$$

Finally, after using equation (14) and combining the above two equations we can have

$$\lambda_{i,j} = \lambda_i\frac{\mathbf{u}_i^T\left[\mathbf{K}_{,j} - \lambda_i^2\mathbf{M}_{,j} + \mathrm{i}\lambda_i\mathbf{C}_{,j}\right]\mathbf{u}_i}{\mathbf{u}_i^T(\lambda_i^2\mathbf{M} + \mathbf{K})\mathbf{u}_i} \tag{20}$$

which is the rate of change of the i-th complex eigenvalue. For the undamped case, when $\mathbf{C} = 0$, $\lambda_i \rightarrow \omega_i$ and $\mathbf{u}_i \rightarrow \mathbf{x}_i$ (ω_i and $\mathbf{x}_i$ are undamped natural frequencies and modes satisfying $\mathbf{K}\mathbf{x}_i = \omega_i^2\mathbf{M}\mathbf{x}_i$), with usual mass normalisation the denominator $\rightarrow 2\omega_i^2$, and we obtain

$$2\omega_i\omega_{i,j} = (\omega_i^2)_{,j} = \mathbf{x}_i^T\left[\mathbf{K}_{,j} - \omega_i^2\mathbf{M}_{,j}\right]\mathbf{x}_i. \tag{21}$$

This is exactly the well-known relationship derived by Fox and Kapoor[2] for the undamped eigenvalue problem. Thus, equation (20) can be viewed as a generalisation of the familiar expression of rates of change of undamped eigenvalues to the damped case. Following observations may be noted from this result

- The derivative of a given eigenvalue requires the knowledge of only the corresponding eigenvalue and eigenvector under consideration, and thus a complete solution of the eigenproblem, or from the experimental point of view, eigensolution determination for *all* the modes is not required.
- Changes in mass and/or stiffness introduce more change in the real part of the eigenvalues whereas changes in the damping introduce more change in the imaginary part.

Since $\lambda_{i,j}$ is complex in equation (20), it can be effectively used to determine the rates of change of Q-factors with respect to the system parameters. For small damping, the Q-factor for the i-th mode is expressed as $Q_i = \Re(\lambda_i)/2\Im(\lambda_i)$, with $\Re(\bullet)$ and $\Im(\bullet)$ denoting real and imaginary parts respectively. Consequently the rate of change can be evaluated from

$$Q_{i,j} = \frac{1}{2}\left[\frac{\Re(\lambda_{i,j})\Im(\lambda_i) - \Re(\lambda_i)\Im(\lambda_{i,j})}{\Im(\lambda_i)^2}\right]. \tag{22}$$

This expression may turn out to be useful since we often directly measure the Q-factors from experiment.

Rates of Change of Eigenvectors

For a small change in the design variables, $\mathbf{\Delta g} \in \mathbb{R}^m$, the i-th complex eigenvector can be expressed as

$$\mathbf{u}_i^{(c)} \approx \mathbf{u}_i + [\nabla \mathbf{u}_i]\,\mathbf{\Delta g} \tag{23}$$

where $\mathbf{u}_i^{(c)} \in \mathbb{C}^N$ denotes the changed complex eigenvector and $[\nabla \mathbf{u}_i] = [\mathbf{u}_{i,1}, \mathbf{u}_{i,2} \cdots \mathbf{u}_{i,m}], \in \mathbb{C}^{N\times m}$, with $\mathbf{u}_{i,j} = \frac{\partial \mathbf{u}_i}{\partial g_j} \in \mathbb{C}^N$ is the i-th complex modal sensitivity matrix. Since $\mathbf{u}_i$ is the first N rows of $\mathbf{z}_i$ (see equation (7)) we first try to derive $\mathbf{z}_{i,j}$ and subsequently obtain $\mathbf{u}_{i,j}$ using their relationship.

Differentiating (6) with respect to g_j one obtains

$$(\mathbf{A} - s_i)\mathbf{z}_{i,j} = -(\mathbf{A}_{,j} - s_{i,j})\mathbf{z}_i. \tag{24}$$

Since it has been assumed that $\mathbf{A}$ has distinct eigenvalues the right eigenvectors, $\mathbf{z}_i$, forms a complete set of vectors. Therefore we can expand $\mathbf{z}_{i,j}$ as

$$\mathbf{z}_{i,j} = \sum_{l=1}^{2N} a_{ijl}\mathbf{z}_l \tag{25}$$

where a_{ijl}, $\forall l = 1, \cdots 2N$ are set of complex constants to be determined. Substituting $\mathbf{z}_{i,j}$ in equation (24) and premultiplying by the left eigenvector $\mathbf{y}_k^T$ one obtains the scalar equation

$$\sum_{l=1}^{2N}(\mathbf{y}_k^T\mathbf{A}\mathbf{z}_l - s_i\mathbf{y}_k^T\mathbf{z}_l)\,a_{ijl} = -\mathbf{y}_k^T\mathbf{A}_{,j}\mathbf{z}_i + s_{i,j}\mathbf{y}_k^T\mathbf{z}_i. \tag{26}$$

Using the orthogonality relationship of left and right eigenvectors from the above equation we obtain

$$a_{ijk} = \frac{\mathbf{y}_k^T\mathbf{A}_{,j}\mathbf{z}_i}{s_i - s_k}; \qquad \forall k = 1, \cdots, 2N; \neq i \tag{27}$$

The a_{ijk} as expressed above is not very useful since it is in terms of the left and right eigenvectors of the first order system. In order to obtain a relationship with the eigenvectors of second order system we assume

$$\mathbf{y}_i = \left\{\begin{array}{c}\mathbf{y_1}_i\\ \mathbf{y_2}_i\end{array}\right\} \tag{28}$$

where $\mathbf{y_1}_i, \mathbf{y_2}_i \in \mathbb{C}^N$. Substituting $\mathbf{y}_i$ in equation (8) and taking transpose one obtains

$$\begin{aligned} s_i\mathbf{y_1}_i &= -\mathbf{K}\mathbf{M}^{-1}\mathbf{y_2}_i\\ s_i\mathbf{y_2}_i &= \mathbf{y_1}_i - \mathbf{C}\mathbf{M}^{-1}\mathbf{y_2}_i\\ \text{or}\quad \mathbf{y_1}_i &= \left[s_i\mathbf{I} + \mathbf{C}\mathbf{M}^{-1}\right]\mathbf{y_2}_i. \end{aligned} \tag{29}$$

Elimination of $\mathbf{y_1}_i$ from the above two equation yields

$$\begin{aligned} s_i\left(s_i\mathbf{y_2}_i + \mathbf{C}\mathbf{M}^{-1}\mathbf{y_2}_i\right) &= -\mathbf{K}\mathbf{M}^{-1}\mathbf{y_2}_i\\ \text{or}\quad \left[s_i^2\mathbf{M} + s_i\mathbf{C} + \mathbf{K}\right]\left(\mathbf{M}^{-1}\mathbf{y_2}_i\right) &= 0. \end{aligned} \tag{30}$$

By comparison of this equation with equation (3) it can be seen that the vector $\mathbf{M}^{-1}\mathbf{y_2}_i$ is parallel to $\mathbf{u}_i$; that is, there exist a non-zero $\beta_i \in \mathbb{C}$ such that

$$\mathbf{M}^{-1}\mathbf{y_2}_i = \beta_i\mathbf{u}_i \quad \text{or} \quad \mathbf{y_2}_i = \beta_i\mathbf{M}\mathbf{u}_i. \tag{31}$$

Now substituting $\mathbf{y_1}_i, \mathbf{y_2}_i$ and using the definition of $\mathbf{z}_i$ from equation (7) into the normalisation condition (10) the scalar *constant* β_i can be obtained as

$$\beta_i = \frac{1}{\mathbf{u}_i^T\left[2s_i\mathbf{M} + \mathbf{C}\right]\mathbf{u}_i}. \tag{32}$$

Using $\mathbf{y_2}_i$ from equation (31) into the second equation of (29) we obtain

$$\mathbf{y}_i = \beta_i\mathbf{P}_i\mathbf{z}_i; \quad \text{where} \quad \mathbf{P}_i = \begin{bmatrix} s_i\mathbf{M} + \mathbf{C} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{M}}{s_i}\end{bmatrix} \in \mathbb{C}^{2N\times 2N}. \tag{33}$$

The above equation along with the definition of $\mathbf{z}_i$ in (7) completely relates the left and right eigenvectors of the first order system to the eigenvectors of the second order system.

The derivative of the system matrix $\mathbf{A}$ can be expressed as

$$\begin{aligned} \mathbf{A}_{,j} &= \begin{bmatrix} \mathbf{0} & \mathbf{0}\\ \left[\mathbf{M}^{-1}\mathbf{K}\right]_{,j} & \left[\mathbf{M}^{-1}\mathbf{C}\right]_{,j}\end{bmatrix}\\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0}\\ -\mathbf{M}^{-2}\mathbf{M}_{,j}\mathbf{K} + \mathbf{M}^{-1}\mathbf{K}_{,j} & -\mathbf{M}^{-2}\mathbf{M}_{,j}\mathbf{C} + \mathbf{M}^{-1}\mathbf{C}_{,j}\end{bmatrix} \end{aligned} \tag{34}$$

from which after some simplifications the numerator of the right hand side of equation (27) can be obtained as

$$\mathbf{y}_k^T\mathbf{A}_{,j}\mathbf{z}_i = -\beta_k\mathbf{u}_k^T\left\{-\mathbf{M}^{-1}\mathbf{M}_{,j}\left[\mathbf{K} + s_i\mathbf{C}\right] + \mathbf{C}_{,j} + \mathbf{K}_{,j}\right\}\mathbf{u}_i. \tag{35}$$

Since $\mathbf{I} = \mathbf{M}\mathbf{M}^{-1}$, $\mathbf{I}_{,j} = \mathbf{M}_{,j}\mathbf{M}^{-1} + \mathbf{M}\left[-\mathbf{M}^{-2}\mathbf{M}_{,j}\right] = 0$ or $\mathbf{M}_{,j}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M}_{,j}$, that is $\mathbf{M}^{-1}$ and $\mathbf{M}_{,j}$ commute in product. Using this property and also from (3) noting that $s_i^2\mathbf{u}_i = -\mathbf{M}^{-1}\left[s_i\mathbf{C} + \mathbf{K}\right]\mathbf{u}_i$ we finally obtain

$$a_{ijk} = -\beta_k\frac{\mathbf{u}_k\left[s_i^2\mathbf{M}_{,j} + s_i\mathbf{C}_{,j} + \mathbf{K}_{,j}\right]\mathbf{u}_i}{s_i - s_k}; \; \forall k = 1, \cdots, 2N; \neq i. \tag{36}$$

This equation relates the a_{ijk} with the complex modes of the second order system.

To obtain a_{iji} we begin with differentiation of the normalisation condition (10) with respect to g_j and obtain the relationship

$$\mathbf{y}_{i,j}^T\mathbf{z}_i + \mathbf{y}_i^T\mathbf{z}_{i,j} = 0. \tag{37}$$

Substitution of $\mathbf{y}_i$ from equation (33) further leads to

$$\beta_i\left\{\mathbf{z}_{i,j}^T\mathbf{P}_i^T\mathbf{z}_i + \mathbf{z}_i^T\mathbf{P}_{i,j}^T\mathbf{z}_i + \mathbf{z}_i^T\mathbf{P}_i^T\mathbf{z}_{i,j}\right\} = 0 \tag{38}$$

where $\mathbf{P}_{i,j}$ can be derived from equation (33) as

$$\mathbf{P}_{i,j} = \begin{bmatrix} s_{i,j}\mathbf{M} + s_i\mathbf{M}_{,j} + \mathbf{C}_{,j} & \mathbf{0}\\ \mathbf{0} & -\frac{\mathbf{M}}{s_i^2}s_{i,j} + \frac{\mathbf{M}_{,j}}{s_i}\end{bmatrix}. \tag{39}$$

Since $\mathbf{P}_i$ is a symmetric matrix, equation (38) can be rearranged as

$$2\left(\beta_i\mathbf{z}_i^T\mathbf{P}_i\right)\mathbf{z}_{i,j} = -\beta_i\mathbf{z}_i^T\mathbf{P}_{i,j}\mathbf{z}_i. \tag{40}$$

Note that the term within the bracket is $\mathbf{y}_i^T$ (see equation (33)). Using the assumed expansion of $\mathbf{z}_{i,j}$ from (27) this equation reads

$$2\mathbf{y}_i^T \sum_{l=1}^{2N} a_{ijl}\mathbf{z}_l = -\beta_i \mathbf{z}_i^T \mathbf{P}_{i,j}\mathbf{z}_i. \tag{41}$$

The left hand side of the above equation can be further simplified

$$\mathbf{z}_i^T \mathbf{P}_{i,j}\mathbf{z}_i = \mathbf{u}_i^T \left[s_{i,j}\mathbf{M} + s_i \mathbf{M}_{,j} + \mathbf{C}_{,j}\right]\mathbf{u}_i + \mathbf{u}_i^T s_i \left[-\frac{\mathbf{M}}{s_i^2}s_{i,j} + \frac{\mathbf{M}_{,j}}{s_i}\right] s_i \mathbf{u}_i = \mathbf{u}_i^T \left[2s_i \mathbf{M}_{,j} + \mathbf{C}_{,j}\right]\mathbf{u}_i. \tag{42}$$

Finally using the orthogonality property of left and right eigenvectors, from equation (41) we obtain

$$a_{iji} = -\frac{1}{2}\frac{\mathbf{u}_i^T \left[2s_i \mathbf{M}_{,j} + \mathbf{C}_{,j}\right]\mathbf{u}_i}{\mathbf{u}_i^T \left[2s_i \mathbf{M} + \mathbf{C}\right]\mathbf{u}_i}. \tag{43}$$

In the above equation a_{iji} is expressed in terms of the complex modes of the second order system. Now recalling the definition of $\mathbf{z}_i$ in (7), from the first N rows of equation (25) one can write

$$\mathbf{u}_{i,j} = a_{iji}\mathbf{u}_i + \sum_{k\neq i}^{2N} a_{ijk}\mathbf{u}_k = -\frac{1}{2}\frac{\mathbf{u}_i^T \left[2s_i \mathbf{M}_{,j} + \mathbf{C}_{,j}\right]\mathbf{u}_i}{\mathbf{u}_i^T \left[2s_i \mathbf{M} + \mathbf{C}\right]\mathbf{u}_i}\mathbf{u}_i - \sum_{k\neq i}^{2N} \beta_k \frac{\mathbf{u}_k \left[s_i^2 \mathbf{M}_{,j} + s_i \mathbf{C}_{,j} + \mathbf{K}_{,j}\right]\mathbf{u}_i}{s_i - s_k}\mathbf{u}_k. \tag{44}$$

We know that for any real symmetric system first order eigenvalues and eigenvectors appear in complex conjugate pairs. Using usual definition of natural frequency, that is, $s_k = \mathrm{i}\lambda_k$ and consequently $s_k^* = -\mathrm{i}\lambda_k^*$, where $(\bullet)^*$ denotes complex conjugate, the above equation can be rewritten in a more convenient form as

$$\mathbf{u}_{i,j} = -\frac{1}{2}\frac{\mathbf{u}_i^T \left[\mathbf{M}_{,j} - \mathrm{i}\mathbf{C}_{,j}/2\lambda_i\right]\mathbf{u}_i}{\mathbf{u}_i^T \left[\mathbf{M} - \mathrm{i}\mathbf{C}/2\lambda_i\right]\mathbf{u}_i}\mathbf{u}_i + \sum_{k\neq i}^{N}\left[\frac{\alpha_k(\mathbf{u}_k^T \tilde{\mathbf{F}}_{i,j}\mathbf{u}_i)\mathbf{u}_k}{\lambda_i - \lambda_k} - \frac{\alpha_k^*({\mathbf{u}_k^*}^T \tilde{\mathbf{F}}_{i,j}^*\mathbf{u}_i^*)\mathbf{u}_k^*}{\lambda_i + \lambda_k^*}\right] \tag{45}$$

where

$$\tilde{\mathbf{F}}_{i,j} = \left[\mathbf{K}_{,j} - \lambda_i^2 \mathbf{M}_{,j} + \mathrm{i}\lambda_i \mathbf{C}_{,j}\right]$$

$$\text{and} \quad \alpha_k = \mathrm{i}\beta_k = \frac{1}{\mathbf{u}_k^T \left[2\lambda_k \mathbf{M} - \mathrm{i}\mathbf{C}\right]\mathbf{u}_k}.$$

This result is a generalisation of the known expression of rates of change of real undamped eigenvectors to complex eigenvectors. The following observations can be made from this result

- Unlike the eigenvalue derivative, the derivative of a given complex eigenvector requires the knowledge of all the other complex eigenvalues and eigenvectors.
- The sensitivity depends very much on the modes whose frequency is close to that of the considered mode.
- Like eigenvalue derivative, changes in mass and/or stiffness introduce more changes in the real part of the eigenvector whereas changes in damping introduce more changes in the imaginary part.

From equation (45), it is easy to see that in the undamped limit $\mathbf{C} \to 0$, and consequently $\lambda_k, \lambda_k^* \to \omega_k$; $\mathbf{u}_k, \mathbf{u}_k^* \to \mathbf{x}_k$; $\tilde{\mathbf{F}}_{i,j}, \tilde{\mathbf{F}}_{i,j}^* \to \left[\mathbf{K}_{,j} - \omega_i^2 \mathbf{M}_{,j}\right]$ and also with usual mass normalisation of the undamped modes $\alpha_k, \alpha_k^* \to \dfrac{1}{2\omega_k}$ reduces the above equation exactly to the corresponding well known expression derived by Fox and Kapoor[2] for derivative of undamped modes.

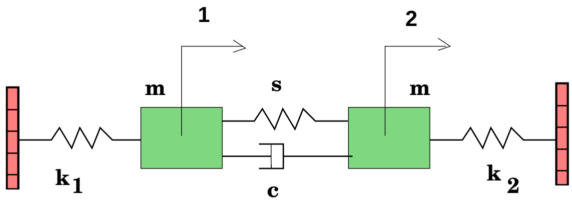

Fig. 1 Two degree-of-system shows veering, $m = 1$ kg, $k_1 = 1000$ N/m, $c = 4.0$ Ns/m

Example: Two Degree of Freedom System

1. Rates of Change of Eigenvalues:

A simple 2 degree-of-freedom system has been considered to illustrate a possible use of the expressions developed so far. Figure 1 shows the example taken together with the numerical values. When eigenvalues are plotted versus a system parameter they create family of 'root loci'. When two loci approach together they may cross or rapidly diverge. The later case is called 'curve veering'. During veering, rapid changes take place in the eigensolutions, as Leissa[16] pointed out ' ... the (eigenfunctions) must undergo violent change – figuratively speaking, a dragonfly one instant, a butterfly the next, and something indescribable in between'. Thus this is an interesting problem for applying the general results derived in this paper.

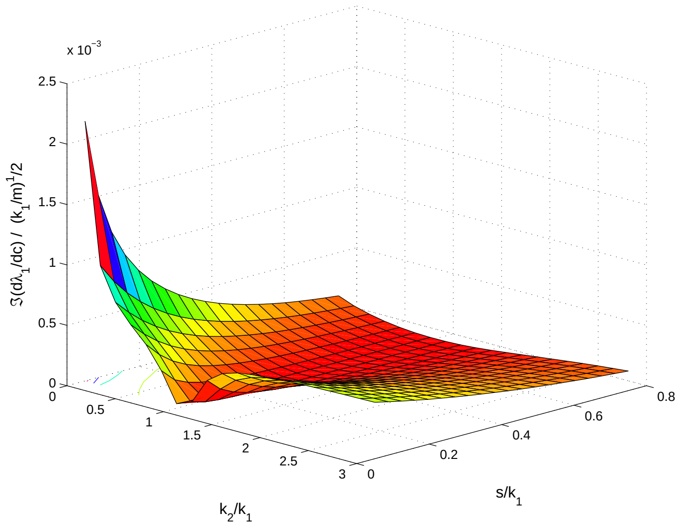

Fig. 2 Imaginary part of rate of change of the first natural frequency, λ_1, with respect to the damping parameter, c

Figure 2 shows the imaginary part (normalised by dividing with $\sqrt{k_1/m}$) of the rate of change of first natural frequency with respect to the damping parameter 'c' over a parameter variation of k_2 and s. This plot was obtained by direct programming of equation (20) in Matlab. The imaginary part has been chosen to be plotted here because a change in damping is expected to contribute a significant change in the imaginary part. The sharp rise of the rate in the low-value region of k_2 and s could be intuitively guessed because there the damper becomes the only 'connecting element' between the two masses and so any change made there is expected to have a strong effect. As we move near to the veering range ($k_2 \approx k_1$ and $s \approx 0$) the story becomes quite different. In the first mode, the two masses move in the same direction, in fact in the limit the motion approaches a 'rigid body mode'. Here, the change is no longer remains sensitive to the changes in connecting the element (*i.e.* only the damper since $s \approx 0$) as hardly any force transmission takes place between the two masses. For this reason we expect a sharp fall in the rate of change as can be noticed along the $s \approx 0$ region of the figure. For the region when s is large, we

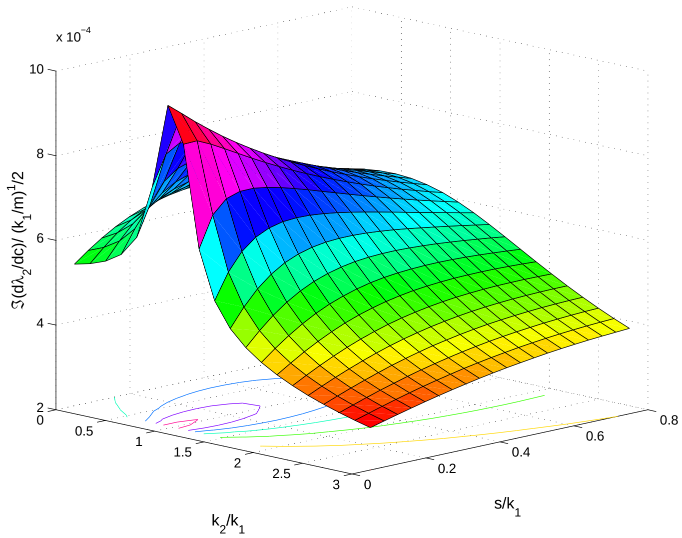

Fig. 3 Imaginary part of rate of change of the second natural frequency, λ_2, with respect to the damping parameter, c

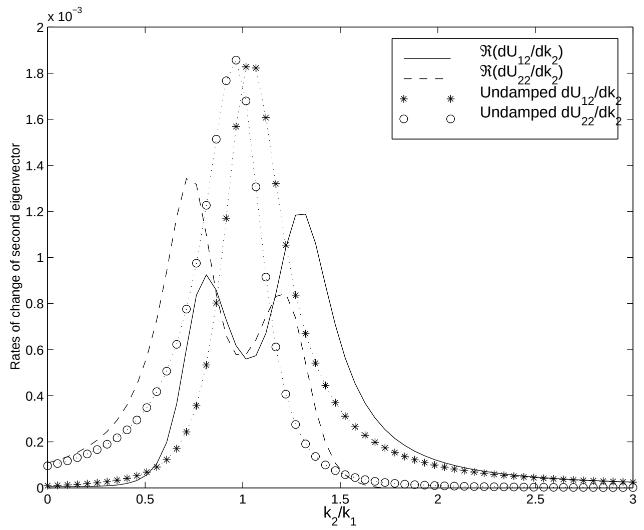

Fig. 5 Real part of rate of change of the second eigenvector with respect to the stiffness parameter k_2

also observe a lower value of rate of change, but the reason there is different. The stiffness element 's' shares most of the force being transmitted between the two masses and hence does not depend much on the change of the value of the damper. A similar plot has been shown in figure 3 for the second natural frequency. Unlike the previous case, here the rate of change increases in the veering range. For the second mode the masses move in the opposite direction and in the veering range the difference between them becomes maximal. Since $s \approx 0$, only the damper is being stretched and as a result of this, a small change there produces a large effect. Thus, the use of equation (20) can provide good physical insight into the problem and can effectively be used in modal updating, damage detection and for design purposes by taking the damping matrix together with the mass and stiffness matrices improving the current practice of using the mass and stiffness matrices only.

Rates of Change of Eigenvectors:

Rates of change of eigenvectors for the problem shown in figure 1

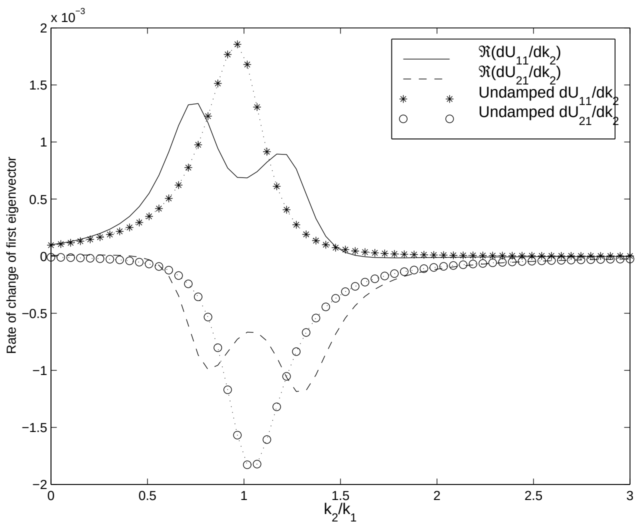

Fig. 4 Real part of rate of change of the first eigenvector with respect to the stiffness parameter k_2

can directly be obtained from equation (45). Here we have focused our attention to calculate the rates of change of eigenvectors with respect to the parameter k_2. Figure 4 shows the real part of rates of change of the first eigenvector normalised by its $\mathcal{L}^2$ norm (that is $\Re\left\{\frac{\mathrm{d}\mathbf{u}_1}{\mathrm{d}k_2}\right\} / \parallel \mathbf{u} \parallel$) plotted over a variation of k_2/k_1 from 0 to 3 for both the coordinates. The value of the spring constant for the connecting spring is kept fixed at $s = 100$ N/m. The real part of the sensitivity of complex eigenvectors has been chosen mainly for two reasons:(a) any change in stiffness is expected to have made more changes in the real part; and (b) to compare it with the corresponding changes of the real undamped modes. Derivative of the first eigenvector (normalised by its $\mathcal{L}^2$ norm) with respect to k_2 corresponding to the undamped system (*i.e* removing the damper) is also shown in the same figure (see the figure legend for details). This is calculated from the expression derived by Fox and Kapoor (1968). Similar plots for the second eigenvector are shown in figure 5. Both of these figures reveal a common feature: around the veering range *i.e.* $0.5 < k_2/k_1 < 1.5$, the damped and the undamped sensitivities show considerable differences while outside this region they almost traces each other. A physical explanation of this phenomenon can be given. For the problem considered here the damper acts as an additional 'connecting element' between the two masses together with the spring 's'. As a result it 'prevents' the system to be close to show a 'strong' veering effect (*i.e.* when $k_2 = k_1$ and the force transmission between the masses is close to zero) and thus reduces the sensitivity of both the modes. However, for the first mode both masses move in the same direction and the damper has less effect compared to second mode where the masses move in the opposite directions and have much greater effect on the sensitivities.

To analyse the results from quantitative point of view at this point it is interesting to look at the variation of the modal Q-factors shown in figure 6. For the first mode Q-factor is quite high (in the order of $\approx 10^3$, *i.e.* quite less damping) near the veering range but still the sensitivities of the undamped mode and that of the real part of the complex mode for both coordinates are quite different. Again, away from the veering range, $k_2/k_1 > 2$, the Q-factor is low but the sensitivities of the undamped mode and that of real part of the complex mode are quite similar. This is opposite to what we normally expect, as the common belief is that, when the Q-factors are high, that is modal dampings are less, the undamped modes and the real part of complex modes should behave similarly and vice versa. For the second mode the Q-factor does not change very much due to a variation of k_2 except becomes bit lower in the vicinity of the veering range. But the difference between the sensitivities of the undamped mode and that of real part of the complex mode for both coordinates changes much more significantly than the Q-factor. For example $Q_2 \approx 9$ for $k_2/k_1 = 1$ and $Q_2 \approx 11$ for $k_2/k_1 = 2$, but the sensitivity of the undamped mode and that of real part of the complex mode is much different when $k_2/k_1 = 1$ and quite similar when $k_2/k_1 = 2$. This demonstrates that even when the Q-factors are similar, the sensitivity of the undamped modes and that of the real part of the complex modes can be significantly different. Thus, use of the expression for derivatives of undamped mode shapes can lead to a significant error even when the damping is very low and the expressions derived in this paper should be used for any kind of study involving such a sensitivity analysis.

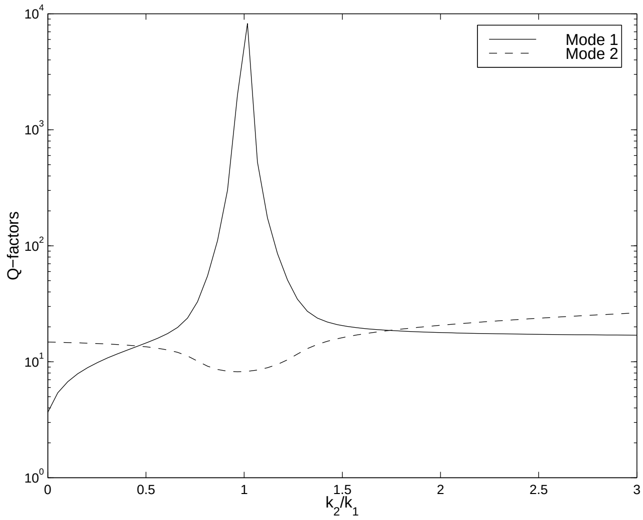

Fig. 6 Q-factors for both the modes

It may be noted that since the expression in equation (20) and (45) has been derived exactly, the numerical results obtained here are also exact within the precision of the arithmetic used for the calculations. The only instance for arriving at an approximate result is when approximate complex frequencies and modes are used in the analysis. However, for this example it was verified that the use of approximate methods to obtain complex eigensolutions in N-space reported in the literature[12,13,14] and the exact ones obtained from the state space method produce negligible discrepancy. Since in most engineering applications we normally do not encounter very high value of damping one can use approximate methods to obtain eigensolusions in N-space in conjunction with the sensitivity expressions derived here. This will allow the analyst to study the rates of change of eigenvalues and eigenvectors of non-classically damped systems in a similar way to those of undamped systems.

Conclusion

Rates of change of eigenvalues and eigenvectors of linear damped discrete systems with respect to the system parameters have been derived. In the presence of general non-proportional viscous damping, the eigenvalues and eigenvectors of the system become complex. The results are presented in terms of changes in mass, damping, stiffness matrices and complex eigensolutions of the second order system so that the state-space representation of equations of motion can be avoided. The expressions derived hereby generalise earlier results on derivatives of eigenvalues and eigenvectors of undamped systems to the damped systems. It was shown through an example problem that use of the expression for derivative of undamped modes can give rise to erroneous results even when the modal damping is quite low. So for a non-classically damped system the expressions for rates of change of eigenvalues and eigenvectors developed in this paper should be used. These complex eigensolution derivatives can be useful in various application areas, for example, finite element model updating, damage detection, design optimisation and system stochasticity analysis relaxing the present restriction to use the real undamped modes only.

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