Transient Dynamics of Stochastically Parametered Beams

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ABSTRACT: The problem of determining the statistics of the transient response of randomly inhomogeneous beams is formulated. This is based on the use of stochastic dynamic stiffness coefficients in conjunction with the fast Fourier transform algorithm. The dynamic stiffness coefficients, in turn, are determined using a stochastic finite-element formulation that employs frequency-dependent shape functions. The approach is illustrated by analyzing the response of a random rod subject to a boxcar type of axial impact and, also, by considering the flexural response of a randomly inhomogeneous beam resting on a randomly varying Winkler's foundation and subjected to the action of a moving force. A discussion on the treatment of system property random fields as being non-Gaussian in nature is presented. Also discussed are the methods for handling nonzero initial conditions within the framework of the frequency domain response analysis employed in the study. Satisfactory comparisons between the analytical results and simulation results are demonstrated.

INTRODUCTION

Problems of structural dynamics with randomly distributed spatial inhomogeneities have been receiving wide research attention. Several mathematical and computational issues lie at the heart of this research; these include random field discretization, characterization of random eigensolutions, random matrix inversion, solutions of stochastic boundary-value problems, and description of random matrix products. Recent reviews on these topics include the works of Ibrahim (1987), Nakagiri (1987), Benaroya and Rehak (1988), Brenner (1991), Ghanem and Spanos (1991), Shinozuka (1991), Der Kiureghian et al. (1991), Liu et al. (1992), Klieber and Hien (1992), and Schueller (1997). An update on the earlier review by Ibrahim (1987) has been recently reported by Manohar and Ibrahim (1999). These studies are motivated by the basic need to improve the rationale of structural reliability assessments and, also, by an aim to gain insights into phenomenological features associated with the effect of structural imperfections on vibration behavior.

Recently, the writers employed frequency-dependent shape functions to discretize random fields for structural dynamic applications (Manohar and Adhikari 1998; Adhikari and Manohar 1999). These studies represent the following:

- The extension of the direct dynamic stiffness method of vibration analysis to problems of structural dynamics with parameter uncertainties
- The generalization of the concept of weighted integrals for discretization of random fields used in static stochastic finite-element analysis to problems of vibration analysis (Shinozuka 1987; Takada 1990; Deodatis and Shinozuka 1991; Bucher and Brenner 1992), this being achieved by the use of frequency-dependent shape functions, which, in turn, make the weighted integrals functions of frequency

This approach, as it presently stands, is basically applicable to the analyses of steady-state harmonic response and stationary random response. A distinguishing feature of the approach is that it does not employ modal series representation for the forced response, and, consequently, the need to perform random free vibration analysis is bypassed. Furthermore, the shape functions used for displacement field discretization are such that, with changes in values of driving frequencies, the shape functions adapt themselves to the spatial variations in waveforms automatically. This provides relief in the selection of mesh size with respect to the frequency range of external excitation. In the present study, the writers extend their earlier work (Manohar and Adhikari 1998; Adhikari and Manohar 1999) to investigate the response of stochastically parametered beams to distributed transient loads. This calls for extending the writers' previous studies from the following two counts:

- The response is represented in time domain in terms of the Fourier transforms of the frequency response functions; a procedure to include the effects of nonzero initial conditions is also outlined.
- The distributed transient loads are represented in terms of equivalent nodal forces using the frequency-dependent shape functions.

Illustrative examples on the dynamics of a rod subjected to a boxcar type of axial impact and the vibration of a beam on an elastic foundation that is traversed by a moving load are presented. The analytical results are shown to compare favorably with results from the Monte Carlo simulations. The studies reported in this paper are relevant to problems involving earthquake loads, moving loads, blasts, and impacts. Some of the earlier works in the existing literature dealing with transient dynamics of randomly parametered structures include the study of statistics of impulse response of single- and multi-degree-of-freedom systems (Chen and Soroka 1973; Prasthofer and Beadle 1975; Udwadia 1987; Lee and Singh 1994), response to seismic inputs (Iwan and Jensen 1993; Katafygiotis and Papadimitriou 1996), response under moving loads (Fryba et al. 1993), and studies on nonlinear systems (Liu et al. 1987; Deodatis and Shinozuka 1988). In these studies, approaches based on perturbations, direct integrations, and Monte Carlo simulations have been employed. Many of the studies involving multi-degree-of-freedom systems employ modal expansion methods (Igusa and Der Kiureghian 1988; Katafygiotis and Papadimitriou 1996). Iwan and Jensen (1993) employed random shape functions to approximate the solution in the spatial domain and in the random space. They derived a set of deterministic ordinary differential equations for the unknown coefficients using weighted residual method. These equations are subsequently solved numerically to characterize the response moments. The study by Liu et al. (1987) employed a mean centered second-order perturbation method in conjunction with direct numerical integration in time to study transient dynamics of linear and nonlinear continua. They discretized the ran-

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dom fields using finite-element shape functions and nodal values of random fields. Deodatis and Shinozuka (1988) studied the transient dynamics of a transversely loaded stochastically inhomogeneous plate on a random nonlinear elastic foundation. They employed Monte Carlo simulations in conjunction with finite-element discretization and time integration techniques. The approach outlined in the present paper represents an alternative to the linear transient dynamic analysis of randomly parameterized continuous systems. This approach avoids the use of modal expansions thereby bypassing the need to perform random eigensolution analysis. It should be noted that the description of structural response to short duration loads requires high resolution in time domain, which, in turn, demands the evaluation of contributions from short wavelength modes. In the context of randomly parameterized systems, this requirement demands that the description of joint statistics of several natural frequencies and mode shapes have to be obtained before forced vibration analysis could be performed. The study reported in the present paper provides a natural framework to avoid these complexities.

STOCHASTIC DYNAMIC STIFFNESS MATRIX AND NODAL FORCE VECTOR

In a recent study, a finite-element-based formulation was developed to obtain the stochastic dynamic element stiffness matrix of a general beam element having randomly inhomogeneous mass density, flexural and axial rigidities, and elastic foundation modulus (Manohar and Adhikari 1998). In this section the procedures followed for derivation of the dynamic stiffness matrix is briefly outlined and additional information on the determination of the equivalent nodal forces is provided. The beam element considered in this study is shown in Fig. 1. It is assumed here that the axial forces are small, to the extent that they do not affect the flexural deformations. It is also assumed that the behavior of the beam follows the Euler-Bernoulli hypotheses and that the beam rests on a Winkler's elastic foundation. First, the system is considered to be initially at rest; a procedure to treat nonzero initial conditions is briefly outlined later in the paper. The governing field equations of motion under these assumptions are given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 Y}{\partial x^2} + c_1\frac{\partial^3 Y}{\partial x^2 \partial t}\right] + m(x)\frac{\partial^2 Y}{\partial t^2} + c_2\frac{\partial Y}{\partial t} + k(x)Y = f(x, t)$$
(1)

$$\frac{\partial}{\partial x}\left[AE(x)\frac{\partial U}{\partial x} + c_3\frac{\partial^2 U}{\partial x \partial t}\right] = m(x)\frac{\partial^2 U}{\partial t^2} + c_4\frac{\partial U}{\partial t} + p(x, t)$$
(2)

where $Y(x, t)$ = transverse flexural displacement; $U(x, t)$ = axial displacement; $EI(x)$ = flexural rigidity; $AE(x)$ = axial rigidity; $m(x)$ = mass per unit length; $k(x)$ = elastic foundation modulus; $f(x, t)$ = distributed time varying transverse force; $p(x, t)$ = distributed time varying axial force; c_1 and c_3 = strain rate-dependent viscous damping coefficients; and c_2 and c_4 = velocity-dependent viscous damping coefficients. The forcing functions $p(x, t)$ and $f(x, t)$ are taken to be deterministic and having finite energy. The quantities $k(x)$, $EI(x)$, $AE(x)$, and $m(x)$ in this study are modeled as jointly homogeneous random fields and are taken to have the following form:

$$k(x) = k_0[1 + \varepsilon_1 f_1(x)]; \quad m(x) = m_0[1 + \varepsilon_2 f_2(x)] \quad (3a,b)$$

$$EI(x) = EI_0[1 + \varepsilon_3 f_3(x)]; \quad AE(x) = AE_0[1 + \varepsilon_4 f_4(x)] \quad (3c,d)$$

where the subscript 0 indicates the mean values; $0 < \varepsilon_i \ll 1$ $(i = 1, \ldots, 4)$ are deterministic constants; and the random fields $f_i(x)$ are taken to have zero mean, unit standard deviation, and covariance $R_{ij}(\xi)$. The following additional restrictions are taken to apply on the random fields $f_i(x)$ $(i = 1, 2, 3, 4)$:
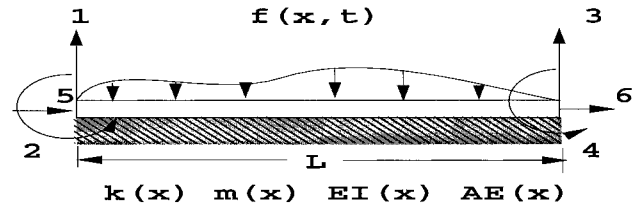
FIG. 1. Randomly Parameterized Beam Element on Winkler's Foundation

1. $f_i(x)$ $(i = 1, 2, 3, 4)$ are mean square bounded; that is, $\langle f_i(x)^2 \rangle < \infty$; here $\langle \cdot \rangle$ is the mathematical expectation operator.
2. $f_3(x)$ is twice differentiable in a mean square sense; this requires that $\partial^4 R_{33}(x_1, x_2)/\partial x_1^2 \partial x_2^2$ must exist for all x_1 and x_2 in the interval $(0, L)$.
3. $f_4(x)$ is differentiable in a mean square sense; this requires that $\partial^2 R_{44}(x_1, x_2)/\partial x_1 \partial x_2$ must exist for all x_1 and x_2 in the interval $(0, L)$.
4. For a specified deterministic function $g(x)$, which is bounded and continuous in $(0, L)$, integrals of the type $\int_0^L g(x)f_i(x)\,dx$ exist in a mean square sense; this requires that $\int_0^L \int_0^L |g(x_1)g(x_2)R_{ii}(x_1, x_2)|\,dx_1\,dx_2 < \infty$ $(i = 1, 2, 3, 4)$.

The terms involved in these assertions are explained in the book by Soong (1973). The first three conditions ensure that the sample realizations of the beam have sufficiently smooth behavior so that the various stress resultants and boundary conditions (such as those at a free edge) are satisfactorily described. The fourth condition is needed in the development of the procedure used in this study and will be explained later.

In view of the assumed linear system behavior, the solution of the field equations can be taken to be of the form

$$Y(x, t) = y(x, \omega)\exp[i\omega t]; \quad U(x, t) = u(x, \omega)\exp[i\omega t] \quad (4a,b)$$

Consequently, the equations governing $y(x, \omega)$ and $u(x, \omega)$ have the form

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 y}{dx^2} + i\omega c_1\frac{d^2 y}{dx^2}\right] + [k(x) - m(x)\omega^2 + c_2 i\omega]y = F(x, \omega)$$
(5)

$$\frac{d}{dx}\left[AE(x)\frac{du}{dx} + i\omega c_3\frac{du}{dx}\right] + [\omega^2 m(x) - i\omega c_4]u = P(x, \omega) \quad (6)$$

where

$$F(x, \omega) = \int_{-\infty}^{\infty} f(x, t)\exp[-i\omega t]\,dt \quad (7a)$$

$$P(x, \omega) = \int_{-\infty}^{\infty} p(x, t)\exp[-i\omega t]\,dt \quad (7b)$$

The above equations, together with the boundary conditions on displacements and forces at $x = 0$ and $x = L$, constitute a set of stochastic boundary-value problems. Furthermore, the presence of the damping terms makes the coefficients in these equations complex valued, which, consequently, makes the solutions also complex valued. Exact solutions to these types of problems are currently not available, and finite-element procedures are employed to arrive at approximate solutions.

Shape Functions

To construct the finite-element approximation, the solutions of (1) and (2) are taken to be of the form

$$Y(x, t) = \sum_{j=1}^{4} \delta_j(t) N_j(x, \omega) \tag{8}$$

$$U(x, t) = \sum_{j=5}^{6} \delta_j(t) N_j(x, \omega) \tag{9}$$

where $\delta_j(t)$ $(j = 1, 6)$ = generalized coordinates representing the nodal displacements; and $N_j(x, \omega)$ = shape functions. Given the well-behaved nature of $p(x, t)$ and $f(x, t)$, and, also, the smooth nature of the random fields $f_i(x)$ $(i = 1, 2, 3, 4)$, the above series representation is considered as a valid form of solutions of (1) and (2). The book by Ghanem and Spanos (1991) provided further details of the mathematical setting under which representations such as those given above are considered to be valid forms of solution of (1) and (2). Furthermore, the mathematical issues pertaining to finite-element methods for stochastic media problems have been recently discussed by Matthies and Bucher (1999) based on recent developments in the field of stochastic partial differential equations.

In this paper, the shape functions are obtained from the field equations representing the undamped free vibration [i.e., (5) and (6) with $c_i = 0$, $\varepsilon_i = 0$ $(i = 1, 4)$, and $F(x, \omega)$ and $P(x, \omega)$ = 0]. Readers are referred to the papers by Manohar and Adhikari (1998) and Adhikari and Manohar (1999) for further details regarding the analytical derivation of the shape functions. The array of the shape functions, $N(x, \omega)$, can be shown to be given by

$$N(x, \omega) = [\Gamma(\omega)]\{s(x, \omega)\} \tag{10}$$

in which

$$\{s(x, \omega)\} = [\{s_f(x, \omega)\}, \{s_a(x, \omega)\}]^T \tag{11}$$

$$\Gamma(\omega) = [R(\omega)^{-1}]^T \tag{12}$$

Here $(\cdot)^T$ represents the matrix transpose and the matrix

$$R(\omega) = \begin{bmatrix} s_{f_1}(0) & s_{f_2}(0) & s_{f_3}(0) & s_{f_4}(0) & 0 & 0 \\ \dfrac{ds_{f_1}}{dx}(0) & \dfrac{ds_{f_2}}{dx}(0) & \dfrac{ds_{f_3}}{dx}(0) & \dfrac{ds_{f_4}}{dx}(0) & 0 & 0 \\ s_{f_1}(L) & s_{f_2}(L) & s_{f_3}(L) & s_{f_4}(L) & 0 & 0 \\ \dfrac{ds_{f_1}}{dx}(L) & \dfrac{ds_{f_2}}{dx}(L) & \dfrac{ds_{f_3}}{dx}(L) & \dfrac{ds_{f_4}}{dx}(L) & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{a_1}(0) & s_{a_2}(0) \\ 0 & 0 & 0 & 0 & s_{a_1}(L) & s_{a_2}(L) \end{bmatrix} \tag{13}$$

The 4×1 array of the basis functions corresponding to the flexural motion $s_f(x, \omega)$, can be defined by Table 1. The 2×1 array of basis functions corresponding to the axial motion can be obtained as

$$\{s_a(x, \omega)\} = [\sin ax, \cos ax]^T \quad \text{with } a^2 = \frac{m_0 \omega^2}{AE_0} \tag{14}$$

Beam Element

The assumed displacement fields [(8) and (9)] can now be used in conjunction with Lagrange's equation of motion to derive the equations governing the generalized coordinates $\delta_j(t)$. Furthermore, to deduce the element dynamic stiffness matrix one must consider the amplitude of nodal harmonic displacements $d_j(\omega)$ defined through the relation $\delta_j(t) = d_j(\omega)\exp[i\omega t]$. These displacements must be shown to be related to the nodal equivalent forces through the equation

$$[D_u(\omega)]d(\omega) = F_{n_u}(\omega) \tag{15}$$

TABLE 1. Basis Functions for Flexural Motion [$\omega^* = \sqrt{k_0/m_0}$, $b^4 = (m_0\omega^2 - k_0)/EI_0$, and $b'^4 = -b^4$]

(1)	$\omega > \omega^*$ (2)	$\omega = \omega^*$ (3)	$\omega < \omega^*$ (4)
s_{f_1}	$\sin bx$	1	$\sin b'x \sinh b'x$
s_{f_2}	$\cos bx$	x	$\sin b'x \cosh b'x$
s_{f_3}	$\sinh bx$	x^2	$\cos b'x \sinh b'x$
s_{f_4}	$\cosh bx$	x^3	$\cos b'x \cosh b'x$

where $F_{n_u}(\omega)$ = frequency-dependent, undamped equivalent nodal force vector, which is defined as

$$F_{n_{u_j}}(\omega) = \int_0^L N_j(x, \omega)F(x, \omega) \, dx \quad \text{for } j = 1, \ldots, 4 \tag{16a}$$

$$F_{n_{u_j}}(\omega) = \int_0^L N_j(x, \omega)P(x, \omega) \, dx \quad \text{for } j = 5, 6 \tag{16b}$$

In these equations, $D_u(\omega)$ is the undamped stochastic dynamic element stiffness matrix given by

$$D_u(\omega) = \bar{D}_u(\omega) + \sum_{l=1}^{13} [\alpha^l(\omega)]X_l(\omega) \tag{17}$$

Here $\bar{D}_u(\omega)$ is the deterministic part, and it can be obtained as follows:

$$\bar{D}_{u_{ij}}(\omega) = \sum_{k=1}^{4} \sum_{r=1}^{4} \Gamma_{ik}(\omega)\Gamma_{jr}(\omega) \int_0^L \left[\{k_0 - m_0\omega^2\}s_k(x, \omega)s_r(x, \omega) \right.$$
$$\left. + EI_0 \frac{d^2 s_k(x, \omega)}{dx^2} \frac{d^2 s_r(x, \omega)}{dx^2} \right] dx \quad \text{for } i, j = 1, 4 \tag{18a}$$

$$\bar{D}_{u_{ij}}(\omega) = \sum_{k=5}^{6} \sum_{r=5}^{6} \Gamma_{ik}(\omega)\Gamma_{jr}(\omega) \int_0^L \left[-m_0\omega^2 s_k(x, \omega)s_r(x, \omega) \right.$$
$$\left. + AE_0 \frac{ds_k(x, \omega)}{dx} \frac{ds_r(x, \omega)}{dx} \right] dx \quad \text{for } i, j = 5, 6 \tag{18b}$$

Also, $[\alpha^l(\omega)]$ $(l = 1, \ldots, 13)$ are 6×6 symmetric matrices of deterministic functions of ω, which are expressed as

$$\alpha_{ij}^l(\omega) = \Gamma_{ik}(\omega)\Gamma_{jr}(\omega) \quad \text{for } l = 1, 5, 8, 10, 11, 13;$$
$$k = r = 1, \ldots, 6 \tag{19a}$$

$$\alpha_{ij}^l(\omega) = \Gamma_{ik}(\omega)\Gamma_{jr}(\omega) + \Gamma_{ir}(\omega)\Gamma_{jk}(\omega) \quad \text{for } l = 2, 3, 4, 6, 7, 9;$$
$$k \neq r; \quad k, r = 1, \ldots, 4 \text{ and } l = 12; \quad k, r = 5, 6 \tag{19b}$$

It may be noted that $X_l(\omega)$ $(l = 1, \ldots, 13)$, appearing in (17), are random in nature and are given by

$$X_1(\omega) = W_{11}(\omega); \quad X_2(\omega) = W_{12}(\omega); \quad X_3(\omega) = W_{13}(\omega);$$

$$X_4(\omega) = W_{14}(\omega); \quad X_5(\omega) = W_{22}(\omega); \quad X_6(\omega) = W_{23}(\omega);$$

$$X_7(\omega) = W_{24}(\omega); \quad X_8(\omega) = W_{33}(\omega); \quad X_9(\omega) = W_{34}(\omega);$$

$$X_{10}(\omega) = W_{44}(\omega) \tag{20}$$

$$X_{11}(\omega) = W_{55}(\omega); \quad X_{12}(\omega) = W_{56}(\omega); \quad X_{13}(\omega) = W_{66}(\omega) \tag{21}$$

with

$$W_{kr}(\omega) = \int_0^L \left[\{k_0\varepsilon_1 f_1(x) - m_0\omega^2\varepsilon_2 f_2(x)\}s_k(x, \omega)s_r(x, \omega) \right.$$
$$\left. + EI_0\varepsilon_3 f_3(x) \frac{d^2 s_k(x, \omega)}{dx^2} \frac{d^2 s_r(x, \omega)}{dx^2} \right] dx \quad \text{for } i, j = 1, \ldots, 4 \tag{22}$$

$$W_{kr}(\omega) = \int_0^L \left[\{-m_0\omega^2\varepsilon_2 f_2(x)\}s_k(x,\,\omega)s_r(x,\,\omega) \right.$$

$$\left. + AE_0\varepsilon_4 f_4(x)\frac{ds_k(x,\,\omega)}{dx}\frac{ds_r(x,\,\omega)}{dx} \right] dx \quad \text{for } i,\, j = 5,\, 6 \tag{23}$$

It may also be noted that $X_l(\omega)$ ($l = 1, \ldots, 13$) are random processes evolving in the frequency parameter ω. Thus, for a fixed value of driving frequency ω, these quantities are random variables, and herein they are termed as dynamic weighted integrals, because they arise as "weighted integrals" of the random fields $f_i(x)$ ($i = 1, \ldots, 4$). It may also be pointed out that, when the above results are specialized to the case of static behavior ($\omega = 0$) of a Euler-Bernoulli beam with no elastic foundations ($k = 0$), the dynamic stiffness matrix reduces to the static stiffness matrix and the weighted integrals listed above reduce to those reported by Deodatis and Shinozuka (1991). It may also be noted that Assumption 4 outlined in the second section guarantees that the weighted integrals exist in a mean square sense. Furthermore, $X_l(\omega)$ are linear functions of the random fields $f_i(x)$, and, if $f_i(x)$ are modeled as jointly Gaussian random fields, it follows from the well-known properties of Gaussian random processes that $X_l(x)$ are also jointly Gaussian [e.g., Theorem 4.6.4. on p. 112 in the book by Soong (1973)]. Because the dynamic stiffness coefficients are a linear function of $X_l(\omega)$, it follows that these coefficients, in turn, are also Gaussian distributed. The analytical results presented in this study are based on the assumption that the dynamic stiffness coefficients are Gaussian distributed. The consequences of treating $f_i(x)$ as being non-Gaussian in nature are discussed later.

The dynamic stiffness matrix and dynamic equivalent nodal force vector, derived thus far, are based on the assumption that the beam element is undamped. Following Manohar and Adhikari (1998), the equation of dynamic equilibrium for a damped element can be written

$$[\mathbf{D}(\omega)]_{6\times 6}\mathbf{d}(\omega)_{6\times 1} = \mathbf{F}_n(\omega)_{6\times 1} \tag{24}$$

where, in general, $\mathbf{F}_n(\omega)$ = complex vector; and $[\mathbf{D}(\omega)]$ = symmetric matrix with a complex deterministic part and a real stochastic part. Thus, in line with (17), $\bar{\mathbf{D}}_u(\omega)$ can be replaced by $\bar{\mathbf{D}}(\omega)$, the deterministic part of the damped dynamic stiffness matrix. The elements of $\bar{\mathbf{D}}(\omega)$, however, can be obtained from (18) with the "damped values" of the following quantities:

$$b^4 = \frac{m_0\omega^2 - k_0 + i\omega c_2}{EI_0 + i\omega c_1}; \quad b'^4 = -b^4 \tag{25a,b}$$

$$a^2 = \frac{m_0\omega^2 - i\omega c_4}{AE_0 + i\omega c_3} \tag{25c}$$

where $i = \sqrt{-1}$. Similarly, using the above values, the damped dynamic equivalent nodal force vector can also be obtained from (16).

The stiffness matrix and nodal force vector derived above can now be used to study the dynamics of skeletal structures. Here, the assembly of the element stiffness matrix and nodal force vectors would follow the same rules as those applicable to the static finite-element analysis. If the interest is focused on steady-state harmonic or stationary random response, the analysis would require the inversion of the global dynamic stiffness matrix. Recently, studies of this type have been carried out, which have involved the development of procedures for inversion of complex valued symmetric random matrices (Adhikari and Manohar 1999). In this paper attention is focused on determining the response when excitations are transient in nature. This is achieved by considering the Fourier transform of the frequency domain response descriptions de-

rived above. The requisite formulary is illustrated in the following sections by considering two specific examples.

EXAMPLE 1—IMPACT ON AXIALLY VIBRATING ROD

In this section, the problem of determining the second-order statistics of time evolution of deformation of a randomly parametered axially vibrating rod subjected to an axial impact is considered. Fig. 2 illustrates the problem considered. In the numerical work one takes the nominal value of axial stiffness $AE_0 = 50 \times 10^6$ N, mass density $m_0 = 1.95$ kg/m, length $L = 2$ m, and the damping values $c_3 = 0.0$ and $c_4 = 794.67$ Ns. The mass and axial stiffness along the length are perturbed by independent, stationary random fields with autocovariances given, respectively, by

$$R_{ii}(\xi) = \sigma_{0i}^2 \exp\{-\alpha_i\xi^2\}; \quad i = 2,\, 4 \tag{26}$$

It can be shown that the correlation length for this model is given by $1/2\sqrt{\pi/\alpha_i}$. It is assumed that ε_2 and $\varepsilon_4 = 0.05$, $\sigma_{02} = \sigma_{04} = 1$, and α_2 and $\alpha_4 = (\pi/4)/\text{m}^2$. The parameters α_2 and α_4 are chosen such that the correlation lengths of the random fields $f_2(x)$ and $f_4(x)$ are equal to half the rod length. For the purpose of illustration, the axial thrust $f(t)$ is assumed to be a box curve with amplitude $F_0 = 1.0$ N and time duration $t_c = 12.8 \times 10^{-3}$ s (Fig. 2). Thus, $f(t)$ can be expressed as

$$f(t) = F_0\{\mathcal{U}(t) - \mathcal{U}(t - t_0)\}$$

where $\mathcal{U}(t)$ = unit step function. The rod is assumed to be at rest before the axial impact is applied. Issues arising out of nonzero initial conditions are briefly discussed later. Response quantity of interest is taken in this example to be the displacement at the right end (node 2).

Response Statistics Calculation—Analytical Methods

Applying (24) to this problem and considering only the axial motion, the displacement at node 2, $\Theta_6(\omega)$ can be obtained as

$$\Theta_6(\omega) = \frac{F_6(\omega)}{D_{6,6}(\omega)} \tag{27}$$

Here

$$F_6(\omega) = \int_{-\infty}^{\infty} f(t)\exp[-i\omega t]\,dt = \frac{F_0}{i\omega}(1 - \exp[-i\omega t_c])$$

and $D_{6,6}(\omega) = \bar{D}_{6,6}(\omega) + V(\omega)$, where the deterministic part $\bar{D}_{6,6}(\omega) = AE_0a\cot aL$, and the random part $V(\omega) = \sum_{l=11}^{13} \alpha_{6,6}^l(\omega)X_l(\omega)$. It is clear that if $f_i(x)$ are taken to be Gaussian distributed, $V(\omega)$ will be Gaussian distributed with $\langle V(\omega)\rangle = 0$ and

$$\langle V^2(\omega)\rangle = \sum_{l=11}^{13}\sum_{m=11}^{13} \alpha_{6,6}^l\alpha_{6,6}^m\langle X_l(\omega)X_m(\omega)\rangle \tag{28}$$
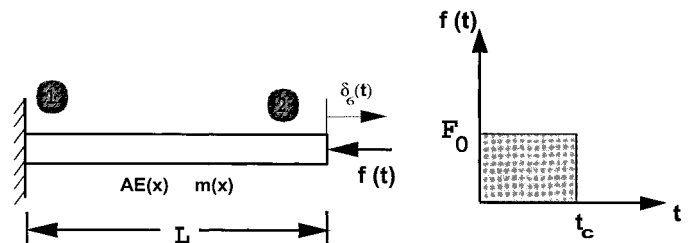

FIG. 2. Randomly Parametered Axially Vibrating Rod under Box Input; $AE_0 = 50 \times 10^6$ N, $L = 2$ m, $m_0 = 1.95$ kg/m, $t_c = 12.8 \times 10^{-3}$ s, $c_3 = 0.0$, $c_4 = 794.67$ Ns, and $F_0 = 1.0$ N

Now, from (27), the time domain response can be obtained by taking the Fourier transform, and, accordingly, the mean of $\theta_6(t)$ can be obtained as

$$\langle\theta_6(t)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_6(\omega) \left\{ \int_{-\infty}^{\infty} \frac{1}{\bar{D}_{6,6}(\omega) + V(\omega)} p_V(V; \omega) \, dV \right\}$$

$$\cdot \exp[i\omega t] \, d\omega \tag{29}$$

where $p_V(V; \omega)$ = probability distribution function of the Gaussian random variable $V(\omega)$, which is completely characterized by its mean and standard deviation defined in (28). The autocorrelation function of $\theta_6(t)$ can similarly be shown to be given by

$$\langle\theta_6(t_1)\theta_6(t_2)\rangle = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\bar{D}_{6,6}(\omega_1) + V_1(\omega_1)} \frac{1}{\bar{D}_{6,6}^*(\omega_2) + V_2(\omega_2)} \right.$$

$$\left. \cdot p_{V_1 V_2}(V_1, V_2; \omega_1, \omega_2) \, dV_1 \, dV_2 \right\} \exp[i(\omega_1 t_1 - \omega_2 t_2)] \, d\omega_1 \, d\omega_2 \tag{30}$$

In the above equation $(\cdot)^*$ denotes the complex conjugation; and $p_{V_1 V_2}(V_1, V_2; \omega_1, \omega_2)$ is the 2D joint Gaussian probability density function of the random variables $V(\omega_1)$ and $V(\omega_2)$. As is well known, this function can be characterized in terms of the mean and covariance of the two random variables $V(\omega_1)$ and $V(\omega_2)$. These moments are further obtainable from (21) and (23) in terms of the covariance functions of the random fields $f_i(x)$ ($i = 2, 4$). It thus follows that the determination of mean and standard deviation of the transient response requires the evaluation of the joint statistics of weighted integrals at two distinct frequencies. This feature is in contrast to the stationary response analysis (Manohar and Adhikari 1998) in which the knowledge of statistics of weighted integrals at single frequencies was sufficient to evaluate the response mean and standard deviation. Given the approximate nature of the analysis presented, it is essential that its acceptability must be verified by using more accurate simulation procedures, and this is considered in the next section.

Response Statistics Calculation— Simulation Methods

The simulation strategy adopted in this study is based on generation of the samples of the spectrum $\Theta_6(\omega)$ given by (27), which is then followed by application of the fast Fourier transform (FFT) algorithm to find the sample response in time domain. As can be seen from (27), this, in turn, requires generation of the samples of dynamic stiffness coefficient $D_{6,6}(\omega)$. It is to be noted that the nodal force $F_6(\omega)$ remains deterministic. The sample solutions for the dynamic stiffness coefficients can be obtained following the method outlined by Manohar and Adhikari (1998) for randomly parametered beams. This study, in turn, is based on an earlier work by Sarkar and Manohar (1996) on dynamics of extensible cables. The solution strategy is based on the conversion of the governing boundary-value problems into a set of equivalent initial-value problems. These initial-value problems, in turn, are solved numerically using a fourth-order Runge-Kutta algorithm. The details of these formulations are omitted here, and readers are referred to the work of Adhikari and Manohar (2000) for the relevant details. It may be noted that (27) is the starting point for the approximate analytical study reported in the previous section and the simulation method being described in this section. This equation is exact in a deterministic sense. In the simulation studies, samples of the dynamic stiffness coefficients $D_{6,6}(\omega)$ are obtained using an approach that is exact within the framework of the accuracy of the Runge-Kutta method and implementation of the FFT algorithm. Thus these solutions are valid as a basis for comparing the results from the approximate analytical procedure described in the previous section.

Numerical Results

The time history of the mean response estimated using the analytical procedure is compared with the results from 500 samples of Monte Carlo simulations in Fig. 3. The results for the deterministic case are also shown in this figure. It is clearly observed that the mean curve closely follows the deterministic curve. Fig. 4 shows the comparison of the analytical result with the simulation results on the variance of the response quantity. Good agreement is observed to exist between the analytical and simulation results, which lends credence to the analysis procedures used in this study. Furthermore, the results show that, for the frequency ranges and randomness parameters considered, the system uncertainties do not affect the mean solution appreciably. This follows from the close agreement that is found to exist between the deterministic solution
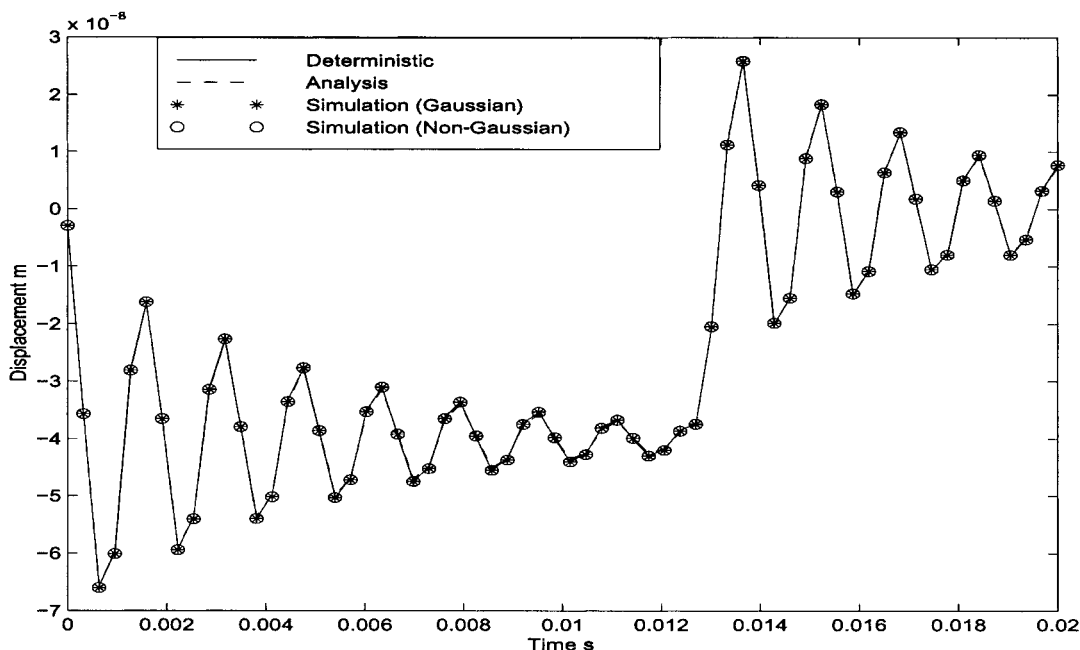

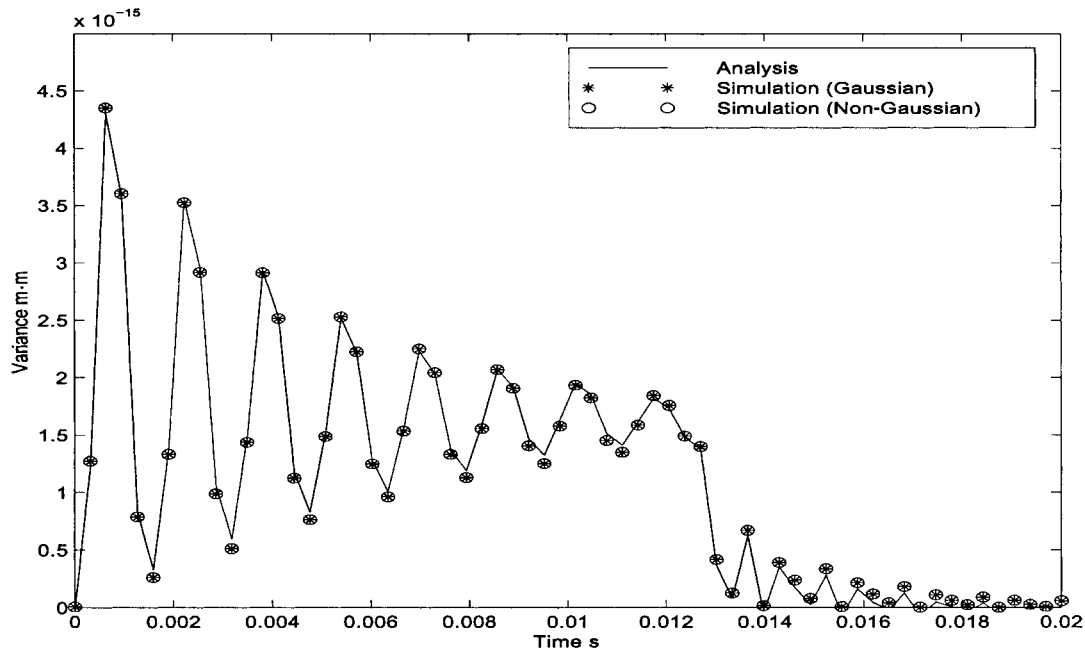**FIG. 3. Mean Displacement at Tip**

FIG. 4. Variance of Displacement at Tip

and the mean solution (Fig. 3). However, this does not mean that the system uncertainties do not have a significant effect on system behavior; in fact, the maximum value of the coefficient of variation of the response can be deduced to be about 0.95 (Figs. 3 and 4).

In the numerical work, the computation of the correlation of the dynamic weighted integrals $X_l(\omega)$ was carried out using a 2D, seventh-order Newton-Cotes integration scheme. Reduction in computational time in carrying out the 2D Fourier transform was achieved by noting that $\langle \Theta_6(\omega_1)\Theta_6^*(\omega_2)\rangle = \langle \Theta_6(\omega_1)\Theta_6(-\omega_2)\rangle$.

EXAMPLE 2—MOVING LOAD ON RANDOM BEAM RESTING ON RANDOM ELASTIC FOUNDATION

An Euler-Bernoulli beam with random flexural rigidity and mass density and resting on a randomly inhomogeneous Winkler's foundation under a moving load is considered in this section. The example considered is shown in Fig. 5. This example serves to illustrate the formulations when the beam is subjected to spatially distributed forcing $f(x, t)$. The beam is taken to be fixed at node 1 and hinged at node 2. It is assumed that $EI_0 = 10.0$, $k_0 = 5.0$, $m_0 = 0.2$, $L = 1.0$, $c_1 = 0.0$, and $c_2 = 1.0$. The flexural rigidity, mass density, and foundation elastic modulus are taken to be independent, homogeneous random fields. It is assumed that $\varepsilon_i = 0.05$ ($i = 1, 2, 3$), and the autocovariance of the processes $f_i(x)$ are taken to be of the form

$$R_{ii}(\xi) = \sigma_{0i}^2 \exp\{-\alpha_i\xi^2\}; \quad i = 1, 2, 3 \tag{31}$$

with $\sigma_{0i} = 1$ per unit length ($i = 1, 2, 3$); and $\alpha_i = \pi/(\text{length})^2$ so that the correlation lengths of $f_i(x)$ ($i = 1, 2, 3$) are half of the beam span. The rotation at node 2 denoted by $\theta_4(t)$ is considered to be the response variable of interest. The moving load is taken to travel from left to right with a constant velocity v. The resulting forcing function $f(x, t)$ can be given by

$$f(x, t) = \bar{P}\delta(x - vt) \quad \text{for } 0 < t < L/v \tag{32}$$

and $f(x, t) = 0$, otherwise. Here, $\delta(\cdot)$ is the Dirac delta function, $\bar{P}$ is the magnitude of moving load, which in the numerical work is taken to be unity, and the velocity v is taken to be 5 units. As in the previous example, it is assumed that the beam
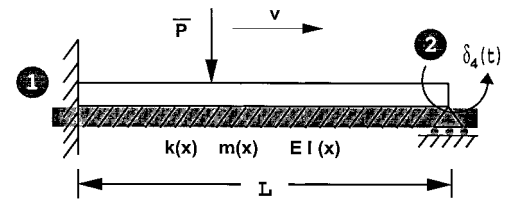

FIG. 5. Randomly Parametered Beam under Moving Force; EI_0 = 10.0, k_0 = 5.0, m_0 = 0.2, L = 1.0, c_1 = 0.0, c_2 = 1.0, ε_i = 0.05 (i = 1, 3) v = 5 m/s, and $\bar{P}$ = 1.0

is at rest before the entry of the load. A procedure to evaluate the effect of nonzero initial conditions is discussed later.

Response Statistics Calculation— Analytical Methods

We begin by noting that the Fourier transform of the forcing function in this case is given by

$$F(x, \omega) = \int_{-\infty}^{\infty} f(x, t)\exp[-i\omega t] \, dt = \frac{\bar{P}}{v} \exp\left[\frac{-i\omega x}{v}\right] \tag{33}$$

Thus, in this example the forcing function is spatially distributed, which is in contrast to the nodal loading considered in the previous example. This distributed loading needs to be converted into an equivalent set of nodal forces as has been described in the second section. Now applying (24) for this problem, $\theta_4(t)$, the time history of rotation at node 2, can be shown to be given by

$$\theta_4(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta_4(\omega)\exp[i\omega t] \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F_{n_4}(\omega)}{D_{4,4}(\omega)} \exp[i\omega t] \, d\omega \tag{34}$$

where $F_{n_4}(\omega)$ can be evaluated using (16) for the damped case. Note that, because the definitions of the shape functions used in this study are different for the three frequency regimes $\omega < \omega^*$, $\omega = \omega^*$, and $\omega > \omega^*$, the corresponding expressions of the equivalent nodal forces will also be different for these three regimes. The dynamic stiffness coefficient $D_{4,4}(\omega)$ is given by $D_{4,4}(\omega) = \bar{D}_{4,4}(\omega) + V(\omega)$, with the deterministic part $\bar{D}_{4,4}(\omega)$ given by

$$\bar{D}_{4,4}(\omega) = \frac{EI(-\cosh bL \sin bL + \cos bL \sinh bL)b}{-1 + \cos bL \cosh bL}$$

for $\omega > \omega^*$ (35a)

$$\bar{D}_{4,4}(\omega) = 4\frac{EI}{L} \quad \text{for } \omega = \omega^* \qquad (35b)$$

$$\bar{D}_{4,4}(\omega) = -2\frac{EI(-\cosh b'L \sinh b'L + \cos b'L \sin b'L)b'}{\cosh^2 b'L - 2 + \cosh^2 b'L}$$

for $\omega < \omega^*$ (35c)

where b and b' are as defined in (25). The random part $V(\omega)$ is given by $V(\omega) = \sum_{l=1}^{10} \alpha_{4,4}^l(\omega)X_l(\omega)$, if $|\omega| > \omega^*$, and $V(\omega) = \sum_{l=1}^{3} \alpha_{4,4}^l(\omega)X_l(\omega)$, if $|\omega| = \omega^*$. Now the statistics of $\theta_4(t)$ can be obtained following the procedure that is essentially similar to the one outlined in the Response Statistics Calculation—Analytical Methods section above.

Response Statistics Calculation—Simulation Methods

Samples of response time histories can be simulated following the approach that is essentially similar to the one mentioned for axially vibrating rods. Additional modifications, however, would be needed to take into account the distributed forces over the element domain. This requires the calculation of the particular integrals. Again, readers are referred to the report by Adhikari and Manohar (2000) for details. The formulation leads to the determination of the response quantity of interest, namely, $\Theta_4(\omega)$. The response in time domain is subsequently obtained by using the FFT algorithm.

Numerical Results

Fig. 6 shows the comparison of analytical results with those from 500 samples of Monte Carlo simulations on the mean of $\theta_4(t)$. It may also be observed from this figure that the mean curve closely follows the deterministic curve. Fig. 7 shows the comparison of analytical results with the simulated results on the variance of $\theta_4(t)$. The agreement between the analytical and simulation results, again, leads to the conclusion that the approximate analysis procedure yields acceptable solutions. As

has been observed in the previous example, in this example the mean values also remain largely unaffected by the system uncertainties, whereas the maximum coefficient of variation of the response is found to be about 1.0 (Figs. 6 and 7).

ADDITIONAL CONSIDERATIONS

In this section some additional features of the problem that have a bearing on the procedures and examples presented in the preceding sections are briefly discussed.

Non-Gaussian Models for Structural Parameters

The stochastic part of the dynamics stiffness matrix $V(\omega)$ arises as a sum of weighted integrals [(17)], and the weighted integrals themselves are linear functions of the system property random fields $f_i(x)$ ($i = 1, \ldots, 4$). Consequently, as has been already pointed out, if the system property random fields $f_i(x)$ are modeled as being Gaussian distributed, the weighted integrals, and, hence, the dynamics stiffness coefficients, become Gaussian distributed. The analytical results presented in this study are in fact based on the assumption that $V(\omega)$ is Gaussian. The choice of Gaussian distributions for system property random fields, strictly speaking, is not acceptable as the system properties such as mass and flexural rigidities are strictly positive. Thus the random fields $f_i(x)$ ($i = 1, \ldots, 4$) must satisfy the condition that $P[\{1 + \varepsilon_i f_i(x)\} < 0] = 0$, which is violated if $f_i(x)$ are Gaussian. This drawback is severe if one is interested in estimating system reliability where accurate modeling of tails of response probability density functions is essential. On the other hand, if interest is limited to estimation of mean and correlation of the response, the Gaussian assumption can be expected to be acceptable. However, this assumption limits magnitudes of the variability parameters ε_i ($i = 1, 4$) to values less than about 0.05. To examine these statements, alternative non-Gaussian models for $f_i(x)$ were considered, and these fields were taken to be distributed uniformly between $-\sqrt{3}$ and $\sqrt{3}$. It must be noted that the random fields $f_i(x)$ were taken to possess first- and second-order properties that were identical to those considered for the Gaussian models used in previous sections. Monte Carlo simulations using 500 samples for the response were performed for the two examples considered in the third and fourth sections. The simulation of the
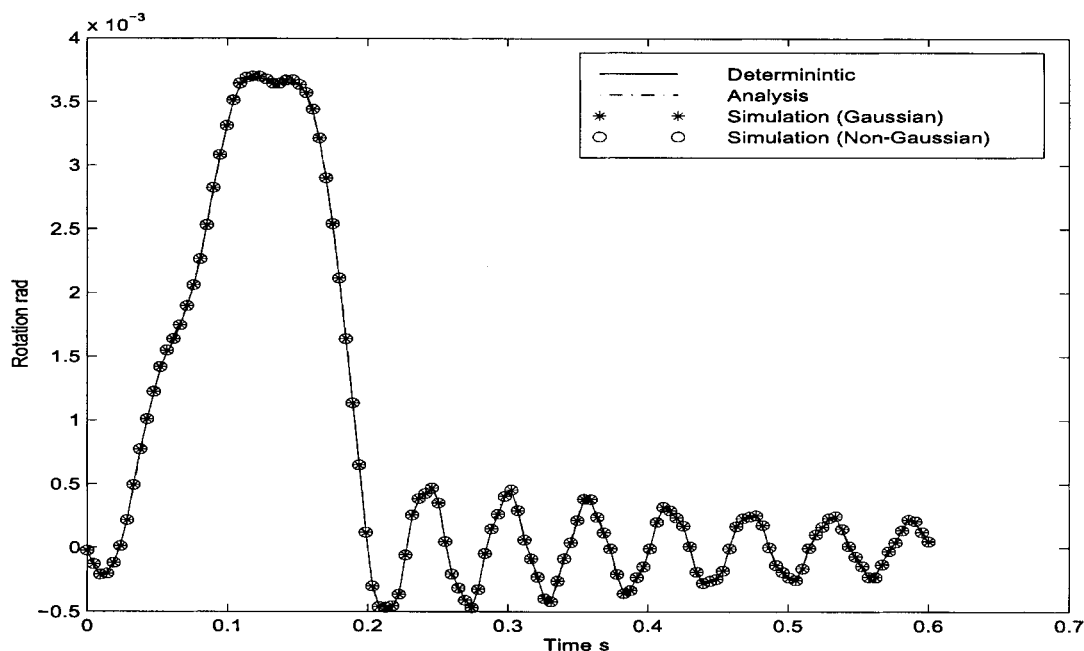
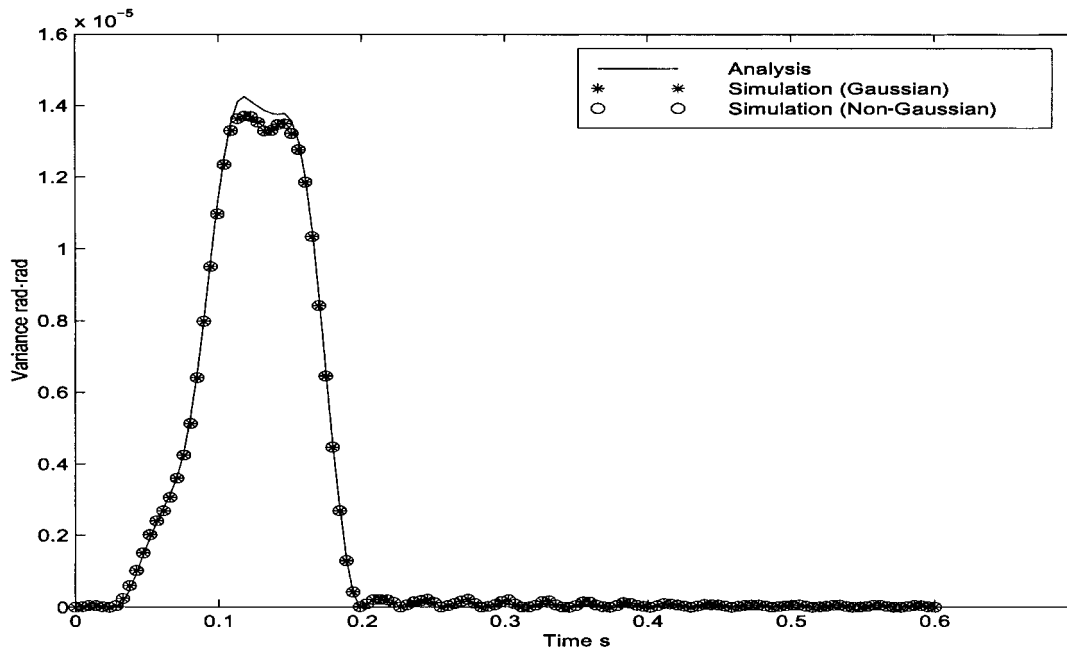

FIG. 6. Mean Rotation at Node 2

FIG. 7.   Variance of Rotation at Node 2

samples of non-Gaussian random fields was based on the procedures described by Grigoriu (1995). The results of this simulation study are also shown in Figs. 3, 4, 6, and 7. As may be observed, the effect of non-Gaussian modeling of $f_i(x)$ has marginal effect on the mean and standard deviations of the response variables considered. It must be noted that, if $f_i(x)$ are taken to be non-Gaussian, then the treatment of $V(\omega)$ as being Gaussian in the analysis amounts to making a Gaussian closure assumption on distribution of $V(\omega)$. This assumption seems plausible because the stiffness coefficients arise as a sum of weighted integrals, and such a sum may be approximated as being Gaussian even when the constituent terms are non-Gaussian. It should also be noted that there are significantly more non-Gaussian distributions (with the same first and second properties), with different tails, that can serve as models for the system property random fields. The assessment of the effects that these non-Gaussian models produce on the response requires further studies.

Nonzero Initial Conditions

Another assumption that has been made in the numerical examples presented in this study is that, at $t = 0$, the system is in a state of rest. It should be pointed out that this assumption is not restrictive as it is possible to consider the effects of nonzero initial conditions within the framework of procedures described in the preceding sections. A brief outline of the steps that need to be taken to achieve this is provided here. Because the forced response, assuming an initial state of rest, has already been determined, it would suffice if it is shown how the free vibration due to nonzero initial conditions can be evaluated. For purposes of illustration, let one consider the example of flexural vibration of beam governed by (1) and take $Y(x, 0) = Y_0(x)$ and $\dot{Y}(x, 0) = \dot{Y}_0(x)$. The requisite solution is obtained using the following steps:

1. Let one seek the solution in the form $Y(x, t) = Y_1(x, t) + Y_2(x, t)$, where $Y_1(x, 0) = 0$, $\dot{Y}_1(x, 0) = \dot{Y}_0(x)$ and $Y_2(x, 0) = Y_0(x)$, $\dot{Y}_2(x, 0) = 0$.
2. The effect of nonzero initial velocity on $Y_1(x, t)$ is equivalent to the application of distributed impulsive force $\delta(t)m(x)\dot{Y}_0(x)$ with zero initial conditions. This would

mean that $Y_1(x, t)$ is obtainable using the methods already described in the study.
3. In fact, by taking $\dot{Y}_1(x) = \delta(x - \xi)$, one can get the system of Green's function $G(x, \xi, t)$ in the time domain.
4. To obtain $Y_2(x, t)$ substitute $Z_2(x, t) = Y_2(x, t) - Y_0(x)$. It follows that the function $Z_2(x, t)$ is again governed by the field equation of the form (1) with zero initial conditions and the right-hand side given by

$$f(x, t) = -\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2}{dx^2}\,Y_0(x)\right]$$

To obtain $Z_2(x, t)$, one can again use the methods outlined previously. It must be noted that the Fourier transform of $f(x, t)$ in this case is expressible in terms of the Dirac delta function. Alternatively, one can also employ Green's function derived in the preceding step in conjunction with the convolution theorem to evaluate the response.
5. The required solution $Y(x, t)$ is thus given by $Y(x, t) = \{Y_1(x, t) + Z_2(x, t)\} + Y_0(x)$.

It can be shown that the implementation of these steps again leads to stochastic algebraic equations of the form $[\mathbf{D}(\omega)]\mathbf{d}(\omega) = \mathbf{F}(\omega)$. This equation is similar to (24) but with one important difference, which is that the elements of stochastic dynamic stiffness matrix $\mathbf{D}(\omega)$ are correlated here with the elements of the force vector $\mathbf{F}(\omega)$. Additionally, the components of the response due to nonzero initial conditions and external forcing also are correlated. The consequent analysis would become more tedious but, in principle, poses no significant difficulty. Details of this analysis, however, is not considered in the present study.

Built-Up Structures

The beam examples considered in the third and fourth sections had special boundary conditions, which ensured that the structure dynamic stiffness matrix in both of the examples had a single element. Consequently, the task of inverting the structure dynamic stiffness matrix became trivially simple. This simplification enabled the applicability of the proposed procedure to be demonstrated without entering into the issues of

inverting stochastic dynamic stiffness matrices. If different boundary conditions are used, or, if the transient dynamics of the built-up structure is considered, an additional step, involving the inversion of the stochastic dynamic stiffness matrix, needs to be taken. It may be noted that the stochastic dynamic stiffness matrix is a complex valued and symmetric matrix and that inversion of these types of matrices has recently been studied by the writers (Adhikari and Manohar 1999) in the context of steady-state dynamics of randomly parametered skeletal structures. This study also can be extended to cover the transient response in a reasonably straightforward manner. To see this, consider the equilibrium equation in the frequency domain of a system with $N \times N$ dynamic stiffness matrix given by $\mathbf{K}(\omega)\mathbf{Z}(\omega) = \mathbf{F}(\omega)$. Here, $\mathbf{F}(\omega)$ is the specified global force vector of size $N \times 1$; $\mathbf{Z}(\omega)$ is the $N \times 1$ displacement vector to be determined; and $\mathbf{K}(\omega)$ is the global stochastic dynamics stiffness matrix of size $N \times N$. All three quantities $\mathbf{K}$, $\mathbf{Z}$, and $\mathbf{F}$ are complex valued. The matrix $\mathbf{K}(\omega)$ is symmetric and can be written as $\mathbf{K}(\omega) = [\mathbf{K}_R^0(\omega) + i\mathbf{K}_I^0(\omega) + \Delta\mathbf{K}(\omega)]$. Here, $\mathbf{K}_R^0(\omega)$ and $\mathbf{K}_I^0(\omega)$ are, respectively, real and imaginary components of the deterministic part of $\mathbf{K}(\omega)$, and $\Delta\mathbf{K}(\omega)$ is the stochastic part. The vector $\mathbf{Z}(\omega)$ can be obtained by inverting the random complex matrix $\mathbf{K}(\omega)$ and can be written as $\mathbf{Z}(\omega) = \mathbf{K}(\omega)^{-1}\mathbf{F}(\omega) = \mathbf{Z}^R + i\mathbf{Z}^I(\omega)$, where $\mathbf{Z}^R(\omega)$ and $\mathbf{Z}^I(\omega)$ are, respectively, the real and imaginary parts of the response vector $\mathbf{Z}(\omega)$. Adhikari and Manohar (1999) outlined procedures based on expansions using eigenfunctions of $\mathbf{K}(\omega)$ and the Neuman expansion method [generalized to account for the complex nature of $\mathbf{K}(\omega)$] to approximately characterize $\mathbf{Z}(\omega)$. When the system property random fields were modeled as Gaussian random fields, and the strength of the randomness is taken to be small, it can be shown that $\mathbf{Z}^R(\omega)$ and $\mathbf{Z}^I(\omega)$ are jointly Gaussian, and their joint probability density function is obtainable. Now taking Fourier transform for any jth element ($1 \leq j \leq N$) of $\mathbf{Z}(\omega)$, one has

$$z_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{z_j^R(\omega) + iz_j^I(\omega)\}\exp[i\omega t] \ d\omega \qquad (36)$$

Using the knowledge that $z_j^R(\omega)$ and $z_j^I(\omega)$ are approximately Gaussian, the quantities $\langle z_j(t)\rangle$ and $\langle z_j(t_1)z_j(t_2)\rangle$ can easily be evaluated. This evaluation would require the joint Gaussian probability density function of $z_j^R(\omega_1)$, $z_j^I(\omega_1)$, $z_k^R(\omega_2)$, and $z_k^I(\omega_2)$. Expressions given by Adhikari and Manohar (1999) can be directly used to obtain the joint probability density function with the difference that now they have to be obtained at two different frequency points. Further studies are needed to examine the accuracy of the analysis as the system becomes larger and more complex.

CONCLUSIONS

The transient dynamics of randomly parametered beam elements has been studied using the stochastic dynamic stiffness approach. This has involved the application of a combination of a stochastic finite-element method using frequency-dependent shape functions to obtain description of response in frequency domain and an FFT algorithm to obtain response in time domain. The random field discretization used in this study involves frequency-dependent weighted integrals. The response is described in terms of time evolution of mean and variance. The illustrative examples presented demonstrate that the approximate analytical results compare favorably with the results from the more exact digital simulation results. The use of a dynamic stiffness matrix approach bypasses the need to perform random free vibration analysis, thereby eliminating a difficult step in the vibration response analysis of randomly parametered systems. This feature is of particular advantage in the context of dynamic response analysis due to short duration

transient loads. In these types of problems, the response is expected to consist of contributions from several modes, and a traditional modal expansion analysis would require an elaborate random eigenvalue analysis. This difficulty is avoided in the present study. Some related issues, which the writers are currently studying, include studies on structural reliability under dynamic loads, studies on the effects of structural nonlinearities, and application of the approach presented here to analyze the transient dynamics of built-up structures.

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