Journal of Applied Mechanics

Technical Briefs

Modal Analysis of Nonviscously Damped Beams

S. Adhikari[1]
e-mail: s.adhikari@bristol.ac.uk

M. I. Friswell
Sir George White Professor of Aerospace Engineering

Department of Aerospace Engineering,
Queens Building,
University of Bristol,
Queens Walk,
Bristol BS8 1TR, UK

Y. Lei
Associate Professor
College of Aerospace and Material Engineering,
National University of Defense Technology,
Changsha 410073, PRC

Linear dynamics of Euler–Bernoulli beams with nonviscous nonlocal damping is considered. It is assumed that the damping force at a given point in the beam depends on the past history of velocities at different points via convolution integrals over exponentially decaying kernel functions. Conventional viscous and viscoelastic damping models can be obtained as special cases of this general damping model. The equation of motion of the beam with such a general damping model results in a linear partial integro-differential equation. Exact closed-form equations of the natural frequencies and mode shapes of the beam are derived. Numerical examples are provided to illustrate the new results.
[DOI: 10.1115/1.2712315]

1 Introduction

Viscous damping is the most common damping model used for linear dynamic systems. However, within the scope of linear theory, more general nonviscous models have been used in the recent past [1–5]. Nonviscous damping models in general have more parameters and therefore are more likely to have a better match with experimental measurements. A linear damped continuous dynamic system in which the displacement variable $u(\mathbf{r},t)$, where $\mathbf{r}$ is the spatial position vector and t is time, specified in some domain $\mathcal{D}$, is governed by a linear partial differential equation

$$\rho(\mathbf{r})\ddot{u}(\mathbf{r},t)+\mathcal{L}_1\dot{u}(\mathbf{r},t)+\mathcal{L}_2u(\mathbf{r},t)=p(\mathbf{r},t);\quad \mathbf{r}\in\mathcal{D},\quad t\in[0,T] \tag{1}$$

with homogeneous linear boundary conditions of the form

$$\mathcal{M}_1u(\mathbf{r},t)=0;\quad \mathbf{r}\in\Gamma_1\quad\text{and}\quad \mathcal{M}_2\dot{u}(\mathbf{r},t)=0;\quad \mathbf{r}\in\Gamma_2 \tag{2}$$

specified on some boundary surfaces Γ_1 and Γ_2. In the above equation $\rho(\mathbf{r})$ is the mass distribution of the system: $p(\mathbf{r},t)$ is the distributed time-varying forcing function; and $\mathcal{L}_2$ is the spatial self-adjoint stiffness operator and $\mathcal{M}_1$ and $\mathcal{M}_2$ are linear operators acting on the boundary. For external (or foundation) damping the operator $\mathcal{L}_1$ can be written in the form

$$\mathcal{L}_1\dot{u}(\mathbf{r},t)=\int_{\mathcal{D}}\int_{-\infty}^{t}C_1(\mathbf{r},\boldsymbol{\xi},t-\tau)\dot{u}(\boldsymbol{\xi},\tau)d\tau\, d\boldsymbol{\xi} \tag{3}$$

where $C_1(\mathbf{r},\boldsymbol{\xi},t)$ is the kernel function. The velocities $\dot{u}(\boldsymbol{\xi},\tau)$ at different time instants and spatial locations are coupled through this kernel function. Lei et al. [5] considered both external and internal damping, and although internal damping is not considered further in this paper, the proposed method may be extended to this case. Kernel functions that serve similar purposes have been described by different names in different subjects (for example, retardation functions, heredity functions, after-effect functions, relaxation functions), and different models have been used to describe them. Equation (1) together with Eq. (3) represents a continuous dynamic system with general linear damping. It may be noted that if $\mathcal{L}_1=0$ in Eq. (1), i.e., an undamped system, or if the system satisfies the criteria given by Caughey and O'Kelly [6], then the system will possess classical normal modes. However, due to the general nature of the operator $\mathcal{L}_1$ as described by Eq. (3), there is no definite reason why the system should have classical normal modes. Thus the mode shapes and natural frequencies of such systems in general will be complex in nature. In this context we wish to note that the system expressed by Eq. (1) and the damping operator defined in Eq. (3) represents a partial integro-differential equation with the boundary conditions given in Eq. (2). In this technical brief we are interested in the natural frequencies and mode shapes of the system. Exact closed-form expressions of such quantities for the general case are difficult to obtain. We make the following general assumptions:

1. The mass and stiffness distributions are homogeneous, that is, they do not vary with the position vector $\mathbf{r}$; and
2. The damping kernel function is separable in space and time so that

$$C_1(\mathbf{r},\boldsymbol{\xi},t-\tau)=C(\mathbf{r})c(\mathbf{r}-\boldsymbol{\xi})g(t-\tau) \tag{4}$$

Depending on the nature of the functions $c(\bullet)$ and $g(\bullet)$, several special cases, starting from the simple viscous model to the more general nonviscous model, may arise [5]. For example, if both $c(\bullet)$

[1]Corresponding author.
Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received February 8, 2006; final manuscript received August 15, 2006. Review conducted by Oliver M. O'Reilly.

Fig. 1 Euler–Bernoulli beam with nonviscous damping patch

and $g(\bullet)$ are delta functions then the result is the "locally reacting" viscous damping model. If only $c(\bullet)$ is a delta function, then the resulting model becomes a locally reacting viscoelastic damping which is also known as the time hysteresis model. For the case when only $g(\bullet)$ is a delta function, the resulting model becomes nonlocal viscous damping or the spatial hysteresis model. Here the general case, that is, when none of these two functions are the delta functions, is considered. Based on the transfer matrix method [7], we propose a new method for modal analysis of a Euler–Bernoulli beam with general linear damping given by Eq. (4).

2 Governing Equation of Motion

The Euler–Bernoulli beam considered in this study is shown in Fig. 1. The left and right coordinates of the beam are denoted by x_L and x_R, respectively. The beam has a nonlocal viscoelastic damping patch between x_1 and x_2. It is assumed that the elastic properties of the beam are uniformly distributed with bending rigidity EI and mass density ρA.

In order to formulate and solve the equation of motion, it is necessary to use some kind of plausible functional form of the kernel functions in space and time. In this study we choose the following functional forms [5]

$$g(t) = g_\infty \mu \exp(-\mu t) \qquad (5a)$$

so that

$$G(s) = \frac{g_\infty \mu}{s+\mu}, \quad g_\infty,\ \mu \geqslant 0 \qquad (5b)$$

and

$$c(x-\xi) = \frac{\alpha}{2}\exp(-\alpha|x-\xi|), \quad C(x)=1 \qquad (6)$$

These models imply that the correlations in both space and time decay exponentially. If $\alpha\to\infty, \mu\to\infty$ one obtains the standard viscous model; if $\alpha\to\infty$ and μ is finite one obtains the time hysteresis model; and if α is finite but $\mu\to\infty$ one obtains the spatial hysteresis model.

The equation of motion of the part with the damping patch can be expressed by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} + \int_{x_1}^{x_2}\int_{-\infty}^{t}\frac{\alpha}{2}\exp(-\alpha|x-\xi|) \times g_\infty\mu\exp[-\mu(t-\tau)]\left.\frac{\partial w(\xi,t)}{\partial t}\right|_{t=\tau} d\xi\, d\tau = 0 \quad \text{when } x\in[x_1,x_2] \qquad (7)$$

The equation of motion of the part outside the damping patch can be expressed by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} + C_0\frac{\partial w(x,t)}{\partial t} = 0 \quad \text{when } x\in(x_L,x_1)\cup(x_2,x_R) \qquad (8)$$

Appropriate boundary conditions must be satisfied at $x=x_L$ and at $x=x_R$. In addition to this we need to satisfy relevant continuity conditions at the internal points x_1 and x_2. No forcing is assumed because the central interest in this study is to obtain the eigensolutions. The function $w(x,t)$ in Eqs. (7) and (8) is smooth and continuously differentiable up to fourth order with respect to x. Here we assume that $w(x,t)$ is continuously differentiable up to fifth order. In what follows, Eqs. (7) and (8) are solved separately and the solutions are combined to obtain the eigensolutions. We begin with the solution of Eq. (7).

3 Solution for the Section With Nonlocal Viscoelastic Damping

Assuming zero initial conditions, the Laplace transform of the displacement (with no external force) satisfies

$$EIW^{IV}(x,s) + s^2\rho A W(x,s) + \frac{\alpha}{2}sG(s)\int_{x_1}^{x_2}\exp(-\alpha|x-\xi|)W(\xi,s)d\xi = 0, \quad x\in(x_1,x_2) \qquad (9)$$

Here s is the complex Laplace parameter; and $W(x,s)$ is the Laplace transform of $w(x,t)$. The roman superscripts, for example $(\bullet)^{IV}$, denote the order of derivative with respect to the spatial variable x. It is useful to separate the contribution arising from the term $|x-\xi|$ in Eq. (9) as

$$EIW^{IV}(x,s) + s^2\rho A W(x,s) + \frac{\alpha}{2}sG(s)\int_{x_1}^{x}\exp[-\alpha(x-\xi)]W(\xi,s)d\xi + \frac{\alpha}{2}sG(s)\int_{x}^{x_2}\exp[\alpha(x-\xi)]W(\xi,s)d\xi = 0 \qquad (10)$$

The function $W(x,s)$ is continuously differentiable up to fifth order with respect to x because $w(x,t)$ is assumed to be continuously differentiable up to fifth order. Differentiating Eq. (10) with respect to the spatial variable x one obtains

$$EIW^{V}(x,s) + s^2\rho A W^{I}(x,s) - \frac{\alpha^2}{2}sG(s)\int_{x_1}^{x}\exp[-\alpha(x-\xi)]W(\xi,s)d\xi + \frac{\alpha^2}{2}sG(s)\int_{x}^{x_2}\exp[\alpha(x-\xi)]W(\xi,s)d\xi = 0 \qquad (11)$$

Differentiating again gives

$$EIW^{VI}(x,s) + s^2\rho A W^{II}(x,s) + \frac{\alpha^3}{2}sG(s)\int_{x_1}^{x}\exp[-\alpha(x-\xi)]W(\xi,s)d\xi + \frac{\alpha^3}{2}sG(s)\int_{x}^{x_2}\exp[\alpha(x-\xi)]W(\xi,s)d\xi - 2\frac{\alpha^2}{2}sG(s)W(x,s) = 0 \qquad (12)$$

Using Eqs. (10) and (12) we have

$$EIW^{VI}(x,s) + s^2\rho A W^{II}(x,s) - \alpha^2[EIW^{IV}(x,s) + s^2\rho A W(x,s)] - \alpha^2 sG(s)W(x,s) = 0 \qquad (13)$$

Equation (13) is a sixth-order ordinary differential equation which can be solved by transforming into the first-order form. Recall that the solution of a sixth-order ordinary differential equation requires six boundary conditions corresponding to $W(x,s),\ldots,W^{VI}(x,s)$. However, the compatibility with the other beam solutions only gives four boundary conditions. The other boundary conditions are implicit in the integro-differential equation. For example, at $x=x_1$, Eq. (10) becomes

$$EIW^{IV}(x_1,s) + s^2\rho A W(x_1,s) + \frac{\alpha}{2}sG(s)\int_{x_1}^{x_2}\exp[\alpha(x-\xi)]W(\xi,s)d\xi = 0 \qquad (14)$$

and Eq. (11) becomes

$$EIW^{V}(x_1,s) + s^2\rho A W^{I}(x_1,s) + \frac{\alpha^2}{2} sG(s)\int_{x_1}^{x_2} \exp[\alpha(x-\xi)]W(\xi,s)d\xi$$

$$= 0 \tag{15}$$

Notice the integrals are the same in Eqs. (14) and (15) and hence combining these equations gives

$$EIW^{V}(x_1,s) - \alpha EIW^{IV}(x_1,s) + s^2\rho A W^{I}(x_1,s) - \alpha s^2 \rho A W(x_1,s) = 0 \tag{16}$$

A similar procedure for $x = x_2$ gives

$$EIW^{V}(x_2,s) + \alpha EIW^{IV}(x_2,s) + s^2\rho A W^{I}(x_2,s) + \alpha s^2 \rho A W(x_2,s) = 0 \tag{17}$$

To find the eigenvalues using the transfer matrix approach, we need to relate the boundary conditions at the ends of the beam segments. For the beam with the damping layer, we define the state vector $\boldsymbol{\eta}(x,s)$ and partition it as

$$\boldsymbol{\eta}(x,s) = \begin{bmatrix} \mathbf{u}(x,s) \\ \mathbf{v}(x,s) \end{bmatrix} \in \mathbb{C}^6 \tag{18}$$

where

$$\mathbf{u}(x,s) = [W(x,s), W^{I}(x,s), W^{II}(x,s), W^{III}(x,s)]^T \in \mathbb{C}^4 \tag{19}$$

and

$$\mathbf{v}(x,s) = [W^{IV}(x,s), W^{V}(x,s)]^T \in \mathbb{C}^2 \tag{20}$$

Using the state vector in Eq. (18), Eq. (13) can be cast in matrix form as

$$\frac{d}{dx}\boldsymbol{\eta}(x,s) = \boldsymbol{\Phi}(s)\boldsymbol{\eta}(x,s), \quad x \in (x_1,x_2) \tag{21}$$

where

$$\boldsymbol{\Phi}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\alpha^2 sG(s) + \alpha^2 s^2 \rho A}{EI} & 0 & -\frac{s^2\rho A}{EI} & 0 & \alpha^2 & 0 \end{bmatrix} \tag{22}$$

The solution of Eq. (21) can be expressed as

$$\boldsymbol{\eta}(x,s) = \exp[\boldsymbol{\Phi}(s)(x-x_1)]\boldsymbol{\eta}_1(s), \quad x \in (x_1,x_2) \tag{23}$$

where $\boldsymbol{\eta}_1(s) = \boldsymbol{\eta}(x_1,s)$. In particular

$$\boldsymbol{\eta}_2(s) = \exp[\boldsymbol{\Phi}(s)(x_2-x_1)]\boldsymbol{\eta}_1(s) = \boldsymbol{\Psi}(s)\boldsymbol{\eta}_1(s) \tag{24}$$

where $\boldsymbol{\eta}_2(s) = \boldsymbol{\eta}(x_2,s)$ and $\boldsymbol{\Psi}(s) = \exp[\boldsymbol{\Phi}(s)(x_2-x_1)]$. We also have from Eqs. (16) and (17) that

$$\mathbf{b}_1(s)^T\boldsymbol{\eta}_1(s) = 0 \tag{25a}$$

and

$$\mathbf{b}_2(s)^T\boldsymbol{\eta}_2(s) = 0 \tag{25b}$$

where

$$\mathbf{b}_1(s) = \begin{Bmatrix} -\alpha s^2\rho A \\ s^2\rho A \\ 0 \\ 0 \\ -\alpha EI \\ EI \end{Bmatrix} \tag{26a}$$

and

$$\mathbf{b}_2(s) = \begin{Bmatrix} \alpha s^2\rho A \\ s^2\rho A \\ 0 \\ 0 \\ \alpha EI \\ EI \end{Bmatrix} \tag{26b}$$

From Eq. (24) and Eq. (25*b*)

$$\mathbf{b}_2(s)^T\boldsymbol{\eta}_2(s) = \mathbf{b}_2(s)^T\boldsymbol{\Psi}(s)\boldsymbol{\eta}_1(s) = 0 \tag{27}$$

Combining Eqs. (25*a*) and (27) we have

$$\mathbf{E}(s)\boldsymbol{\eta}_1(s) = \mathbf{0} \tag{28}$$

where

$$\mathbf{E}(s) = \begin{bmatrix} \mathbf{b}_1(s)^T \\ \mathbf{b}_2(s)^T\boldsymbol{\Psi}(s) \end{bmatrix}$$

We partition $\mathbf{E}(s)$ as

$$\mathbf{E}(s) = [\mathbf{E}_1(s) \quad \mathbf{E}_2(s)] \tag{29}$$

where $\mathbf{E}_1(s)$ is 2×4 and $\mathbf{E}_2(s)$ is 2×2. From Eq. (28) we have

$$\mathbf{E}_1(s)\mathbf{u}(x_1,s) + \mathbf{E}_2(s)\mathbf{v}(x_1,s) = \mathbf{0} \tag{30}$$

or

$$\mathbf{v}(x_1,s) = -\mathbf{E}_2(s)^{-1}\mathbf{E}_1(s)\mathbf{u}(x_1,s) \tag{31}$$

Using Eq. (24) we finally have

$$\mathbf{u}(x_2,s) = \mathbf{T}(s)\mathbf{u}(x_1,s) \tag{32}$$

where

$$\mathbf{T}(s) = [\mathbf{I}_{4\times4} \quad \mathbf{0}_{4\times2}]\boldsymbol{\Psi}(s)\begin{bmatrix} \mathbf{I}_{4\times4} \\ -\mathbf{E}_2(s)^{-1}\mathbf{E}_1(s) \end{bmatrix} \tag{33}$$

4 Eigensolutions of the Complete Beam

Applying the Laplace transform to the equation of motion for the viscously damped segments Eq. (8), and considering the state vector $\mathbf{u}(x,s)$ we can obtain

$$\frac{d}{dx}\mathbf{u}(x,s) = \bar{\boldsymbol{\Phi}}(s)\mathbf{u}(x,s), \quad x \in (x_L,x_1) \cup (x_2,x_R) \tag{34}$$

where

$$\bar{\boldsymbol{\Phi}}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{s^2\rho A + sC_0}{EI} & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

The transfer function matrices can be obtained in the usual manner from Eq. (34) as discussed before. The boundary conditions of the beam can be expressed as

$$\mathbf{M}(s)\mathbf{u}(x_L,s) + \mathbf{N}(s)\mathbf{u}(x_R,s) = \mathbf{0} \tag{36}$$

where $\mathbf{M} \in \mathbb{C}^{4\times4}$ and $\mathbf{N} \in \mathbb{C}^{4\times4}$ are the boundary matrices (see, for example, Ref. [7]). Using the transfer matrices corresponding to the three parts, $\mathbf{u}(x_R,s)$ can be expressed in terms of $\mathbf{u}(x_L,s)$ as

$$\mathbf{u}(x_R,s) = \mathbf{T}_R(s)\mathbf{T}(s)\mathbf{T}_L(s)\mathbf{u}(x_L,s) \tag{37}$$

where $\mathbf{T}(s)$ is defined in Eq. (33) and

$$\mathbf{T}_R(s) = \exp[\bar{\boldsymbol{\Phi}}(s)(x_R-x_2)] \tag{38a}$$

and

Table 1 The first five eigenvalues of the damped pinned-pinned beam

	$\lambda_j \times 10^{-3}$			
j	Proportional viscous	$\mu=100,\ \alpha=10$	$\mu=100,\ \alpha=0.1$	$\mu=1,\ \alpha=0.01$
1	$-0.1161\pm0.0725i$	$-0.0604\pm0.1369i$	$-0.0214\pm0.0702i$	$-0.0214\pm0.0698i$
2	$-0.1161\pm0.2901i$	$-0.0622\pm0.2949i$	$-0.0585\pm0.2853i$	$-0.0585\pm0.2853i$
3	$-0.1161\pm0.6528i$	$-0.0696\pm0.6490i$	$-0.0703\pm0.6453i$	$-0.0703\pm0.6453i$
4	$-0.1161\pm1.1606i$	$-0.0574\pm1.1575i$	$-0.0576\pm1.1550i$	$-0.0576\pm1.1550i$
5	$-0.1161\pm1.8134i$	$-0.0505\pm1.8124i$	$-0.0505\pm1.8113i$	$-0.0505\pm1.8113i$

$$\mathbf{T}_L(s) = \exp[\bar{\mathbf{\Phi}}(s)(x_1 - x_L)] \tag{38b}$$

Substituting Eq. (37) into Eq. (36), one concludes that the eigenvalues are the roots of characteristic equation

$$\det[\mathbf{M}(s) + \mathbf{N}(s)\mathbf{T}_R(s)\mathbf{T}(s)\mathbf{T}_L(s)] = 0 \tag{39}$$

Assuming λ_j are the eigenvalues, the corresponding mode shapes can be determined by

$$\boldsymbol{\psi}_j(x) = \mathbf{u}(x,\lambda_j) = \begin{cases} \exp[\bar{\mathbf{\Phi}}(\lambda_j)(x - x_L)]\mathbf{u}_0(\lambda_j), & x_L \leqslant x \leqslant x_1 \\ \mathbf{T}_m(x,\lambda_j)\mathbf{T}_L(\lambda_j)\mathbf{u}_0(\lambda_j), & x_1 \leqslant x \leqslant x_2 \\ \exp[\bar{\mathbf{\Phi}}(\lambda_j)(x - x_2)]\mathbf{T}(\lambda_j)\mathbf{T}_L(\lambda_j)\mathbf{u}_0(\lambda_j), & x_2 \leqslant x \leqslant x_R \end{cases} \tag{40}$$

Here

$$\mathbf{T}_m(x,\lambda_j) = [\mathbf{I}_{4\times4} \quad \mathbf{0}_{4\times2}]\exp[\mathbf{\Phi}(\lambda_j)(x - x_1)]\begin{bmatrix} \mathbf{I}_{4\times4} \\ -\mathbf{E}_2(\lambda_j)^{-1}\mathbf{E}_1(\lambda_j) \end{bmatrix} \tag{41}$$

and $\mathbf{u}_0(\lambda_j)\,\forall j$ is a vector in the null space of the matrix in Eq. (39) evaluated at $s=\lambda_j$.

5 Numerical Example

A damped pinned–pinned beam similar to that shown in Fig. 1 is used to illustrate the proposed method. The numerical values used are as follows: $x_R=0$ m, $x_L=1$ m, $x_1=0.25$ m, and $x_2=0.75$ m, $E=70$ GN/m^2, $\rho=2700$ kg/m^3, $C_0=g_\infty=15.667$ Ns/m, and the cross section is 5×5 mm^2. The first five eigenvalues for different values of the relaxation parameter, including the case when the whole beam is uniformly viscously damped with parameter C_0, is shown in Table 1.

The real parts of the first four modes are shown in Fig. 2 for the four sets of parameter values given in Table 1.

The corresponding imaginary parts of the first four modes are shown in Fig. 3.

Note that, unlike the eigenvalues, μ and α do significantly affect the eigenvectors.

6 Conclusions

The increasing use of advanced composite materials and active control mechanisms demand sophisticated treatment of damping forces within a distributed parameter system. This technical brief proposes a new method to obtain the natural frequencies and mode shapes of Euler–Bernoulli beams with general linear damping models. It was assumed that the damping force at a given point in the beam depends on the past history of velocities at different points via convolution integrals over exponentially decaying kernel functions. Due to the general nature of the damping forces, the equation of motion becomes an integro-differential equation which couples the deflections at different time instants and spatial locations. The conventional transfer matrix method was extended to such integro-differential equations. Commonly used viscous and viscoelastically damped systems can be considered as special cases of the general formulation derived in the paper. The method was applied to a uniform beam with a nonlocal viscoelastic damping patch. Future work will discuss computational issues and forced vibration problems.

Acknowledgment

Sondipon Adhikari gratefully acknowledges the support of the Engineering and Physical Sciences Research Council through the award of an Advanced Research Fellowship. Michael Friswell gratefully acknowledges the support of the Royal Society through

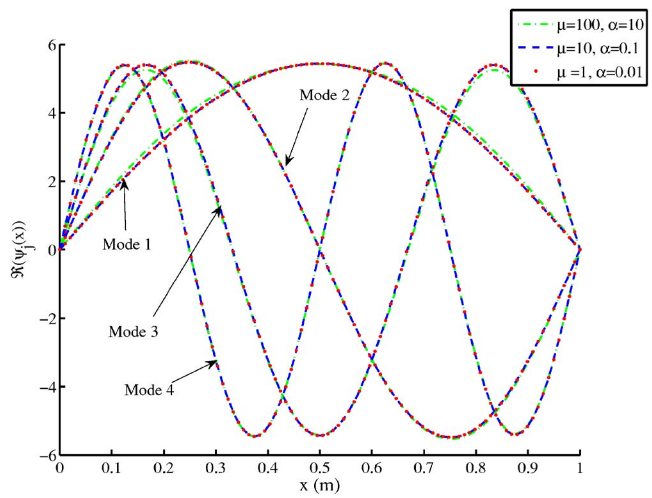

Fig. 2 The real parts of the first four modes

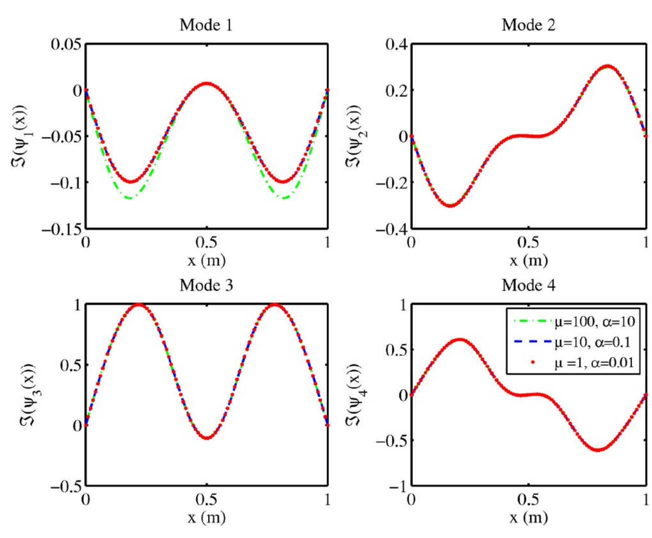

Fig. 3 The imaginary parts of the first four modes

a Royal Society-Wolfson Research Merit Award. Y. Lei gratefully acknowledges the support of China Scholarship Council through a Scholarship Fund Award.

References

[1] Banks, H. T., and Inman, D. J., 1991, "On Damping Mechanisms in Beams," ASME J. Appl. Mech., **58**, pp. 716–723.
[2] Woodhouse, J., 1998, "Linear Damping Models for Structural Vibration," J. Sound Vib., **215**(3), pp. 547–569.
[3] Adhikari, S., 2002, "Dynamics of Non-Viscously Damped Linear Systems," J. Eng. Mech., **128**(3), pp. 328–339.
[4] Wagner, N., and Adhikari, S., 2003, "Symmetric State-Space Formulation for a Class of Non-viscously Damped Systems," AIAA J., **41**(5), pp. 951–956.
[5] Lei, Y., Friswell, M. I., and Adhikari, S., 2006, "A Galerkin Method for Distributed Systems with Non-Local Damping," Int. J. Solids Struct., **43**(11–12), pp. 3381–3400.
[6] Caughey, T. K., and O'Kelly, M. E. J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," ASME J. Appl. Mech., **32**, pp. 583–588.
[7] Yang, B., and Tan, C. A., 1992, "Transfer Functions of One-Dimensional Distributed Parameter Systems," Trans. ASME, J. Appl. Mech., **59**(4), pp. 1009–1014.