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Optimal parameters of viscoelastic tuned-mass dampers

A. Batou [a,*], S. Adhikari [b]

[a] School of Engineering, The University of Liverpool, Liverpool, UK
[b] College of Engineering, Swansea University, Bay Campus, Swansea, SA1 8EN, UK

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ABSTRACT

A vibration absorber, also known as a tuned mass damper (TMD), is a passive vibration control device. This is achieved by attaching a secondary oscillator to a primary oscillator. In general, the aim is to reduce the vibration of the primary oscillator by suitably choosing the parameters of the secondary oscillator. The effectiveness of a TMD depends on (a) optimised the value of the tuned parameters, and (b) the nature of ambient damping of the absorber. They theory of TMD when the secondary and the primary oscillators are undamped or viscously damped is well developed. This paper presents an analytical approach to obtain optimal parameters of a TMD when the vibration absorber is viscoelastically damped. Classical results on viscously damped vibration absorbers can be obtained as a special case of the general results reduced in the paper. It is shown that by using a viscoelastically damped TMD, it is possible to obtain superior vibration absorption compared to an equivalent viscously damped TMD.

1. Introduction

Tuned mass damper (TMD) is a classical device concept for passive vibration control. Early works can be seen in Ref. [1]. A comprehensive details regarding the physics and design of tuned mass dampers, also know as vibration absorbers, can be found in the 1956 book of Den Hartog [2]. Since the original theoretical concepts, there have been many practical high profile implementation of tuned mass dampers in high-rise buildings and towers. We refer to the book by Soong and Dargush [3] for comprehensive details.

The central idea behind the tuned mass dampers is to attach a carefully chosen secondary oscillator to a primary oscillator such that a part of the vibratory energy of the primary oscillator is transferred for certain forcing to the secondary oscillator which will dissipate this energy. It is generally considered that the primary oscillator is 'given' so that its parameters, that is, the mass, stiffness and damping (if present) are fixed. The mathematical problem of tuning the mass damper, therefore, comes down to selecting the parameters of the secondary oscillator. For practical reasons, the mass of the secondary oscillator is often fixed apriori. We suppose $\mu \leq 1$ is the ratio of the mass between the primary and the secondary oscillator. When the primary oscillator is undamped, it was proved [2] that the optimal tuning ratio (the ratio of the natural frequency between the secondary oscillator the primary oscillator) is given by $\sqrt{1/(1+\mu)}$. This remarkable simple result was derived based on a curious observation by Den Hartog [2] which came to be known as the 'fixed-point theory'. This pertains to the fact that there are two fixed frequencies at which the frequency response of the primary oscillator is independent of the damping of the secondary system. This observation results into certain simplifications in the subsequent mathematical derivations, which in turn gives rise to the simple closed-form expression of the optimal tuning parameter. For complex primary structures, the optimal parameters

* Corresponding author.
E-mail addresses: anas.batou@liverpool.ac.uk (A. Batou), S.Adhikari@swansea.ac.uk (S. Adhikari).

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and position of the TMD can be determined by introducing and solving appropriate non-convex optimisation problems (see for instance Refs. [4,5]).

When damping is introduced in the primary structure, the existence of the fixed points, as they were known, is lost. Consequently, the derivation of the optimal tuning parameter becomes less straightforward compared the classical undamped case. The path of investigation for the damped case mainly follow two routes - (a) rigorous mathematical optimisation based exact numerical methods, and (b) approximate but simple analytical methods. Some examples of the first category include a numerical optimisation scheme proposed by Randall et al. [6,7], an optimal design of vibration absorbers using nonlinear programming techniques by Ref. [8], and an optimum design using a frequency locus method by Thompson [9]. Pennestri [10] obtained optimal solutions using the min-max Chebyshev's criterion. Numerical studies based on minimax optimisation are reported in Refs. [11,12]. Works within the second category include empirical formulae for optimum stiffness and damping of a vibration absorber based on the minimization of the acceleration response by Ioi and Ikeda [13] and a perturbation technique by Fujino and Abe [14]. A particularity noteworthy observation, made by Ghosh and Basu [15], was that even when the primary structure is damped, the so-called fixed points remain within a reasonably close proximity to the original fixed points corresponding to the undamped primary structure. This lead to the derivation of approximate closed-form expression of the optimal tuning parameter for the damped case. Later approximate fined point theory was employed to obtain optimal parameters of a piezoelectric energy harvesting dynamic vibration absorber [16]. It was shown that as a special case when the piezoelectric coupling goes to zero, the optimal tuning parameter reduces to the one derived by Den Hartog [2].

Viscous damping model is one of the many possible physically damping models that can be used for dynamic systems [17,18]. The aim of this paper is to explore the use of more general viscoelastic damping in the context of tuned mass damper. The motivation behind this investigation is that by using a more generalised damper, it may be possible to achieve a passive vibration control which is beyond the limit of a purely viscous damper.

The paper is organised as follows. A brief overview of general viscoelastic damping models are given in Section 2. The equation of motion of the viscoelastically damped vibration absorber in given in Section 3. In Section 4 we derive the optimal parameters of the vibration absorber.

2. Brief overview of viscoelastic models

In classical elasticity, instantaneous stress within a material is a function of instantaneous strain only. In contrast, in viscoelasticity, instantaneous stress is considered to be a function of strain history. When a linear viscoelastic model is employed, the stress at some point of a structure can be expressed as a convolution integral over a kernel function [19] as

$$\sigma(t) = \int_{-\infty}^{t} g(t-\tau)\frac{d\epsilon(\tau)}{d\tau}\,\tau \tag{1}$$

Here $t \in \mathbb{R}^+$ is the time, $\sigma(t)$ is stress and $\epsilon(t)$ is strain. The kernel function $g(t)$ also known as 'hereditary function', 'relaxation function' or 'after-effect function' in the context of different subjects. The stress-strain relationship in (1) can be directly applied to dynamic analysis of a solid body. For example, it is applied to a uniform rod, Equation (1) can be multiplied by the area and the equation can in tern be expressed in terms of the force and displacement rate (or velocity). In practice, the kernel function is often defined in the frequency domain (or Laplace domain). Taking the Laplace transform of Equation (1), we have

$$\overline{\sigma}(s) = s\overline{G}(s)\overline{\epsilon}(s) \tag{2}$$

Here $\overline{\sigma}(s)$, $\overline{\epsilon}(s)$ and $\bar{G}(s)$ are Laplace transforms of $\sigma(t)$, $\epsilon(t)$ and $g(t)$ respectively and $s \in \mathbb{C}$ is the (complex) Laplace domain parameter. There are two broad ways by which the kernel function $g(t)$ can be constructed, namely by a physics based approach or a more general mathematical approach.

2.1. Physics based representation of the kernel function

In a physics based approach, the kernel function appearing in the viscoelastic constitutive relationship can arise from a combination of springs and dashpots. This can be achieved in various ways. Four main cases are shown in Fig. 1.

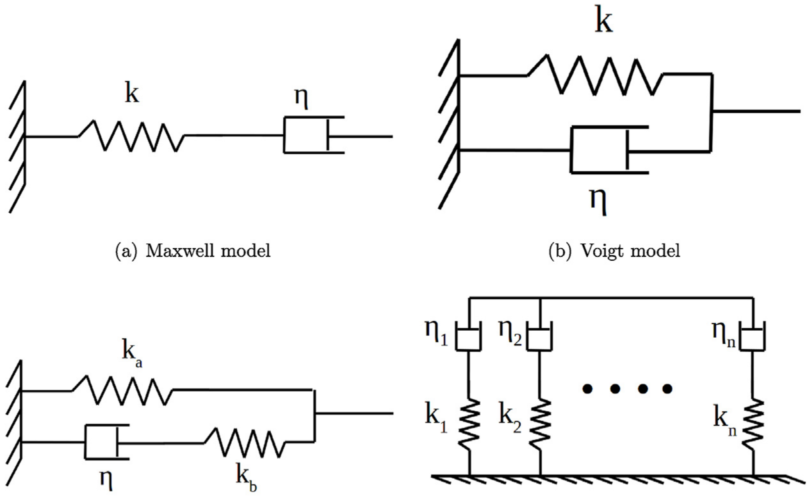
We define the unit step function $\mathcal{U}(t)$ and Dirac delta function $\delta(t)$ as below

$$\mathcal{U}(t) = \begin{cases} 1 & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0. \end{cases} \quad \text{and} \quad \delta(t) = \begin{cases} 0 & \text{if} \quad t \neq 0, \\ \displaystyle\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 & \end{cases} \tag{3}$$

Using these functions, the viscoelastic kernel function can be expressed [19–22] for the four models as

- *Maxwell model:*

$$g(t) = ke^{-(k/\eta)t}\,\mathcal{U}(t) \tag{4}$$



(a) Maxwell model (b) Voigt model

(c) Standard linear model (d) Generalised Maxwell model

Fig. 1. Springs and dashpots based models viscoelastic materials.

- *Voigt model:*

$$g(t) = \eta\delta(t) + k\,\mathcal{U}(t) \tag{5}$$

- *Standard linear model:*

$$g(t) = \left[k_a + k_b\, e^{-(k_b/\eta)t}\right]\mathcal{U}(t) \tag{6}$$

- *Generalised Maxwell model:*

$$g(t) = \left[\sum_{j=1}^{n} k_j e^{-(k_j/\eta_j)t}\right]\mathcal{U}(t) \tag{7}$$

Models similar to this is also known as the Pony series model.

These functions can be constructed by considering the equilibrium of forces arising by stretching the springs and dashpots appearing in Fig. 1.

2.2. Mathematical representation of the kernel function

The kernel function in Equation (2) is a complex function in the frequency domain. For notational convenience we denote

$$\overline{G}(s) = \overline{G}(\mathrm{i}\omega) = G(\omega) \tag{8}$$

where $\omega \in \mathbb{R}^+$ is the frequency. The complex modulus $G(\omega)$ can be expressed in terms of its real and imaginary parts or in terms of its amplitude and phase as follows

$$G(\omega) = G'(\omega) + \mathrm{i}G''(\omega) = |G(\omega)|e^{\mathrm{i}\phi(\omega} \tag{9}$$

The real and imaginary parts of the complex modulus, that is, $G'(\omega)$ and $G''(\omega)$ are also known as the storage and loss moduli respectively. One of the main restriction on the form of the kernel function comes from the fact that the response of the structure must to start before the application of the forces. This causality condition imposes a mathematical relationship between real and imaginary parts of the complex modulus, known as Kramers-Kronig relations (see for example [23] for recent discussions). Kramers-Kronig relations specifies that the real and imaginary parts should be related by a Hilbert transform pair, but can be general otherwise. Mathematically this can be expressed as

$$\begin{aligned} G'(\omega) &= G_\infty + \frac{2}{\pi}\int_0^\infty \frac{uG''(u)}{\omega^2 - u^2}\mathrm{d}u \\ G''(\omega) &= \frac{2\omega}{\pi}\int_0^\infty \frac{G'(u)}{u^2 - \omega^2}\mathrm{d}u \end{aligned} \tag{10}$$

Table 1
Complex modulus for viscoelastic models in the frequency domain.

Viscoelastic model	Complex modules	Main references
Biot model	$G(\omega) = G_0 + \sum_{k=1}^{n} \frac{a_k \mathrm{i}\omega}{\mathrm{i}\omega + b_k}$	Biot [27,28]
Fractional derivative	$G(\omega) = \frac{G_0 + G_\infty (\mathrm{i}\omega\tau)^\beta}{1 + (\mathrm{i}\omega\tau)^\beta}$	Bagley and Torvik [29]
GHM	$G(\omega) = G_0 \left[1 + \sum_k \alpha_k \frac{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega}{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega + \omega_k^2}\right]$	Golla and Hughes [30] and McTavish and Hughes [31]
ADF	$G(\omega) = G_0 \left[1 + \sum_{k=1}^{n} \Delta_k \frac{\omega^2 + \mathrm{i}\omega\Omega_k}{\omega^2 + \Omega_k^2}\right]$	Lesieutre and Mingori [32]
Step-function	$G(\omega) = G_0 \left[1 + \eta \frac{1 - e^{-st_0}}{st_0}\right]$	Adhikari [33]
Half cosine model	$G(\omega) = G_0 \left[1 + \eta \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}\right]$	Adhikari [33]
Gaussian model	$G(\omega) = G_0 \left[1 + \eta\, e^{\omega^2/4\mu} \left\{1 - \mathrm{erf}\left(\frac{\mathrm{i}\omega}{2\sqrt{\mu}}\right)\right\}\right]$	Adhikari and Woodhouse [34]

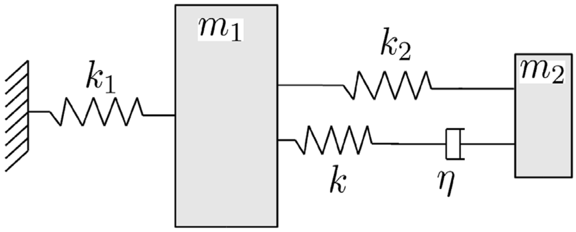


Fig. 2. A schematic representation of the viscoelastic tuned mass damper.

where the unrelaxed modulus $G_\infty = G(\omega \to \infty) \in \mathbb{R}$. Equivalent relationships linking the modulus and the phase of $G(\omega)$ can expressed as

$$\begin{aligned} \ln|G'(\omega)| &= \ln|G_\infty| + \frac{2}{\pi}\int_0^\infty \frac{u\phi(u)}{\omega^2 - u^2}\mathrm{d}u \\ \phi(\omega) &= \frac{2\omega}{\pi}\int_0^\infty \frac{\ln|G(u)|}{u^2 - \omega^2}\mathrm{d}u \end{aligned} \tag{11}$$

It should be noted that complex modulus derived using the physics based principled discussed above automatically satisfy these conditions. However, there can be many other function which would also satisfy these conditions. It is possible to determine $G(\omega)$ from experimental measurements (see Refs. [23–26]) which satisfy these conditions. In Table 1 we show some functions which have been used in literature. Among various possible viscoelastic models, the Standard linear model (which is equivalent to a Biot model with $n = 1$) is considered here. Other viscolelastic models could considered for the TMD (see for instance Ref. [26]). Nevertheless, the use of a Standard linear model enables a closed-form solution to be calculated (and thus avoids the solving of a non-convex inverse problem), which is one of the objectives of this paper.

3. Equation of motion of the coupled system

The viscoelastic tuned mass damper considered here is shown in Fig. 2. The primary structure is assumed to be an undamped single degree of freedom system. The secondary oscillator or the vibration absorber is considered to be coupled with the primary oscillator through a Standard linear model type as discussed in the previous section. Damping is the primary structure is ignored here as this would help to reduce the vibration and therefore could mask any added benefit arising due to the viscoelastic absorber which we want to investigate. Also, this assumption is necessary to derive a methodology for optimizing the viscoelastic TMD since it is a condition for fixed-points to exist.

The mass and stiffness of the primary oscillator are represented as m_1 and k_1 respectively. The vibration absorber has a mass and viscoelastic damping function as m_2, $k_2 + ke^{-(k/\eta)t}\mathcal{U}(t)$. The dynamics of the primary mass (m_1) and the absorber mass (m_2) can be expressed by two coupled ordinary differential equations as

$$m_1\ddot{X}_1(t) + k_1 X_1(t) + k_2\left(X_1(t) - X_2(t)\right) + \int_0^\infty ke^{-(k/\eta)(t-\tau)}\left(\dot{X}_1(\tau) - \dot{X}_2(\tau)\right)\mathrm{d}\tau = F(t), \tag{12}$$

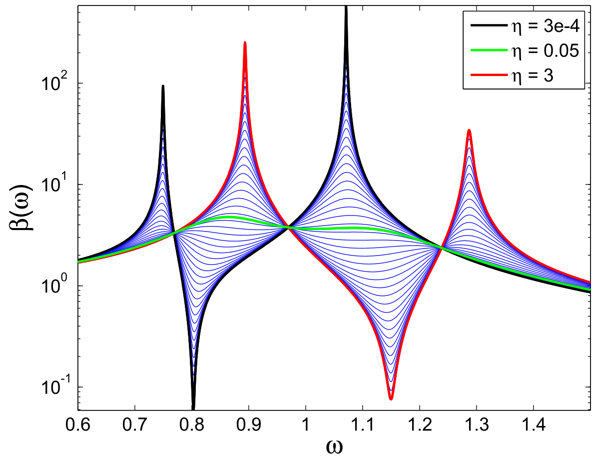


Fig. 3. The amplification factor $\beta(\omega)$ for different values of η.

$$m_2\ddot{X}_2 + k_2\left(X_2(t) - X_1(t)\right) + \int_0^{\infty} k e^{-(k/\eta)(t-\tau)}\left(\dot{X}_2(\tau) - \dot{X}_1(\tau)\right) d\tau = 0. \tag{13}$$

where $X_1(t)$ and $X_2(t)$ are the displacement of the primary mass m_1 and absorber mass m_2 respectively. The primary structure is assumed to be driven by an excitation force $F(t)$. Transforming Equations (12) and (13) in the frequency domain we have

$$\begin{aligned}&(-\omega^2 m_1 + k_1 + k_2 + i\omega\frac{\eta k}{k + i\omega\eta})x_1(\omega) - (k_2 + i\omega\frac{\eta k}{k + i\omega\eta})x_2(\omega) = f(\omega),\\ &(-\omega^2 m_2 + k_2 + i\omega\frac{\eta k}{k + i\omega\eta})\left(x_2(\omega) - x_1(\omega)\right) = 0.\end{aligned} \tag{14}$$

Here $x_1(\omega)$, $x_2(\omega)$ and $f(\omega)$ are Fourier transforms of $X_1(t)$, $X_2(t)$ and $F(t)$ respectively. We introduce the following notations for convenience

$$\begin{aligned}&\mu = \frac{m_2}{m_1}, \quad \omega_1 = \sqrt{\frac{k_1}{m_1}}, \quad \Omega(\omega) = \frac{\omega}{\omega_1},\\ &\widetilde{k}(\omega) = k_2 + \frac{\omega^2\eta^2 k}{k^2 + \omega^2\eta^2}, \quad \widetilde{c}(\omega) = \frac{\eta k^2}{k^2 + \omega^2\eta^2},\\ &\omega_2(\omega) = \sqrt{\frac{\widetilde{k}(\omega)}{m_2}}, \quad \xi(\omega) = \frac{\widetilde{c}(\omega)}{2\,m_2\omega_1}, \quad q(\omega) = \frac{\omega_2^2(\omega)}{\omega_1^2}.\end{aligned} \tag{15}$$

Using these notations it can be shown that the modulus of $x_1(\omega)$ solution of Eq. (14) reads

$$|x_1(\omega)| = \beta(\omega)|f(\omega)|, \tag{16}$$

where

$$\beta(\omega) = \sqrt{\frac{(2\,\xi(\omega)\,\Omega(\omega))^2 + (\Omega(\omega)^2 - q(\omega))^2}{(2\,\xi(\omega)\,\Omega(\omega))^2(\Omega(\omega)^2 - 1 + \mu\,\Omega(\omega)^2)^2 + (\mu\,q(\omega)\,\Omega(\omega)^2 - (\Omega(\omega)^2 - 1)(\Omega(\omega)^2 - q(\omega)))^2}}\;, \tag{17}$$

is the amplification factor for which, contrary to the classical TMD case, ξ and q depend on the frequency ω.

4. Derivation of the tuning parameters

4.1. Description of the general methodology

For the case $m_1 = 1$ kg, $m_2 = 0.1$ kg, $k_1 = 1$ N/m, $k_2 = 0.0645$ N/m, $k = 0.0678$ N/m, the function $\beta(\omega)$ is plotted in Fig. 3 for different values of the damping coefficient η.

It can be seen in this figure that:

1. For very small values of η, as for classical TMDs, the viscoelastic TMD presents two large resonance peaks. Indeed in this case, the link between m_1 and m_2 is equivalent to the unique spring k_2 without damping.
2. For very large values of η, the damper locks and then the link between m_1 and m_2 is equivalent to only two parallel springs k_2 and k without damping. Therefore contrary to the case of a classical TMDs, no complete lock for mass m_2 appears when increasing the damping η.
3. There are three points for which the response amplitude does not depends on the damping η. For classical TMDs, such fixed-points have also been observed but there were only two of them. The classical methods for deriving the optimal parameters of a classical TMDs are based on these fixed-points. For viscoelastic TMDs, we now have three fixed-points. We therefore need to revisit the methodology for finding the optimal parameters.

Based on Fig. 3, we propose to derive the optimal parameters in three steps:

- **Step 1**: Calculate the positions of the three fixed-points.
- **Step 2**: Determine the optimal values for k_2 and k such that the values at the fixed-points (which are independent of the damping) have prescribed relative positions.
- **Step 3**: Determine the damping η that gives a symmetric response with respect to the central fixed point.

4.2. Positions of the three fixed-points

To calculate the position of the three fixed-points, we introduce a methodology similar to the one used for classical TMDs: we search the frequencies for which the amplification factor β has the same values for $\eta = 0$ and $\eta = +\infty$. Indeed for these two values, the expression of β in (17) has a simple expression. It can be shown that $\widetilde{k}(\omega;\eta=0)=k_2$, $\widetilde{k}(\omega;\eta=+\infty)=k_2+k$, $\widetilde{c}(\omega;\eta=0)=\widetilde{c}(\omega;\eta=+\infty)=0$, and then $q_0(\omega) = q(\omega;\eta = 0) = k_2/(\mu k_1)$, $q_\infty(\omega) = q(\omega;\eta = +\infty) = (k_2 + k)/(\mu k_1)$, $\xi(\omega;\eta = 0) = \xi(\omega;\eta = +\infty) = 0$. Finally equating the amplification factor β for $\eta = 0$ and $\eta = +\infty$ yields the equation

$$\frac{\Omega^2 - q_0}{\mu\, q_0\Omega^2 - (\Omega^2-1)(\Omega^2-q_0)} = -\frac{\Omega^2 - q_\infty}{\mu\, q_\infty\Omega^2 - (\Omega^2-1)(\Omega^2-q_\infty)}. \tag{18}$$

The minus sign in the above equation is arising from the sign change when taking the absolute value. This equation can be rearranged to obtain the following equation

$$\Omega^6 - \frac{1}{2}((q_0+q_\infty)(2+\mu)+2)\Omega^4 + ((q_0+q_\infty+q_0q_\infty(\mu+1))\Omega^2 - q_0q_\infty = 0. \tag{19}$$

This equation is cubic in Ω^2. Therefore, eliminating negative values, we can derive the three closed-form solutions $\Omega_l(q_0,q_\infty)$, $\Omega_c(q_0,q_\infty)$, and $\Omega_r(q_0,q_\infty)$ which correspond to left, central and right fixed-points respectively (See Appendix A for the closed-form solutions). For parameters corresponding to Fig. 3, it is found $\Omega_l = 0.77$, $\Omega_c = 0.97$, and $\Omega_l = 1.24$ (for this particular case $\omega_1 = 1$ rad/s and then $\Omega = \omega$).

4.3. Calculation of the stiffnesses k_2 and k

In order to obtain a good mitigation of the around the resonance frequency of the primary mass m_1, we propose to calculate k_2 and k as the values that (1) provides equal values for the end fixed-point Ω_l and Ω_r and (2) allows to control the amplitude of the central fixed-point Ω_c with respect to the amplitudes at Ω_l and Ω_r. The advantage of this methodology is that the obtained values are independent of the damping coefficient that can then be fixed later. Let be $\omega_l = \omega_1\Omega_l$, $\omega_c = \omega_1\Omega_c$ and $\omega_r = \omega_1\Omega_r$. We then introduce the two following equations

$$\begin{aligned} \beta(\omega_l;\eta=0) &= \alpha\beta(\omega_c;\eta=0),\\ \beta(\omega_r;\eta=+\infty) &= \alpha\beta(\omega_c;\eta=+\infty), \end{aligned} \tag{20}$$

in which α is a proportion parameter. For $\alpha = 1$, the amplitude is the same for the three fixed-points. We then obtain the two equations

$$\begin{aligned} &q_0^2 + \gamma_1(\Omega_l,\Omega_c;\alpha)q_0 + \gamma_2(\Omega_l,\Omega_c,\alpha),\\ &q_\infty^2 + \gamma_1(\Omega_r,\Omega_c;\alpha)q_\infty + \gamma_2(\Omega_r,\Omega_c;\alpha) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \gamma_1(u,v;\alpha) &= \frac{(1+\alpha)\mu u^2v^2 - (u^2+v^2)((1+\alpha)-(\alpha u^2+v^2))}{1+\alpha-(1+\mu)(\alpha u^2+v^2)},\\ \gamma_2(u,v;\alpha) &= \frac{u^2v^2((1+\alpha)-(\alpha u^2+v^2)}{1+\alpha-(1+\mu)(\alpha u^2+v^2)}, \end{aligned} \tag{22}$$

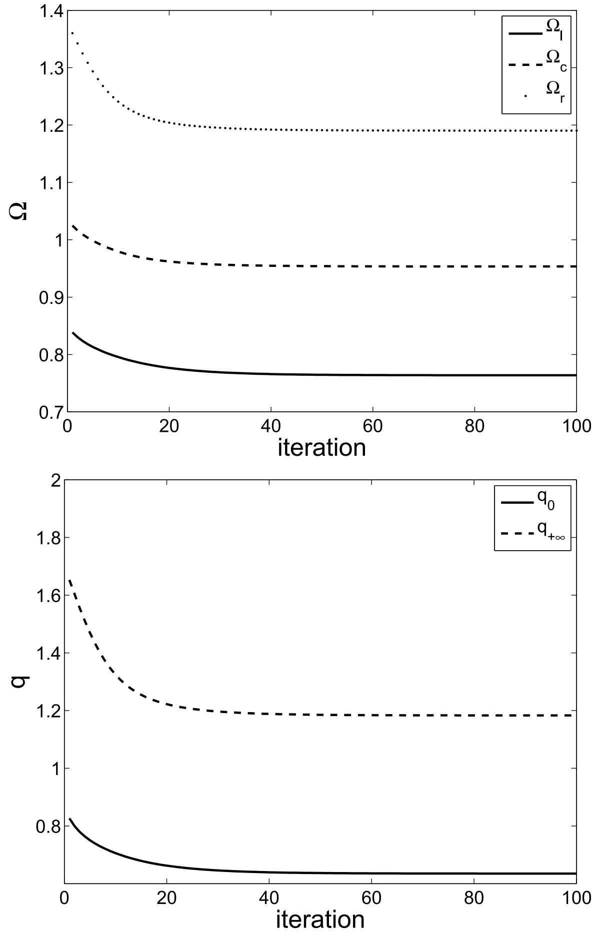


Fig. 4. The convergence for $\Omega_l, \Omega_c, \Omega_c$ (solutions of Eq. (19)), q_0 and q_∞ (see Eq. (23)) with number of iterations.

By eliminating non physical solutions, only one solution remains for each equation, yielding

$$q_0 = \frac{-\gamma_1(\Omega_l, \Omega_c; \alpha) - \sqrt{\gamma_1((\Omega_l, \Omega_c; \alpha))^2 - 4\gamma_2((\Omega_l, \Omega_c; \alpha)}}{2},$$
$$q_\infty = \frac{-\gamma_1(\Omega_r, \Omega_c; \alpha) + \sqrt{\gamma_1((\Omega_r, \Omega_c; \alpha))^2 - 4\gamma_2((\Omega_r, \Omega_c; \alpha)}}{2}. \quad (23)$$

These solutions depend on Ω_l, Ω_c and Ω_c which themselves depend on q_0 and q_∞. Any attempt to eliminate Ω_l, Ω_c and Ω_c or q_0 and q_∞ yields very complex equations that cannot not be solved explicitly. To circumvent this difficulty, we propose to solve Eqs. (19) and (21) iteratively. For the initialization, the parameters q_0 and q_∞ are set such that $k_2 = k = k_0$ in which k_0 is the optimal stiffness for a classical TMD, i.e., $k_0 = \mu k_1/(1+\mu)^2$. Then Eqs. (19) and (21) are successively and repeatedly solved until convergence. For the case $m_1 = 1$ kg, $m_2 = 0.1$ kg, $k_1 = 1$ N/m, and $\alpha = 1$, the convergence curves of parameters Ω_l, Ω_c, Ω_c, q_0 and q_∞ for 100 iterations are plotted in Fig. 4. It can be seen a good convergence after 50 iterations. Once the q_0 and q_∞ calculated, the stiffnesses k_2 and k are calculated such that

$$k_2 = q_0\, k_1\, \mu, \quad k = q_\infty\, k_1\, \mu - k_2; \quad (24)$$

For the optimal values of k_2 and k, the function $\beta(\omega)$ is plotted on Fig. 5 for different values of the damping coefficient η.

4.4. Calculation of the damping coefficient η

The effect of the damping coefficient η on the amplitude of the response can be seen in Fig. 5. This parameter directly impacts the symmetry of function $\beta(\omega)$ with respect to the central fixed-point Ω_c. It appears clear that the optimal curve corresponds to

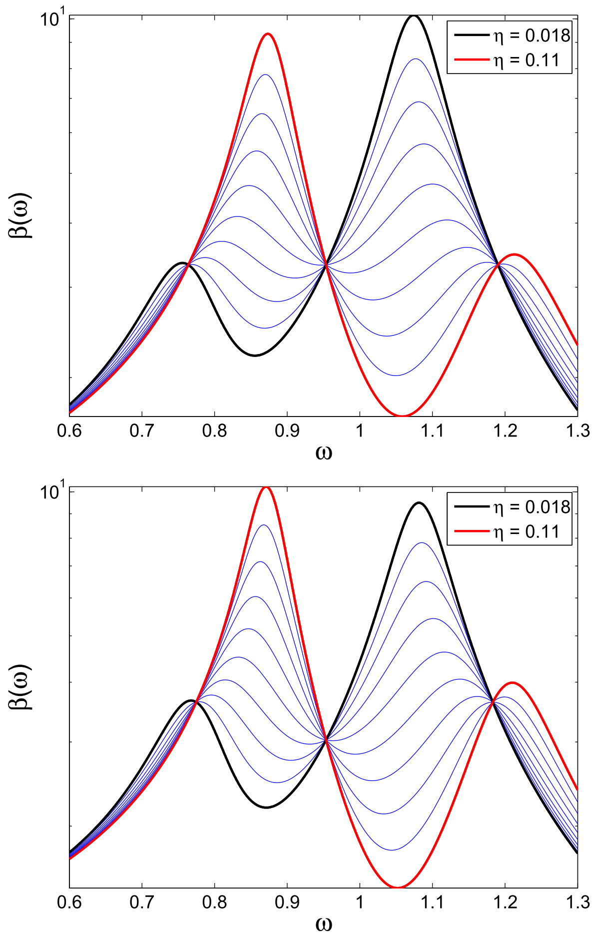


Fig. 5. The amplification factor $\beta(\omega)$ for different values of η. Top: $\alpha = 1$. Bottom: $\alpha = 1.2$.

the one that will present the best symmetry with respect to Ω_c. A possible way to control this symmetry consists in controlling the slopes of function $\beta(\omega)$ at the fixed-points Ω_l and Ω_r and try to make these slopes opposite. A direct method would consists in calculating the derivative $\beta'(\omega)$ of the function $\beta(\omega)$ with respect to ω and then calculate then calculate the slopes at frequencies Ω_l and Ω_r. Then the optimal value for η_{opt} is such that

$$\beta'(\omega_l; \eta_{opt}) = -\beta'(\omega_r; \eta_{opt}). \tag{25}$$

Unfortunately, the function $\beta'(\omega)$ is very complex and this method does not allow to obtain a closed-form expression for η_{opt}. We then propose an alternative approach consisting in approximating the derivative. Then Eq. (15) is modified using a forward approximation at ω_l and backward approximation at ω_r, i.e.,

$$\frac{\beta(\omega_l + \Delta\omega; \eta_{opt}) - \beta(\omega_l; \eta_{opt})}{\Delta\omega} = -\frac{\beta(\omega_r; \eta_{opt}) - \beta(\omega_r - \Delta\omega; \eta_{opt})}{\Delta\omega}, \tag{26}$$

where $\Delta\omega$ is a small frequency increment. Then since $\beta(\omega_l; \eta_{opt}) = \beta(\omega_r; \eta_{opt})$, solving Eq. (15) consists in searching the zero of the function $\epsilon(\eta)$ defined by

$$\epsilon(\eta) = \beta(\omega_l + \Delta\omega; \eta) - \beta(\omega_r - \Delta\omega; \eta). \tag{27}$$

Finally, this approximation consists in equalling the value of the amplitudes just after ω_l and just before ω_r. The zero of function $\epsilon(\eta)$ can be search graphically or using any devoted algorithm. For the previous case, with $\alpha = 1$ and $\Delta\omega = 0.01$, function $\epsilon(\eta)$ is plotted in Fig. 6. In can be seen in this figure that the zero is reached for $\eta_{opt} = 0.045$ N/m. The corresponding function $\beta(\omega)$ is plotted in Fig. 7. In Fig. 8, the mitigation obtained using the optimal viscoelastic TMD is compared to the one obtained using a

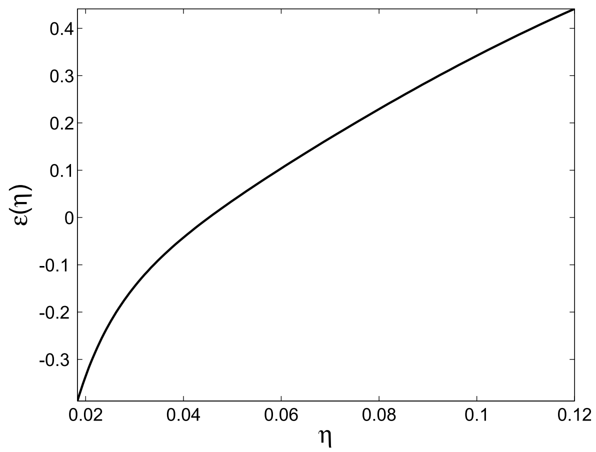


Fig. 6. The function $\epsilon(\eta)$ (see Eq. (27)).

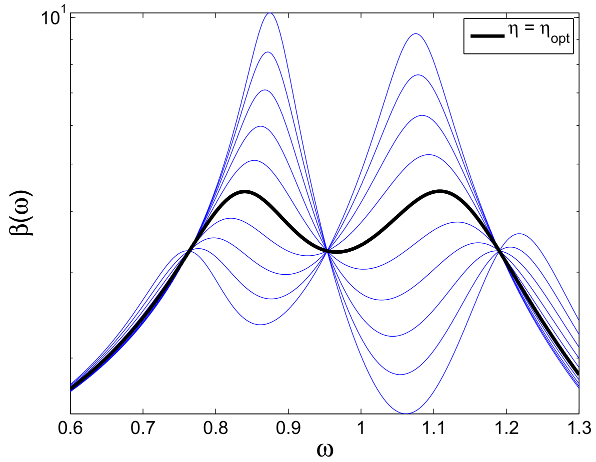


Fig. 7. The amplification factor $\beta(\omega)$ for different values of the damping coefficient η.

classical optimal TMD. We can see in this figure that a viscoelastic TMD gives a better mitigation than a classical optimal TMD for a wide range of frequencies including around the resonance frequencies.

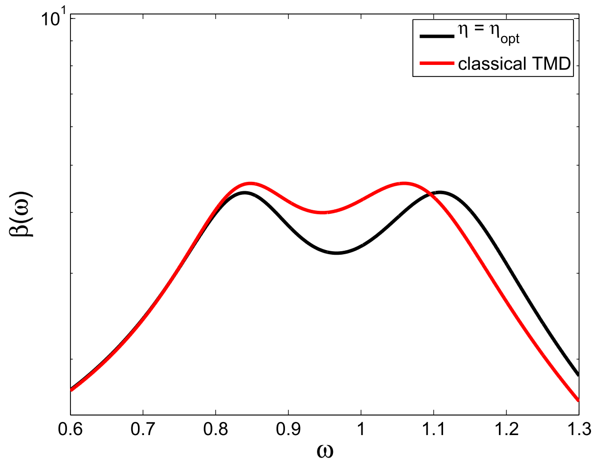


Fig. 8. A comparison of the amplification factor $\beta(\omega)$ between the optimal viscoelastic TMD and the classical optimal TMD.

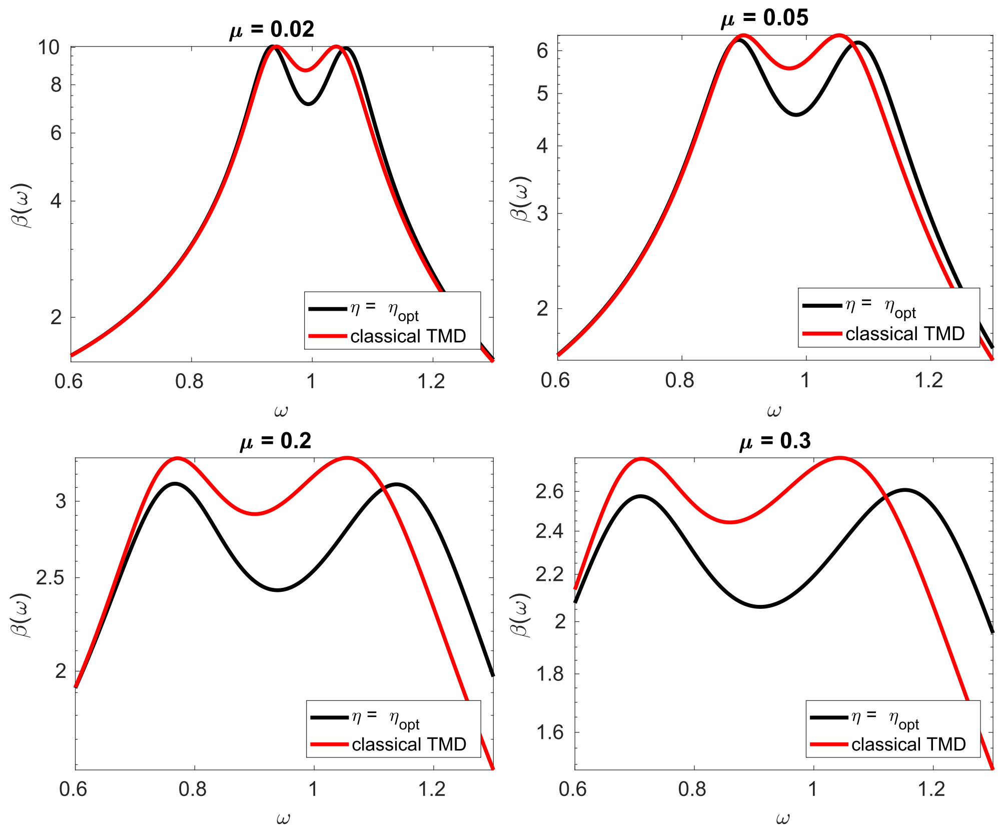


Fig. 9. Comparison of the amplification factor $\beta(\omega)$ between the optimal viscoelastic TMD and the classical optimal TMD for different values of the mass ratio μ.

The comparison is also performed for different value of the mass ratio μ. The results are plotted in Fig. 9. It can be seen in this figure that the gain provided by a viscoelastic TMD increases with the mass ratio.

5. Summary and discussions

Tuned mass dampers (TMD) are classical passive vibration control concepts achieved by attaching a secondary oscillator to a primary oscillator. A TMD with viscoelastic properties has been considered in this paper as a generalisation of the classical viscously damped TMDs. The widely-used Biot model of viscoelasticity with one term is used. This gives rise to two additional parameters (the stiffness and the time constant) to be determined compared to an undamped TMD. Some of the main points of this paper are:

- The fixed-points theory has been generalised to the optimal parameters of the viscoelastic TMD.
- A TMD with a standard linear viscoelastic model shows three fixed-points instead of the two fixed-points for a classical TMD.
- The two stiffnesses are obtained by controlling the vibration amplitude at the three fixed-points.
- The damping coefficient is obtained by controlling the symmetry of the of the amplitude function with respect with respect to the central fixed point.
- The optimised viscoelastic TMD offers more flexibility and allows to reach a better vibration mitigation compared to classical viscously damped TMD.

Representative numerical results with non-dimensional parameters are given. A semi-analytical iterative procedure for calculating the optimal parameters has been proposed. Through numerical examples. It has been demonstrated that the iterative approached converges rapidly to the final value. It is assumed that there is no damping for the primary structure. In the presence of damping, similarly to the classical TMDs, the dynamics won't exhibit fixed-points any more. In the case of reasonably small damping, it is expected, as shown for classical TMDs (see Ref. [15]) that the fixed points will remain close to the fixed-points corresponding to the undamped primary structure. It is also assumed that the primary structure is linear. In the case of a nonlinear behaviour, a temporally averaged linearisation technique, as suggested in Ref. [35], could be applied as a prior step before applying the method proposed in the present paper. The methodology has been derived for the standard (Biot) linear viscoelastic model. The use of a Standard linear model enables a closed-form solution to be calculated (and thus avoids the solving of a non-convex inverse problem), which is one of the objectives of this paper. Future research is necessary to determine if there would still be fixed-points for a more general viscoelastic model.

Appendix A. Closed-form solution for the fixed-points equation

Let a_0, a_1 and a_2 be the coefficients associated with the cubic (in Ω^2) Eq. (19):

$$a_0 = -q_0 q_\infty, \quad a_1 = (q_0 + q_\infty) + q_0 q_\infty(\mu + 1), \quad a_2 = -((q_0 + q_\infty)(2 + \mu) + 2)/2. \tag{A.1}$$

In this case the three solutions Ω_l, Ω_c and Ω_d can be calculated as

$$\Omega_l = \sqrt{r_1}, \quad \Omega_c = \sqrt{r_2}, \quad \Omega_r = \sqrt{r_3}, \tag{A.2}$$

where

$$\begin{aligned} r_1 &= -a_2/3 - (S+T)/2 + i\sqrt{3}(S-T)/2, \\ r_2 &= -a_2/3 - (S+T)/2 - i\sqrt{3}(S-T)/2, \\ r_3 &= -a_2/3 + S + T, \end{aligned} \tag{A.3}$$

in which

$$S = \left(R + \sqrt{Q^3 + R^2}\right)^{1/3}, \quad T = \left(R - \sqrt{Q^3 + R^2}\right)^{1/3}, \tag{A.4}$$

and such that

$$Q = (3a_1 - a_2^2)/9, \quad R = (9a_2 a_1 - 27a_0 - 2a_2^3)/54, \tag{A.5}$$

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