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Vibration analysis of beams with non-local foundations using the finite element method

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SUMMARY

In this paper, a non-local viscoelastic foundation model is proposed and used to analyse the dynamics of beams with different boundary conditions using the finite element method. Unlike local foundation models the reaction of the non-local model is obtained as a weighted average of state variables over a spatial domain *via* convolution integrals with spatial kernel functions that depend on a distance measure. In the finite element analysis, the interpolating shape functions of the element displacement field are identical to those of standard two-node beam elements. However, for non-local elasticity or damping, nodes remote from the element do have an effect on the energy expressions, and hence the damping and stiffness matrices. The expressions of these direct and cross-matrices for stiffness and damping may be obtained explicitly for some common spatial kernel functions. Alternatively numerical integration may be applied to obtain solutions. Numerical results for eigenvalues and associated eigenmodes of Euler–Bernoulli beams are presented and compared (where possible) with results in literature using exact solutions and Galerkin approximations. The examples demonstrate that the finite element technique is efficient for the dynamic analysis of beams with non-local viscoelastic foundations.

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1. INTRODUCTION

Beam and beam-like structures resting on elastic or viscoelastic foundations have wide application in modern engineering practices, such as, railway tracks, highway pavements, and continuously

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supported pipelines. As a result, numerous studies have been performed to investigate the static deflection, the dynamic response, and the dynamic stability of beams on elastic or viscoelastic foundations. However, in most of these studies, the foundation was idealized as a one-parameter Winkler model, where the system is modelled as infinitely close linear springs and the foundation pressure at any point is proportional to the deflection at that point. Several more complicated models have been reported in the literature, such as the two-parameter Pasternak model, the three-parameter Kerr model, the Vlasov model, or fractional derivative viscoelastic models.

Yankelevsky and Eisenberge [1] used the analytical expressions for the deflection of a finite beam with unit end loads as interpolating shape functions for a finite element to analyse the static deflection of a beam–column on an elastic foundation. Thambiratnam and Zhuge [2] also used the finite element method to investigate the free vibration of variable thickness thin beams supported on elastic foundations. Sun [3] presented the closed-form solution for an Euler–Bernoulli beam on a viscoelastic foundation excited by harmonic line loads using a Green's Function approach. Kim [4] applied the Fourier transform to obtain the solution for the vibration and stability of axial loaded beams on elastic foundations with moving harmonic loads. Other authors analysed the static and dynamic response of other structural components (rods, plates, shells) [5–8]. In most of these studies, the foundation is idealized as a Winkler (one-parameter) model. In the Winkler model, the foundation is modelled as a system composed of infinitely close linear springs, and assumes that the foundation pressure at any point is proportional to the deflection at that point. The Winkler model does not accurately represent the characteristics of many practical foundations. One of the most important shortcomings of this model is that it assumes no interactions between the springs or discontinuities in the foundation. To overcome these problems, several more complicated models have been suggested in the literature, such as the Pasternak (two-parameter), Kerr (three-parameter), or Vlasov models. For example, Yokoyama [9] proposed a finite element method to determine the vibration characteristics of a uniform Timoshenko beam–column supported on a two-parameter elastic foundation. Eisenberger [10] introduced an exact method to compute the natural frequencies of Euler beams on a two-parameter elastic foundation. Chen *et al.* [11] combined the state-space and differential quadrature methods to develop a mixed method for the bending and free vibration of a beam resting on a Pasternak elastic foundation. Similar problems for uniform beams on elastic foundations were treated by Chen [12, 13] using the differential quadrature method. de Rosa and Maurizi [14] investigated the influence of a concentrated mass and the Pasternak soil foundation model on the free vibration of Euler–Bernoulli beams. Avramidis and Morfidis [15] studied the static bending of Timoshenko beams on Kerr-type, three-parameter elastic foundations. Ayvaz and Ozgan [16] considered the modified Vlasov model to analyse the free vibration of beams resting on elastic foundations, and analysed the effects on the natural frequencies of the subsoil depth, the beam length, their ratio and the value of the vertical deformation parameter within the subsoil. Atanackovic and Stankovic [17] and Fenander [18] investigated the stability and dynamic response of elastic beams on viscoelastic foundations modelled using fractional derivatives.

All of the papers described above are formulated in terms of local field theory, where the foundation reactions are directly proportional to the state variables (displacement, rotation, curvature) of the beam deflection. For the Winkler-type one-parameter model, the relationship between the foundation reaction $Q_{\mathrm{W}}(x,t)$ and the deflection on the foundation surface $w(x,t)$ is [2]

$$Q_{\mathrm{W}}(x,t) = k_1(x) w(x,t) \tag{1}$$

where $k_1(x)$ is the stiffness coefficient of the elastic foundation model, and x and t are spatial and time variables, respectively.

For the Pasternak-type two-parameter model, the foundation reaction $Q_P(x,t)$ is [9]

$$Q_P(x,t) = k_1(x)w(x,t) - G_1(x)\frac{\partial w^2(x,t)}{\partial x^2} \tag{2}$$

where $k_1(x)$ and $G_1(x)$ are the stiffness coefficients of the spring and shear layers of the Pasternak foundation model, respectively.

For the Kerr-type three-parameter model, the foundation reaction $Q_K(x,t)$ is given by [15, 19]

$$\left(1+\frac{k_2(x)}{k_1(x)}\right)Q_K(x,t) - \frac{G_1(x)}{k_1(x)}\frac{\partial^2 Q_K(x,t)}{\partial x^2} = k_2(x)w(x,t) - G_1(x)\frac{\partial w^2(x,t)}{\partial x^2} \tag{3}$$

where $k_1(x)$, $k_2(x)$ and $G_1(x)$ are the stiffness coefficients of the two spring layers and the middle shear layer for the Kerr foundation model, respectively.

However, Flugge [20] pointed out that

> The reaction $q(x_1)$ at any point x_1, of course, not only upon the local value $w(x_1)$ of the deflection, but also upon that of neighbour points x_2, their influence decreasing as the distance $|x_1 - x_2|$ increases.

Obviously, the Winkler foundation model described by Equation (1) is a local reaction model and the influence of deflections close to the point of interest have not been considered. Equations (2) and (3) give relatively accurate models for soil foundations compared to the Winkler foundation model. In the Pasternak model, the influence of the deflection at neighbouring points are considered implicitly through the second-order derivatives of the deflection with respect to the spatial variable. However, this model does not consider attenuation effects with increasing spatial distance. The Kerr model is a special type of non-local foundation model because of the intermediate layer.

The purpose of this paper is to develop and analyse general non-local models for elasticity and damping. The development considers general kernel functions, and common kernel functions are highlighted and used in the examples. The main innovation in this paper is the development of finite element models for beams with non-local foundations. Non-local damping and stiffness are considered, and much of the analysis considers the free and forced vibration response. However, static problems of beams with non-local foundation stiffness may also be analysed using this approach.

2. THE NON-LOCAL FOUNDATION MODEL

In what follows, the elastic Winkler foundation model is generalized to a non-local Kelvin viscoelastic foundation model using non-local field theory [21–23]. Non-local elasticity theory has been investigated, however, few authors have addressed non-local viscoelastic media [24, 25]. In one-dimensional non-local elasticity, the foundation reaction is given as

$$Q_N(x,t) = \int_{x_1}^{x_2} K(x,\xi)w(\xi,t)\,\mathrm{d}\xi + \int_{x_1}^{x_2}\int_{-\infty}^{t} C(x,\xi,t-\tau)\frac{\partial w(\xi,\tau)}{\partial \tau}\,\mathrm{d}\tau\,\mathrm{d}\xi \tag{4}$$

where w is the bending deflection of the beam, $K(x,\xi)$ is the elastic spatial kernel, $C(x,\xi,t)$ is the damping kernel, and the spatial integrations are over the length of the foundation, whose extent is denoted by x_1 and x_2 in Figure 1.

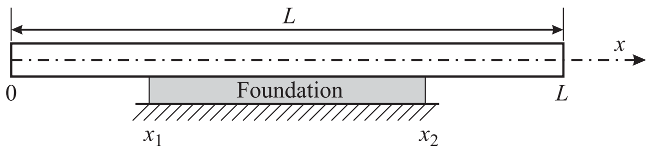

Figure 1. A beam with a partial non-local foundation.

The equation of motion for a non-uniform Euler–Bernoulli beam on a non-local viscoelastic foundation between x_1 and x_2 is then

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)+\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}+Q_{\mathrm{N}}(x,t)[H(x-x_1)-H(x-x_2)]=f(x,t) \tag{5}$$

where $f(x,t)$ is the excitation force, $EI(x)$ and $\rho A(x)$ are the bending stiffness and mass per unit length of the beam and $H(\cdot)$ is the Heaviside step function. The beam is initially assumed to be at rest and standard boundary conditions are applied at the two ends.

Equation (5) is an integro-differential equation, and obtaining closed-form solutions is difficult. Adhikari *et al.* [26] presented a closed-form solution for beams with non-local damping by the transfer function method [27]. Lei *et al.* [28] presented approximate solutions using a Galerkin method for uniform beams with typical kernel functions. To treat more complicated problems with variable foundation stiffness, non-uniform section properties or with intermediate supports, a finite element method is developed.

2.1. *Typical kernel functions*

For a given foundation the kernel function is fixed and could, in principle, be measured. Banks and Inman [29] considered different damping mechanisms for composite beams, and found that a non-local spatial model combined with viscous air damping gave the best quantitative agreement with experimental time histories. The important spatial property of the kernel functions is that the influence reduces with distance. The following examples are commonly used kernel functions, and they may be combined in weighted sums to match experimental data to arbitrary accuracy. We assume that the damping kernel function $C(x,\xi,t)$ is separable in space and time. Thus, the kernel function takes the form

$$C(x,\xi,t-\tau)=C_0[H(\xi-x_1)-H(\xi-x_2)]c(x-\xi)g(t-\tau) \tag{6}$$

Physically, this model represents non-local viscoelastic damping, for example, in a beam with a viscoelastic foundation. Viscous damping has the form $g(t-\tau)=\delta(t-\tau)$, and local damping $c(x-\xi)=\delta(x-\xi)$.

The function $g(t)$ is the relaxation kernel function of the non-viscous element and is often approximated as

$$g(t)=\sum_{i=1}^{m}\frac{g_i}{\tau_i}\mathrm{e}^{-t/\tau_i} \tag{7}$$

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where g_i and τ_i are positive constants representing the damping coefficients and relaxation times, respectively. The Laplace transform of the relaxation kernel is then

$$G(s) = \sum_{i=1}^{m} \frac{g_i}{\tau_i s + 1} \tag{8}$$

If $\tau_i \to 0, \forall i$, then one obtains the standard viscous damping model. Alternatively, fractional derivative or GHM models could be used [30–34].

The spatial kernel function, $c(x-\xi)$ in Equation (6) is normalized to satisfy the condition

$$\int_{-\infty}^{\infty} c(x)\,\mathrm{d}x = 1 \tag{9}$$

Common choices for this kernel function are the exponential decay

$$c(x-\xi) = \frac{\alpha}{2} \mathrm{e}^{-\alpha|x-\xi|} \tag{10}$$

the Gaussian (error) function

$$c(x-\xi) = \frac{\alpha}{\sqrt{2\pi}} \mathrm{e}^{-\alpha^2 (x-\xi)^2/2} \tag{11}$$

or the triangular function

$$c(x-\xi) = \begin{cases} \dfrac{\alpha}{2}\left(1 - \dfrac{\alpha}{2}|x-\xi|\right) & \text{for } |x-\xi| \leqslant \dfrac{2}{\alpha} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Other models are also possible [28]. In the kernel definitions, Equations (10)–(12), α is a parameter of the model that determines the rate of decay of the influence of the non-local stiffness or damping with distance. If $\alpha \to \infty$ then one obtains the standard local model.

The elastic spatial kernel function $K(x,\xi)$ will be similar, except that the time dependence is assumed to be negligible. Thus,

$$K(x,\xi) = K_0[H(\xi - x_1) - H(\xi - x_2)]k(x-\xi) \tag{13}$$

where $k(x-\xi)$ has the same form as $c(x-\xi)$.

3. THE FINITE ELEMENT MODEL

Generally, the approach adopted when developing models using finite element analysis is to approximate the deformation within an element using nodal values of displacement and rotation. The kinetic and strain energy for each element is then computed and the contributions of each element added together to obtain a global model of the structure. The damping matrix is obtained in a similar way using the dissipation function. One key aspect of finite element analysis is that the energy contributions from each element only depend on the displacements and rotations at the nodes associated with that element. Clearly for non-local elasticity this will not be the case, although the form of the exponential kernel function given by Equation (10) does lead to considerable

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simplification. This paper only considers a non-local foundation, and the development of the mass and stiffness matrices for the beam follows the standard procedure, which will be outlined briefly to establish notation.

3.1. Shape function definition and beam elements

A standard beam element is modelled using two nodes (at the ends of the beam element), and two degrees of freedom per node (translation and rotation), as shown in Figure 2. This section considers a beam modelled using Euler–Bernoulli beam theory; Section 3.5 considers Timoshenko beam theory. The deformation within the eth element, $w_e(\eta, t)$, is approximated using cubic shape shape functions, for $\eta \in [0, \ell_e]$, as

$$w_e(\eta, t) = \mathbf{N}_e(\eta)\mathbf{q}_e(t) \tag{14}$$

where

$$\mathbf{N}_e(\eta) = [N_{e1}(\eta) \;\; N_{e2}(\eta) \;\; N_{e3}(\eta) \;\; N_{e4}(\eta)], \quad \mathbf{q}_e(t) = \begin{Bmatrix} w_{e1}(t) \\ \psi_{e1}(t) \\ w_{e2}(t) \\ \psi_{e2}(t) \end{Bmatrix} \tag{15}$$

and

$$\begin{aligned} N_{e1}(\eta) &= 1 - 3\frac{\eta^2}{\ell_e^2} + 2\frac{\eta^3}{\ell_e^3}, \quad N_{e2}(\eta) = \eta - 2\frac{\eta^2}{\ell_e} + \frac{\eta^3}{\ell_e^2} \\ N_{e3}(\eta) &= 3\frac{\eta^2}{\ell_e^2} - 2\frac{\eta^3}{\ell_e^3}, \quad N_{e4}(\eta) = -\frac{\eta^2}{\ell_e} + \frac{\eta^3}{\ell_e^2} \end{aligned} \tag{16}$$

The kinetic energy, for a beam of length L with M elements, is then approximated as

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A(x) \left(\frac{\partial w(x,t)}{\partial t}\right)^2 \mathrm{d}x = \frac{1}{2}\sum_{e=1}^{M} \int_0^{\ell_e} \rho A(x_e + \eta) \left(\frac{\partial w_e(\eta,t)}{\partial t}\right)^2 \mathrm{d}\eta \\ &= \frac{1}{2}\sum_{e=1}^{M} \dot{\mathbf{q}}_e(t)^\top \mathbf{M}_e \dot{\mathbf{q}}_e(t) \end{aligned} \tag{17}$$

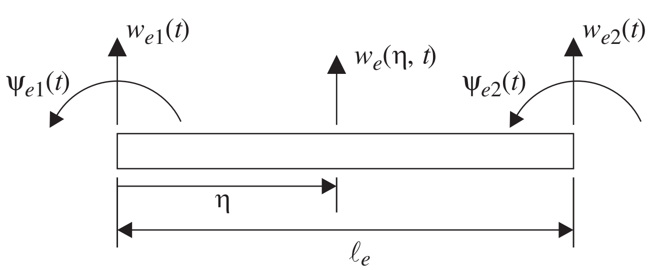

Figure 2. The degrees of freedom of a beam finite element.

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where the eth node, corresponding to node 1 of the eth element, is located at x_e, and the element mass matrix is

$$\mathbf{M}_e = \int_0^{\ell_e} \rho A(x_e + \eta)\mathbf{N}_e(\eta)^\top \mathbf{N}_e(z)\,\mathrm{d}\eta \tag{18}$$

3.2. Non-local stiffness and damping

For non-local stiffness the global stiffness matrix is obtained in a similar way using the strain energy. Assume that the foundation covers the whole length of the beam; the case of a partial foundation is an easy extension. Only the contribution from the foundation will be considered here, and this is given by

$$\begin{aligned} U_\mathrm{F} &= \frac{1}{2}\int_0^L Q_\mathrm{N}(x,t)w(x,t)\,\mathrm{d}x = \frac{1}{2}\int_0^L \int_0^L K(x,\xi)w(\xi,t)w(x,t)\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{i,j=1}^{M} \mathbf{q}_i^\top \mathbf{K}_{ij}\mathbf{q}_j \end{aligned} \tag{19}$$

where

$$\mathbf{K}_{ij} = \int_0^{\ell_j}\int_0^{\ell_i} k(x_j + \hat{x} - x_i - \hat{\xi})\mathbf{N}_i^\top(\hat{\xi})\mathbf{N}_j(\hat{x})\,\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x} \tag{20}$$

and $\hat{x}$ and $\hat{\xi}$ are local co-ordinates, for example $x = x_j + \hat{x}$.

For non-local damping the global damping matrix is obtained in a similar way using the energy dissipation function. The dissipation function is

$$\begin{aligned} \mathscr{F}(t) &= \frac{1}{2}\int_0^L \left\{\int_0^L \int_{-\infty}^t C(x,\xi,t-\tau)\dot{w}(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi\right\}\dot{w}(x,t)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^t g_e(t-\tau)\sum_{i,j=1}^{M} \dot{\mathbf{q}}_i(t)^\top \mathbf{C}_{ij}\dot{\mathbf{q}}_j(\tau)\,\mathrm{d}\tau \end{aligned} \tag{21}$$

where

$$\mathbf{C}_{ij} = \int_0^{\ell_j}\int_0^{\ell_i} c(x_j + \hat{x} - x_i - \hat{\xi})\mathbf{N}_i^\top(\hat{\xi})\mathbf{N}_j(\hat{x})\,\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x} \tag{22}$$

In general, the degrees of freedom at all nodes will be coupled within a region where the structure has non-local stiffness and damping behaviour. This means that the global stiffness and damping matrices will be full, and each term must be determined independently. The problem simplifies significantly if the beams are modelled using beam elements of equal length, so that $\mathbf{N}_i = \mathbf{N}$, $\ell_i = \ell$, $\forall i$, and the foundation is uniform. Because the kernel function only involves $x_j - x_i$ the matrices $\mathbf{K}_{ij}$ or $\mathbf{C}_{ij}$ are equal for a fixed $j - i$. Thus, if the foundation covers M elements, only M of the element matrices $\mathbf{K}_{ij}$ or $\mathbf{C}_{ij}$ have to be calculated. Figure 3 illustrates this for a foundation modelled with three elements, which are defined using four nodes (or eight degrees of

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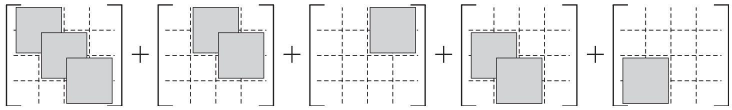

Figure 3. Illustration of the assembly of the global matrices for the non-local foundation. For each term depicted by the global matrix, each submatrix denoted by the shaded square is equal.

freedom). Each 4×4 block matrix in the 8×8 global matrix is denoted by a shaded square and for each term these block matrices are identical. For general kernel functions the matrices must be calculated by numerical integration. However, the integrations may be performed explicitly for uniform foundations with exponential or Gaussian (error function) kernels. Further simplification is possible for the exponential kernel.

3.3. The exponential kernel

The exponential kernel function given in Equation (10) leads to a significant simplification in the stiffness and damping matrices, and only requires the calculation of two element matrices for a uniform foundation and elements of equal length. When $i = j$, the direct stiffness matrix is

$$\mathbf{K}_{ii} = \frac{K_0\alpha}{2} \int_0^\ell \int_0^\ell \mathrm{e}^{-\alpha|\hat{x}-\hat{\xi}|} \mathbf{N}^\top(\hat{\xi}) \mathbf{N}(\hat{x}) \, \mathrm{d}\hat{\xi} \, \mathrm{d}\hat{x} \tag{23}$$

If $j>i$

$$\mathbf{K}_{ij} = \frac{K_0\alpha}{2} \int_0^\ell \int_0^\ell \mathrm{e}^{-\alpha(x_j - x_i + \hat{x} - \hat{\xi})} \mathbf{N}^\top(\hat{\xi}) \mathbf{N}(\hat{x}) \, \mathrm{d}\hat{\xi} \, \mathrm{d}\hat{x} = \mathrm{e}^{-\alpha(x_j - x_i)} \bar{\mathbf{K}} = \mathrm{e}^{-\alpha(j-i)\ell} \bar{\mathbf{K}} \tag{24}$$

where the cross-stiffness matrix is

$$\bar{\mathbf{K}} = \frac{K_0\alpha}{2} \int_0^\ell \int_0^\ell \mathrm{e}^{-\alpha(\hat{x}-\hat{\xi})} \mathbf{N}^\top(\hat{\xi}) \mathbf{N}(\hat{x}) \, \mathrm{d}\hat{\xi} \, \mathrm{d}\hat{x} = \frac{K_0\alpha}{2} \left(\int_0^\ell \mathrm{e}^{\alpha\hat{\xi}} \mathbf{N}^\top(\hat{\xi}) \, \mathrm{d}\hat{\xi} \right) \left(\int_0^\ell \mathrm{e}^{-\alpha\hat{x}} \mathbf{N}(\hat{x}) \, \mathrm{d}\hat{x} \right) \tag{25}$$

Similarly if $j<i$,

$$\mathbf{K}_{ij} = \mathrm{e}^{-\alpha(i-j)\ell} \bar{\mathbf{K}}^\top \tag{26}$$

Thus, only two element stiffness matrices need to be computed and then the complete global stiffness matrix is easily derived. These matrices may be obtained explicitly by integrating Equations (23) and (25), and the resulting matrices are given in Appendix A.1.

As an example to demonstrate how the stiffness matrix is assembled, consider that a uniform non-local foundation is modelled, where the three elements have equal length. Then the stiffness matrix will be zero apart from the eight degrees of freedom associated with the foundation. Figure 4 illustrates the assembly and shows the non-zero terms. The global damping matrix for the non-local foundation may be assembled in a similar way.

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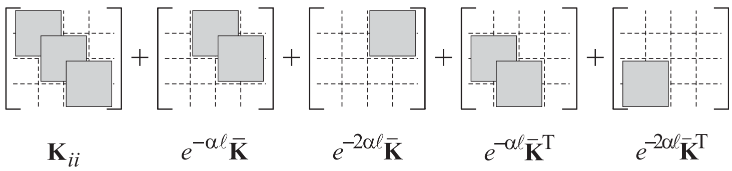

Figure 4. Illustration of the assembly of the global stiffness matrix for a non-local foundation with an exponential kernel.

3.4. The Gaussian kernel

For a uniform foundation, the error function (or Gaussian) kernel function given in Equation (11) gives the direct stiffness matrix as

$$\mathbf{K}_{ii} = \frac{K_0\alpha}{\sqrt{2\pi}} \int_0^{\ell} \int_0^{\ell} \mathrm{e}^{-\alpha^2(\hat{x}-\hat{\xi})^2/2} \mathbf{N}^{\top}(\hat{\xi})\mathbf{N}(\hat{x})\,\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x} \tag{27}$$

This integral may be determined in closed form and is given in Appendix A.2.

For a uniform foundation and elements of equal length, the cross-stiffness matrix, $\mathbf{K}_{ij}$ depends on the value of $|j-i|$. Thus, if $j>i$ and $d=j-i$ then

$$\mathbf{K}_{ij} = \frac{K_0\alpha}{\sqrt{2\pi}} \int_0^{\ell} \int_0^{\ell} \mathrm{e}^{-\alpha^2(\hat{x}-\hat{\xi}+d\ell)^2/2} \mathbf{N}^{\top}(\hat{\xi})\mathbf{N}(\hat{x})\,\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x} \tag{28}$$

Note also that

$$\mathbf{K}_{ij} = \mathbf{K}_{ji}^{\top} \tag{29}$$

and hence the cross-stiffness matrices for $j<i$ are obtained immediately. The resulting closed-form expressions are quite complex and are not given here. However, these matrices are easily obtained using symbolic software.

3.5. Timoshenko beam theory

The development thus far has considered Euler–Bernoulli beam theory. For thick beams shear effects are important and Timoshenko beam theory is often used [35]. The shape functions corresponding to a uniform Timoshenko beam element are

$$\begin{aligned}
N_{e1}(\eta) &= \frac{1}{1+\Phi_e}\left(1+\Phi_e-\Phi_e\frac{\eta}{\ell_e}-3\frac{\eta^2}{\ell_e^2}+2\frac{\eta^3}{\ell_e^3}\right)\\
N_{e2}(\eta) &= \frac{\ell_e}{1+\Phi_e}\left(\frac{2+\Phi_e}{2}\frac{\eta}{\ell_e}-\frac{4+\Phi_e}{2}\frac{\eta^2}{\ell_e^2}+\frac{\eta^3}{\ell_e^3}\right)\\
N_{e3}(\eta) &= \frac{1}{1+\Phi_e}\left(\Phi_e\frac{\eta}{\ell_e}+3\frac{\eta^2}{\ell_e^2}-2\frac{\eta^3}{\ell_e^3}\right)\\
N_{e4}(\eta) &= \frac{\ell_e}{1+\Phi_e}\left(-\frac{\Phi_e}{2}\frac{\eta}{\ell_e}-\frac{2-\Phi_e}{2}\frac{\eta^2}{\ell_e^2}+\frac{\eta^3}{\ell_e^3}\right)
\end{aligned} \tag{30}$$

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where $\Phi_e = 12EI_e/\kappa_e G_e A_e \ell_e^2$, EI_e is the flexural rigidity, G_e is the shear modulus, and A_e is the cross-sectional area of the beam. κ_e is the shear constant that depends on the cross-section of the beam. For a rectangular section, the shear constant is $\kappa_e = 10(1+\nu_e)/(12+11\nu_e)$ where ν_e is the Poisson ratio. These shape functions give the standard element mass and stiffness matrices for a Timoshenko beam. If $\Phi_e = 0$, shear effects are neglected, and the result is an Euler–Bernoulli beam element.

Since the shape functions for the beam are also used in the development of the element matrices for the foundation, the foundation matrices need to be reformulated. This is done by substituting the shape functions given in Equation (30) into the general expressions for the element matrices, Equations (20) and (22). Alternatively these shape functions are substituted into Equations (23) and (25) for the exponential kernel, or Equations (27) and (28) for the Gaussian kernel. These element matrices are easily generated using symbolic software, or calculated by numerical integration.

3.6. The eigenvalue problem

Taking the Laplace transform gives the equations of motion in the form

$$[s^2\mathbf{M} + sG(s)\mathbf{C} + \mathbf{K}]\mathbf{q} = \mathbf{0} \tag{31}$$

where $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ are the global mass, stiffness and damping matrices with respect to the generalized co-ordinates of the beam model. Remember that the stiffness matrix has a contribution from the foundation as well as the beam itself. In Equation (31), the time response of the viscoelastic foundation has been assumed to arise from a single material, leading to a single $G(s)$ term. If different materials are present a damping term will appear for each different relaxation kernel, $G(s)$. Also in Equation (31), all of the damping is assumed to arise in the foundation; internal damping within the beam is easily incorporated by introducing extra terms.

The eigenvalues are obtained by solving

$$\det|s^2\mathbf{M} + sG(s)\mathbf{C} + \mathbf{K}| = 0 \tag{32}$$

The corresponding mode shape functions are obtained by substituting the eigenvalues into Equation (31) and computing the null space of the matrix.

Alternatively, for kernel functions where $G(s)$ is a rational polynomial, the internal variable approach may be used to produce a linear eigenvalue problem, where the size of the matrices is larger than those in Equation (32) [36, 37]. Suppose the kernel is defined by Equation (8) and that $\mathbf{C} = \mathbf{R}\mathbf{R}^\top$, where $\mathbf{R}$ is full rank and will be rectangular if $\mathbf{C}$ is singular. Then, introducing the velocity vector $\mathbf{p} = s\mathbf{q}$ and the internal state variables

$$\mathbf{r}_i = \frac{1}{\tau_i s + 1}\mathbf{R}^\top\mathbf{p} \quad \text{for } i = 1, \ldots, m \tag{33}$$

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results in the linear eigenvalue problem

$$s\begin{bmatrix} \mathbf{0} & \mathbf{M} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \tau_1\mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau_2\mathbf{I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{Bmatrix} \mathbf{q} \\ \mathbf{p} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} & g_1\mathbf{R} & g_2\mathbf{R} & \cdots \\ \mathbf{0} & -\mathbf{M} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & -\mathbf{R}^\top & \mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{0} & -\mathbf{R}^\top & \mathbf{0} & \mathbf{I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{Bmatrix} \mathbf{q} \\ \mathbf{p} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{Bmatrix} = \mathbf{0} \tag{34}$$

This form of the state-space eigenvalue problem is not unique, and other formulations may have benefits. For example, the GHM method [32–34] is based on the second-order form of the equations of motion and is able to retain symmetry in the matrices.

3.7. The frequency response

The equations of motion for the forced response may be obtained by extending Equation (31) to give

$$[-\omega^2\mathbf{M} + \mathrm{j}\omega G(\mathrm{j}\omega)\mathbf{C} + \mathbf{K}]\mathbf{q}(\omega) = \mathbf{Q}(\omega) \tag{35}$$

where $\mathbf{Q}$ is the force applied to the beam degrees of freedom and $\mathrm{j} = \sqrt{-1}$. The frequency response function is then defined as

$$\mathbf{H}(\omega) = [-\omega^2\mathbf{M} + \mathrm{j}\omega G(\mathrm{j}\omega)\mathbf{C} + \mathbf{K}]^{-1} \tag{36}$$

4. EXAMPLES

A number of examples are given to demonstrate the approach. Where possible the finite element solutions are compared to other analytical or approximate methods. Adhikari *et al.* [26] described the transfer function method and Lei *et al.* [28] described the Galerkin method.

4.1. A full elastic foundation

A simply supported uniform beam of length $L = 6.096\,\mathrm{m}$ resting on a uniform non-local elastic foundation is considered, as shown in Figure 1 with $x_1 = 0$ and $x_2 = L$ (that is the foundation supports the whole beam). This example was analysed by Lai *et al.* [38] and Thambiratnam and Zhuge [2]. The beam has Young's modulus 24.82 GPa, second moment of area $1.439 \times 10^{-3}\,\mathrm{m}^4$ and mass per unit length $446.3\,\mathrm{kg\,m}^{-1}$. The stiffness of the foundation is $16.55\,\mathrm{MN\,m}^{-2}$. Using the finite element method developed in this paper, the first four natural frequencies of free vibration were obtained with 6, 8 and 10 elements, for different values of α for the exponential kernel, Equation (10). The results are shown in Table I together with those from the analytical solution for local foundation stiffness ($\alpha = \infty$) [39], given by

$$\omega_i = \sqrt{\frac{EI}{\rho A}}\sqrt{\frac{i^4\pi^4}{L^4} + \frac{K_0}{EI}} \tag{37}$$

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Table I. The natural frequencies (Hz) of vibration for the simple beam on a non-local elastic foundation with an exponential kernel.

α (m^{-1})	2	2	2	5	10	50	∞	∞
N_e	6	8	10	10	10	10	10	Analytical
Mode 1	32.137	32.137	32.137	32.758	32.862	32.897	32.898	32.898
Mode 2	55.310	55.287	55.281	56.495	56.728	56.808	56.812	56.808
Mode 3	110.89	110.62	110.54	111.61	111.86	111.95	111.95	111.90
Mode 4	194.85	193.36	192.92	193.74	193.98	194.07	194.08	193.76

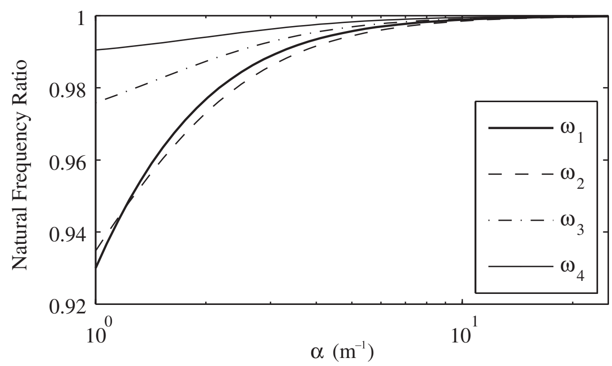

Figure 5. The variation of the first four natural frequencies with α for the fully supported beam and a exponential kernel. The natural frequencies have been normalized using the values at $\alpha = \infty$.

Table II. The natural frequencies (Hz) of vibration for the simple beam, modelled with 10 finite elements, on a non-local elastic foundation with an error function kernel.

α (m^{-1})	2	5	10	50	∞
Mode 1	32.470	32.825	32.880	32.898	32.898
Mode 2	55.862	56.644	56.769	56.810	56.812
Mode 3	110.95	111.76	111.90	111.95	111.95
Mode 4	193.15	193.88	194.03	194.07	194.08

Figure 5 shows the effect of varying α on these natural frequencies for the finite element model with 10 elements.

Consider now that the kernel consists of the Gaussian or error function, Equation (11). Table II shows the first four natural frequencies for the finite element method with 10 elements and Figure 6 shows these results graphically. Interestingly, the amplitude of α required for the error function kernel to approximate the Dirac delta function (local elasticity) is lower than that required for the exponential kernel. This is because of the shape of the kernels close to the centre point, as shown in Figure 7. Note also that the ω_1 and ω_2 lines cross in Figure 5 and do not cross in Figure 6.

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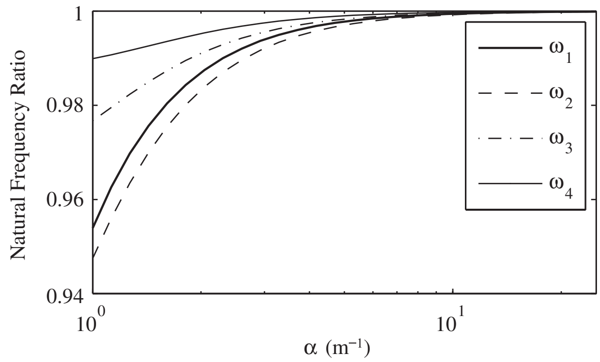

Figure 6. The variation of the first four natural frequencies with α for the fully supported beam and an error function kernel. The natural frequencies have been normalized using the values at $\alpha=\infty$.

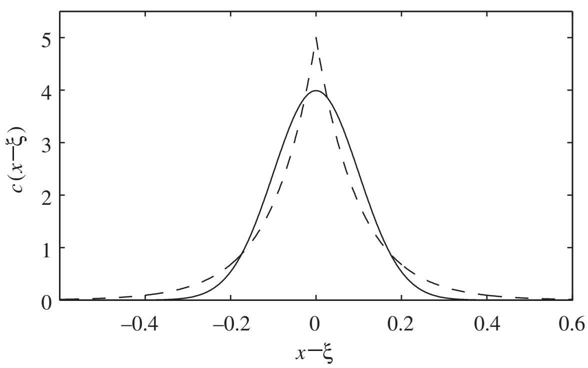

Figure 7. The exponential kernel (dashed), Equation (10), and error function kernel (solid), Equation (11), for $\alpha=10\,\mathrm{m}^{-1}$.

Table III. The first four eigenvalue pairs for the simple beam, modelled with 10 finite elements, on a non-local viscously damped foundation with an exponential function kernel with $C_0=1\,\mathrm{kNs\,m}^{-2}$.

α (m^{-1})	2	10	∞	Analytical
	-1.0613 ± 75.125j	-1.1175 ± 75.125j	-1.1203 ± 75.125j	-1.1203 ± 75.124j
	-0.9157 ± 300.561j	-1.1089 ± 300.560j	-1.1203 ± 300.560j	-1.1203 ± 300.528j
	-0.7443 ± 676.553j	-1.0950 ± 676.553j	-1.1203 ± 676.553j	-1.1203 ± 676.191j
	-0.5891 ± 1204.11j	-1.0761 ± 1204.11j	-1.1203 ± 1204.11j	-1.1203 ± 1202.12j

This change occurs because of differences in the relationships between the rate of decay in the two spatial kernel functions and the mode-shape deformation.

Table III shows the effect of a viscously damped foundation with an exponential kernel for various values of α ($K_0=0$, $C_0=1\,\mathrm{kNs\,m}^{-2}$). The beam is modelled with 10 finite elements.

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4.2. A partial viscoelastic foundation

A pinned–pinned beam will be used as an example to demonstrate the use of the finite element methods for a partial non-local viscoelastic foundation (that is the foundation stiffness is zero, $K_0 = 0$). The results will also be compared to those from the Galerkin method. The ith mode of the undamped beam is $\phi_i(x) = A_i \sin(i\pi x/L)$ where A_i is calculated to give mass normalized mode shapes. These undamped mode shapes will be used as the admissible functions in the Galerkin approach. The dimensions are (see Figure 1) $L = 200$ mm, $x_1 = 50$ mm, and $x_2 = 150$ mm. Young's modulus is $E = 70\,\mathrm{GN\,m^{-2}}$, the density is $\rho = 2700\,\mathrm{kg\,m^{-3}}$, and the cross-section is 5×5 mm. Only one term is used in the relaxation kernel function given by Equation (8), and the constants are defined as $\tau = \tau_1$ and $g_1 = g_\infty$.

Table IV shows the first three eigenvalue pairs for the pinned–pinned beam example with viscous damping ($\tau = 0$, $g_\infty = 1$) and an exponential spatial kernel ($\alpha = 1\,\mathrm{m^{-1}}$, $C_0 = 200\,\mathrm{Ns\,m^{-2}}$) for the Galerkin method with 10 admissible functions and also for the finite element method with 4, 8 and 40 elements.

Table V shows the first three eigenvalue pairs for the pinned–pinned beam example with a variety of non-viscous and non-local damping, that is as τ and α vary, for the finite element method with eight elements. It is interesting to observe that the natural frequency corresponding to the first complex pair of eigenvalues increases significantly around $\alpha = 10\,\mathrm{m^{-1}}$ and $\tau = 0.001$ s. Figure 8 shows the variation of the first natural frequency with τ for $\alpha = 10\,\mathrm{m^{-1}}$. Between $\tau = 10^{-4}$ and 10^{-2} s there is clearly a significant interaction with the internal variables of the foundation that leads to a significant increase in the stiffness of the structure.

Suppose now that the beam is clamped at one end and free at the other. All other dimensions are identical to the pinned–pinned beam example. Table VI shows the first three eigenvalue pairs for this cantilever beam example with a variety of non-viscous and non-local damping,

Table IV. The first three eigenvalue pairs for the pinned–pinned beam example obtained from the Galerkin method with 10 admissible functions and the finite element (FE) method with 4, 8 and 40 elements.

Galerkin	FE (four elements)	FE (eight elements)	FE (40 elements)
−58.176 ± 1812.4j	−58.174 ± 1812.9j	−58.176 ± 1812.5j	−58.176 ± 1812.4j
−0.72086 ± 7253.5j	−0.72080 ± 7282.1j	−0.72086 ± 7255.4j	−0.72086 ± 7253.5j
−6.7383 ± 16 320j	−6.5458 ± 16 618j	−6.7359 ± 16 341j	−6.7384 ± 16 320j

Note: $\alpha = 1\,\mathrm{m^{-1}}$, $g_\infty = 1$, $C_0 = 200\,\mathrm{Ns\,m^{-2}}$, $\tau = 0$.

Table V. The first three eigenvalue pairs for the pinned–pinned beam example obtained from the finite element method with eight elements for different values of τ and α. $g_\infty = 1$, $C_0 = 200\,\mathrm{Ns\,m^{-2}}$.

τ (s)	0	0.001	0.001	0
α ($\mathrm{m^{-1}}$)	1	1	10	10
	−58.176 ± 1812.5j	−13.366 ± 1838.1j	−92.216 ± 2006.0j	−447.62 ± 1757.7j
	−0.72086 ± 7255.4j	−0.013438 ± 7255.5j	−0.94877 ± 7262.3j	−50.996 ± 7255.2j
	−6.7359 ± 16 341j	−0.025152 ± 16 342j	−0.26562 ± 16 346j	−70.624 ± 16 338j

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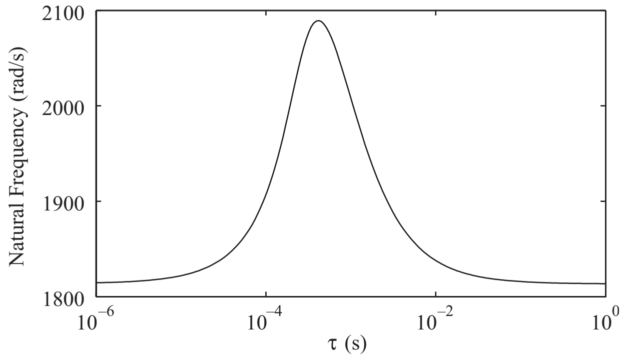

Figure 8. The variation of the first natural frequency with τ for the fully supported beam and an exponential kernel: $\alpha = 10\,\mathrm{m}^{-1}$, $g_\infty = 1$, $C_0 = 200\,\mathrm{Ns\,m}^{-2}$.

Table VI. The first three eigenvalue pairs for the cantilever beam example obtained from the finite element method with eight elements for different values of τ and α. $g_\infty = 1$, $C_0 = 200\,\mathrm{Ns\,m}^{-2}$.

τ (s)	0	0.001	0.001	0
α (m^{-1})	1	1	10	10
	-17.841 ± 645.83j	-12.672 ± 654.28j	-102.77 ± 721.46j	-141.58 ± 634.22j
	-45.254 ± 4048.1j	-2.6009 ± 4059.3j	-20.277 ± 4132.1j	-353.66 ± 4009.6j
	$-1.0206 \pm 11\,343$j	$-0.007875 \pm 11\,343$j	$-0.47410 \pm 11\,348$j	$-61.492 \pm 11\,342$j

that is as τ and α vary, for the finite element method with eight elements. Clearly, both the non-local and non-viscous properties of the beam have a significant effect on the damping within the structure.

4.3. The effect of beam shear

The beam example of Section 4.2 will be used to demonstrate the effect of incorporating shear into the beam model. It should be emphasized that the model of the foundation is unchanged, but the shape functions used to approximate the effect of the foundation do change. Table VII shows the effect of the different beam models, and the use of different shape functions to approximate the foundation effects, on the example of Section 4.2 with $\tau = 0$, $\alpha = 10\,\mathrm{m}^{-1}$ and $C_0 = 200\,\mathrm{Ns\,m}^{-2}$. The results corresponding to Section 4.2 have a beam thickness of $h = 5\,\mathrm{mm}$; also shown are results for a beam thickness of 20 mm, where the shear effects will be more important. Since the foundation only provides damping, the imaginary part of the eigenvalue is only affected by the model used for the beam. The shape functions used for the foundation do affect the real part (i.e. the damping), although this effect is small. Using inconsistent shape functions (that is a Timoshenko beam model with Euler–Bernoulli shape functions for the foundation) does increase the errors, although the effect is small. Furthermore, as expected, the effect of shear is more important for thicker beams.

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Table VII. The first three eigenvalue pairs for the pinned–pinned beam example obtained from the finite element method with eight elements for different beam and foundation approximations. $\tau = 0$, $\alpha = 10\,\mathrm{m}^{-1}$ and $C_0 = 200\,\mathrm{Ns\,m}^{-2}$.

	Beam shape functions		
	Euler–Bernoulli	Timoshenko	Timoshenko
	Foundation shape functions		
	Euler–Bernoulli	Euler–Bernoulli	Timoshenko
Beam thickness, $h = 5$ mm	-447.62 ± 1757.7j -50.996 ± 7255.2j $-70.624 \pm 16\,338$j	-447.62 ± 1756.2j -51.002 ± 7233.6j $-70.634 \pm 16\,236$j	-447.62 ± 1756.2j -50.996 ± 7233.6j $-70.614 \pm 16\,236$j
Beam thickness, $h = 20$ mm	-111.86 ± 7252.7j $-12.749 \pm 29\,021$j $-17.703 \pm 65\,365$j	-111.86 ± 7164.3j $-12.771 \pm 27\,726$j $-17.743 \pm 59\,617$j	-111.86 ± 7164.3j $-12.748 \pm 27\,726$j $-17.676 \pm 59\,617$j

5. CONCLUSION

In this paper, a new method of analysing beams with a non-local foundation was proposed based on the finite element method. The non-local foundation model is a generalization of the Kevin–Voigt viscoelastic foundation model, where the foundation reaction at a given point depends on the time history and the velocities within a spatial domain. The finite element models for an exponential foundation, for either the stiffness or damping matrices, require only two matrices, one for the cross-element terms and one for the direct terms. The cross-matrix to model non-local effects is a novel concept introduced in this paper, and these matrices are zero for local damping and elasticity models. Numerical solutions have been obtained for beams with a variety of boundary conditions and foundations. It has been demonstrated that the form of the non-local foundation model has a significant impact on the dynamic characteristics of structures.

APPENDIX A

A.1. The exponential kernel

This appendix gives the explicit expressions for the direct and cross-stiffness matrices for the exponential foundation kernel. The direct stiffness matrix is given by Equation (23). The integral over $\hat{\xi}$ involving the term $|\hat{x} - \hat{\xi}|$ can be expressed as

$$\mathbf{K}_{ii} = \frac{K_0\alpha}{2}\int_{\hat{x}=0}^{\ell}\left\{\int_{\hat{\xi}=0}^{x} \mathrm{e}^{-\alpha(\hat{x}-\hat{\xi})}\mathbf{N}^{\mathrm{T}}(\hat{\xi})\mathbf{N}(\hat{x})\,\mathrm{d}\hat{\xi} + \int_{\hat{\xi}=x}^{\ell} \mathrm{e}^{\alpha(\hat{x}-\hat{\xi})}\mathbf{N}^{\mathrm{T}}(\hat{\xi})\mathbf{N}(\hat{x})\,\mathrm{d}\hat{\xi}\right\}\mathrm{d}\hat{x} \tag{A1}$$

Evaluating the above integral and simplifying we have

$$\mathbf{K}_{ii} = \frac{K_0\mathrm{e}^{-\alpha\ell}}{2\ell^6\alpha^7}\mathbf{A} + \frac{K_0}{420\ell^6\alpha^7}\mathbf{B} \tag{A2}$$

where the elements of the symmetric matrices **A** and **B** may be obtained in closed form as

$$A_{11} = -288 - 288\alpha\ell - 72\alpha^2\ell^2 + 24\alpha^3\ell^3 + 12\alpha^4\ell^4$$

$$A_{12} = -2\ell(72 + 72\alpha\ell + 24\alpha^2\ell^2 - \alpha^4\ell^4)$$

$$A_{13} = 288 + 288\alpha\ell + 72\alpha^2\ell^2 - 24\alpha^3\ell^3 - 12\alpha^4\ell^4 + \alpha^6\ell^6$$

$$A_{14} = -\ell(144 + 144\alpha\ell + 48\alpha^2\ell^2 - 4\alpha^4\ell^4 - \alpha^5\ell^5)$$

$$A_{22} = -4\ell^2(18 + 18\alpha\ell + 7\alpha^2\ell^2 + \alpha^3\ell^3)$$

$$A_{23} = -A_{14}$$

$$A_{24} = -\ell^2(72 + 72\alpha\ell + 32\alpha^2\ell^2 + 8\alpha^3\ell^3 + \alpha^4\ell^4)$$

$$A_{33} = A_{11}$$

$$A_{34} = -A_{12}$$

$$A_{44} = A_{22}$$

and

$$B_{11} = 60\,480 - 15\,120\alpha^2\ell^2 + 2520\alpha^4\ell^4 - 504\alpha^5\ell^5 - 210\alpha^6\ell^6 + 156\alpha^7\ell^7$$

$$B_{12} = 2\ell(15\,120 - 2520\alpha^2\ell^2 + 420\alpha^4\ell^4 - 126\alpha^5\ell^5 + 11\alpha^7\ell^7)$$

$$B_{13} = -60\,480 + 15\,120\alpha^2\ell^2 - 2520\alpha^4\ell^4 + 504\alpha^5\ell^5 + 54\alpha^7\ell^7$$

$$B_{14} = \ell(30\,240 - 5040\alpha^2\ell^2 + 420\alpha^4\ell^4 - 42\alpha^5\ell^5 - 13\alpha^7\ell^7)$$

$$B_{22} = 2\ell^2(7560 - 840\alpha^2\ell^2 + 105\alpha^4\ell^4 - 28\alpha^5\ell^5 + 2\alpha^7\ell^7)$$

$$B_{23} = -B_{14}$$

$$B_{24} = \ell^2(15\,120 - 840\alpha^2\ell^2 + 14\alpha^5\ell^5 - 3\alpha^7\ell^7)$$

$$B_{33} = B_{11}$$

$$B_{34} = -B_{12}$$

$$B_{44} = B_{22}$$

The cross-stiffness matrix in Equation (25) can be expressed as

$$\bar{\mathbf{K}} = \frac{K_0}{2\ell^6\alpha^7}(\mathbf{A}e^{-\alpha\ell} + \mathbf{B}e^{\alpha\ell} + \mathbf{C}) \tag{A3}$$

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where the elements of the non-symmetric symmetric matrices **A**, **B** and **C** can be expressed as

$$A_{11} = -144 - 144\alpha\ell - 36\alpha^2\ell^2 + 12\alpha^3\ell^3 + 6\alpha^4\ell^4$$

$$A_{12} = -\ell(72 + 84\alpha\ell + 36\alpha^2\ell^2 + 6\alpha^3\ell^3)$$

$$A_{13} = 144 + 144\alpha\ell + 36\alpha^2\ell^2$$

$$A_{14} = -\ell(72 + 60\alpha\ell + 12\alpha^2\ell^2)$$

$$A_{21} = -\ell(72 + 60\alpha\ell + 12\alpha^2\ell^2 - 6\alpha^3\ell^3 - 2\alpha^4\ell^4)$$

$$A_{22} = -\ell^2(36 + 36\alpha\ell + 14\alpha^2\ell^2 + 2\alpha^3\ell^3)$$

$$A_{23} = -A_{14}$$

$$A_{24} = -\ell^2(36 + 24\alpha\ell + 4\alpha^2\ell^2)$$

$$A_{31} = 144 + 144\alpha\ell + 36\alpha^2\ell^2 - 24\alpha^3\ell^3 - 12\alpha^4\ell^4 + \alpha^6\ell^6$$

$$A_{32} = \ell(72 + 84\alpha\ell + 36\alpha^2\ell^2 - 4\alpha^4\ell^4 - \alpha^5\ell^5)$$

$$A_{33} = A_{11}$$

$$A_{34} = -A_{21}$$

$$A_{41} = -A_{32}$$

$$A_{42} = -\ell^2(36 + 48\alpha\ell + 28\alpha^2\ell^2 + 8\alpha^3\ell^3 + \alpha^4\ell^4)$$

$$A_{43} = -A_{12}$$

$$A_{44} = A_{22}$$

$$B_{11} = -144 + 144\alpha\ell - 36\alpha^2\ell^2 - 12\alpha^3\ell^3 + 6\alpha^4\ell^4$$

$$B_{12} = \ell(-72 + 60\alpha\ell - 12\alpha^2\ell^2 - 6\alpha^3\ell^3 + 2\alpha^4\ell^4)$$

$$B_{13} = 144 - 144\alpha\ell + 36\alpha^2\ell^2 + 24\alpha^3\ell^3 - 12\alpha^4\ell^4 + \alpha^6\ell^6$$

$$B_{14} = -\ell(72 - 84\alpha\ell + 36\alpha^2\ell^2 - 4\alpha^4\ell^4 + \alpha^5\ell^5)$$

$$B_{21} = -\ell(72 - 84\alpha\ell + 36\alpha^2\ell^2 - 6\alpha^3\ell^3)$$

$$B_{22} = \ell^2(-36 + 36\alpha\ell - 14\alpha^2\ell^2 + 2\alpha^3\ell^3)$$

$$B_{23} = -B_{14}$$

$$B_{24} = -\ell^2(36 - 48\alpha\ell + 28\alpha^2\ell^2 - 8\alpha^3\ell^3 + \alpha^4\ell^4)$$

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$$B_{31} = 144 - 144\alpha\ell + 36\alpha^2\ell^2$$

$$B_{32} = \ell(72 - 60\alpha\ell + 12\alpha^2\ell^2)$$

$$B_{33} = B_{11}$$

$$B_{34} = -B_{21}$$

$$B_{41} = -B_{32}$$

$$B_{42} = -\ell^2(36 - 24\alpha\ell + 4\alpha^2\ell^2)$$

$$B_{43} = -B_{12}$$

$$B_{44} = B_{22}$$

and

$$C_{11} = 288 - 72\alpha^2\ell^2 + 12\alpha^4\ell^4 - \alpha^6\ell^6$$

$$C_{12} = \ell(144 + 24\alpha\ell - 24\alpha^2\ell^2 + 4\alpha^4\ell^4 + \alpha^5\ell^5)$$

$$C_{13} = -288 + 72\alpha^2\ell^2 - 24\alpha^3\ell^3 - 12\alpha^4\ell^4$$

$$C_{14} = 2\ell(72 - 12\alpha\ell - 12\alpha^2\ell^2 + 6\alpha^3\ell^3 + \alpha^4\ell^4)$$

$$C_{21} = \ell(144 - 24\alpha\ell - 24\alpha^2\ell^2 + 4\alpha^4\ell^4 - \alpha^5\ell^5)$$

$$C_{22} = \ell^2(72 - 8\alpha^2\ell^2 + \alpha^4\ell^4)$$

$$C_{23} = -C_{14}$$

$$C_{24} = 4\ell^2(18 - 6\alpha\ell - \alpha^2\ell^2 + \alpha^3\ell^3)$$

$$C_{31} = -288 + 72\alpha^2\ell^2 + 24\alpha^3\ell^3 - 12\alpha^4\ell^4$$

$$C_{32} = -2\ell(72 + 12\alpha\ell - 12\alpha^2\ell^2 - 6\alpha^3\ell^3 + \alpha^4\ell^4)$$

$$C_{33} = C_{11}$$

$$C_{34} = -C_{21}$$

$$C_{41} = -C_{32}$$

$$C_{42} = 4\ell^2(18 + 6\alpha\ell - \alpha^2\ell^2 - \alpha^3\ell^3)$$

$$C_{43} = -C_{12}$$

$$C_{44} = C_{22}$$

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A.2. The Gaussian kernel

Evaluating the integral given in Equation (27), we can express the direct stiffness matrix, $\mathbf{K}_{ii}$, as

$$\mathbf{K}_{ii} = \frac{\mathrm{erf}(\alpha\ell/\sqrt{2})}{420\ell\alpha^2}\mathbf{A} + \frac{\sqrt{2}\mathrm{e}^{-\alpha^2\ell^2/2}}{420\alpha^7\ell^6\sqrt{\pi}}\mathbf{B} + \frac{\sqrt{2}}{210\alpha^7\ell^6\sqrt{\pi}}\mathbf{C} \tag{A4}$$

where the elements of the symmetric matrices **A**, **B** and **C** are given by

$$A_{11} = -252 + 156\alpha^2\ell^2$$

$$A_{12} = 2\ell(-63 + 11\alpha^2\ell^2)$$

$$A_{13} = 252 + 54\alpha^2\ell^2$$

$$A_{14} = -\ell(21 + 13\alpha^2\ell^2)$$

$$A_{22} = 4\ell^2(-7 + \alpha^2\ell^2)$$

$$A_{23} = -A_{14}$$

$$A_{24} = \ell^2(7 - 3\alpha^2\ell^2)$$

$$A_{33} = A_{11}$$

$$A_{34} = -A_{12}$$

$$A_{44} = A_{22}$$

$$B_{11} = -576 + 720\alpha^2\ell^2 - 408\alpha^4\ell^4 + 156\alpha^6\ell^6$$

$$B_{12} = 2\ell(-144 + 96\alpha^2\ell^2 - 74\alpha^4\ell^4 + 11\alpha^6\ell^6)$$

$$B_{13} = 576 - 720\alpha^2\ell^2 + 408\alpha^4\ell^4 + 54\alpha^6\ell^6$$

$$B_{14} = -\ell(288 - 192\alpha^2\ell^2 + 8\alpha^4\ell^4 + 13\alpha^6\ell^6)$$

$$B_{22} = 4\ell^2(-36 + 10\alpha^2\ell^2 - 8\alpha^4\ell^4 + \alpha^6\ell^6)$$

$$B_{23} = -B_{14}$$

$$B_{24} = -\ell^2(144 + 16\alpha^2\ell^2 - 10\alpha^4\ell^4 + 3\alpha^6\ell^6)$$

$$B_{33} = B_{11}$$

$$B_{34} = -B_{12}$$

$$B_{44} = B_{22}$$

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and

$$
\begin{aligned}
C_{11} &= 288 - 504\alpha^2\ell^2 + 420\alpha^4\ell^4 - 105\alpha^6\ell^6\\
C_{12} &= 4\ell(36 - 42\alpha^2\ell^2 + 35\alpha^4\ell^4)\\
C_{13} &= -288 + 504\alpha^2\ell^2 - 420\alpha^4\ell^4\\
C_{14} &= 2\ell(72 - 84\alpha^2\ell^2 + 35\alpha^4\ell^4)\\
C_{22} &= \ell^2(72 - 56\alpha^2\ell^2 + 35\alpha^4\ell^4)\\
C_{23} &= -C_{14}\\
C_{24} &= 4\ell^2(18 - 7\alpha^2\ell^2)\\
C_{33} &= C_{11}\\
C_{34} &= -C_{12}\\
C_{44} &= C_{22}
\end{aligned}
$$

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