PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2005) **461**, 2269–2288
doi:10.1098/rspa.2005.1485
Published online 16 June 2005

Qualitative dynamic characteristics of a non-viscously damped oscillator

By Sondipon Adhikari

Department of Aerospace Engineering, University of Bristol, Queens Building, Bristol BS8 1TR, UK
(s.adhikari@bristol.ac.uk)

This paper considers the linear dynamics of a single-degree-of-freedom non-viscously damped oscillator. It is assumed that the non-viscous damping force depends on the history of velocity via a convolution integral over an exponentially decaying kernel function. Classical qualitative dynamic properties known for viscously damped oscillators have been generalized to such non-viscously damped oscillators. The following questions of fundamental interest have been addressed: (i) under what conditions can a non-viscously damped oscillator sustain oscillatory motions? (ii) how does the natural frequency of a non-viscously damped oscillator compare with that of an equivalent undamped oscillator? and (iii) how does the decay rate compare with that of an equivalent viscously damped oscillator? Introducing two non-dimensional factors, namely, the viscous damping factor and the non-viscous damping factor, we provide answers to these questions. Wherever possible, attempts are made to relate the new results with equivalent classical results for a viscously damped oscillator.

Keywords: linear oscillator; non-viscous damping; eigenvalue problem

1. Introduction

The increasing use of modern composite materials and active control mechanisms in aerospace and automotive industries demands sophisticated treatment of dissipative forces for proper analysis and design. Viscous damping is the most common model for modelling of vibration damping in linear systems. This model, first introduced by Lord Rayleigh (1877), assumes that the instantaneous generalized velocities are the only relevant variables that determine damping. Viscous damping models are used widely for their simplicity and mathematical convenience, even though the energy dissipation behaviour of real structural materials may not be accurately represented by simple viscous models. It is well recognized that a physically realistic model of damping will generally not be viscous. Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are non-viscous damping models. Mathematically, any causal model that makes the energy dissipation functional non-negative is a possible candidate for a non-viscous damping model. Clearly, a wide range of choice is possible, either based on the physics of the problem, or by a priori selecting a model and fitting its parameters from

Received 28 September 2004
Accepted 31 March 2005

experiments. Among various damping models, the 'exponential damping model' is particularly promising and has been used by many authors. With this model, the damping force is expressed as

$$f_{\rm d}(t) = \int_0^t c\mu {\rm e}^{-\mu(t-\tau)} \dot{u}(\tau) {\rm d}\tau. \tag{1.1}$$

Here, c is the viscous damping constant, μ is the relaxation parameter and $u(t)$ is the displacement as a function of time. In the context of viscoelastic materials, the physical basis for exponential models has been well established (e.g. Cremer & Heckl 1973). A selected literature review including the justifications for considering exponential damping model may be found in Wagner & Adhikari (2003). Adhikari & Woodhouse (2001*b*) have proposed some methods by which the damping parameters in equation (1.1) can be obtained from experimental measurements. Recently, Adhikari & Woodhouse (2003) proposed four non-viscosity indices in order to quantify non-viscous damping in multiple degree-of-freedom dynamic systems. Moreover, studies reported by Adhikari & Woodhouse (2001*a*) show that the use of a viscous damping model when the 'true' damping model is non-viscous can lead to modelling errors.

Methods for the analysis of linear multiple-degree-of-freedom (MDOF) systems with damping of the form (1.1) have been considered by many authors, for example, McTavish & Hughes (1993), Woodhouse (1998), Adhikari (2002) and Wagner & Adhikari (2003). Muravyov & Hutton (1997, 1998) have considered this kind of system where the exponential kernel function is associated with the stiffness matrix. Although these publications provide excellent analytical and numerical tools for the analysis of non-viscously damped systems, most of the physical understandings are still from the point of view of a viscously damped oscillator. In this paper, we address the issue of how far one can extend the classical concepts borrowed from viscously damped systems to non-viscously damped systems. Historically, the majority of vibration engineers are trained from a viscous damping perspective and most vibration analysis textbooks, finite element packages and modal analysis software only allow viscous damping models. For these reasons, in this paper, we attempt to relate the newly developed results to a viscously damped oscillator and point out any conceptual differences. The present study is limited to a single-degree-of-freedom (SDOF) oscillator with energy dissipation characteristics given by equation (1.1). This paper is structured as follows. In §2 the equation of motion is introduced and the exact analytical solutions of the eigenvalues are derived. The conditions for sustainable oscillatory motion are discussed in §3. The critical damping factors of a non-viscously damped oscillator are discussed in §4. The nature of the eigenvalues of the oscillator is considered in §5. Finally, our main findings are summarized in §6.

2. The equation of motion

An SDOF system with non-viscous damping is shown in figure 1. The equation of motion of the system with damping characteristics given by equation (1.1) can be

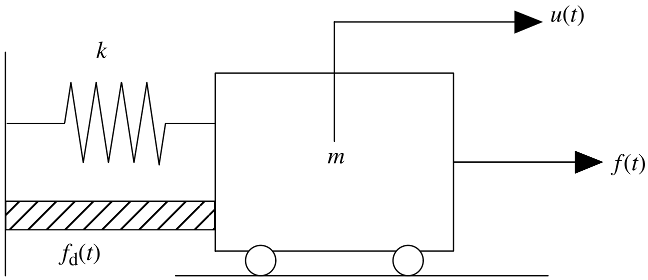

Figure 1. Single degree-of-freedom non-viscously damped oscillator, $f_{\rm d}(t)=\int_0^t c\mu {\rm e}^{-\mu(t-\tau)}\dot{u}(\tau){\rm d}\tau$.

expressed as

$$m\ddot{u}(t)+\int_0^t c\mu {\rm e}^{-\mu(t-\tau)}\dot{u}(\tau){\rm d}\tau+ku(t)=f(t), \tag{2.1}$$

together with the initial conditions

$$u(0)=u_0 \quad \text{and} \quad \dot{u}(0)=\dot{u}_0. \tag{2.2}$$

Here, m is the mass of the oscillator, k is the spring stiffness, $f(t)$ is the applied forcing and $(\dot{\bullet})$ represents derivative with respect to time. Transforming equation (2.1) into the Laplace domain, one obtains

$$s^2 m\bar{u}(s)+sc\left(\frac{\mu}{s+\mu}\right)\bar{u}(s)+k\bar{u}(s)=\bar{f}(s)+m\dot{u}_0+\left(sm+c\frac{\mu}{s+\mu}\right)u_0, \tag{2.3}$$

where s is the complex Laplace domain parameter and $\overline{(\bullet)}$ is the Laplace transform of $(\bullet)$. For convenience, we introduce the constants $\omega_{\rm n}$, ζ and β as follows

$$\omega_{\rm n}=\sqrt{\frac{k}{m}}, \quad \zeta=\frac{c}{2\sqrt{km}} \quad \text{and} \quad \beta=\frac{\omega_{\rm n}}{\mu}. \tag{2.4}$$

Here, $\omega_{\rm n}$ is the undamped natural frequency, ζ is the viscous damping factor and β is the non-viscous damping factor. When $\zeta \to 0$, the oscillator is effectively undamped. When $\beta \to 0$, then $\mu \to \infty$ and the oscillator is effectively viscously damped. We will use these limiting cases frequently to develop our physical understandings of the results to be derived in this paper. Using the constants in equation (2.4), equation (2.3) can be rewritten as

$$\bar{d}(s)\bar{u}(s)=\bar{p}(s), \tag{2.5}$$

where the dynamic stiffness coefficient $\bar{d}(s)$ and the equivalent forcing function $\bar{p}(s)$ are given by

$$\bar{d}(s)=s^2+s2\zeta\omega_{\rm n}\left(\frac{\omega_{\rm n}}{s\beta+\omega_{\rm n}}\right)+\omega_{\rm n}^2 \tag{2.6}$$

and

$$\bar{p}(s) = \frac{\bar{f}(s)}{m} + \dot{u}_0 + \left(s + 2\zeta\omega_{\mathrm{n}} \frac{\omega_{\mathrm{n}}}{s\beta + \omega_{\mathrm{n}}}\right) u_0. \tag{2.7}$$

The aim of a dynamic analysis is often to obtain the dynamic response, either in the time domain or in the frequency domain. For an SODF oscillator, it is a relatively simple task; one can either directly integrate equation (2.1) with the initial conditions (2.2) or, alternatively, one can invert the coefficient associated with $\bar{u}(s)$ in equation (2.5). Such an approach is not suitable for MDOF systems with non-proportional damping and may not provide much physical insight. We pursue an approach involving eigensolutions of the oscillator. The eigenvalues are the zeros of the dynamic stiffness coefficient and can be obtained by setting $\bar{d}(s) = 0$. Therefore, using equation (2.6), the eigenvalues are the solutions of the characteristics equation

$$\beta s^3 + \omega_{\mathrm{n}} s^2 + (\beta\omega_{\mathrm{n}}^2 + 2\zeta\omega_{\mathrm{n}}^2)s + \omega_{\mathrm{n}}^3 = 0. \tag{2.8}$$

In contrast to a viscously damped oscillator where one obtains a quadratic equation, this is a cubic equation.

The three roots of equation (2.8) can appear in two distinct forms: (i) one root is real and the other two roots are in a complex conjugate pair; or (ii) all roots are real. Case (i) represents an *underdamped oscillator*, which usually arises when the 'small damping' assumption is made. The complex conjugate pair of roots corresponds to the 'vibration' of the oscillator while the third root corresponds to a purely dissipative motion. Case (ii) represents an *overdamped oscillator* in which the system cannot sustain any oscillatory motion. For simplicity, we introduce a non-dimensional frequency parameter

$$r = \frac{s}{\omega_{\mathrm{n}}}, \tag{2.9}$$

and transform the characteristics equation (2.8) to

$$\beta r^3 + r^3 + (\beta + 2\zeta)r + 1 = 0, \tag{2.10}$$

or

$$r^3 + \sum_{j=0}^{2} a_j r^j = 0. \tag{2.11}$$

The constants associated with the powers of r are given by

$$a_0 = \frac{1}{\beta}, \quad a_1 = 1 + 2\frac{\zeta}{\beta} \quad \text{and} \quad a_2 = \frac{1}{\beta}. \tag{2.12}$$

The cubic equation (2.11) can be precisely solved in closed form (e.g. Abramowitz & Stegun 1965, section 3.8). Define the following constants:

$$Q = \frac{3a_1 - a_2^2}{9} = \frac{(3\beta^2 + 6\beta\zeta - 1)}{9\beta^2} \tag{2.13}$$

and

$$R = \frac{9a_2 a_1 - 27a_0 - 2a_2^3}{54} = -\frac{(9\beta^2 - 9\beta\zeta + 1)}{27\beta^3}. \tag{2.14}$$

From these, calculate the negative of the discriminant

$$D = Q^3 + R^2 = \frac{1}{27\beta^4}(\beta^4 + 6\beta^3\zeta + 2\beta^2 + 12\beta^2\zeta^2 - 10\beta\zeta + 1 + 8\beta\zeta^3 - \zeta^2), \tag{2.15}$$

and define two new constants

$$S = \sqrt[3]{R + \sqrt{D}} \text{ and } T = \sqrt[3]{R - \sqrt{D}}. \tag{2.16}$$

Using these constants, the three roots of equation (2.11) can be expressed by Cardanos formula as

$$r_1 = -\frac{a_2}{3} - \frac{1}{2}(S+T) + \mathrm{i}\frac{\sqrt{3}}{2}(S-T), \tag{2.17}$$

$$r_2 = -\frac{a_2}{3} - \frac{1}{2}(S+T) - \mathrm{i}\frac{\sqrt{3}}{2}(S-T), \tag{2.18}$$

and

$$r_3 = -\frac{a_2}{3} + (S+T). \tag{2.19}$$

These are the normalized eigenvalues of the system. The actual eigenvalues, that is, the solutions of equation (2.8), can be obtained as $s_j = \omega_n r_j$, $j = 1, 2, 3$. Throughout the paper, we use these normalized eigenvalues to investigate the dynamic properties of the non-viscously damped oscillator. If the non-viscous damping factor β is zero, then equation (2.10) reduces to the quadratic equation

$$r^2 + 2\zeta r + 1 = 0. \tag{2.20}$$

As expected, this is the characteristic equation for a viscously damped oscillator. For this special case, the two solutions of equation (2.20) are given by

$$r_1 = -\zeta + \mathrm{i}\sqrt{1-\zeta^2} \text{ and } r_2 = -\zeta - \mathrm{i}\sqrt{1-\zeta^2}. \tag{2.21}$$

Because the nature of these solutions is very well understood, throughout the paper, we compare them with the new results. In particular, we will point out the ways in which the normalized eigensolutions of the non-viscously damped oscillator given by equations (2.17)–(2.19) differ from the normalized eigensolutions of the corresponding viscously damped oscillator given by equation (2.21).

3. Conditions for oscillatory motion

We consider the following questions of fundamental interest:

(i) Under what conditions can a non-viscously damped oscillator sustain oscillatory motions?

(ii) Is there any 'critical damping factor' for a non-viscously damped oscillator so that, beyond this value, the oscillator becomes overdamped?

For a viscously damped oscillator, the answers to the above questions are well known. From equation (2.21), it is clear that if the viscous damping factor ζ is more than 1, then the oscillator becomes overdamped and, consequently, it will not be able to sustain any oscillatory motions. This simple fact is no longer true for a non-viscously damped oscillator. We will show that there is a precisely defined parameter space where the oscillator becomes overdamped. We will also derive an expression of critical viscous damping factor ζ as a function of non-viscous damping factor β in §4.

Roots r_1 and r_2 in equations (2.17) and (2.18) will be in a complex conjugate pair provided $S-T\neq 0$. The motion corresponding to the complex conjugate roots r_1 and r_2 is oscillatory (and decaying) in nature, whereas the motion corresponding to the real root r_3 is a pure non-oscillatory decay. Considering the expressions of S and T in equation (2.16), it is easily observed that the system will oscillate provided $D>0$. Therefore, the critical condition is given by

$$D = 0. \tag{3.1}$$

From the expression of D in equation (2.15), this condition can be rewritten as

$$27\beta^4 D = 8\beta\zeta^3 + (12\beta^2 - 1)\zeta^2 + (6\beta^3 - 10\beta)\zeta + (1 + 2\beta^2 + \beta^4) = 0. \tag{3.2}$$

In figure 2, $D(\zeta, \beta)=0$ is plotted for $0\leq\zeta\leq 6$ and $0\leq\beta\leq 0.5$. This plot shows the parameter domain where the system can have oscillatory motion. For a viscously damped oscillator $\beta=0$, which is represented by the x-axis of figure 2. Along the x-axis, when $\zeta>1$, the oscillatory motion is not possible; this is well known. However, the scenario changes in an interesting way for non-zero β (i.e. for a non-viscously damped oscillator). For example, if $\beta\approx 0.1$, then the system can have oscillatory motion even when $\zeta>2$, which is more than twice the critical viscous damping factor. Conversely, there are also regions where the system may not have oscillatory motion, even when $\zeta<1$. Perhaps the most interesting observation from figure 2 is that if β is more than about 0.2, then the oscillator will *always* have oscillatory motion, no matter what the value of the viscous damping factor is. We explain these observations by solving a constrained optimization problem.

In view of figure 2, we want to find the maximum value of β for which $D(\zeta,\beta)=0$. To solve this constrained optimization problem, we construct the Lagrangian

$$\mathcal{L}(\zeta,\beta) = \beta + \gamma D(\zeta,\beta), \tag{3.3}$$

where γ is the Lagrange multiplier. Differentiating the Lagrangian with respect to ζ and β and setting them to zero, the optimality conditions can be expressed by

$$\frac{\partial\mathcal{L}(\zeta,\beta)}{\partial\zeta} = \gamma\frac{\partial D(\zeta,\beta)}{\partial\zeta} = 0 \tag{3.4}$$

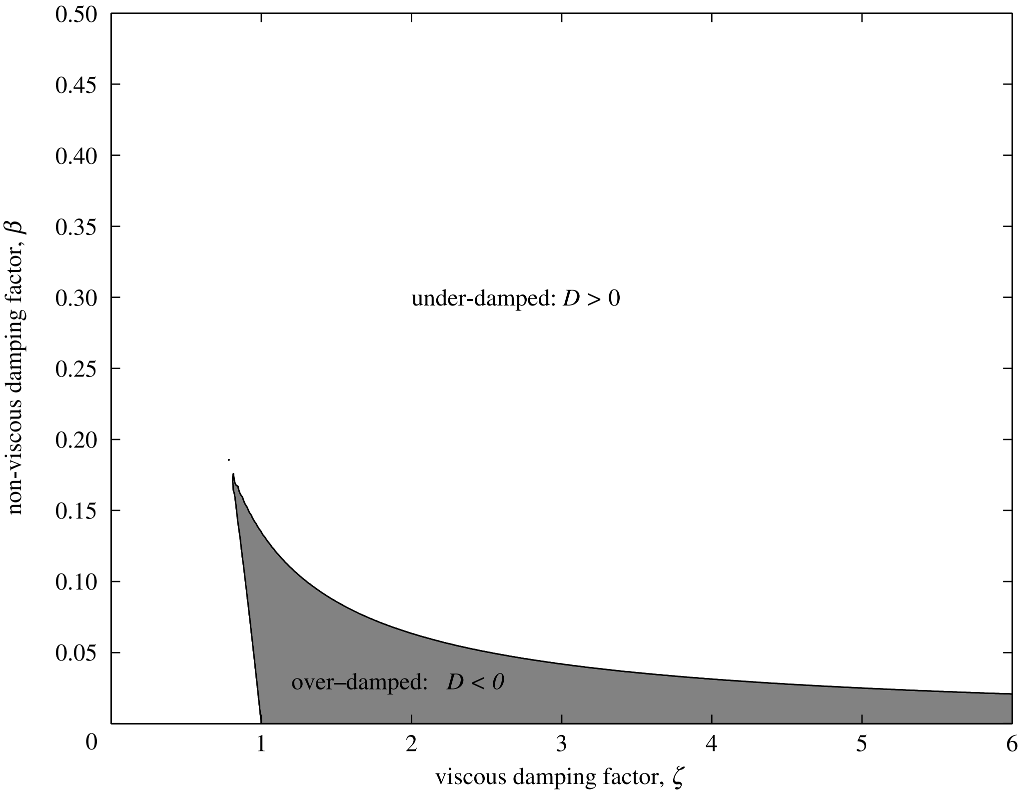

Figure 2. The boundary between oscillatory and non-oscillatory motion.

and

$$\frac{\partial \mathcal{L}(\zeta, \beta)}{\partial \beta} = 1 + \gamma \frac{\partial D(\zeta, \beta)}{\partial \beta} = 0. \tag{3.5}$$

Since the Lagrange multiplier cannot be zero, from equation (3.4), we obtain $\partial D/\partial \zeta = 0$. Differentiating equation (3.2), this condition can be expressed as

$$12\beta\zeta^2 + (12\beta^2 - 1)\zeta + 3\beta^3 - 5\beta = 0. \tag{3.6}$$

Solving this quadratic equation, the positive value of ζ is given by

$$\zeta = \frac{(1 - 12\beta^2) + \sqrt{1 + 216\beta^2}}{24\beta}. \tag{3.7}$$

We need to satisfy the constrain $D(\zeta, \beta) = 0$. Substituting ζ from equation (3.7) in the expression of D in equation (3.2) and then simplifying, we obtain

$$5832\beta^4 + 540\beta^2 - 1 - (1 + 216\beta^2)^{3/2} = 0. \tag{3.8}$$

Letting

$$z = 27\beta^2, \tag{3.9}$$

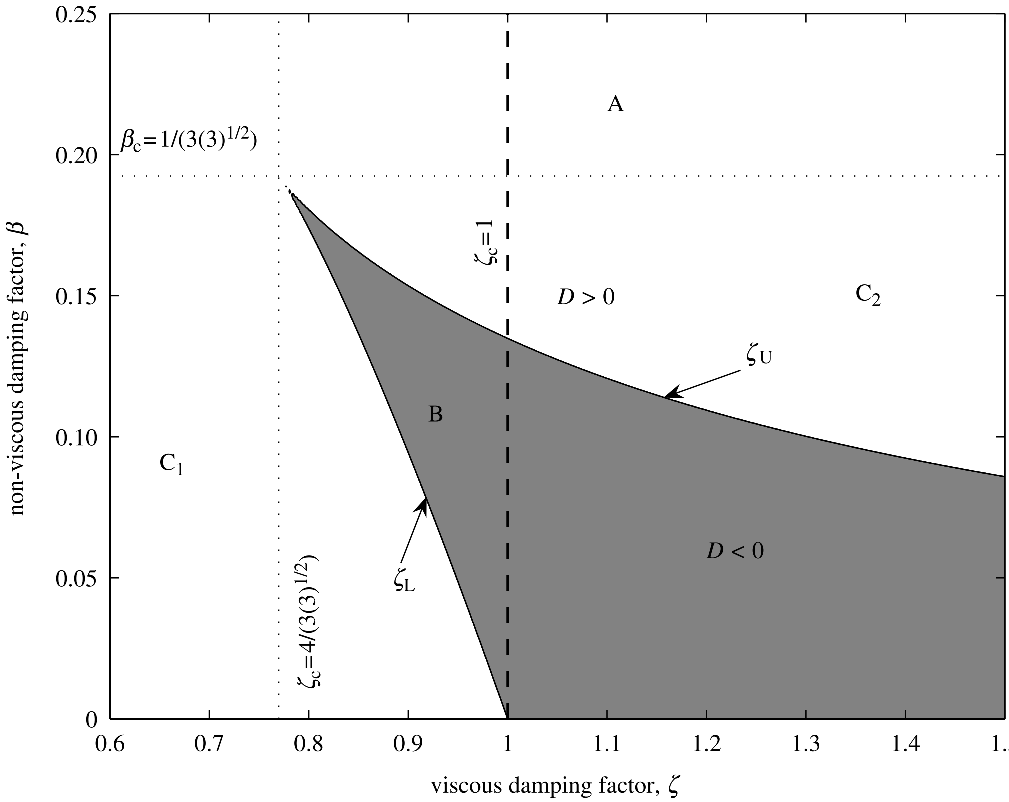

Figure 3. Critical values of ζ and β for oscillatory motion.

equation (3.8) can be expressed as

$$8z^2+20z-1-(1+8z)^{3/2}=0 \quad \text{or } 8z^2+20z-1=(1+8z)^{3/2}. \tag{3.10}$$

This equation has only one positive solution, which is precisely at $z=1$. Therefore, using equation (3.9), the maximum value of β so that $D=0$ is obtained as

$$\beta_c=\frac{1}{3\sqrt{3}}=0.1925. \tag{3.11}$$

Substituting this value of β in the expression of ζ in equation (3.7), the critical value of ζ can be obtained as

$$\zeta_c=\frac{4}{3\sqrt{3}}=0.7698. \tag{3.12}$$

In figure 3, we have (again) plotted the surface $D=0$ concentrating on the derived critical values of ζ and β. The shaded region corresponds to the parameter combinations for which oscillatory motion is not possible. Based on this analysis, it can be said that a non-viscously damped oscillator will *always* have oscillatory motions if $\zeta<\zeta_c$ and/or $\beta>\beta_c$ (parameter regions C_1 and A in figure 3). If $\beta<\beta_c$, then it is possible to have overdamped motion, even if $\zeta<1$ as in the parameter region B shown in figure 3. When $\beta<\beta_c$, there are two distinct parameter regions (shown as C_1 and C_2 in the figure) in which oscillatory motion is possible. Therefore, there are two critical damping factors for a non-viscously damped oscillator. We investigate this issue in §4. From this analysis, we have the following fundamental result:

Theorem 3.1. *An exponential non-viscously damped oscillator will have oscillatory motions if* $\zeta < 4/(3\sqrt{3})$ *or* $\beta > 1/(3\sqrt{3})$.

It is often useful to express the critical value of β in equation (3.11) in terms of the relaxation time and the natural time period of the corresponding undamped oscillator. Using the definition of β in equation (2.4), it follows that a non-viscously damped oscillator will always have oscillatory motion if

$$\frac{2\pi/T_\mathrm{n}}{\mu} > \frac{1}{3\sqrt{3}} \quad \text{or} \quad \frac{1}{\mu} > \frac{1}{3\sqrt{3}}\frac{T_\mathrm{n}}{2\pi}, \tag{3.13}$$

that is, if the relaxation time is more than $1/(3\sqrt{3})$ times the natural time period of the corresponding undamped oscillator. An equivalent result for a viscoelastic system has been obtained in an insightful paper by Muravyov & Hutton (1998). In their study, the hereditary term was associated with the stiffness coefficient as opposed to the damping coefficient. Using the properties of the roots of the characteristic equation, they showed that oscillatory motion is always possible when the relaxation time $1/\alpha$ is greater than $(1/\sqrt{3})(T_\mathrm{n}/2\pi)$. This implies that Muravyov and Hutton's limiting value of the relaxation time is three times more than what has been derived in this paper.

4. Critical damping factors

In figure 3, it was noted that when $\beta < \beta_\mathrm{c}$, there are two distinct parameter regions for which oscillatory motion is possible (regions C_1 and C_2 in the figure). Therefore, for each value of β, two critical damping factors can be defined. Using the notations ζ_L and ζ_U, the oscillator will have overdamped motion when $\zeta_\mathrm{L} < \zeta < \zeta_\mathrm{U}$. We call ζ_L 'the lower critical damping factor' and ζ_U 'the upper critical damping factor'.

To obtain the critical damping factors, $D=0$ must be solved for ζ. From equation (3.2), note that $D=0$ is a cubic equation and, again, we use Cardanos formula to obtain the explicit solutions. Rewrite equation (3.2) as

$$\zeta^3 + \sum_{j=0}^{2} b_j \zeta^j = 0, \tag{4.1}$$

where the coefficients associated with the powers of ζ are given by

$$b_0 = \frac{1 + 2\beta^2 + \beta^4}{8\beta}, \quad b_1 = \frac{3\beta^2 - 5}{4} \quad \text{and} \quad b_2 = \frac{12\beta^2 - 1}{8\beta}. \tag{4.2}$$

Obtain the constants

$$Q_\zeta = \frac{3b_1 - b_2^2}{9} = -\frac{216\beta^2 + 1}{576\beta^2} \tag{4.3}$$

and

$$R_\zeta = \frac{9b_2 b_1 - 27b_0 - 2b_2^3}{54} = -\frac{5832\beta^4 + 540\beta^2 - 1}{13\,824\beta^3}, \tag{4.4}$$

and calculate

$$D_\zeta = Q_\zeta^3 + R_\zeta^2 = \frac{-2187\beta^4 + 81\beta^2 - 1 + 19\,683\beta^6}{110\,592\beta^4}. \tag{4.5}$$

It can be easily verified that when $\beta=\beta_c$, $D_\zeta=0$ and when $\beta<\beta_c$, $D_\zeta<0$. This implies that all three solutions of equation (3.2) will be real when $\beta<\beta_c$. Defining

$$\theta = \text{arc}\cos(R_\zeta/\sqrt{-Q_\zeta^3}) = \text{arc}\cos\left(-\frac{5832\beta^4 + 540\beta^2 - 1}{(216\beta^2 + 1)^{3/2}}\right), \tag{4.6}$$

the three solutions of equation (4.1) can be given by

$$\zeta_1 = 2\sqrt{-Q_\zeta}\cos\left(\frac{\theta}{3}\right) - b_2/3, \tag{4.7}$$

$$\zeta_2 = 2\sqrt{-Q_\zeta}\cos\left(\frac{2\pi + \theta}{3}\right) - b_2/3 \tag{4.8}$$

and

$$\zeta_3 = 2\sqrt{-Q_\zeta}\cos\left(\frac{4\pi + \theta}{3}\right) - b_2/3. \tag{4.9}$$

It turns out that $\zeta_1>\zeta_3$ and ζ_2 is always negative when $\beta<\beta_c$. Therefore, we discard ζ_2 and assign $\zeta_L=\zeta_3$ and $\zeta_U=\zeta_1$. After some simplification, the lower and the upper critical damping factors for a non-viscously damped oscillator can be expressed as

$$\zeta_L = \frac{1}{24\beta}(1 - 12\beta^2 + 2\sqrt{1 + 216\beta^2} + \cos((4\pi + \theta)/3)) \tag{4.10}$$

and

$$\zeta_U = \frac{1}{24\beta}(1 - 12\beta^2 + 2\sqrt{1 + 216\beta^2} + \cos(\theta/3)). \tag{4.11}$$

The above two equations are plotted in figure 3. When $\beta\to\beta_c$, the critical damping factors approach each other and eventually, when $\beta=\beta_c$, both critical damping factors become the same and equal to ζ_c. When $\beta<\beta_c$, the absolute value of the argument of across in equation (4.6) will always be less than 1 so that the computation of θ is always possible. The existence of two critical damping factors is a significantly different property from a viscously damped oscillator. In the limiting case when $\beta\to 0$, it can be verified that $\zeta_L\to 1$ and $\zeta_U\to\infty$. Indeed, this implies that a viscously damped oscillator has only one critical damping factor: $\zeta=1$. The results obtained in this section can be summarized in the following theorem:

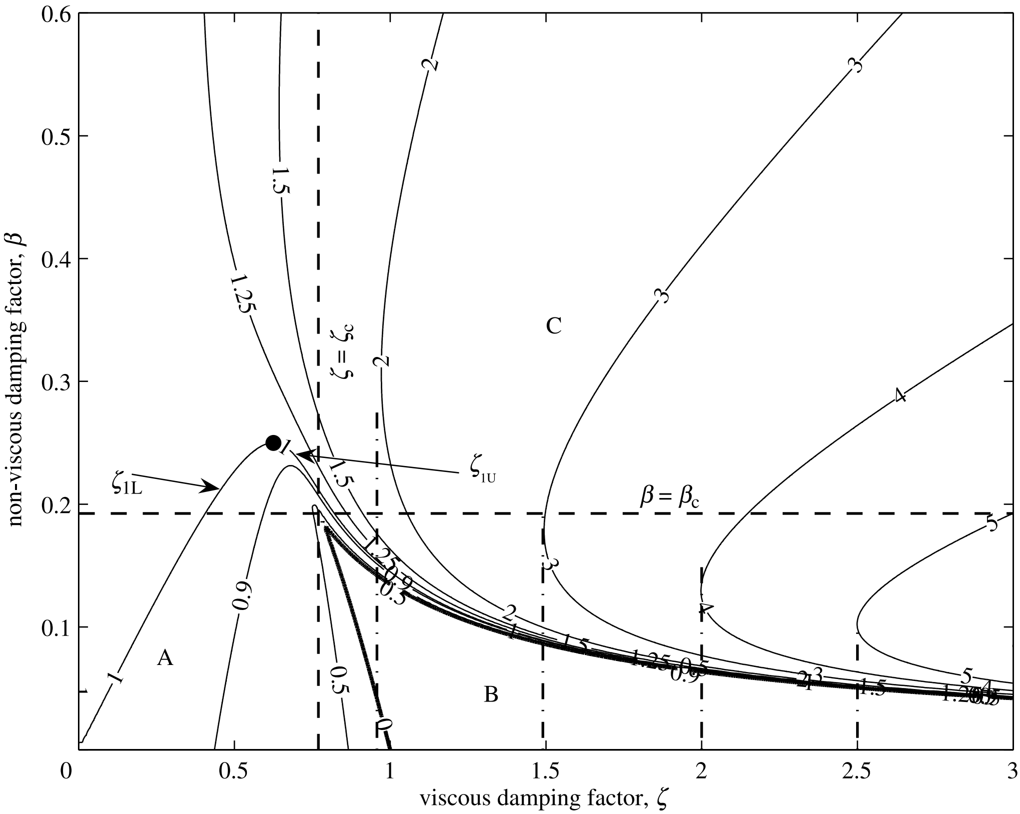

Figure 4. Contours of the normalized natural frequency ($\Im(r_1)$).

Theorem 4.1. *When* $\beta < 1/(3\sqrt{3})$, *an exponential non-viscously damped oscillator will have oscillatory motions if and only if* $\zeta \notin [\zeta_L, \zeta_U]$.

5. Characteristics of the eigenvalues

Dynamic properties of linear systems are characterized by their eigensolutions. The normalized eigenvalues of the non-viscously damped oscillator are expressed in terms of the non-dimensional parameters ζ and β in equations (2.17)–(2.19). We are interested in the natural frequency and the decay rate corresponding to the oscillating eigenvalues r_1 and r_2 given by equations (2.17) and (2.18). The decay rate of the non-oscillating eigenvalue r_3, given by equation (2.19), is also a quantity of interest in this study.

(*a*) *Characteristics of the natural frequency*

The normalized natural frequency is given by the imaginary part of r_1 (or r_2) in equation (2.17). The contours of $\Im(r_1(\zeta, \beta))$, when $0 \le \zeta \le 3$ and $0 \le \beta \le 0.6$, are shown in figure 4. These parameters are selected based on numerical studies and it is within these parameters that the majority of interesting features arise. There are three distinct regions (marked by the letters A, B and C) in figure 4, which are consistent with our findings in the previous sections. In region A where

$\zeta < \zeta_c$, the system has always non-zero natural frequency. For increasing ζ while $\beta < \beta_c$, the natural frequency decreases and eventually, in region B, the natural frequency becomes zero (i.e. the system cannot oscillate). In region C where $\beta > \beta_c$, the normalized natural frequency is always greater than 1. For a fixed value of ζ, the natural frequency decreases for increasing β. An interesting fact that emerges from figure 4 is that for a fixed value of β in region C, the natural frequency increases for increasing ζ. This is exactly the opposite behaviour to a viscously damped oscillator, where the damped natural frequency decreases for increasing ζ. A physical explanation of these facts can be given.

For convenience, express the damping force in equation (1.1) as

$$f_{\mathrm{d}}(t) = \int_0^t g(t-\tau)\dot{u}(\tau)\mathrm{d}\tau, \quad \text{where} \quad g(t) = c\mu \mathrm{e}^{-\mu t}. \tag{5.1}$$

Integrating by parts, one obtains

$$\begin{aligned} f_{\mathrm{d}}(t) &= |u(\tau)g(t-\tau)|_0^t + \int_0^t \dot{g}(t-\tau)u(\tau)\mathrm{d}\tau \\ &= u(t)g(0) - u(0)g(t) + \int_0^t \dot{g}(t-\tau)u(\tau)\mathrm{d}\tau. \end{aligned}$$

Assuming zero initial conditions and utilizing the expression of $g(t)$, this equation can be expressed as

$$f_{\mathrm{d}}(t) = c\mu u(t) - \mu \int_0^t c\mu \mathrm{e}^{-\mu(t-\tau)} u(\tau)\mathrm{d}\tau = \frac{1}{m}\left(\frac{\zeta}{\beta}\right) u(t) - \mu \int_0^t c\mu \mathrm{e}^{-\mu(t-\tau)} u(\tau)\mathrm{d}\tau.$$

It is evident that the term $(1/m)(\zeta/\beta)$ would associate with the stiffness term in the equation of motion (2.1). This implies that for higher values of ζ, the effective stiffness (hence, the natural frequency) of the oscillator would increase and it would decrease for higher values of β. This analysis explains the observations in figure 4.

The contour line 1 in figure 4 is particularly interesting. For these parameter combinations, the damped natural frequency will be the same as the natural frequency of the equivalent undamped oscillator. Viscously damped systems do not show this behaviour, which is unique to non-viscously damped systems. In order to obtain these parameter combinations analytically, we substitute

$$r = -\sigma + \mathrm{i}, \tag{5.2}$$

in the characteristic equation (2.11). Here, σ is an arbitrary positive real number. This substitution ensures that the normalized natural frequency is equal to 1. Carrying out the substitution in equation (5.2) and separating the real and imaginary parts of the resulting equation, one obtains

$$\beta\sigma^2 - 2\beta - \sigma + 2\zeta = 0 \tag{5.3}$$

and

$$3\beta\sigma^2 - 2\sigma + 2\zeta = 0, \tag{5.4}$$

when $\zeta \neq 0$ and $\beta \neq 0$. Eliminating σ from these equations, we have

$$8\beta\zeta^2 - (1+24\beta^2)\zeta + 2\beta(1+9\beta^2) = 0. \tag{5.5}$$

The system parameters must satisfy this equation for the damped natural frequency to be the same as the undamped natural frequency. The two solutions for ζ can be obtained from equation (5.5) as

$$\zeta_{1L} = (1 + 24\beta^2 - \sqrt{1 - 16\beta^2})/16\beta \tag{5.6}$$

and

$$\zeta_{1U} = (1 + 24\beta^2 + \sqrt{1 - 16\beta^2})/16\beta. \tag{5.7}$$

These equations are shown in figure 4. Note that, in general, $\zeta_{1L} < \zeta_{1U}$. A real value of ζ can only be obtained from equations (5.6) and (5.7) if

$$1 - 16\beta^2 \geq 0 \quad \text{or} \quad \beta \leq \frac{1}{4}. \tag{5.8}$$

This implies that for a non-viscously damped oscillator, the damped natural frequency cannot be the same as the undamped natural frequency if $\beta > (1/4)$. When $\beta < (1/4)$, such a situation may arise for two values of ζ given by equations (5.6) and (5.7). When $\beta = (1/4)$, from equations (5.6) and (5.7), one obtains $\zeta_{1L} = \zeta_{1U} = 5/8$. This point is shown by a dot in figure 4. For a given value of β, if $\zeta < \zeta_{1L}$ or $\zeta > \zeta_{1U}$, then the damped natural frequency will be more than the undamped natural frequency and vice versa. We therefore obtain the following results.

Theorem 5.1. *If $\beta > 1/4$, then the natural frequency of an exponential non-viscously damped oscillator will be more than that of an equivalent undamped oscillator.*

Theorem 5.2. *When $\beta < 1/4$, the natural frequency of an exponential non-viscously damped oscillator will be more than that of an equivalent undamped oscillator if and only if $\zeta \notin [\zeta_{1L}, \zeta_{1U}]$.*

In order to further understand the normalized natural frequency, we substitute

$$r = -\sigma + \mathrm{i}\lambda \tag{5.9}$$

in the characteristic equation (2.11). Here, λ is an arbitrary positive real number. Carrying out the substitution in equation (5.9) and separating the real and imaginary parts of the resulting equation, one obtains

$$(3\beta\sigma - 1)\lambda^2 - \beta\sigma^3 - 2\zeta\sigma + \sigma^2 - \beta\sigma + 1 = 0 \tag{5.10}$$

and

$$3\beta\sigma^2 - \beta\lambda^2 - 2\sigma + \beta + 2\zeta = 0. \tag{5.11}$$

Eliminating σ from these equations, we obtain

$$h_\lambda(\zeta, \beta) = 0, \tag{5.12}$$

where

$$h_\lambda = 8\beta\zeta^3 + (12\beta^2 - 36\lambda^2\beta^2 - 1)\zeta^2 + (12\beta\lambda^2 + 6\beta^3 - 36\beta^3\lambda^2 - 10\beta + 48\beta^3\lambda^4)\zeta + (1 - \lambda^2 - 8\beta^2\lambda^4 + 2\beta^2 - 9\beta^4\lambda^2 + \beta^4 - 16\beta^4\lambda^6 + 6\beta^2\lambda^2 + 24\beta^4\lambda^4). \tag{5.13}$$

For the normalized natural frequency to be λ, the constants ζ and β must satisfy equation (5.12). All contour lines in figure 4 are plots of equation (5.12) for different values of λ. When $\lambda > 2$, note that there is a minimum value of ζ for the natural frequency to be λ. To obtain the minimum values of ζ for every λ, we need to solve the following constrained optimization problem

$$\min \zeta \text{ subject to } h_\lambda(\zeta, \beta) = 0. \tag{5.14}$$

Constructing the Lagrangian

$$\mathcal{L}_\lambda(\zeta, \beta) = \zeta + \gamma_\lambda h_\lambda(\zeta, \beta), \tag{5.15}$$

and setting $(\partial\mathcal{L}_\lambda(\zeta,\beta))/\partial\zeta = (\partial\mathcal{L}_\lambda(\zeta,\beta))/\partial\beta = 0$, it is possible to solve this problem in closed form. Owing to the complexity of $h_\lambda(\zeta, \beta)$ in equation (5.13), the resulting expressions are complicated. It can be shown that the optimization problem (5.14) has real solutions only if $\lambda > 1$. This can also be observed from figure 4. We will pursue a simplified approach based on the second derivative of the Lagrangian. The optimal conditions will be satisfied if $(\partial^2\mathcal{L}_\lambda(\zeta,\beta))/\partial\zeta^2 > 0$ and $(\partial^2\mathcal{L}_\lambda(\zeta,\beta))/\partial\beta^2 > 0$. Because the Lagrange multiplier $\gamma_\lambda \neq 0$, the critical condition can be expressed by $(\partial^2 h_\lambda(\zeta,\beta))/\partial\zeta^2 = 0$, that is

$$12(3\lambda^2 - 1)\beta^2 + 24\zeta\beta - 1 = 0. \tag{5.16}$$

Solving for β, we have

$$\beta = \frac{\zeta \pm \sqrt{36\zeta^2 - 9\lambda^2 + 3}}{3\lambda^2 - 1}. \tag{5.17}$$

To obtain a real solution from this equation, we need $36\zeta^2 - 9\lambda^2 + 3 > 0$, that is

$$\zeta > \sqrt{(\lambda/2)^2 - 1/12}. \tag{5.18}$$

In the above formulation, we have not solved the true optimization problem. Consequently, equation (5.18) does not produce accurate results for all $\lambda > 1$. Numerical calculations show that equation (5.18) works well when $\lambda \geq 2$. For large λ, the term $\sqrt{(\lambda/2)^2 - 1/12}$ can be approximated as $(\lambda/2)$ without significant loss of accuracy. Thus, for all practical purposes, we have obtained the following general result.

Theorem 5.3. *The normalized natural frequency of an exponential non-viscously damped oscillator will be more than* λ *if and only if* $\zeta > \sqrt{(\lambda/2)^2 - 1/12} \approx \lambda/2$ *for all* $\lambda \geq 2$.

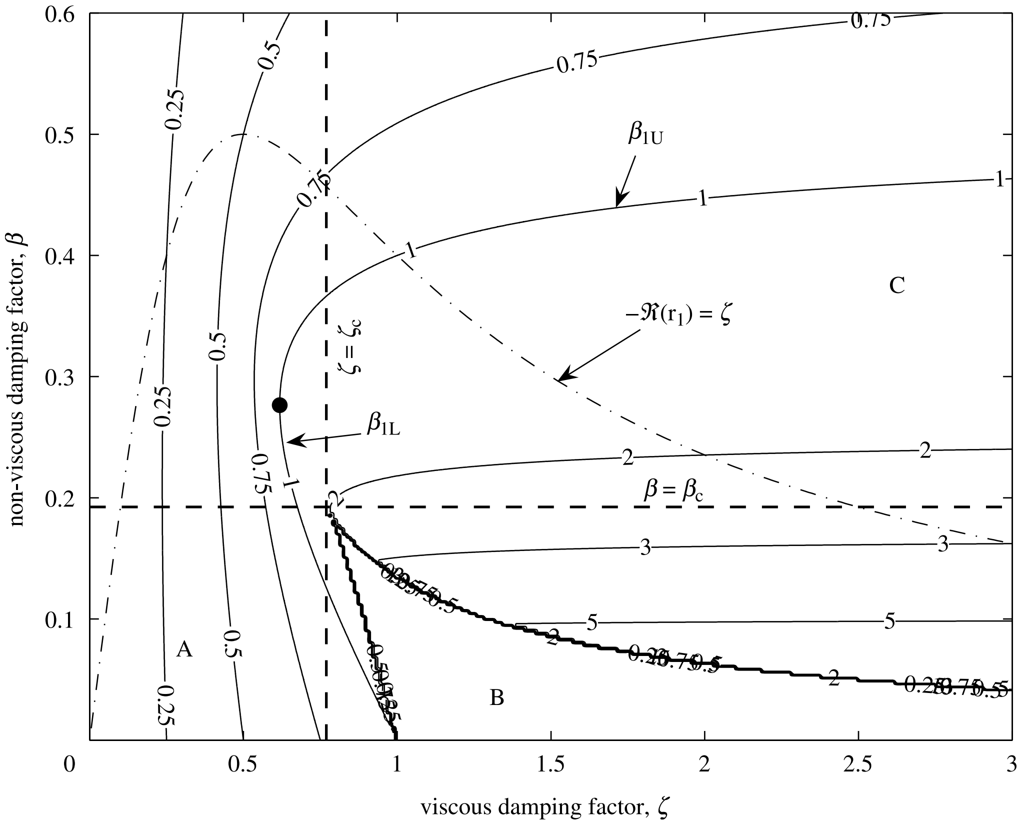

Figure 5. Contours of the normalized decay rate ($-\Re(r_1)$) corresponding to the oscillating mode.

The minimum values of ζ required for different values of λ using this theorem are shown by a dash-dot line in figure 4. From the numerical results, the applicability of theorem 5.3 can be confirmed.

(*b*) *Characteristics of the decay rate corresponding to the oscillating mode*

The normalized decay rate corresponding to the oscillating mode is given by the real part of r_1 or r_2 in equations (2.17) and (2.18). Figure 5 shows the contours of $-\Re(r_1(\zeta,\beta))$ when $0\leq\zeta\leq 3$ and $0\leq\beta\leq 0.6$. To understand the characteristics of the decay rate, it is helpful to divide figure 5 into three regions as before. In region A, where $\zeta<\zeta_c$, the decay rate is less than 1 for the values of β considered. In region B, where $\zeta<\zeta_c$ and $\beta<\beta_c$, the system does not oscillate. In region C, where $\beta>\beta_c$ and $\zeta>1$, the decay rate decreases for increasing β. Interestingly, if β is close to 0.5 or more, then the decay rate can be less than 1, even when $\zeta\geq 3$. This is in sharp contrast with a viscously damped oscillator where the normalized decay rate is always ζ. For a better understanding of these observations, we again substitute $r=-\sigma+\mathrm{i}\lambda$ in the characteristic equation (2.11). Eliminating λ from equations (5.10) and (5.11), we have

$$(4\sigma^3+\sigma)\beta^2+(2\zeta\sigma-4\sigma^2)\beta+\sigma-\zeta=0. \tag{5.19}$$

For the normalized decay rate to be σ, the constants ζ and β must satisfy equation (5.19). For a given value of ζ, equation (5.19) can be solved for β as

$$\beta_{\sigma\mathrm{L}} = \frac{2\sigma - \zeta - \sqrt{\zeta^2 - 1 + \zeta/\sigma}}{1 + 4\sigma^2} \tag{5.20}$$

and

$$\beta_{\sigma\mathrm{U}} = \frac{2\sigma - \zeta + \sqrt{\zeta^2 - 1 + \zeta/\sigma}}{1 + 4\sigma^2}. \tag{5.21}$$

It follows that for a known value of ζ, a particular decay rate corresponding to the oscillating mode can occur for two values of β given by equations (5.20) and (5.21). This fact can be verified from figure 5 by considering any contour level. Considering $\sigma = 1$, the plots of $\beta_{1\mathrm{L}}$ and $\beta_{1\mathrm{U}}$ are shown in figure 5.

When β is known, by solving equation (5.19) for ζ, we have

$$\zeta_\sigma = \frac{\sigma(4\beta^2\sigma^2 + \beta^2 - 4\sigma\beta + 1)}{1 - 2\sigma\beta}, \quad \beta \neq 1/2\sigma. \tag{5.22}$$

This implies that, for a known value of β, a particular decay rate corresponding to the oscillating mode can occur only for a unique value of ζ. This can also be verified from figure 5. From the contour levels in figure 5, it can be observed that there is a minimum value of ζ for a particular decay rate. To obtain the minimum value of ζ, we consider the expressions of $\beta_{\sigma\mathrm{L}}$ and $\beta_{\sigma\mathrm{U}}$. Real values of $\beta_{\sigma\mathrm{L}}$ and $\beta_{\sigma\mathrm{U}}$ can be obtained from equations (5.20) and (5.21) if and only if

$$\zeta^2 - 1 + \zeta/\sigma \geq 0. \tag{5.23}$$

Considering the positive solution, the condition in equation (5.23) can be expressed as

$$\zeta \geq \left(\sqrt{1 + 4\sigma^2} - 1\right)/2\sigma. \tag{5.24}$$

It follows that, no matter what the value of β is, the decay rate will always be less than σ if ζ is less than $(\sqrt{1 + 4\sigma^2} - 1)/2\sigma$. To illustrate this point, consider the $\sigma = 1$ curve in figure 5. From equation (5.24), the minimum value of ζ for which σ can be 1 is obtained as $\zeta = (\sqrt{5} - 1)/2$. This point is shown by a dot in figure 5.

Another interesting observation from figure 5 is that when $\zeta > 1$ the decay rate in general decreases for increasing values of β, and there is a critical value of β beyond which a particular decay rate cannot occur. Considering asymptotic expansion of the upper value $\beta_{\sigma\mathrm{U}}$ given in equation (5.21), one obtains

$$\beta_{\sigma\mathrm{U}} = \frac{2\sigma - \zeta + \sqrt{\zeta^2 - 1 + \zeta/\sigma}}{1 + 4\sigma^2} \approx \frac{1}{2\sigma} - \frac{1}{8\sigma^2}\frac{1}{\zeta} + \frac{1}{16\sigma^3}\frac{1}{\zeta^2} - \frac{5 + 4\sigma^2}{128\sigma^4}\frac{1}{\zeta^3} + \cdots. \tag{5.25}$$

Therefore, when $\zeta \to \infty$, we have

$$(\beta_\sigma)_{\max} = \lim_{\zeta \to \infty} \frac{2\sigma - \zeta + \sqrt{\zeta^2 - 1 + \zeta/\sigma}}{1 + 4\sigma^2} = \frac{1}{2\sigma}. \tag{5.26}$$

This analysis indicates that, no matter what the value of ζ is, the decay rate will always be less than σ if β is more than $1/2\sigma$. To illustrate this, again consider the $\sigma=1$ curve in figure 5 and observe that the decay rate is always less than 1 if $\beta>1/2$. From these observations, we have the following universal results.

Theorem 5.4. *The normalized decay rate corresponding to the oscillating mode of an exponential non-viscously damped oscillator will be less than σ if $\beta>1/2\sigma$ or $\zeta<(\sqrt{1+4\sigma^2}-1)/2\sigma$ for any $\sigma\in\mathbb{R}^+$.*

Theorem 5.5. *When $\zeta>(\sqrt{1+4\sigma^2}-1)/2\sigma$, the normalized decay rate corresponding to the oscillating mode of an exponential non-viscously damped oscillator will be less than σ if and only if $\beta\notin[\beta_{\sigma\mathrm{L}}, \beta_{\sigma\mathrm{U}}]$ for any $\sigma\in\mathbb{R}^+$.*

It is useful to compare the decay rate of a non-viscously damped oscillator with that of an equivalent viscously damped oscillator. From equation (2.21), recall that the decay rate of a viscously damped oscillator is ζ. Therefore, substituting $\sigma=\zeta$ in equation (5.19) and considering $\zeta\neq0$ and $\beta\neq0$, we have

$$4\zeta^2\beta+\beta-2\zeta=0. \tag{5.27}$$

Solving this equation for β, we have

$$\beta_\zeta=\frac{2\zeta}{1+4\zeta^2}. \tag{5.28}$$

If this condition is satisfied, then the decay rate of a non-viscously damped oscillator will be the same as that of an equivalent viscously damped oscillator. The parameter combinations that satisfy equation (5.28) are plotted by a dash-dotted line in figure 5. If $\beta>2\zeta/(1+4\zeta^2)$, then the decay rate will be less than ζ and vice versa. Observe that there is a maximum value of β beyond which the decay rate will always be less than ζ. To obtain this critical value of β, we solve equation (5.27) for ζ and obtain

$$\zeta_{\zeta\mathrm{L}}=\left(1-\sqrt{1-4\beta^2}\right)/4\beta, \tag{5.29}$$

and

$$\zeta_{\zeta\mathrm{U}}=\left(1+\sqrt{1-4\beta^2}\right)/4\beta. \tag{5.30}$$

A real solution can be obtained only if $(1-4\beta^2)\geq0$, that is, when

$$\beta\leq\frac{1}{2}. \tag{5.31}$$

This result implies that if $\beta>1/2$ then the decay rate will always be less than ζ. From equations (5.29) and (5.30), it is also clear that for a given value of $\beta<1/2$ the normalized decay rate will be more than ζ if $\zeta_{\zeta\mathrm{L}}<\zeta<\zeta_{\zeta\mathrm{U}}$ and vice versa. Based on these findings, we have the following theorems.

Theorem 5.6. *If $\beta>1/2$, then the decay rate corresponding to the oscillating mode of an exponential non-viscously damped oscillator will be less than that of an equivalent viscously damped oscillator.*

Theorem 5.7. *When $\beta<1/2$, then the decay rate corresponding to the oscillating mode of an exponential non-viscously damped oscillator will be less than that of an equivalent viscously damped oscillator if and only if $\zeta\notin[\zeta_{\zeta\mathrm{L}}, \zeta_{\zeta\mathrm{U}}]$* or *$\beta>2\zeta/(1+4\zeta^2)$.*

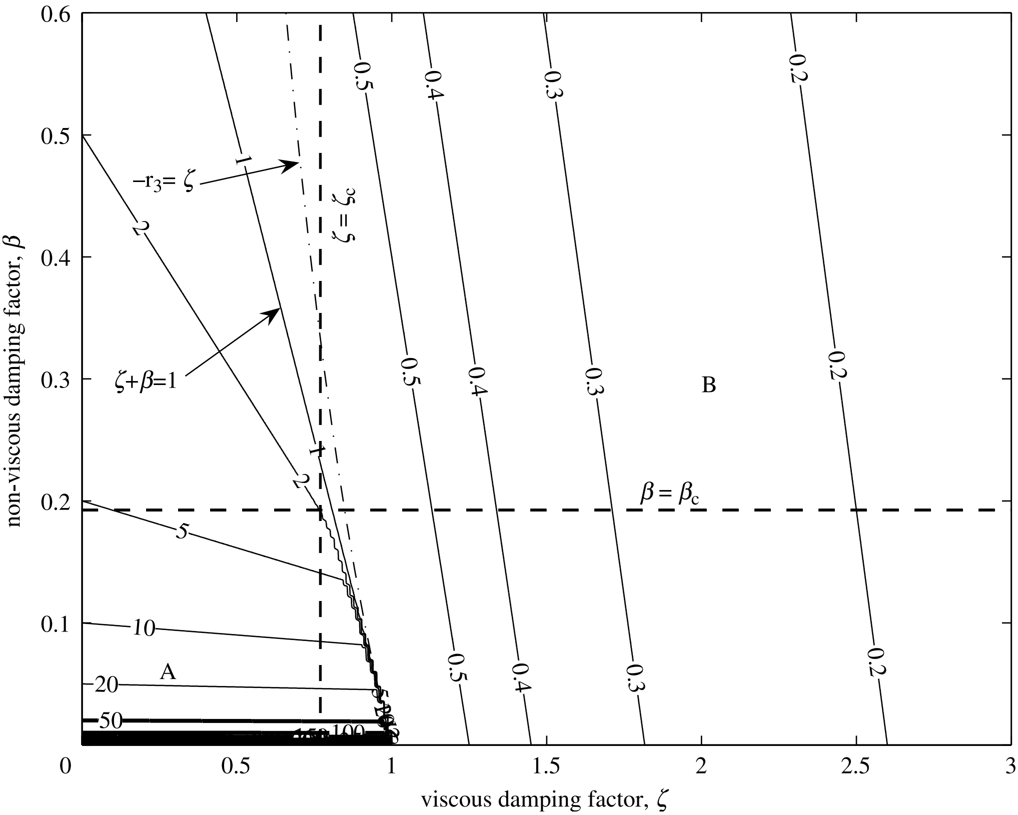

Figure 6. Contours of the normalized decay rate corresponding to the overdamped eigenvalue $-r^3$ as a function of ζ and β.

(*c*) *Characteristics of the decay rate corresponding to the non-oscillating mode*

The overdamped or non-oscillating eigenvalue r_3 is given by equation (2.19). The contours of $-r_3(\zeta, \beta)$ when $0\leq\zeta\leq 3$ and $0 \leq\beta\leq 0.6$ are shown in figure 6. There are two distinct regions, denoted by A and B in the figure. In region A, where $\zeta<1$ and β is small, the decay rate is very high. This is expected because this eigenvalue would not have existed for a viscously damped oscillator (i.e. when $\beta=0$). Interestingly, in region B, where $\zeta>1$, the decay rate is not high, even for small values of β. Moreover, for increasing values of ζ, the decay rate decreases, which appears to be counter-intuitive. Also note the very sharp change in the values of r_3 around $\zeta=1$ for small values of β ($\beta<0.05$). The reasons for this behaviour are not entirely clear to us, and further research is required for proper explanations.

To compare the decay rate of the overdamped mode with the decay rate of a viscously damped oscillator, we need to know the conditions under which the decay rate will be ζ. Substituting $r=-\zeta$ in the characteristic equation (2.11) and solving for β, we have

$$\beta = \frac{1-\zeta^2}{\zeta(1+\zeta^2)}. \tag{5.32}$$

This equation is plotted in figure 6 by a dash-dotted line. Clearly, to obtain a positive value of β from equation (5.32), we need $\zeta < 1$. From this analysis, we have the following result.

Theorem 5.8. *The decay rate corresponding to the non-oscillating mode of an exponential non-viscously damped oscillator will be less than that of an equivalent viscously damped oscillator if* $\beta > (1-\zeta^2)/(\zeta(1+\zeta^2))$.

For a general value of the decay rate, we substitute $r = -\sigma$ in the characteristic equation (2.11). After some simplification, the required condition can be expressed as

$$\frac{\beta}{1/\sigma^2} + \frac{\zeta}{1/2(\sigma + 1/\sigma)} = 1. \tag{5.33}$$

This is an equation of a straight line. To illustrate this result, consider $\sigma = 1$. According to equation (5.33), this can occur if $\zeta + \beta = 1$. This line is shown in figure 6. From the figure, it is evident that if $\zeta + \beta > 1$ the decay rate will be less than 1. From this discussion, we have the following generalization.

Theorem 5.9. *The normalized decay rate corresponding to the non-oscillating mode of an exponential non-viscously damped oscillator will be less than* σ *if* $\beta(1/\sigma^2) + \zeta(1/2(\sigma + 1/\sigma)) > 1$ *for any* $\sigma \in \mathbb{R}^+$.

Another useful result that follows from this theorem is as follows.

Corollary 5.10. *If* $\zeta + \beta > 1$*, then the normalized decay rate corresponding to the non-oscillating mode of an exponential non-viscously damped oscillator will be less than* 1.

6. Summary and conclusions

Dynamic characteristics of a non-viscously damped linear SDOF oscillator have been discussed. The non-viscous damping force was expressed by an exponentially fading memory kernel. It was shown that the dynamic properties of the oscillator are governed by two non-dimensional factors, namely, the viscous damping factor ζ and the non-viscous damping factor β. The system considered reduces to the classical viscously damped oscillator when the non-viscous damping factor is zero. Several new properties that characterize the dynamics of a non-viscously damped oscillator have been discovered. A non-viscously damped oscillator has three eigenvalues, one of which is always non-oscillating in nature. The conditions for the occurrence of a pair of oscillating eigenvalues were derived. Characteristics of the natural frequency and decay rates were discussed in detail. The findings in the paper were described as 12 theorems. Here, we summarize some important outcomes as follows.

(i) An exponential non-viscously damped oscillator will have oscillatory motion if $\zeta < 4/(3\sqrt{3})$ or $\beta > 1/(3\sqrt{3})$.

(ii) If $\beta < 1/(3\sqrt{3})$, then the oscillator will have oscillatory motion if and only if $\zeta \notin [\zeta_L, \zeta_U]$. ζ_L and ζ_U, given in equations (4.10) and (4.11), are the lower and upper critical damping factors, respectively.

(iii) If $\beta > 1/4$, then the natural frequency of an exponential non-viscously damped oscillator will be more than that of an equivalent undamped oscillator.
(iv) If $\beta > 1/2$, then the decay rate corresponding to the oscillating mode will be less than that of an equivalent viscously damped oscillator.
(v) If $\zeta + \beta > 1$, then the normalized decay rate corresponding to the non-oscillating mode will be less than 1.

There are some results in the paper (theorems 5.3, 5.4, 5.5, 5.9 and corollary 5.10) that discuss the conditions of occurrence (or non-occurrence) of particular values of natural frequency and decay rates. Using these results, one can understand the nature of the eigenvalues without actually solving the eigenvalue problem. These concepts will be particularly useful in dealing with MDOF systems.

The studies reported in this paper show that the classical concepts based on a viscously damped oscillator can be extended to a non-viscously damped system only under certain conditions. In general, if $\beta > 1/(3\sqrt{3})$ (i.e. when the relaxation time is more than $1/(3\sqrt{3})$ times the natural time period of the corresponding undamped oscillator), then the associated dynamics will be significantly different from those of a classical viscously damped oscillator. The results derived in this paper are expected to be valid for a proportionally damped MDOF system with a single exponential kernel. However, further research is needed to extend these results to systems with multiple exponential kernels and non-proportional damping.

The author acknowledges the support of the EPSRC through the award of an advanced research fellowship, grant number GR/T03369/01.

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