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Rayleigh Quotient and Dissipative Systems

Rayleigh quotients in the context of linear, nonconservative vibrating systems with viscous and nonviscous dissipative forces are studied in this paper. Of particular interest is the stationarity property of Rayleigh-like quotients for dissipative systems. Stationarity properties are examined based on the perturbation theory. It is shown that Rayleigh quotients with stationary properties exist for systems with proportional viscous and nonviscous damping forces. It is also shown that the stationarity property of Rayleigh quotients in the case of nonproportional damping (viscous and nonviscous) is conditional upon the diagonal dominance of the modal damping matrix. [DOI: 10.1115/1.2910898]

1 Introduction

In his classical treatise on the theory of sound [1], Rayleigh has introduced the notion of a quotient of two quadratics representing the potential and kinetic energies of a vibrating system. Since then, Rayleigh quotient has been widely applied in the analysis of many vibrating systems and their associated linear algebraic eigenvalue problems. Rayleigh quotient provides a variational approach to estimate the eigenvalues of an algebraic, generalized eigenvalue problem, as in the case of determining the natural frequencies of a vibrating system. Numerical methods to solve eigenvalue problems such as the shifted inverse power method rely on the properties of Rayleigh quotients for speedier convergence [2]. Thus, the practical utility of the Rayleigh quotient is wide ranging.

Traditionally, and in many textbooks on vibration analysis [3,4] and linear algebra [2,5], a Rayleigh quotient is defined as a ratio of two quadratics. In the case of a generalized eigenvalue problem involving two real and symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the Rayleigh quotient is defined as follows:

$$R(\boldsymbol{u}) = \frac{\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T\boldsymbol{B}\boldsymbol{u}}, \quad \boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{B}\boldsymbol{v} \quad \text{(EVP)} \tag{1}$$

where the eigenvalue problem is abbreviated as EVP.

The stationarity properties of this "classical" Rayleigh quotient are well established [2]. The objective of the present investigation is to explore whether similar Rayleigh-like quotients with stationary properties exist for a vibrating system with dissipation. Discrete vibrating systems are chosen here for the purpose of illustration; generalization of the results to continuous systems is straightforward.

This paper is presented as follows. Rayleigh quotients for discrete systems are defined in Sec. 2. Three quotients are introduced in the case of a viscously damped system and their stationary properties are investigated in Secs. 3 and 4. Rayleigh quotients in the context of nonviscously damped systems are studied in Sec. 5. The importance of Rayleigh quotients studied here is illustrated in Sec. 6, and main conclusions emerging from this study are summarized in Sec. 7. Throughout this study, the terms modes and eigenvectors are used interchangeably.

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2 Rayleigh Quotients for Discrete Systems

Small oscillations of a discrete, linear vibrating system with viscous damping about its equilibrium position are governed by the following equations of motion:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f} \tag{2}$$

where the matrices $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{C}$ are, respectively, the mass, stiffness, and damping matrices and the vectors $\boldsymbol{x}$ and $\boldsymbol{f}$ denote the displacement response and applied forces, respectively. In the absence of damping and applied forces, the above equation simplifies to

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0} \tag{3}$$

The above equation leads to a linear, algebraic eigenvalue problem for the natural frequencies of free vibration, denoted by ω, given as follows:

$$\boldsymbol{K}\boldsymbol{u} = \lambda \boldsymbol{M}\boldsymbol{u} \tag{4}$$

where the eigenvalue λ is related to the frequency via $\omega = \sqrt{\lambda}$. Here, the positive branch of the square-root operation is assumed. $\boldsymbol{u}$ is the eigenvector (mode shape) associated with the eigenvalue λ (or vibration mode with natural frequency ω). For linear systems that obey Rayleigh's reciprocity principle, the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are symmetric. This implies that the solutions of the eigenvalue problem in Eq. (4), λ and $\boldsymbol{u}$, are real.

In the context of vibration analysis of undamped systems, the two quadratic functions in the Rayleigh quotient assume the physical meaning of the kinetic and potential energies. Thus, associated with any admissible deformation vector ($\boldsymbol{\phi}$), one can define the following quantities:

$$\mathcal{U} = \boldsymbol{\phi}^T\boldsymbol{K}\boldsymbol{\phi}, \quad \mathcal{T} = \boldsymbol{\phi}^T\boldsymbol{M}\boldsymbol{\phi}$$

$$R(\boldsymbol{\phi}) = \frac{\mathcal{U}}{\mathcal{T}} = \frac{\boldsymbol{\phi}^T\boldsymbol{K}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\boldsymbol{M}\boldsymbol{\phi}} \tag{5}$$

where $\mathcal{T}$ and $\mathcal{U}$ are the kinetic and potential energies of the system and R is the classical Rayleigh quotient.

However, when systems with dissipation are considered, one is faced with three quadratics. In this situation, one can define *three* quotients as follows:

$$\mathcal{U} = \boldsymbol{\phi}^T\boldsymbol{K}\boldsymbol{\phi}, \quad \mathcal{T} = \boldsymbol{\phi}^T\boldsymbol{M}\boldsymbol{\phi}, \quad \mathcal{D} = \boldsymbol{\phi}^T\boldsymbol{C}\boldsymbol{\phi}$$

$$R_1(\boldsymbol{\phi}) = \frac{\mathcal{U}}{\mathcal{T}} = \frac{\boldsymbol{\phi}^T\boldsymbol{K}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\boldsymbol{M}\boldsymbol{\phi}}$$

$$R_2(\boldsymbol{\phi}) = \frac{\mathcal{D}}{\mathcal{T}} = \frac{\boldsymbol{\phi}^T\boldsymbol{C}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\boldsymbol{M}\boldsymbol{\phi}}$$

$$R_3(\boldsymbol{\phi}) = \frac{\mathcal{D}}{\mathcal{U}} = \frac{\boldsymbol{\phi}^T \boldsymbol{C} \boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{K} \boldsymbol{\phi}} \tag{6}$$

Note that for Rayleigh quotient to be finite, the denominator terms in the above equation should not be equal to zero. This requires that $\boldsymbol{M}$ be positive definite for R_1 and R_2 to be finite, and $\boldsymbol{K}$ be positive definite for R_3 to be finite. For majority of vibrating systems, $\boldsymbol{M}$ is positive definite while $\boldsymbol{K}$ need not be. Thus, the existence of R_3 is case specific.

It is the objective of this work to investigate the stationarity properties of the quotients defined in Eqs. (5) and (6). The proof of stationary property of the quotient defined in Eq. (5) is well known [2,5,6]. However, it is repeated here for the sake of completeness and also since the proof of stationarity for other quotients closely follows a similar procedure.

2.1 Stationarity of $R(\boldsymbol{\phi})$. Let a vector $\boldsymbol{\phi}$ be chosen such that it is close to one of the eigenvectors (modes) $\boldsymbol{u}_r$ of the system so that we can express $\boldsymbol{\phi}$ as

$$\boldsymbol{\phi} = \sum_i c_i \boldsymbol{u}_i = \boldsymbol{u}_r + \sum_{i\neq r} \epsilon_i \boldsymbol{u}_i, \quad \epsilon_i = \frac{c_i}{c_r} \ll 1 \tag{7}$$

where ϵ_i is a small real quantity. Now, the Rayleigh quotient reads

$$R(\boldsymbol{\phi}) = \frac{\boldsymbol{\phi}^T \boldsymbol{K} \boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi}} = \frac{\boldsymbol{u}_r^T \boldsymbol{K} \boldsymbol{u}_r + 2\sum_{i\neq r} \epsilon_i \boldsymbol{u}_i^T \boldsymbol{K} \boldsymbol{u}_r + O(\epsilon^2)}{\boldsymbol{u}_r^T \boldsymbol{M} \boldsymbol{u}_r + 2\sum_{i\neq r} \epsilon_i \boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_r + O(\epsilon^2)} \tag{8}$$

here, the symmetry of $\boldsymbol{M}$ and $\boldsymbol{K}$ is assumed. Due to the orthogonality properties of the eigenvectors [2,5],

$$\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_j = \delta_{ij}, \quad \boldsymbol{u}_i^T \boldsymbol{K} \boldsymbol{u}_j = 0, \quad \boldsymbol{u}_i^T \boldsymbol{K} \boldsymbol{u}_i = \lambda_i \tag{9}$$

Equation (8) simplifies to

$$R(\boldsymbol{\phi}) = \frac{\lambda_r + O(\epsilon^2)}{1 + O(\epsilon^2)} = \lambda_r(1 + O(\epsilon^2)) \tag{10}$$

The above result proves the stationarity of the Rayleigh quotient, i.e., first order changes in $\boldsymbol{\phi}$ lead to second order changes in $R(\boldsymbol{\phi})$. When $\boldsymbol{\phi}$ is close to one of the eigenvectors, the corresponding value of the quotient is stationary. Further choosing the first eigenvector as the trial vector $\boldsymbol{\phi}$ leads to a minimum value of $R(\boldsymbol{\phi})$. $R(\boldsymbol{\phi})$ is maximum when the trial vector is close to the eigenvector corresponding to the highest eigenvalue. For intermediate eigenvectors, $R(\boldsymbol{\phi})$ is neither a minimum nor a maximum, i.e., $R(\boldsymbol{\phi})$ is at a saddle point. A mini-max (or inf-sup) principle due to Courant and Fischer applies in this case [2,6].

3 Proportional Damping

We consider first the case of proportional damping. Here, proportional damping is defined in the sense that the same vector $\boldsymbol{\phi}$ simultaneously diagonalizes the three quadratics $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{D}$. In other words, the three matrices $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{C}$ can be simultaneously diagonalized. Although a viscous damping matrix of the form $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$ is the most widely understood model of a proportional damping model, it is only a subset of a wider class of models [7]. The necessary and sufficient conditions for proportional damping are established in Ref. [7] and revisited in Refs. [8–11]. Adhikari [10] showed that viscously damped linear systems will have classical normal modes if and only if the damping matrix can be represented by

(a) $\boldsymbol{C}=\boldsymbol{M}\beta_1(\boldsymbol{M}^{-1}\boldsymbol{K})+\boldsymbol{K}\beta_2(\boldsymbol{K}^{-1}\boldsymbol{M})$
or
(b) $\boldsymbol{C}=\beta_3(\boldsymbol{K}\boldsymbol{M}^{-1})\boldsymbol{M}+\beta_4(\boldsymbol{M}\boldsymbol{K}^{-1})\boldsymbol{K}$

where $\beta_i(\bullet)$ are smooth analytic functions in the neighborhood of all the eigenvalues of their argument matrices. Rayleigh's result can be obtained directly from this "generalized proportional damping" as a special case by choosing each matrix function $\beta_i(\bullet)$ as a real scalar times an identity matrix, that is $\beta_i(\bullet)=\alpha_i\boldsymbol{I}$. In the case of proportionally damped systems, the eigenvectors are real but the eigenvalues are not, i.e., the undamped modes are also the modes of the proportionally damped system. Thus, the proof of stationarity of the first Rayleigh quotient $R_1(\boldsymbol{\phi})$ is the same as that given in Sec. 2.1.

We consider the second Rayleigh quotient associated with any admissible deformation vector $\boldsymbol{\phi}$ as defined in Eq. (7),

$$R_2(\boldsymbol{\phi}) = \frac{\boldsymbol{\phi}^T \boldsymbol{C} \boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi}} = \frac{\boldsymbol{u}_r^T \boldsymbol{C} \boldsymbol{u}_r + 2\sum_{i\neq r} \epsilon_i \boldsymbol{u}_i^T \boldsymbol{C} \boldsymbol{u}_r + O(\epsilon^2)}{\boldsymbol{u}_r^T \boldsymbol{M} \boldsymbol{u}_r + 2\sum_{i\neq r} \epsilon_i \boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_r + O(\epsilon^2)} \tag{11}$$

here, the symmetry of $\boldsymbol{M}$ and $\boldsymbol{C}$ is assumed. Due to the orthogonality properties of the eigenvectors,

$$\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_j = \delta_{ij} \tag{12}$$

We define

$$\boldsymbol{u}_i^T \boldsymbol{C} \boldsymbol{u}_j = C'_{ij}, \quad \boldsymbol{u}_i^T \boldsymbol{C} \boldsymbol{u}_i = C'_{ii} \tag{13}$$

With the above definition, Eq. (11) can be expressed as

$$R_2(\boldsymbol{\phi}) = \frac{C'_{rr} + 2\Sigma_{i\neq r}\epsilon_i C'_{ir} + O(\epsilon^2)}{1 + O(\epsilon^2)} \tag{14}$$

When damping is proportional, the matrix $\boldsymbol{C}'_{ir}$ is diagonal, i.e., $\boldsymbol{C}'_{ir}=0$ for $i\neq r$. In this case, the above equation simplifies to

$$R_2(\boldsymbol{\phi}) = \frac{C'_{rr} + O(\epsilon^2)}{1 + O(\epsilon^2)} = C'_{rr}(1 + O(\epsilon^2)) \tag{15}$$

which proves the stationarity of Rayleigh quotient in the case of a proportionally damped system.

Similar proof can be constructed for $R_3(\boldsymbol{\phi})$. The equation corresponding to Eq. (15) in this case will read as

$$R_3(\boldsymbol{\phi}) = \frac{C'_{rr} + O(\epsilon^2)}{\omega_r^2 + O(\epsilon^2)} = \frac{C'_{rr}}{\omega_r^2}(1 + O(\epsilon^2)) \tag{16}$$

4 Nonproportional Damping

We consider the case of nonproportional damping wherein the damping matrix $\boldsymbol{C}$ cannot be diagonalized simultaneously with $\boldsymbol{M}$ and $\boldsymbol{K}$ matrices. Consequently, the vector $\boldsymbol{\phi}$ is not necessarily real. Vibrating systems with nonproportional damping are known to possess complex modes in general. Physically, the complex modes represent *nearly* standing waves. For systems with small dissipation, a perturbation theory originally due to Rayleigh [1] can be used to represent the complex modes in terms of the real modes of the undamped system.

According to the first order perturbation theory [12], the complex modes of a viscously damped system are related to the corresponding undamped modes by

$$z_n \approx \boldsymbol{u}_n + \iota \sum_{k\neq n} \alpha_{kn} \boldsymbol{u}_k \quad \text{where } \alpha_{kn} = \frac{\omega_n C'_{kn}}{\omega_n^2 - \omega_k^2} < 1 \tag{17}$$

The undamped modes are mass normalized i.e., $\boldsymbol{u}_n^T \boldsymbol{M} \boldsymbol{u}_n=1$. In the above equation, $\boldsymbol{C}'$ is the damping matrix in modal coordinates, i.e., $C'_{kn}=\boldsymbol{u}_k^T \boldsymbol{C} \boldsymbol{u}_n$. The assumption in the perturbation theory is that the terms of the order α_{kn}^2 are very small and hence negligible.

When k and n refer to two adjacent modes, the coefficient α_{kn} can be related to the modal overlap factor defined as $\mu_{kn} \equiv \zeta_n\omega_n/(\omega_k-\omega_n)$ and the ratio $\gamma_{kn}=C'_{kn}/C'_{nn}$ by $\alpha_{kn} \approx (1/2)\mu_{kn}\gamma_{kn}$. Notice that γ_{kn} is a measure of the diagonal dominance of the $\boldsymbol{C}'$ matrix. μ_{kn} is a measure of the spacing of adjacent modes normalized with respect to the half power bandwidth of each mode. Thus, significantly complex modes are to be expected when the modal damping matrix is not diagonally dominant and the modal overlap is not small. Unless the modal overlap factor is of the order of unity, the second and higher order powers of the α_{kn} can be safely ignored. If not, then the perturbation expansion has to be extended suitably until the imaginary part of the complex mode converges. Adequacy of the first order theory for systems with small damping has been shown in Ref. [12].

Since the complex eigenvectors z_i, $i=1\cdots n$ form the complete basis in an n dimensional complex vector space, any arbitrary complex vector $\boldsymbol{\psi}$ can be written as

$$\boldsymbol{\psi}=\sum_i c_i z_i \tag{18}$$

We select a vector close to z_r, which can be written as

$$\boldsymbol{\psi}=z_r+\sum_{i\neq r}\epsilon_i z_i, \quad |\epsilon_i|=\left|\frac{c_i}{c_r}\right|\ll 1. \tag{19}$$

We consider the first real valued Rayleigh quotient associated with the above trial vector,

$$R_1(\boldsymbol{\psi})=\frac{\boldsymbol{\psi}^H\boldsymbol{K}\boldsymbol{\psi}}{\boldsymbol{\psi}^H\boldsymbol{M}\boldsymbol{\psi}}=\frac{z_r^H\boldsymbol{K}z_r+2\sum_{i\neq r}\Re(\epsilon_i)z_i^H\boldsymbol{K}z_r+O(|\epsilon|^2)}{z_r^H\boldsymbol{M}z_r+2\sum_{i\neq r}\Re(\epsilon_i)z_i^H\boldsymbol{M}z_r+O(|\epsilon|^2)} \tag{20}$$

Noting the orthogonality properties given in Eq. (9), one can deduce the following equations:

$$z_r^H\boldsymbol{M}z_r=1+O(\alpha^2) \tag{21}$$

and

$$z_i^H\boldsymbol{M}z_r=O(\alpha^2) \tag{22}$$

Similarly with $\boldsymbol{K}$, one can show

$$z_r^H\boldsymbol{K}z_r=\omega_r^2+O(\alpha^2) \tag{23}$$

and

$$z_i^H\boldsymbol{K}z_r=O(\alpha^2) \tag{24}$$

Substituting Eqs. (21)–(24) in Eq. (20), one obtains

$$R_1(\boldsymbol{\psi})=\frac{\omega_r^2+O(\alpha^2)+O(\epsilon^2)}{1+O(\alpha^2)+O(\epsilon^2)}\approx\omega_r^2(1-O(\epsilon^2)) \tag{25}$$

which proves the stationarity of $R_1(\boldsymbol{\psi})$.

We consider the second real valued Rayleigh quotient defined as follows:

$$R_2(\boldsymbol{\psi})=\frac{z^H\boldsymbol{C}z}{z^H\boldsymbol{M}z} \tag{26}$$

Substituting Eq. (17) in the above equation leads to

$$R_2(\boldsymbol{\psi})=\frac{\boldsymbol{\psi}^H\boldsymbol{C}\boldsymbol{\psi}}{\boldsymbol{\psi}^H\boldsymbol{M}\boldsymbol{\psi}}=\frac{z_r^H\boldsymbol{C}z_r+2\sum_{i\neq r}\Re(\epsilon_i)z_i^H\boldsymbol{C}z_r+O(|\epsilon|^2)}{z_r^H\boldsymbol{M}z_r+2\sum_{i\neq r}\Re(\epsilon_i)z_i^H\boldsymbol{M}z_r+O(|\epsilon|^2)} \tag{27}$$

Now $z_r^H\,\boldsymbol{C}z_r$ can be expanded as

$$\begin{aligned}z_r^H\boldsymbol{C}z_r&=\left[\boldsymbol{u}_r^T-\iota\sum_{k\neq r}\alpha_{kr}\boldsymbol{u}_k^T\right]\boldsymbol{C}\left[\boldsymbol{u}_r+\iota\sum_{k\neq r}\alpha_{kr}\boldsymbol{u}_k\right]\\&=C'_{rr}-\iota\sum_{k\neq r}\alpha_{kr}[\boldsymbol{u}_k^T\boldsymbol{C}\boldsymbol{u}_r-\boldsymbol{u}_r^T\boldsymbol{C}\boldsymbol{u}_k]+O(\alpha^2)=C'_{rr}+O(\alpha^2)\end{aligned} \tag{28}$$

Note that $\boldsymbol{C}$ is assumed to be symmetric in simplifying the above equation. Similarly, one can write

$$\begin{aligned}z_i^H\boldsymbol{C}z_r&=\left[\boldsymbol{u}_i^T-\iota\sum_{k\neq i}\alpha_{ki}\boldsymbol{u}_k^T\right]\boldsymbol{C}\left[\boldsymbol{u}_r+\iota\sum_{k\neq r}\alpha_{kr}\boldsymbol{u}_k\right]\\&=C'_{ir}-\iota\sum_{k\neq i}\alpha_{ki}C'_{kr}+\iota\sum_{k\neq r}\alpha_{kr}C'_{ik}+O(\alpha^2)\\&=C'_{ir}+O(\alpha)+O(\alpha^2)\end{aligned} \tag{29}$$

Substituting Eqs. (21) and (22), and Eqs. (28) and (29) in Eq. (27), one obtains

$$R_2(\boldsymbol{\psi})=\frac{C'_{rr}+2\sum_{i\neq r}\Re(\epsilon_i)C'_{ir}+O(\epsilon)O(\alpha)+O(\epsilon^2)+O(\alpha^2)}{1+O(\epsilon^2)+O(\alpha^2)} \tag{30}$$

It can be seen that first order changes in $\boldsymbol{\psi}$ lead to first order changes in $R_2(\boldsymbol{\psi})$. However, if the modal damping matrix is diagonally dominant, i.e.,

$$\frac{C'_{ir}}{C'_{rr}}\ll 1 \tag{31}$$

then first order changes in $\boldsymbol{\psi}$ lead to second order changes in $R_2(\boldsymbol{\psi})$. In this case, stationarity of the Rayleigh quotient is obtained.

Returning to the third quotient $R_3(\boldsymbol{\psi})$,

$$\begin{aligned}R_3(\boldsymbol{\psi})&=\frac{\boldsymbol{\psi}^H\boldsymbol{C}\boldsymbol{\psi}}{\boldsymbol{\psi}^H\boldsymbol{K}\boldsymbol{\psi}}\\&=\frac{C'_{rr}+2\sum_{i\neq r}\Re(\epsilon_i)C'_{ir}+O(\epsilon)O(\alpha)+O(\epsilon^2)+O(\alpha^2)}{\omega_r^2+O(\epsilon^2)+O(\alpha^2)}\end{aligned} \tag{32}$$

The above quotient is not stationary in general. However, when the modal damping matrix is diagonally dominant in accordance with Eq. (31), stationarity of $R_3(\boldsymbol{\psi})$ can be shown as earlier (see Sec. 3).

In the case of a complex vector $\boldsymbol{\psi}$, one is also tempted to define complex valued Rayleigh quotients by replacing the Hermitian transpose (complex conjugate transpose) with the ordinary transpose operator. The stationarity property of these complex valued quotients, however, cannot be shown. Hence, the discussion of these quotients will not be pursued any further.

5 Nonviscous Damping

In this section, we consider general linear damping models, described by convolution integrals of the generalized coordinates over appropriate kernel functions. The equation of motion of a N degrees-of-freedom nonviscously damped system is given by

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t)+\int_{-\infty}^{t}\boldsymbol{G}(t-\tau)\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau+\boldsymbol{K}\boldsymbol{x}(t)=\boldsymbol{f}(t) \tag{33}$$

Here, $\boldsymbol{G}(t)$ is a $N\times N$ matrix of kernel functions. It will be assumed that $\boldsymbol{G}(t)$ is a symmetric matrix so that reciprocity automatically holds. In the special case when $\boldsymbol{G}(t)=\boldsymbol{C}\delta(t)$, where $\delta(t)$ is the Dirac delta function and $\boldsymbol{C}$ is a $N\times N$ matrix, Eq. (33) reduces to the standard form for viscous damping.

Taking the Fourier transform of Eq. (33), the eigenvalue equation can be expressed as

$$-\lambda_n^2\boldsymbol{M}z_n+\iota\lambda_n\boldsymbol{G}(\lambda_n)z_n+\boldsymbol{K}z_n=\boldsymbol{0} \tag{34}$$

where $\boldsymbol{G}(\lambda)$ is the Fourier transform of $\boldsymbol{G}(t)$. In general, $\boldsymbol{G}(\lambda)$ is a complex valued function of λ. For viscously damped system, $\boldsymbol{G}(\lambda)=\boldsymbol{C},\forall\lambda$. Equation (34) is a nonlinear eigenvalue problem. In contrast with the viscously damped case, the number of eigenvalues will not necessarily be equal to $2N$, since additional eigenvalues may be introduced by the form of the functions $\boldsymbol{G}(\lambda_n)$. Woodhouse [12] and Adhikari [13] have treated this problem using a first order perturbation method assuming the damping to be small. We suppose the undamped problem has eigenvalues (natural frequencies) ω_n and eigenvectors (modes) $\boldsymbol{u}_n$. The complex eigenvalues can then be expressed as

$$\lambda_n \approx \pm \omega_n + \iota G'_{nn}(\pm \omega_n)/2 \tag{35}$$

where $G'_{kl}(\omega_n)=\boldsymbol{u}_k^T \boldsymbol{G}(\omega_n)\boldsymbol{u}_l$ is the frequency dependent damping matrix expressed in normal coordinates. Since the inverse Fourier transform of $\boldsymbol{G}(\omega)$ must be real, it must satisfy the condition $\boldsymbol{G}(-\omega)=\boldsymbol{G}(\omega)^*$, where $(\bullet)^*$ denotes complex conjugation. It follows that the eigenvalues of the generally damped system appear in pairs λ and $-\lambda^*$ (unless λ is purely imaginary). The first order approximate expression for the complex eigenvectors can be obtained in a way similar to that used for the viscously damped system (as was first given by Rayleigh [1]). The result is

$$z_n \approx \boldsymbol{u}_n + \iota \sum_{k\neq n} \beta_{kn}\boldsymbol{u}_k \quad \text{where } \beta_{kn} = \frac{\omega_n G'_{kn}(\omega_n)}{(\omega_n^2 - \omega_k^2)} \tag{36}$$

Note that the eigenvectors also appear in complex conjugate pairs. Since, in general, $G'_{kn}(\omega_n)$ will be complex, in contrast to the viscously damped case, the real part of natural frequencies and mode shapes do not coincide with the undamped ones. Adequacy of the first order theory for systems with small damping has been investigated in Refs. [12,13].

Since the complex eigenvectors $z_i, i=1\cdots N$ form the complete basis of an N-dimensional complex vector space, any arbitrary complex vector $\boldsymbol{\psi}$ can be expressed as

$$\boldsymbol{\psi} = \sum_i c_i z_i \tag{37}$$

We consider a vector close to z_r, which can be written as

$$\boldsymbol{\psi} = z_r + \sum_{i\neq r} \epsilon_i z_i, \quad |\epsilon_i| = \left|\frac{c_i}{c_r}\right| \ll 1 \tag{38}$$

Replacing the matrix $\boldsymbol{G}(\omega_r)$ with $\boldsymbol{M}$ and noting the orthogonality properties given in Eq. (9), one obtains

$$z_r^H \boldsymbol{M} z_r = 1 + O(|\beta|^2) \tag{39}$$

$$z_i^H \boldsymbol{M} z_r = O(|\beta|^2) \tag{40}$$

Similarly with $\boldsymbol{K}$, one obtains

$$z_r^H \boldsymbol{K} z_r = \omega_r^2 + O(|\beta|^2) \tag{41}$$

$$z_i^H \boldsymbol{K} z_r = O(|\beta|^2) \tag{42}$$

We consider the first Rayleigh quotient

$$R_1(\boldsymbol{\psi}) = \frac{\boldsymbol{\psi}^H \boldsymbol{K}\boldsymbol{\psi}}{\boldsymbol{\psi}^H \boldsymbol{M}\boldsymbol{\psi}} = \frac{z_r^H \boldsymbol{K} z_r + 2\sum_{i\neq r} \Re(\epsilon_i) z_i^H \boldsymbol{K} z_r + O(|\epsilon|^2)}{z_r^H \boldsymbol{M} z_r + 2\sum_{i\neq r} \Re(\epsilon_i) z_i^H \boldsymbol{M} z_r + O(|\epsilon|^2)} \tag{43}$$

Substituting Eqs. (39)–(42) in the above equation, one obtains

$$R_1(\boldsymbol{\psi}) = \frac{\omega_r^2 + O(|\beta|^2) + O(\epsilon^2)}{1 + O(|\beta|^2) + O(\epsilon^2)} \approx \omega_r^2(1 - O(\epsilon^2)) \tag{44}$$

which proves the stationarity of $R_1(\boldsymbol{\psi})$.

The second and third Rayleigh quotients involving the damping term need to be carefully defined. The difference between the viscous and the nonviscous case is that the (effective) damping matrix for the nonviscous case is complex valued and a function of frequency. Therefore, in order to define a meaningful Rayleigh quotient, we need to select a value of frequency. If we are interested in studying the stationary behavior of rth mode, then it is logical to select the frequency value as ω_r. We define the real valued Rayleigh quotient for a nonviscously damped system as

$$R_2(\boldsymbol{\psi}, \omega_r) = \frac{|\boldsymbol{\psi}^H \boldsymbol{G}(\omega_r)\boldsymbol{\psi}|}{\boldsymbol{\psi}^H \boldsymbol{M}\boldsymbol{\psi}} \tag{45}$$

For a viscously damped system $\boldsymbol{G}(\omega_r)=\boldsymbol{C}, \forall\, r$ and because $\boldsymbol{C}$ is a real matrix, Eq. (45) reduces to Eq. (26) as a special case. Therefore, Eq. (45) can be viewed as a generalization of the Rayleigh quotient defined in Eq. (26).

Substituting Eq. (36) in the above equation leads to

$$\begin{aligned} R_2(\boldsymbol{\psi}) &= \frac{\boldsymbol{\psi}^H \boldsymbol{G}(\omega_r)\boldsymbol{\psi}}{\boldsymbol{\psi}^H \boldsymbol{M}\boldsymbol{\psi}} \\ &= \frac{|z_r^H \boldsymbol{G}(\omega_r) z_r + 2\sum_{i\neq r} \Re(\epsilon_i) z_i^H \boldsymbol{G}(\omega_r) z_r + O(|\epsilon|^2)|}{z_r^H \boldsymbol{M} z_r + 2\sum_{i\neq r} \Re(\epsilon_i) z_i^H \boldsymbol{M} z_r + O(|\epsilon|^2)} \end{aligned} \tag{46}$$

The first term in numerator can be expressed as

$$\begin{aligned} z_r^H \boldsymbol{G}(\omega_r) z_r &= \left[\boldsymbol{u}_r^T - \iota\sum_{k\neq r} \beta_{kr}^* \boldsymbol{u}_k^T\right]\boldsymbol{G}(\omega_r)\left[\boldsymbol{u}_r + \iota\sum_{k\neq r}\beta_{kr}\boldsymbol{u}_k\right] \\ &= \boldsymbol{u}_r^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r - \iota\sum_{k\neq r}(\beta_{kr}^* - \beta_{kr})[\boldsymbol{u}_k^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r \\ &\quad - \boldsymbol{u}_r^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_k] + O(|\beta|^2) \\ &= G'_{rr} + 2\iota\sum_{k\neq r}\Im(\beta_{kr})[\boldsymbol{u}_k^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r \\ &\quad - \boldsymbol{u}_r^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_k] + O(|\beta|^2) \\ &= G'_{rr} + O(\Im(\beta)) + O(|\beta|^2) \end{aligned} \tag{47}$$

where $G'_{rr}=\boldsymbol{u}_r^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r$. Note that $\boldsymbol{G}(\omega_r)$ is assumed to be symmetric in simplifying the above equation. From the second term in the numerator of Eq. (46), one has

$$\begin{aligned} z_i^H \boldsymbol{G}(\omega_r) z_r &= \left[\boldsymbol{u}_i^T - \iota\sum_{k\neq i} \beta_{ki}^* \boldsymbol{u}_k^T\right]\boldsymbol{G}(\omega_r)\left[\boldsymbol{u}_r + \iota\sum_{k\neq r}\beta_{kr}\boldsymbol{u}_k\right] \\ &= \boldsymbol{u}_i^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r - \iota\sum_{k\neq i}\beta_{ki}^* \boldsymbol{u}_k^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r \\ &\quad + \iota\sum_{k\neq r}\beta_{kr}\boldsymbol{u}_i^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_k + O(|\beta|^2) \\ &= G'_{ir} + O(\beta) + O(|\beta|^2) \end{aligned} \tag{48}$$

where $G'_{ir}=\boldsymbol{u}_i^T \boldsymbol{G}(\omega_r)\boldsymbol{u}_r$.

Substituting Eqs. (47), (46), (45), (44), (43), (42), (41), and (40) in Eq. (46), one obtains

$$R_2(\boldsymbol{\psi}) = \frac{|G'_{rr} + 2\sum_{i\neq r}\Re(\epsilon_i)G'_{ir} + O(\epsilon)O(\beta) + O(\epsilon^2) + O(\Im(\beta)) + O(|\beta|^2)|}{1 + O(\epsilon^2) + O(|\beta|^2)}$$
$$< \frac{|G'_{rr}| + 2\Sigma_{i\neq r}\Re(\epsilon_i)|G'_{ir}| + O(\epsilon)O(\beta) + O(\epsilon^2) + O(\Im(\beta)) + O(|\beta|^2)}{1 + O(\epsilon^2) + O(|\beta|^2)} \quad (49)$$

The last line in the above equation follows from the triangle inequality. The terms involving $O(\Im(\beta))$ are smaller than $O(|\beta|)$ terms. Moreover, for lightly nonviscous systems, the terms involving $O(\Im(\beta))$ are expected to be smaller than the $O(\Re(\beta))$ terms [14]. As a result, one expects to have the inequality

$$O(\Im(\beta)) < O(\Re(\beta)) < O(|\beta|) \quad (50)$$

From Eq. (49), it can be seen that first order changes in $\boldsymbol{\psi}$ lead to first order changes in $R_2(\boldsymbol{\psi})$. However, if the complex modal damping matrix is diagonally dominant, i.e.,

$$\frac{|G'_{ir}|}{|G'_{rr}|} \ll 1 \quad (51)$$

then first order changes in $\boldsymbol{\psi}$ lead to second order changes in $R_2(\boldsymbol{\psi})$. In this case, stationarity of the Rayleigh quotient is obtained.

Returning to the third quotient, the equation corresponding to Eq. (46) in the case of $R_3(\boldsymbol{\psi})$ is

$$R_3(\boldsymbol{\psi}) = \frac{|\boldsymbol{\psi}^H \boldsymbol{G}(\omega_r)\boldsymbol{\psi}|}{\boldsymbol{\psi}^H \boldsymbol{K}\boldsymbol{\psi}} < \frac{|G'_{rr}| + 2\sum_{i\neq r}\Re(\epsilon_i)|G'_{ir}| + O(\epsilon)O(\beta) + O(\epsilon^2) + O(\Im(\beta)) + O(|\beta|^2)}{\omega_r^2 + O(\epsilon^2) + O(|\beta|^2)} \quad (52)$$

The above quotient is not stationary. However, when the modal damping matrix is diagonally dominant in accordance with Eq. (31), stationarity of $R_3(\boldsymbol{\psi})$ holds.

6 Application of Rayleigh Quotients

In the case of a single degree of freedom system with viscous damping the three quotients simplify to $R_1 = \omega^2$, $R_2 = 2\zeta\omega$, and $R_3 = 2\zeta/\omega$, where ω and ζ denote the natural frequency and the critical damping factor, respectively. The response of the system in the time domain is described by $\exp(-\zeta\omega t - \imath\omega\sqrt{1-\zeta^2}t)$. We note that R_2 governs the decay rate (or real part of the complex eigenvalue) of vibration in the time domain. The same will be true for a multidegree of freedom system, provided that its response can be decomposed into a single degree of freedom system using modal summation i.e., damping is proportional [3].

The Rayleigh quotient R_1 and its usefulness in solving the eigenvalue problem associated with the undamped system are well documented [2,3,5]. Consequent to the stationary property of R_1, a theorem originally due to Rayleigh, known as Rayleigh's principle or interlacing theorem, gives the influence of constraints. It states that the eigenvalues of the constrained system (ω') interlace with the eigenvalues of the unconstrained system (ω) such that $\omega_n \leq \omega'_n \leq \omega_{n+1}$.

Similar results follow from the stationarity of R_2 and R_3. In this context, we refer to Rayleigh's original statement in Sec. 88 of Ref. [1]: "… theorems, of importance in other branches of science, may be stated for systems such that only T and F, or only V and F, are sensible." We note that $T \equiv \mathcal{T}$, $V \equiv \mathcal{U}$, and $F \equiv \mathcal{D}$ in the notation of the present paper. Thus, stationarity of R_2 implies that the decay rates of each normal mode are stationary. The interlacing theorem would suggest that the decay rates of each normal mode also interlace when a constraint is applied. The interlacing property was discussed in Sec. 88 of Ref. [1] and a less known work of Rayleigh [15]. The present study extends these ideas to the general case of nonconservative systems with viscous or nonviscous dissipative processes.

In a viscoelastic system, one deals with elastic potentials and dissipative potentials. Stationarity of R_3 has important consequences for such problems, especially in conjunction with the interlacing theorem. A noteworthy work on applying the Rayleigh quotients to determine the elastic and material loss constants of orthotropic sheet materials was undertaken in Refs. [16,17].

Our primary aim in this work has been to show the range of applicability of stationarity principles in nonconservative viscous and nonviscous systems. Further application of these results remains to be explored in future studies.

7 Conclusions

Rayleigh quotients are revisited in the context of dissipative systems. The study of their stationarity properties leads to the following conclusions.

1. In the case of a proportionally damped viscous system, the three Rayleigh quotients associated with the damped system are stationary.
2. In the case of a nonproportionally damped system, the Rayleigh quotient involving mass and stiffness matrix is stationary while the remaining two involving damping matrix are not. Stationarity in this case is subject to the diagonal dominance of the modal damping matrix. For an arbitrarily chosen viscous damping matrix, the stationarity property does not hold true. However, this negative conclusion is to be balanced by the wide variety of practical engineering structures where the modal damping is diagonally dominant; consequently, Rayleigh quotients are stationary.
3. In the case of a nonviscously damped system, the Rayleigh quotient involving mass and stiffness matrix is still stationary while the remaining two involving the frequency dependent damping matrix are not. Stationarity in this case is subject to (a) the diagonal dominance of the absolute value of the frequency dependent complex modal damping matrix, and (b) light nonviscous damping. For an arbitrarily chosen nonviscous damping function, the stationarity property does not hold true.

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