Dynamic Response Characteristics of a Nonviscously Damped Oscillator

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The characteristics of the frequency response function of a nonviscously damped linear oscillator are considered in this paper. It is assumed that the nonviscous damping force depends on the past history of velocity via a convolution integral over an exponentially decaying kernel function. The classical dynamic response properties, known for viscously damped oscillators, have been generalized to such nonviscously damped oscillators. The following questions of fundamental interest have been addressed: (a) Under what conditions can the amplitude of the frequency response function reach a maximum value?, (b) At what frequency will it occur?, and (c) What will be the value of the maximum amplitude of the frequency response function? Introducing two nondimensional factors, namely, the viscous damping factor and the nonviscous damping factor, we have provided exact answers to these questions. Wherever possible, attempts have been made to relate the new results with equivalent classical results for a viscously damped oscillator. It is shown that the classical concepts based on viscously damped systems can be extended to a nonviscously damped system only under certain conditions. [DOI: 10.1115/1.2755096]

1 Introduction

The characterization of dissipative forces is crucial for the design of safety critical engineering structures subjected to dynamic forces. Viscous damping is the most common approach for the modeling of dissipative or damping forces in engineering structures. This model assumes that the instantaneous generalized velocities are the only relevant variables that determine damping. Viscous damping models are used widely for their simplicity and mathematical convenience, even though the energy dissipation behavior of real structural materials may not be accurately represented by simple viscous models. Increasing use of modern composite materials, high-damping elements, and active control mechanisms in the aerospace and automotive industries in recent years demands sophisticated treatment of the dissipative forces for proper analysis and design. It is well known that, in general, a physically realistic model of damping in such cases will not be viscous. Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are nonviscous damping models.

Recognizing the need to incorporate generalized dissipative forces within the equations of motion, several authors have used nonviscous damping models. Within the scope of linear models, the damping force can, in general [1], be expressed by

$$f_d(t) = \int_0^t g(t-\tau)\dot{u}(\tau)\mathrm{d}\tau \tag{1}$$

Bagley and Torvik [2], Torvik and Bagley [3], Gaul et al. [4] and Maia et al. [5] have considered damping modeling in terms of fractional derivatives of the displacements, which can be obtained by properly choosing the damping kernel function $g(t)$ in Eq. (1). This type of problem has also been treated extensively within the viscoelasticity literature; see, for example, the books by Bland [6] and Christensen [7] and references therein. Among various other nonviscous damping models, the "Biot model" [8] or "exponential damping model" is particularly promising and has been used by many authors [9–14]. With this model, the damping force is expressed as

$$f_d(t) = \sum_{k=1}^{n} c_k \int_0^t \mu_k e^{-\mu_k(t-\tau)}\dot{u}(\tau)\mathrm{d}\tau \tag{2}$$

Here, c_k are the damping constants, μ_k are the relaxation parameters, n is the number of relaxation parameters required to describe the damping behavior, and $u(t)$ is the displacement as a function of time. In the context of viscoelastic materials, the physical basis for exponential models has been well established; see, for example, Ref. [15]. A selected literature review including the justifications for considering the exponential damping model may be found in Ref. [13]. Adhikari and Woodhouse [16] proposed a few methods by which the damping parameters in Eq. (2) can be obtained from experimental measurements.

Methods for the analysis of linear systems with damping of the form (2) have been considered by many authors; for example [1,9–13,17,18]. Although these publications provide excellent analytical and numerical tools for the analysis of nonviscously damped systems, most of the physical understandings are still from the point of view of a viscously damped oscillator. In this paper, we address the dynamic response characteristics of a nonviscously damped oscillator with energy dissipation characteristics given by Eq. (2) with $n=1$. The outline of the paper is as follows. In Sec. 2, the equation of motion is introduced and the exact analytical solutions of the eigenvalues are derived. The conditions for sustainable oscillatory motion are discussed in Sec. 3.1. The critical damping factors of a nonviscously damped oscillator are discussed in Sec. 3.2. The frequency response function of the system is derived in Sec. 4. The characteristics of the response amplitude are discussed in Sec. 5. In Sec. 6, a simplified analysis of dynamic response is proposed. Finally, our main findings are summarized in Sec. 7.

2 Background

The equation of motion of the system with damping characteristics given by Eq. (2) with $n=1$ can be expressed as

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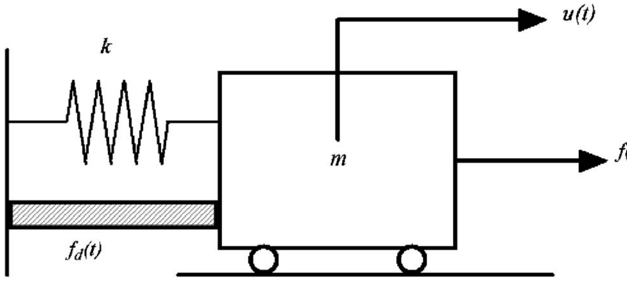

Fig. 1 A single-degree-of-freedom nonviscously damped oscillator with the damping force $f_d(t)=\int_0^t c\mu e^{-\mu(t-\tau)}\dot{u}(\tau)\mathrm{d}\tau$

$$m\ddot{u}(t)+\int_0^t c\mu e^{-\mu(t-\tau)}\dot{u}(\tau)\mathrm{d}\tau+ku(t)=f(t) \tag{3}$$

together with the initial conditions

$$u(0)=u_0 \quad \text{and } \dot{u}(0)=\dot{u}_0 \tag{4}$$

The system is shown in Fig. 1. Here, m is the mass of the oscillator, k is the spring stiffness, $f(t)$ is the applied forcing, and $(\dot{\bullet})$ represents a derivative with respect to time. Qualitative properties of the eigenvalues of this system have been discussed in detail by Adhikari [19]. Here, we review some basic results.

Transforming Eq. (3) into the Laplace domain, one obtains

$$s^2m\bar{u}(s)+sc\left(\frac{\mu}{s+\mu}\right)\bar{u}(s)+k\bar{u}(s)=\bar{f}(s)+m\dot{u}_0+\left(sm+c\frac{\mu}{s+\mu}\right)u_0 \tag{5}$$

where s is the complex Laplace domain parameter and $\overline{(\bullet)}$ is the Laplace transform of $(\bullet)$. For convenience, we introduce the constants ω_n, ζ, and β as follows:

$$\omega_n=\sqrt{\frac{k}{m}} \quad \zeta=\frac{c}{2\sqrt{km}} \quad \beta=\frac{\omega_n}{\mu} \tag{6}$$

Here, ω_n is the undamped natural frequency, ζ is the viscous damping factor, and β is the nonviscous damping factor. When $\zeta\to 0$, the oscillator is effectively undamped. When $\beta\to 0$, then $\mu\to\infty$, and the oscillator is effectively viscously damped. We will use these limiting cases frequently to develop our physical understandings of the results to be derived in this paper. In the context of multiple-degree-of-freedom dynamic systems, Adhikari and Woodhouse [20] have proposed four nonviscosity indices in order to quantify nonviscous damping. The nonviscous damping factor β proposed here also serves a similar purpose. Using the constants in (6), Eq. (5) can be rewritten as

$$\bar{d}(s)\bar{u}(s)=\bar{p}(s) \tag{7}$$

where the dynamic stiffness coefficient $\bar{d}(s)$ and the equivalent forcing function $\bar{p}(s)$ are given by

$$\bar{d}(s)=s^2+s2\zeta\omega_n\left(\frac{\omega_n}{s\beta+\omega_n}\right)+\omega_n^2 \tag{8}$$

and

$$\bar{p}(s)=\frac{\bar{f}(s)}{m}+\dot{u}_0+\left(s+2\zeta\omega_n\frac{\omega_n}{s\beta+\omega_n}\right)u_0 \tag{9}$$

The aim of a dynamic analysis is often to obtain the dynamic response, either in the time domain or in the frequency domain. For a single-degree-of-freedom (SDOF) oscillator, it is a relatively simple task; one can either directly integrate Eq. (3) with the initial conditions (4), or alternatively can invert the coefficient associated with $\bar{u}(s)$ in Eq. (7). Such an approach is not suitable for multiple degree-of-freedom systems with nonproportional damping and may not provide much physical insight. We pursue an approach that involves eigensolutions of the oscillator. The eigenvalues are the zeros of the dynamic stiffness coefficient and can be obtained by setting $\bar{d}(s)=0$. Therefore, using Eq. (8), the eigenvalues are the solutions of the characteristic equation:

$$\beta s^3+\omega_n s^2+(\beta\omega_n^2+2\zeta\omega_n^2)s+\omega_n^3=0 \tag{10}$$

In contrast to a viscously damped oscillator where one obtains a quadratic equation.

The three roots of Eq. (10) can appear in two distinct forms: (a) One root is real and the other two roots are in a complex conjugate pair, or (b) all roots are real. Case (a) represents an *underdamped oscillator*, which usually arises when the "small damping" assumption is made. The complex conjugate pair of roots corresponds to the "vibration" of the oscillator, while the third root corresponds to a purely dissipative motion. Case (b) represents an *overdamped oscillator* in which the system cannot sustain any oscillatory motion. For simplicity, we introduce a nondimensional frequency parameter

$$r=\frac{s}{\omega_n}\in\mathbb{C} \tag{11}$$

and transform the characteristics of Eq. (10) to

$$\beta r^3+r^2+(\beta+2\zeta)r+1=0 \tag{12}$$

or

$$r^3+\sum_{j=0}^{2}a_jr^j=0 \tag{13}$$

The constants associated with the powers of r are given by

$$a_0=\frac{1}{\beta} \quad a_1=1+2\frac{\zeta}{\beta} \quad a_2=\frac{1}{\beta} \tag{14}$$

The cubic Eq. (13) can be solved exactly in closed form; see, for example [[21] Sec. 3.8]. Define the following constants

$$Q=\frac{3a_1-a_2^2}{9}=\frac{(3\beta^2+6\beta\zeta-1)}{9\beta^2} \tag{15}$$

and

$$R=\frac{9a_2a_1-27a_0-2a_2^3}{54}=-\frac{(9\beta^2-9\beta\zeta+1)}{27\beta^3} \tag{16}$$

From these, calculate the negative of the discriminant

$$D=Q^3+R^2=\frac{1}{27\beta^4}(\beta^4+6\beta^3\zeta+2\beta^2+12\beta^2\zeta^2-10\beta\zeta+1+8\beta\zeta^3-\zeta^2) \tag{17}$$

and define two new constants

$$S=\sqrt[3]{R+\sqrt{D}} \quad \text{and } T=\sqrt[3]{R-\sqrt{D}} \tag{18}$$

Using these constants, the roots of Eq. (13) can be expressed by the Cardanos formula as

$$r_1=-\frac{a_2}{3}-\frac{1}{2}(S+T)+\mathrm{i}\frac{\sqrt{3}}{2}(S-T) \tag{19}$$

$$r_2=-\frac{a_2}{3}-\frac{1}{2}(S+T)-\mathrm{i}\frac{\sqrt{3}}{2}(S-T) \tag{20}$$

and

$$r_3=-\frac{a_2}{3}+(S+T) \tag{21}$$

These are the normalized eigenvalues of the system. The actual eigenvalues, that is the solutions of Eq. (10), can be obtained as

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$\lambda_j = \omega_n r_j$, $j=1,2,3$. If the nonviscous damping factor β is zero, Eq. (12) reduces to the quadratic equation

$$r^2 + 2\zeta r + 1 = 0 \tag{22}$$

which, as expected, is the characteristic equation of a viscously damped oscillator. For this special case, the two solutions of Eq. (22) are given by

$$r_1 = -\zeta + \mathrm{i}\sqrt{1-\zeta^2} \quad r_2 = -\zeta - \mathrm{i}\sqrt{1-\zeta^2} \tag{23}$$

Since the nature of these solutions is very well understood, we will compare the new results with them.

3 Characteristic of the Eigenvalues

3.1 Conditions for Oscillatory Motion. The conditions for oscillatory motion have been discussed by Muravyov and Hutton [10] and more recently by Muller [22] and Adhikari [19]. Here, we briefly review the answers to the following questions of fundamental interest:

- Under what conditions can a nonviscously damped oscillator sustain oscillatory motions?
- Is there any critical damping factor for a nonviscously damped oscillator so that, beyond this value, the oscillator becomes overdamped?

For a viscously damped oscillator, the answer to the above questions is well known. From Eq. (23) it is clear that if the viscous damping factor ζ is more than 1, then the oscillator becomes overdamped and consequently it will not be able to sustain any oscillatory motions. This simple fact is no longer true for a nonviscously damped oscillator.

Roots r_1 and r_2 in Eqs. (19) and (20), respectively, will be in a complex conjugate pair, provided $S-T \neq 0$. The motion corresponding to the complex conjugate roots r_1 and r_2 is oscillatory (and decaying) in nature, while the motion corresponding to the real root r_3 is a pure nonoscillatory decay. Considering the expressions of S and T in Eq. (18), it is easy to observe that the system can oscillate provided $D>0$. Therefore, the critical condition is given by

$$D(\zeta,\beta) = 0 \tag{24}$$

From the expression of D in (17), this condition can be rewritten as

$$8\beta\zeta^3 + (12\beta^2 - 1)\zeta^2 + (6\beta^3 - 10\beta)\zeta + (1 + 2\beta^2 + \beta^4) = 0 \tag{25}$$

In Fig. 2, the surface $D(\zeta,\beta)=0$ is plotted for $0 \leqslant \zeta \leqslant 6$ and $0 \leqslant \beta \leqslant 0.5$. This plot shows the parameter domain where the system can have oscillatory motion. For a viscously damped oscillator, $\beta=0$, which is represented by the X-axis of Fig. 2. Along the X-axis when $\zeta>1$, the oscillatory motion is not possible, which is well known. But the scenario changes in an interesting way for nonzero β (i.e., for a nonviscously damped oscillator). For example, if $\beta \approx 0.1$, the system can have oscillatory motion even when $\zeta>2$, which is more than twice the critical viscous damping factor! Conversely, there are also regions where the system may not have oscillatory motion even when $\zeta<1$. Perhaps the most interesting observation from Fig. 2 is that if β is more than about 0.2, then the oscillator will *always* have oscillatory motions, no matter what the value of the viscous damping factor is. Therefore, there is a critical value of ζ, say ζ_c, *below* which the system will always have an oscillatory motion. Similarly, there is a critical value of β, say β_c, *above* which the system will always have an oscillatory motion. In the previous work [19], the exact critical values of ζ and β were obtained and the following basic result was proved:

THEOREM 3.1. *A nonviscously damped oscillator will have oscillatory motions if* $\zeta<4/(3\sqrt{3})$ or $\beta>1/(3\sqrt{3})$.

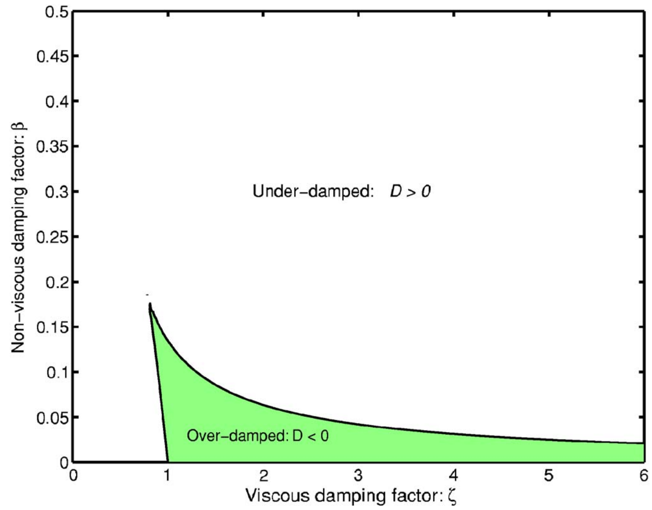

Fig. 2 The boundary between oscillatory and nonoscillatory motion

In the next section, the precise parameter region, where oscillatory motion is possible, is defined using the concept of critical damping factors.

3.2 Critical Damping Factors. In Fig. 3, we have (again) plotted the surface $D(\zeta,\beta)=0$ concentrating around the critical values of ζ and β. The shaded region corresponds to the parameter combinations for which oscillatory motion is not possible. A nonviscously damped oscillator will *always* have oscillatory motions if $\zeta<\zeta_c$ and/or $\beta>\beta_c$ (parameter regions C_1 and A in the figure). If $\beta<\beta_c$, then it is possible to have overdamped motion even if $\zeta<1$, as in the parameter region B, shown in Fig. 3. When $\beta<\beta_c$, there are two distinct parameter regions (shown as C_1 and C_2 in the figure) in which oscillatory motion is possible. Therefore, one can think of two critical damping factors for a nonviscously damped oscillator.

Using the notations ζ_L and ζ_U, the oscillator will have overdamped motion when $\zeta_L<\zeta<\zeta_U$. We call ζ_L the lower critical damping factor and ζ_U the upper critical damping factor.

To obtain the critical damping factors, it is required to solve $D=0$ for ζ, which is a cubic equation in ζ. In the previous work

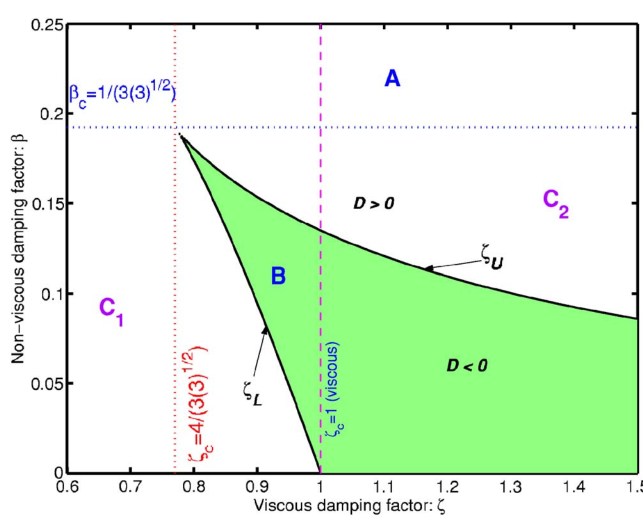

Fig. 3 Critical values of ζ and β for oscillatory motion

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[19], it was proved that the lower and the upper critical damping factors of a nonviscously damped oscillator are given by

$$\zeta_L = \frac{1}{24\beta}\{1 - 12\beta^2 + 2\sqrt{1+216\beta^2} + \cos[(4\pi + \theta_c)/3]\} \quad (26)$$

and

$$\zeta_U = \frac{1}{24\beta}[1 - 12\beta^2 + 2\sqrt{1+216\beta^2} + \cos(\theta_c/3)] \quad (27)$$

where

$$\theta_c = \arccos\left[\frac{1 - 5832\beta^4 - 540\beta^2}{(216\beta^2+1)^{3/2}}\right] \quad (28)$$

Equations (26) and (27) are plotted in Fig. 3. When $\beta \to \beta_c$, the critical damping factors approach each other and eventually when $\beta = \beta_c$, both critical damping factors become the same and equal to ζ_c. The existence of two critical damping factors is a new concept compared to a viscously damped oscillator. In the limiting case when $\beta \to 0$, it can be verified that $\zeta_L \to 1$ and $\zeta_U \to \infty$. This indeed implies that a viscously damped oscillator has only one critical damping factor, and that is $\zeta = 1$. These results can be summarized in the following theorem:

THEOREM 3.2. *When* $\beta < 1/(3\sqrt{3})$, *a nonviscously damped oscillator will have oscillatory motions if and only if* $\zeta \notin [\zeta_L, \zeta_U]$.

4 The Frequency Response Function

The results given in the previous section define the conditions under which a nonviscously damped oscillator can sustain oscillatory motions. The rest of the paper is aimed at gaining insights into the nature of the dynamic response. The frequency response function of linear systems contains complete information regarding the dynamic response. The direct computation of the frequency response function of a SDOF system is a trivial task. To gain further insight into the dynamic response characteristics, it is often useful to express the frequency response function in terms of the eigenvalues of the system. The aim of this section is to establish a connection to the results given in the previous section, which gives the expression of the eigenvalues as a function of ζ and β.

We begin with the normalized frequency response function $\bar{h}(s)$, which is defined as the solution of Eq. (7) with the forcing function $\bar{p}(s) = 1$. Therefore, from Eq. (7) one obtains

$$\bar{h}(s) = \frac{1}{\bar{d}(s)} \quad \text{where } \bar{d}(s) = s^2 + s2\zeta\omega_n\left(\frac{\omega_n}{s\beta + \omega_n}\right) + \omega_n^2 \quad (29)$$

Noting that $\bar{d}(s)$ has zeros at $s = \lambda_j$, $j = 1, 2, 3$, where the eigenvalues $\lambda_j = \omega_n r_j$, the frequency response function can be conveniently expressed by the pole-residue form as

$$\bar{h}(s) = \sum_{j=1}^{3} \frac{R_j}{s - \lambda_j} \quad (30)$$

Here, the residues

$$R_j = \lim_{s\to\lambda_j} \frac{s - \lambda_j}{\bar{d}(s)} = \frac{1}{\partial\bar{d}(s)/\partial s|_{s=\lambda_j}} = \frac{1}{2}\frac{1}{\lambda_j + \zeta\omega_n[\omega_n/(\beta\lambda_j + \omega_n)]^2} \quad (31)$$

Because λ_1 and λ_2 appear in a complex conjugate pair, it is convenient to write $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^*$, where $(\bullet)^*$ denotes the complex conjugation. We denote the real eigenvalue $\lambda_3 = \nu$. Using these notations and substituting $s = \mathrm{i}\omega$, the frequency response function in Eq. (30) can be expressed as

$$\bar{h}(\mathrm{i}\omega) = \frac{R_\lambda}{\mathrm{i}\omega - \lambda} + \frac{R_\lambda^*}{\mathrm{i}\omega - \lambda^*} + \frac{R_\nu}{\mathrm{i}\omega - \nu} \quad (32)$$

where

$$R_\lambda = \frac{1}{2}\frac{1}{\lambda + \zeta\omega_n[1 + (\beta\lambda/\omega_n)]^{-2}} \quad R_\nu = \frac{1}{2}\frac{1}{\nu + \zeta\omega_n[1 + (\beta\nu/\omega_n)]^{-2}} \quad (33)$$

For the special cases when the system is undamped ($\zeta = 0$), or viscously damped ($\beta = 0$), Eq. (32) reduces to its corresponding familiar forms as follows:

- For undamped systems, $\zeta = 0$ and ν does not exist. The eigenvalue λ is purely imaginary so that $\lambda = \mathrm{i}\omega_n$. From Eq. (33), one obtains $R_\lambda = 1/(2\mathrm{i}\omega_n)$. Substitution of these values in Eq. (32) results in

$$\bar{h}(\mathrm{i}\omega) = \frac{1}{2\mathrm{i}\omega_n}\frac{1}{\mathrm{i}\omega - \mathrm{i}\omega_n} - \frac{1}{2\mathrm{i}\omega_n}\frac{1}{\mathrm{i}\omega + \mathrm{i}\omega_n} = \frac{1}{2\mathrm{i}\omega_n}\left[\frac{1}{\mathrm{i}\omega - \mathrm{i}\omega_n} - \frac{1}{\mathrm{i}\omega + \mathrm{i}\omega_n}\right] = \frac{1}{\omega_n^2 - \omega^2} \quad (34)$$

- For viscously damped systems, $\beta = 0$ and ν does not exist. The eigenvalue λ can be expressed as

$$\lambda = -\zeta\omega_n + \mathrm{i}\omega_d \quad \text{where } \omega_d = \omega_n\sqrt{1 - \zeta^2} \quad (35)$$

From Eq. (33), one obtains $R_\lambda = 1/[2(-\zeta\omega_n + \mathrm{i}\omega_d + \zeta\omega_n)] = 1/(2\mathrm{i}\omega_d)$. Substituting these in Eq. (32), one obtains

$$\bar{h}(\mathrm{i}\omega) = \frac{1}{2\mathrm{i}\omega_d}\frac{1}{\mathrm{i}\omega - (-\zeta\omega_n + \mathrm{i}\omega_d)} - \frac{1}{2\mathrm{i}\omega_d}\frac{1}{\mathrm{i}\omega - (-\zeta\omega_n - \mathrm{i}\omega_d)} = \frac{1}{2\mathrm{i}\omega_d}\left[\frac{2\mathrm{i}\omega_d}{(\zeta\omega_n + \mathrm{i}\omega)^2 - (\mathrm{i}\omega_d)^2}\right] = \frac{1}{\omega_n^2 + 2\mathrm{i}\omega\zeta\omega_n - \omega^2} \quad (36)$$

In the time domain, the impulse response function can be obtained by taking the inverse Laplace transform of $\bar{h}(s)$ as

$$h(t) = \mathrm{Re}\left\{\frac{e^{\lambda t}}{\lambda + \zeta\omega_n[1 + (\beta\lambda/\omega_n)]^{-2}}\right\} + \frac{1}{2}\frac{e^{\nu t}}{\nu + \zeta\omega_n[1 + (\beta\nu/\omega_n)]^{-2}} \quad (37)$$

The first term in Eq. (37) is oscillating in nature because λ is complex, while the second term is purely decaying in nature as ν is real and negative.

It is convenient to define a nondimensional driving frequency parameter

$$\tilde{\omega} = \frac{\omega}{\omega_n} \quad (38)$$

Substituting $s = \mathrm{i}\omega = \mathrm{i}\tilde{\omega}\omega_n$ in Eq. (29), one has

$$\bar{h}(\mathrm{i}\omega) = \frac{1}{\omega_n^2}\left[\frac{1}{-\tilde{\omega}^2 + 2\mathrm{i}\zeta\tilde{\omega}(1/\mathrm{i}\beta\tilde{\omega} + 1) + 1}\right] \quad (39)$$

Separating the real and imaginary parts, the nondimensional frequency response function can be expressed as

$$G(\mathrm{i}\omega) = \omega_n^2\bar{h}(\mathrm{i}\omega) = \frac{1 + \mathrm{i}\beta\tilde{\omega}}{(1 - \tilde{\omega}^2) + \mathrm{i}\tilde{\omega}(2\zeta + \beta - \beta\tilde{\omega}^2)} \quad (40)$$

From Eq. (40), the amplitude of vibration can be obtained as

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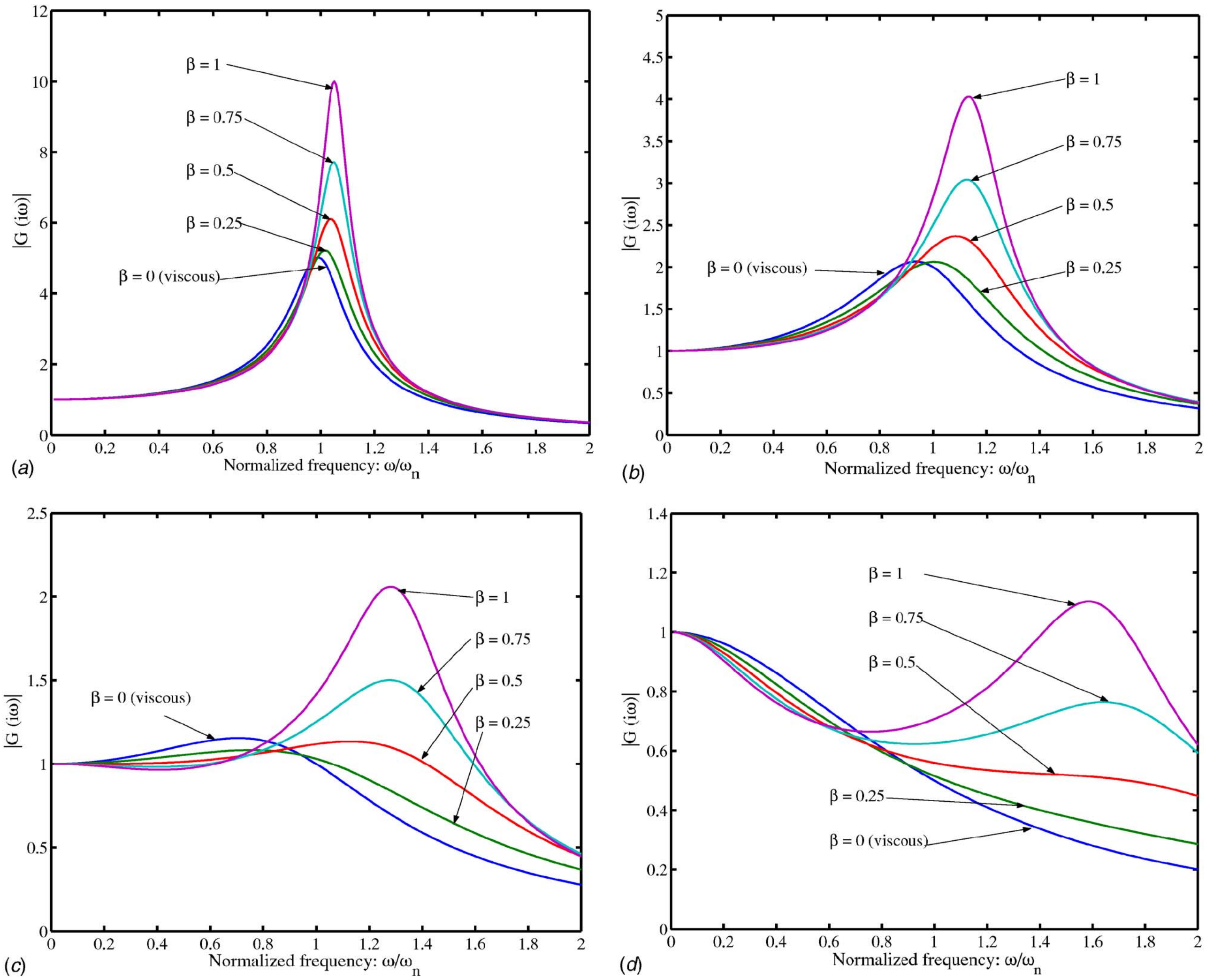

Fig. 4 Amplitude of the nondimensional frequency response $|G(\mathrm{i}\omega)|$ as a function of the normalized frequency ω/ω_n for different values of ζ and β. (*a*) $\zeta=0.1$. (*b*) $\zeta=0.25$. (*c*) $\zeta=0.5$. (*d*) $\zeta=1.0$.

$$|G(\mathrm{i}\omega)| = \sqrt{G(\mathrm{i}\omega)G^*(\mathrm{i}\omega)} = \sqrt{\frac{1+\beta^2\tilde{\omega}^2}{(1-\tilde{\omega}^2)^2+\tilde{\omega}^2(2\zeta+\beta-\beta\tilde{\omega}^2)^2}} \tag{41}$$

Figure 4 shows the amplitude of the nondimensional frequency response $|G(\mathrm{i}\omega)|$ as a function of the normalized frequency ω/ω_n. The numerical values of β and ζ are selected such that Fig. 4 represents the general overall behavior. In the static case, that is when $\omega/\omega_n=0$, the amplitude of vibration is 1. Therefore, as the frequency changes, the values of $|G(\mathrm{i}\omega)|$ in Eq. (41) can be regarded as the amplification factors.

When $\beta<\beta_c=1/(3\sqrt{3})$, the frequency response function is similar to that of the viscously damped system. This is expected because the value of β is relatively small. The amplitude of the peak response of the nonviscously damped system is more than that of the viscously damped system. In general, the higher the values of β, the higher the values of the amplitudes of the peak response. Another interesting fact can be seen from Fig. 4 is that the dynamic response amplitude has a peak even when $\zeta>1/\sqrt{2}$. For example, in Fig. 4(*d*), the viscously damped system does not have any response peak as $\zeta=1$ (critical viscous damping). However, for the nonviscously damped system, the response amplitude has a peak when $\beta=1$ or $\beta=0.75$, but not if $\beta<0.5$. These interesting response behaviors are explored further in the next section.

5 Characteristics of the Response Amplitude

The maximum vibration amplitude of a linear system near the resonance is of fundamental engineering interest because it can lead to damage or even failure of a structure. For a viscously damped system, it is well known that if $\zeta<1/\sqrt{2}$, then the frequency response function has a peak when $\omega/\omega_n=\sqrt{1-2\zeta^2}$. At this frequency, the amplitude of the maximum dynamic response is given by

$$|G|_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{42}$$

Recently, Vinokur [23] derived a closed-form expression of the frequency point where the vibration amplitude of a hysteretically damped system reaches its maximum value. We are interested in the equivalent results for nonviscously damped systems. Specifically, we ask the following questions of fundamental engineering interest:

- Under what conditions can the amplitude of the frequency response function reach a maximum value?
- At what frequency will it occur?
- What will be the value of the maximum amplitude of the frequency response function?

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5.1 The Frequency for the Maximum Response Amplitude. For notational convenience, denoting

$$x = \tilde{\omega}^2 = \frac{\omega^2}{\omega_n^2} \tag{43}$$

from Eq. (41), the amplitude of the dynamic response can be expressed as

$$|G|^2 = \frac{1 + \beta^2 x}{(1 - x)^2 + x(2\zeta + \beta - \beta x)^2} \tag{44}$$

For the maximum value of $|G|$, we set

$$\frac{\partial |G|^2}{\partial x} = 0 \tag{45}$$

or

$$\frac{2x^2\beta^2 - 2\beta^2 x - 2\beta^3\zeta x^2 - \beta^4 x^2 + \beta^4 x^3 - 1 + x + 2\zeta^2 + 2\zeta\beta - 4\zeta x\beta}{[(1 - x)^2 + x(2\zeta + \beta - \beta x)^2]^2} = 0 \tag{46}$$

At the solution point, it is also required that

$$\frac{\partial^2 |G|^2}{\partial x^2} < 0 \tag{47}$$

that in turn implies satisfying

$$\begin{aligned} &3\beta^6 x^5 + (9\beta^4 - 12\zeta\beta^5 - 6\beta^6)x^4 + (9\beta^2 + 12\zeta^2\beta^4 - 18\beta^4 - 40\beta^3\zeta \\ &+ 12\zeta\beta^5 + 3\beta^6)x^3 + (60\beta^2\zeta^2 + 9\beta^4 + 3 + 60\beta^3\zeta - 24\zeta\beta \\ &- 18\beta^2)x^2 + (9\beta^2 - 24\beta^3\zeta - 48\zeta^3\beta + 12\zeta^2 + 36\zeta\beta - 6 \\ &- 72\beta^2\zeta^2)x + 3 + 4\beta^3\zeta - 16\zeta^2 - 12\zeta\beta + 20\beta^2\zeta^2 + 32\zeta^3\beta \\ &+ 16\zeta^4 < 0 \end{aligned} \tag{48}$$

The numerator of Eq. (46) is a cubic equation in x and can be expressed as

$$x^3 + \sum_{j=0}^{2} c_j x^j = 0 \tag{49}$$

where

$$c_0 = \frac{2\zeta\beta + 2\zeta^2 - 1}{\beta^4} \quad c_1 = \frac{1 - 2\beta^2 - 4\zeta\beta}{\beta^4} \quad c_2 = \frac{2 - 2\zeta\beta - \beta^2}{\beta^2} \tag{50}$$

The three roots of Eq. (49) can either be all real or one real and one complex conjugate pair. The nature of the roots depends on the discriminant, which can be obtained from the constants

$$Q_x = \frac{3c_1 - c_2^2}{9} = -\frac{1}{9\beta^4}(1 + 2\zeta\beta + \beta^2)^2 \tag{51}$$

and

$$\begin{aligned} R_x &= \frac{9c_2c_1 - 27c_0 - 2c_2^3}{54} \\ &= \frac{1}{27\beta^6}[8\zeta^3\beta^3 + (12\beta^4 - 15\beta^2)\zeta^2 + (-15\beta^3 - 21\beta + 6\beta^5)\zeta \\ &\quad + 3\beta^4 + 3\beta^2 + 1 + \beta^6] \end{aligned} \tag{52}$$

as

$$\begin{aligned} D_x = Q_x^3 + R_x^2 = -\frac{\zeta}{27\beta^{11}}&[16\beta^4\zeta^4 + (13\beta^3 + 40\beta^5)\zeta^3 \\ &+ (18\beta^4 + 36\beta^6 - 18\beta^2)\zeta^2 + (-13\beta - 12\beta^3 + 15\beta^5 + 14\beta^7)\zeta \\ &+ 2 + 8\beta^2 + 12\beta^4 + 2\beta^8 + 8\beta^6] \end{aligned} \tag{53}$$

If $D_x > 0$, then Eq. (49) has one complex conjugate pair and only one real solution. It turns out that when $D_x > 0$, the real solution is always negative and, therefore, is not of interest in this study. However, when $D_x < 0$, all the roots of Eq. (49) become real. We define an angle θ as

$$\cos(\theta) = (R_x/\sqrt{-Q_x^3}) = \frac{8\zeta^3\beta^3 + (12\beta^4 - 15\beta^2)\zeta^2 + (6\beta^5 - 15\beta^3 - 21\beta)\zeta + 3\beta^4 + 3\beta^2 + 1 + \beta^6}{(1 + \beta^2 + 2\zeta\beta)^3} \tag{54}$$

Using θ, the three real solutions of Eq. (49) can be given using Dickson's formula [24] as

$$x_1 = 2\sqrt{-Q_x}\cos\left(\frac{\theta}{3}\right) - c_2/3 \tag{55}$$

$$x_2 = 2\sqrt{-Q_x}\cos\left(\frac{2\pi + \theta}{3}\right) - c_2/3 \tag{56}$$

and

$$x_3 = 2\sqrt{-Q_x}\cos\left(\frac{4\pi + \theta}{3}\right) - c_2/3 \tag{57}$$

Among the above three solutions, we need to choose a positive solution that also satisfies Eq. (48). From numerical calculations, it turns out that only x_1 in Eq. (55) satisfies the condition in Eq. (48). Substituting Q_x from (51) and c_2 from (50) into Eq. (55), the normalized excitation frequency for which the amplitude of the frequency response function reaches its maximum value is given by

$$x_{\max} = \frac{1}{3\beta^2}\{(1 + 2\zeta\beta + \beta^2)[2\cos(\theta/3) + 1] - 3\} \tag{58}$$

For convenience, we define the notation $\omega_{\max}$ as

$$x_{\max} = \frac{\omega_{\max}^2}{\omega_n^2} \tag{59}$$

we have

$$\omega_{\max} = \frac{\omega_n}{\beta}\sqrt{(1 + 2\zeta\beta + \beta^2)[2\cos(\theta/3) + 1]/3 - 1} \tag{60}$$

This is the extension of the well known result for viscously damped systems for which $\omega_{\max} = \omega_n\sqrt{1 - 2\zeta^2}$.

Figure 5 shows the contours of $\omega_{\max}/\omega_n$ obtained from Eq. (60), as a function of ζ and β. The value of $\omega_{\max}$ is the frequency where the amplitude of the frequency response function reaches its maximum value.

For a better understanding, Fig. 5 is divided into three regions. In region A where $\zeta \lesssim 0.5$ and β is small, $\omega_{\max}/\omega_n < 1$. This implies that in this parameter region, the frequency at which the amplitude of the frequency response function reaches its maximum appears *below* the system's natural frequency. Contour line 0

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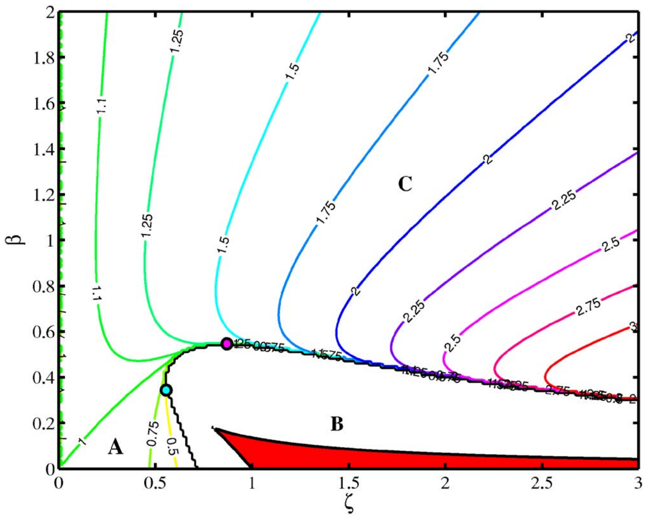

Fig. 5 Contours of the normalized excitation frequency corresponding to the maximum value of the amplitude of the frequency response function ω_{max}/ω_n as a function of ζ and β

separates the region B from A and C. In region B, $\beta \lesssim 0.5$ and $\zeta \gtrsim 1/\sqrt{2}$ and the amplitude of the frequency response function does not have any maximum value. This implies that within this parameter region, it is not possible to find a positive real solution of the cubic Eq. (49) and the system response decays gradually, as in 4(d) for $\beta=0$ and $\beta=0.25$. The shaded portion inside region B (shown before in Figs. 2 and 3) corresponds to the parameter region where the system cannot have any oscillatory motions. Clearly, within this overdamped region, it is not possible for the dynamic response amplitude to reach a maximum value. In region C, where $\zeta \gtrsim 1$ and $\beta \gtrsim 0.5$, observe that $\omega_{max}/\omega_n > 1$. The contour plots in Fig. 5 also show a general trend that ω_{max}/ω_n increases for increasing values of ζ and β.

An interesting contour line in Fig. 5 is line 1. For these parameter combinations of ζ and β, the frequency at which the amplitude of the frequency response function reaches the maximum value coincides exactly with the undamped natural frequency. This surprising observation implies that the system may be heavily damped ($\zeta > 0.5$), but still can have a peak at ω_n, for some appropriate values of β. Another interesting fact observed from Fig. 5 is that there exist a critical value of ζ, say ζ_{mL}, *below* which the amplitude of the frequency response will always have a maximum value for any values of β. Similarly, there is also a critical value of β, say β_{mU}, *above* which the amplitude of the frequency response will always have a maximum value for any values of ζ. The explanation of these observations, including the derivation of the exact values of ζ_{mL} and β_{mU}, are considered in the next subsections.

5.1.1 Critical Parameter Values for the Maximum Response Amplitude. Suppose a general complex solution of Eq. (49) is expressed as

$$x = \sigma + \mathrm{i}\psi \tag{61}$$

for arbitrary $\sigma, \psi \in \mathbb{R}$. Substituting x from the above equation in (49) and separating the real and imaginary parts, we have

$$-2\beta^4\sigma^3 + (4\beta^3\zeta + 2\beta^4 - 4\beta^2)\sigma^2 + (6\beta^4\psi^2 - 2 + 4\beta^2 + 8\zeta\beta)\sigma - 4\zeta\beta - (4\beta^3\zeta + 2\beta^4 - 4\beta^2)\psi^2 - 4\zeta^2 + 2 = 0 \tag{62}$$

and

$$-6\beta^4\sigma^2\psi + (-8\psi\beta^2 + 8\psi\beta^3\zeta + 4\psi\beta^4)\sigma + 2\beta^4\psi^3 + 4\psi\beta^2 + 8\psi\zeta\beta - 2\psi = 0 \tag{63}$$

Eliminating σ from Eqs. (62) and (63) and substituting $\psi=0$ (because we are interested only in the real solution) in the resulting equation, after some algebra one has

$$\mathcal{M}(\zeta,\beta) = 0 \tag{64}$$

where

$$\mathcal{M}(\zeta,\beta) = 16\beta^3\zeta^3 + (24\beta^4 - 3\beta^2)\zeta^2 + (12\beta^5 - 15\beta - 3\beta^3)\zeta + 2 + 6\beta^2 + 6\beta^4 + 2\beta^6 \tag{65}$$

The parameters ζ and β must satisfy Eq. (64) in order to have a real solution. Therefore, in view of Fig. 5, the values of β_{mU} and ζ_{mL} can be obtained from the following optimization problems, respectively:

$$\beta_{mU}\text{: max } \beta \text{ subject to } \mathcal{M}(\zeta,\beta) = 0 \tag{66}$$

and

$$\zeta_{mL}\text{: min } \zeta \text{ subject to } \mathcal{M}(\zeta,\beta) = 0 \tag{67}$$

First, consider the constrained optimization problem in Eq. (66). Using the Lagrange multiplier γ_1, we construct the Lagrangian

$$\mathcal{L}_1(\zeta,\beta) = \beta + \gamma_1\mathcal{M}(\zeta,\beta) \tag{68}$$

The optimization problem shown in Eq. (66) can be solved by setting

$$\frac{\partial \mathcal{L}_1}{\partial \zeta} = 0 \tag{69a}$$

and

$$\frac{\partial \mathcal{L}_1}{\partial \beta} = 0 \tag{69b}$$

Differentiating the Lagrangian in Eq. (68), the above two conditions result

$$\gamma_1[48\beta^3\zeta^2 + (-6\beta^2 + 48\beta^4)\zeta - 15\beta + 12\beta^5 - 3\beta^3] = 0 \tag{70}$$

and

$$1 + \gamma_1[48\beta^2\zeta^3 + (-6\beta + 96\beta^3)\zeta^2 + (-15 + 60\beta^4 - 9\beta^2)\zeta + 12\beta + 24\beta^3 + 12\beta^5] = 0 \tag{71}$$

Because the Lagrange multiplier γ_1 cannot be zero, solving Eq. (70) one has

$$\zeta = -\frac{1+\beta^2}{2\beta} \tag{72a}$$

or

$$\zeta = \frac{5-4\beta^2}{8\beta} \tag{72b}$$

Ignoring the first solution, which is always negative, and substituting $\zeta=(5-4\beta^2)/8\beta$ in the constraint Eq. (64) and simplifying we have

$$\beta^4 + 2\beta^2 - 11/16 = 0 \tag{73}$$

There is only one feasible solution to the above equation, which can be obtained as

$$\beta_{mU} = \tfrac{1}{2}\sqrt{3\sqrt{3}-4} = 0.5468 \tag{74}$$

For this value of β, the value of ζ can be obtained from Eq. (72b) as

$$\zeta_{mU} = \tfrac{3}{4}\sqrt{(12\sqrt{3}-6)/11} = 0.8695 \tag{75}$$

The point (ζ_{mU}, β_{mU}) is shown by a dot in Fig. 5. From this plot, it can be observed that if $\beta > \beta_{mU}$, then there always exists a

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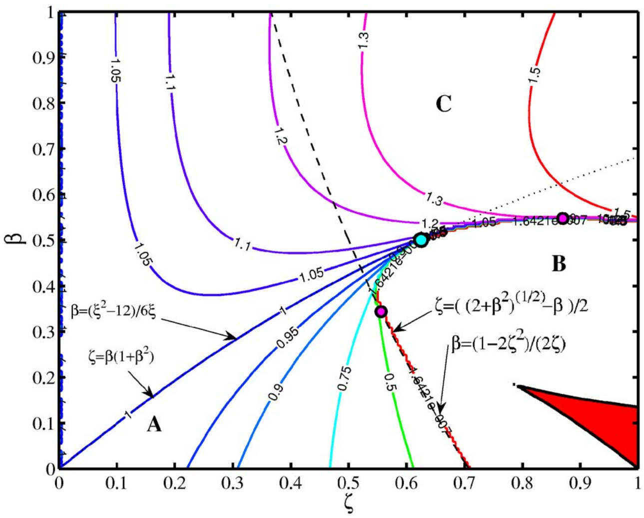

Fig. 6 Contours of the normalized excitation frequency corresponding to the maximum value of the amplitude of the frequency response function, ω_{max}/ω_n, as a function of ζ and β. Equations corresponding to $\omega_{max}/\omega_n=0$ (dashed line) and $\omega_{max}/\omega_n=1$ (dotted line) are shown in the figure. These equations are valid in the region A only. The function ξ is defined in Eq. (81).

driving frequency for which the amplitude of the frequency response function will reach a maximum value.

The value of ζ_{mL} can be obtained from the optimization problem (67) by constructing the Lagrangian

$$\mathcal{L}_2(\zeta,\beta) = \zeta + \gamma_2 \mathcal{M}(\zeta,\beta) \tag{76}$$

where γ_2 is the Lagrange multiplier. Following a similar procedure, it can be shown that the optimal value of ζ is given by

$$\zeta_{mL} = \tfrac{1}{2}\sqrt{\sqrt{5}-1} = 0.5559 \tag{77}$$

For this value of ζ, the value of β can be obtained as

$$\beta_{mL} = \tfrac{1}{2}\sqrt{2\sqrt{5}-4} = 0.3436 \tag{78}$$

The point (ζ_{mL},β_{mL}) is shown by a dot in Fig. 5. From this plot, it can be observed that if $\zeta<\zeta_{mL}$, then there always exists a driving frequency for which the amplitude of the frequency response function will reach a maximum value. From the preceding discussions, we have the following fundamental results:

THEOREM 5.1. *The amplitude of the frequency response function of a nonviscously damped oscillator can reach a maximum value if* $\zeta<\frac{1}{2}\sqrt{\sqrt{5}-1}$ *or* $\beta>\frac{1}{2}\sqrt{3\sqrt{3}-4}$.

THEOREM 5.2. *If* $\zeta<\frac{1}{2}\sqrt{\sqrt{5}-1}$ *or* $\beta>\frac{1}{2}\sqrt{3\sqrt{3}-4}$, *then the amplitude of the frequency response function of a nonviscously damped oscillator reaches a maximum value when the driving frequency* $\omega=\omega_n(\sqrt{(1+2\zeta\beta+\beta^2)[2\cos(\theta/3)+1)/3-1]})/\beta$.

5.1.2 Parameter Relationships for $\omega_{max}=\omega_n$. The contour line $\omega_{max}/\omega_n=1$ in Fig. 5 is of special interest. For these particular parameter combinations, the maximum amplitude of the frequency response function of the damped system occurs *exactly* at the undamped natural frequency. This surprising fact occurs only in a nonviscously damped system and it is not possible for viscously damped systems. For a more detailed analysis, Fig. 6 again shows the contours of ω_{max}/ω_n when $\zeta\leqslant 1$ and $\beta\leqslant 1$.

In Fig. 6, when $\beta=0$, then ω_{max}/ω_n can be equal to 1 if and only if $\zeta=0$ (that is, when the system is undamped). The conditions for $\omega_{max}/\omega_n=1$ can be obtained by enforcing $x_{max}=1$. Thus, substituting $x=1$ in Eq. (49) and considering that $\zeta\neq 0$, we have

$$\beta + \beta^3 - \zeta = 0 \tag{79}$$

Solving this, the required condition can be given by

$$\zeta = \beta(1+\beta^2) \tag{80}$$

when β is known, or

$$\beta = (\xi^2-12)/6\xi \quad \text{where } \xi = \sqrt[3]{108\zeta + 12\sqrt{12+81\zeta^2}} \tag{81}$$

when ζ is known. Equation (80) is plotted in Fig. 6. The same curve can also be obtained by plotting Eq. (81). One interesting fact emerging from Fig. 6 is that beyond certain values of ζ and β, the maximum dynamic response amplitude cannot occur at $\omega_{max}/\omega_n=1$. To obtain these limiting values, we substitute ζ from Eq. (80) into the condition of real solution given in Eq. (64). After some algebra, the resulting equation becomes

$$16\beta^{12} + 72\beta^{10} + 105\beta^8 + 45\beta^6 - 15\beta^4 - 9\beta^2 + 2 = 0 \tag{82}$$

The only positive real solution of the above equation is

$$\beta = 1/2 \tag{83}$$

Substituting this value β in Eq. (80), one obtains

$$\zeta = 5/8 \tag{84}$$

The point $(5/8, 1/2)$ is shown in Fig. 6 by a dot. From this diagram, it is clear that ω_{max}/ω_n can be equal to one, if and only if $\zeta<5/8$ and $\beta<1/2$. When $x_{max}=1$, the maximum value of the amplitude of the frequency response function can be obtained from Eq. (44) as

$$|G|_{x_{max}=1} = \frac{\sqrt{1+\beta^2}}{2\zeta} \tag{85}$$

From this discussion, we have the following useful results:

THEOREM 5.3. *The maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator will occur below the undamped natural frequency if and only if* $\zeta<5/8$ *and* $\beta<1/2$.

THEOREM 5.4. *The maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator will occur above the undamped natural frequency if* $\zeta>5/8$ *or* $\zeta<\beta(1+\beta^2)$ *and* $\beta>1/2$ *or* $\beta>(\xi^2-12)/6\xi$.

Another curious feature of Fig. 6 is the flatness of ω_{max}/ω_n around the contour line 1. This implies that for a wide range of parameter combinations, it is possible to observe a damped resonance very close to the undamped natural frequency. For a viscously damped system, this can happen only if the damping is very small ($\zeta\lesssim 0.05$). But for a nonviscously damped system, this can happen even when ζ is as large as 0.6.

It was shown that the amplitude of the frequency response function cannot reach a maximum value for some combinations of ζ and β (the parameter region B in Figs. 5 and 6). Considering small values of ζ and β so that $\zeta\leqslant\zeta_{mL}$ and $\beta\leqslant\beta_{mL}$, we aim to derive a simple analytical expression for the existence of $|G|_{max}$. Because $x=\tilde{\omega}^2$, the condition for existence of the maximum amplitude of the frequency response function can be expressed as

$$x_{max} \geqslant 0 \tag{86}$$

Therefore, the critical condition can be obtained by substituting $x=0$ in Eq. (49) as

$$1 - 2\zeta\beta - 2\zeta^2 = 0 \tag{87}$$

Solving this equation for ζ, the condition for existence of $|G|_{max}$ can be expressed by

$$\zeta < \tfrac{1}{2}(\sqrt{2+\beta^2} - \beta) \tag{88}$$

when

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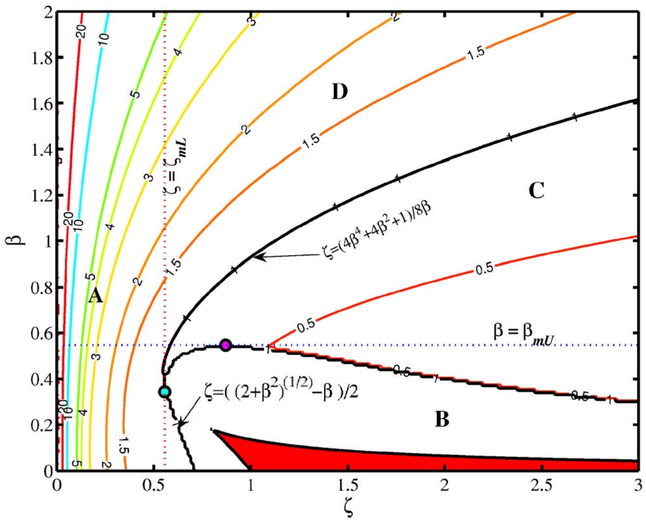

Fig. 7 Contours of the maximum amplitude of the normalized frequency response function $|G|_{max}$ as a function of ζ and β

$$\beta \leq \frac{1}{2}\sqrt{2\sqrt{5}-4} \tag{89}$$

For the special case when only viscous damping is present, substituting $\beta=0$ in Eq. (88), one obtains the required condition as $\zeta \leq 1/\sqrt{2}$, which is well known for viscously damped systems. This condition can alternatively be expressed in terms of ζ by solving Eq. (87) for β as

$$\beta < \frac{1-2\zeta^2}{2\zeta} \tag{90}$$

when

$$\zeta \leq \frac{1}{2}\sqrt{\sqrt{5}-1} \tag{91}$$

The validity of Eqs. (88) and (90) can be verified from Fig. 6. When $\zeta \leq \zeta_{mL}$ and $\beta \leq \beta_{mL}$, Eqs. (88) and (90) match perfectly with the zero line obtained from the expression of x_{max} in Eq. (58). Observe that these equations become invalid when $\zeta \geq \zeta_{mL}$ and $\beta \geq \beta_{mL}$. From this discussion, we have the following result:

THEOREM 5.5. *If $\zeta \leq \frac{1}{2}\sqrt{\sqrt{5}-1}$ and $\beta \leq \frac{1}{2}\sqrt{2\sqrt{5}-4}$, the amplitude of the frequency response function of a nonviscously damped oscillator can reach a maximum value if and only if $\zeta < (\sqrt{2+\beta^2}-\beta)/2$ or $\beta < (1-2\zeta^2)/2\zeta$.*

5.2 The Amplitude of the Maximum Dynamic Response. The maximum value of the amplitude of the frequency response function is a useful quantity because it can be related to the structural failure and design. Figure 7 shows the contours of the maximum amplitude of the normalized frequency response function $|G|_{max}$ as a function of ζ and β. The values of $|G|_{max}$ are calculated from Eq. (44) by substituting x_{max} from Eq. (58) in place of x. This diagram is divided into four regions for discussions. In region A, where $\zeta < \zeta_{mL}$, the amplitude of the frequency response function of the system will always have a maximum value. The values of $|G|_{max}$ are higher for smaller values of ζ, as expected. A useful fact to be noted is that for a fixed value of ζ, the value of $|G|_{max}$ is higher for higher values of β. This can also be verified from Fig. 4. This fact may have undesirable consequences, especially if β is large. In region B, the amplitude of the frequency response function does not have a maximum value. The shaded portion inside region B (shown before in Figs. 2 and 3) corresponds to the parameter region, where the system cannot have any oscillatory motions. Clearly, within this overdamped region, it is not possible for the dynamic response amplitude to reach a maximum value. In region C where $\beta > \beta_{mU}$, the amplitude of the frequency response function of the system will always have a maximum value, but the value of the maximum response is less than 1. In region D, observe that $\beta > \beta_{mU}$, but unlike region C, the value of the maximum response is more than 1. In general, for a fixed value of ζ, the values of $|G|_{max}$ increase with the increasing values of β. The numerical values of $|G|_{max}$ in regions C and D are, however, smaller compared to those in region A. From this discussion, we have the following general result:

THEOREM 5.6. *For a given value of ζ, the maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator increases with increasing values of β.*

The contour line "1" in Fig. 7 is of special interest because $|G|_{max} > 1$ implies that the maximum dynamic response amplitude is more than the static response. For the parameter combinations in the left side of the contour line 1, the amplitude of the maximum dynamic response is always greater than 1. In the the region to the right, the amplitude of the maximum dynamic response is less than the static response amplitude of the system. The exact parameter combinations for which $|G|_{max}$ is more than 1 is considered next.

Substituting x_{max} from Eq. (58) in the expression of $|G|^2$ in Eq. (44), we can obtain the expression of $|G|^2_{max}$. Equating the resulting expression to 1 and simplifying, we have

$$\begin{aligned}&[8\zeta^3\beta^3 + (12\beta^2 + 12\beta^4)\zeta^2 + (12\beta^3 + 6\beta + 6\beta^5)\zeta + 3\beta^2 + 1 + \beta^6 \\ &\quad + 3\beta^4][8\cos^3(\theta/3) - 12\cos^2(\theta/3)] + 18[8\zeta^2\beta^2 + (-2\beta^5 + 4\beta \\ &\quad + 4\beta^3)\zeta - \beta^4 - \beta^6]\cos(\theta/3) + 32\zeta^3\beta^3 + (48\beta^4 + 12\beta^2)\zeta^2 \\ &\quad + (-48\beta + 6\beta^5 - 24\beta^3)\zeta + 4 + 3\beta^4 - 5\beta^6 + 12\beta^2 = 0\end{aligned} \tag{92}$$

This is a cubic equation in $\cos(\theta/3)$ and it can be solved exactly to obtain

$$\cos(\theta/3) = \frac{1-\zeta\beta-\beta^2/2}{1+\beta^2+2\zeta\beta} \tag{93}$$

or

$$\cos(\theta/3) = \frac{\beta^2+2\zeta\beta+(1\pm 3\kappa)/4}{1+\beta^2+2\zeta\beta} \tag{94}$$

where

$$\kappa = \sqrt{4\beta^4+4\beta^2-8\zeta\beta+1} \tag{95}$$

Among the above three solutions, any one of the two solutions given in Eq. (94) turns out be more useful. In order to obtain the relationship between ζ and β so that $|G|_{max}=1$, it is required to relate the expression of $\cos(\theta/3)$ in Eq. (94) to the expression of $\cos(\theta)$ in Eq. (54). Using the identity

$$\cos(\theta) = 4\cos^3(\theta/3) - 3\cos(\theta/3) \tag{96}$$

and substituting the expression of $\cos(\theta)$ from Eq. (54) and $\cos(\theta/3)$ from Eq. (94), we have

$$\begin{aligned}&4(4\beta-\kappa\beta)\zeta^2 + (2\kappa - 4\beta^2\kappa - 8\beta^4 - 2)\zeta - 4\beta^3 - \beta - 2\beta^3\kappa - 4\beta^5 \\ &\quad - \kappa\beta = 0\end{aligned} \tag{97}$$

or

$$\kappa = -\frac{(8\beta^4+2)\zeta + 4\beta^5 + 4\beta^3 + \beta - 16\zeta^2\beta}{4\zeta^2\beta + (4\beta^2-2)\zeta + 2\beta^3 + \beta} \tag{98}$$

Equating the right-hand sides of Eqs. (95) and (98) and simplifying we have

$$\begin{aligned}&16\zeta^4\beta^2 + (14\beta - 8\beta^5 + 24\beta^3)\zeta^3 + (-4\beta^2 - 8\beta^4 - 2 - 16\beta^6)\zeta^2 \\ &\quad - (9\beta + 14\beta^3 + 8\beta^7 + 12\beta^5)\zeta + 1 + 5\beta^2 + 8\beta^4 + 4\beta^6 = 0\end{aligned} \tag{99}$$

The two real and positive solutions of ζ of the preceding equation are given by

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$$\zeta = (\sqrt{2+\beta^2} - \beta)/2 \tag{100}$$

or

$$\zeta = (4\beta^4 + 4\beta^2 + 1)/8\beta \tag{101}$$

If the expression of $\cos(\theta)$ in Eq. (93) was used in place of that in Eq. (94), then one would obtain only the condition in Eq. (100). The expression of $\cos(\theta)$ in Eq. (94) was selected because it produces more general results. Interestingly, the condition given in Eq. (100) was also identified as the condition for the existence of the maximum value of the frequency response function in Eq. (88). The value of ζ given in Eqs. (100) and (101) are shown in Fig. 7. Equation (100) is valid when $\beta \leq \frac{1}{2}\sqrt{2\sqrt{5}-4}$ and Eq. (101) is valid when $\beta > \frac{1}{2}\sqrt{2\sqrt{5}-4}$. From this analysis, we have the following fundamental result:

THEOREM 5.7. *The maximum amplitude of the normalized frequency response function of a nonviscously damped oscillator will be more than 1 if and only if* $\zeta < (\sqrt{2+\beta^2}-\beta)/2$ *when* $\beta \leq \frac{1}{2}\sqrt{2\sqrt{5}-4}$ *and* $\zeta < (4\beta^4+4\beta^2+1)/8\beta$ *when* $\beta \geq \frac{1}{2}\sqrt{2\sqrt{5}-4}$.

From this result, one practical question that naturally arises is, What is the critical value of ζ below which the maximum amplitude of the normalized frequency response function will always be more than 1? To answer this question, we look for the minimum value of ζ given by Eq. (101). Differentiating Eq. (101) with respect to β, the optimal value can be obtained from

$$\frac{4\beta^2 + 12\beta^4 - 1}{8\beta^2} = 0 \tag{102}$$

The only real and positive solution of this equation is

$$\beta = \frac{1}{\sqrt{6}} \tag{103}$$

Substituting this value of β in Eq. (101), the optimal value of ζ can be obtained as

$$\zeta = 2\sqrt{6}/9 \tag{104}$$

From this discussion we have the following theorem:

THEOREM 5.8. *The maximum amplitude of the normalized frequency response function of a nonviscously damped oscillator will be more than 1 if* $\zeta < 2\sqrt{6}/9$.

The converse statement of Theorem 5.8 is, however, not always true. The value of $|G|_{\max}$ can be more than 1 even if $\zeta > 2\sqrt{6}/9$, as can be seen in region C in Fig. 7.

6 Simplified Analysis of the Frequency Response Function

Dynamic characteristics of the frequency response function of a nonviscously damped SDOF system have been elucidated in the previous section. The frequency at which the amplitude of the frequency response function reaches its maximum value can be obtained from Eq. (58). Although this is an exact expression, it is difficult to gain much physical insight due to its complexity. Here, we derive some simple expressions considering that ζ and β are small.

In Fig. 6, it was noted that for a wide range of values of ζ and β, the amplitude of the frequency response function reaches its maximum value when the normalized excitation frequency is close to 1. For this reason, we assume that

$$x_{\max} = 1 - \delta \tag{105}$$

Substituting this in place of x in Eq. (49) and simplifying, one obtains:

$$\beta^4\delta^3 + (-2\beta^4 + 2\beta^3\zeta - 2\beta^2)\delta^2 + (\beta^4 + 2\beta^2 - 4\beta^3\zeta - 4\zeta\beta + 1)\delta + 2\zeta\beta + 2\beta^3\zeta - 2\zeta^2 = 0 \tag{106}$$

This is a cubic equation in δ, which needs to be solved to obtain the frequency where $|G|^2$ reaches its maximum value. Since δ is expected to be small for small values of ζ and β, neglecting the coefficients associated with δ^2 and δ^3 in Eq. (106) and solving the resulting linear equation we obtain

$$\delta \approx \frac{2\zeta^2 - 2\zeta\beta(1+\beta^2)}{(1+\beta^2)(1+\beta^2-4\zeta\beta)} \tag{107}$$

Substituting δ in Eq. (105), the frequency corresponding to the maximum value of the amplitude of the frequency response function can be approximately obtained as

$$\tilde{\omega}_{\max} = \sqrt{x_{\max}} = \frac{\omega_{\max}}{\omega_n} \approx \sqrt{1 - \frac{2\zeta^2 - 2\zeta\beta(1+\beta^2)}{(1+\beta^2)(1+\beta^2-4\zeta\beta)}} \tag{108}$$

For the special case when only viscous damping is present, substituting $\beta=0$ in Eq. (108), one obtains $\tilde{\omega}_{\max} = \sqrt{1-2\zeta^2}$, which is well known for viscously damped systems.

Substituting $x=x_{\max}$ from (105) into the expression of $|G|^2$ in Eq. (44) and retaining only up to quadratic terms in δ, one has

$$|G|^2_{\max} \approx \frac{1+\beta^2-\beta^2\delta}{4\zeta^2 + (4\zeta\beta - 4\zeta^2)\delta + (\beta^2+1-4\zeta\beta)\delta^2} \tag{109}$$

Substituting δ from (107) into the preceding equation and retaining only up to cubic terms in β, one has

$$|G|_{\max} \approx \frac{1}{2\zeta}\sqrt{\frac{(1+\beta^2)}{(1+2\beta^2)}\frac{(1-4\zeta\beta+(3-2\zeta^2)\beta^2-6\beta^3\zeta)}{(1-2\zeta\beta-\zeta^2)}} \tag{110}$$

For the special case when only viscous damping is present, substituting $\beta=0$ in Eq. (110) results in the exact corresponding expression $|G|_{\max} = 1/(2\zeta\sqrt{1-\zeta^2})$, as given in Eq. (42). To verify the accuracy of the approximate formulas (108) and (110), we calculate the percentage error with respect to the exact solutions obtained in the previous section. The percentage error is calculated, for example, as

$$100 \times \frac{(\tilde{\omega}_{\max})_{\text{exact}} - (\tilde{\omega}_{\max})_{\text{approx}}}{(\tilde{\omega}_{\max})_{\text{exact}}} \tag{111}$$

Figures 8 and 9, respectively, show the contours of percentage errors arising due to the use of approximate Eqs. (108) and (110).

For $\tilde{\omega}_{\max}$ calculated from Eq. (108), the error is less than 2% when $\zeta, \beta \leq 0.5$. The error in the calculation of $|G|_{\max}$ from Eq.

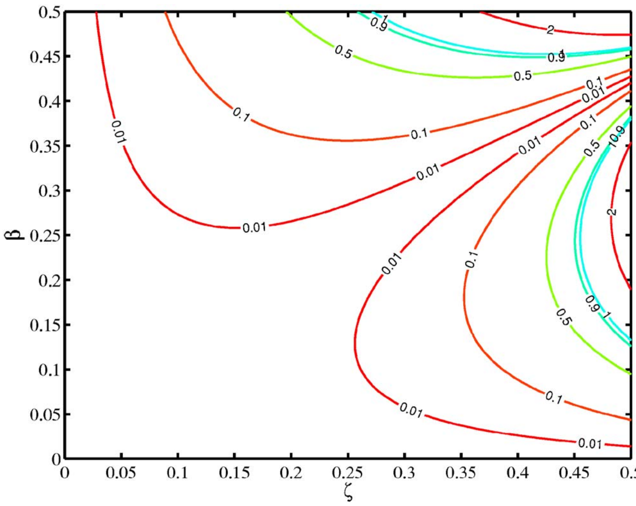

Fig. 8 Contours of percentage error in the approximate calculation of $\omega_{\max}/\omega_n$ from Eq. (108) as a function of ζ and β

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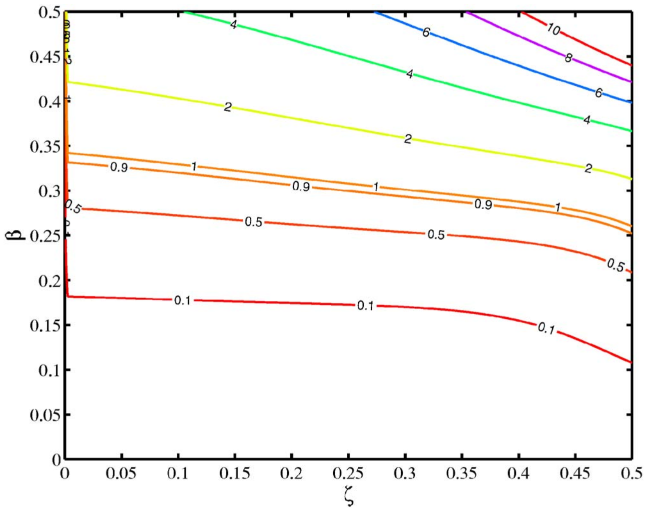

Fig. 9 Contours of percentage error in the approximate calculation of $|G|_{\max}$ from Eq. (110) as a function of ζ and β

(110) is somewhat more. When $\zeta, \beta \approx 0.5$, the error is close to 10%. From Fig. 9, it can be observed that the error in the calculation of $|G|_{\max}$ increases with the increasing values of β, but it is relatively insensitive with respect to ζ.

The approximate expressions (108) and (110) will break down if $\zeta > \zeta_{mL}$ and $\beta > \beta_{mU}$. For such parameter values, it is not possible to extend a perturbation type method based on a viscous damped system, as proposed here. Nevertheless, if a system is moderately nonviscously damped (say $\beta < 0.5$), the dynamics can be explained using the proposed approximations (108) and (110).

7 Summary and Concluding Remarks

Dynamic response characteristics of a nonviscously damped linear single-degree-of-freedom oscillator have been discussed. The nonviscous damping force was expressed by a viscoelastic type exponentially fading memory kernel. It was shown that the dynamic response properties of the oscillator are governed by two nondimensional factors; namely, the viscous damping factor ζ and the nonviscous damping factor β. The system considered reduces to the classical viscously damped oscillator when the nonviscous damping factor is zero. Several fundamental properties that characterize the dynamic response of a nonviscously damped oscillator have been discovered. A nonviscously damped oscillator has three eigenvalues, one of which is always nonoscillating in nature. The conditions for the occurrence of the maximum value of the amplitude of the dynamic response were reviewed. The characteristics of the driving frequency corresponding to the maximum amplitude of the frequency response function and the value of the maximum response amplitude were discussed in detail. The main findings of the paper are:

1. A nonviscously damped oscillator will have oscillatory motions if $\zeta < 4/(3\sqrt{3})$ or $\beta > 1/(3\sqrt{3})$.
2. If $\beta < 1/(3\sqrt{3})$, the oscillator will have oscillatory motions if and only if $\zeta \notin [\zeta_L, \zeta_U]$. ζ_L and ζ_U given in Eqs. (26) and (27) are the lower and upper critical damping factors, respectively.
3. The amplitude of the frequency response function of a nonviscously damped oscillator can reach a maximum value if $\zeta < \frac{1}{2}\sqrt{\sqrt{5}-1}$ or $\beta > \frac{1}{2}\sqrt{3\sqrt{3}-4}$.
4. If $\zeta < \frac{1}{2}\sqrt{\sqrt{5}-1}$ or $\beta > \frac{1}{2}\sqrt{3\sqrt{3}-4}$, then the amplitude of the frequency response function of a nonviscously damped oscillator reaches a maximum value when the driving frequency $\omega = \omega_n\{\sqrt{(1+2\zeta\beta+\beta^2)[2\cos(\theta/3)+1]/3-1}\}/\beta$.
5. The maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator will occur below the undamped natural frequency if and only if $\zeta < 5/8$ and $\beta < 1/2$.
6. The maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator will occur above the undamped natural frequency if $\zeta > 5/8$ or $\zeta < \beta(1+\beta^2)$ and $\beta > 1/2$ or $\beta > (\xi^2-12)/6\xi$.
7. If $\zeta \leqslant \frac{1}{2}\sqrt{\sqrt{5}-1}$ and $\beta \leqslant \frac{1}{2}\sqrt{2\sqrt{5}-4}$, the amplitude of the frequency response function of a nonviscously damped oscillator can reach a maximum value if and only if $\zeta < (\sqrt{2+\beta^2}-\beta)/2$ or $\beta < (1-2\zeta^2)/2\zeta$.
8. For a given value of ζ, the maximum amplitude of the frequency response function (if it exists) of a nonviscously damped oscillator increases with increasing values of β.
9. The maximum amplitude of the normalized frequency response function of a nonviscously damped oscillator will be more than 1 if and only if $\zeta < (\sqrt{2+\beta^2}-\beta)/2$ when $\beta \leqslant \frac{1}{2}\sqrt{2\sqrt{5}-4}$ and $\zeta < (4\beta^4+4\beta^2+1)/8\beta$ when $\beta \geqslant \frac{1}{2}\sqrt{2\sqrt{5}-4}$.
10. The maximum amplitude of the normalized frequency response function of a nonviscously damped oscillator will be more than 1 if $\zeta < 2\sqrt{6}/9$.

Using these results, one can understand the nature of the dynamic response without actually solving the problem. These concepts will be particularly useful in dealing with multiple-degree-of-freedom systems. The studies reported in this paper show that the classical concepts based on viscously damped oscillators can be extended to nonviscously damped systems only under certain conditions. In general, if $\beta > \frac{1}{2}\sqrt{3\sqrt{3}-4}$, the dynamic response characteristics will be significantly different from a classical viscously damped oscillator. The results derived in this paper are expected to be valid for a proportionally damped multiple-degree-of-freedom system with a single exponential kernel. However, formal results are necessary in this direction. Further research is needed to extend these results to systems with multiple exponential kernels and nonproportional damping.

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