

# Direct time-domain integration method for exponentially damped linear systems

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## Abstract

Time-domain analysis of multiple-degree-of-freedom linear non-viscously damped systems is considered. It is assumed that the non-viscous damping forces depend on the past history of velocities via convolution integrals over exponentially decaying kernel functions. A direct time-domain integration method is proposed. The proposed approach is based on an extended state-space representation of the equations of motion. The state-space method, in turn, is based on introduction of a set of internal variables. The numerical method for the calculation of the displacements eliminates the need for explicit calculation of the velocities and usually large number of internal variables at each time step. This fact particularly makes this method numerically efficient. The proposed method is illustrated by two numerical examples.

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## 1. Introduction

In structural dynamics often viscous damping model (proposed by Lord Rayleigh [1] in 1877) is used to model the energy dissipation. It is however, well known that the classical viscous damping model is a mathematical idealization and the ‘true’ damping model is likely to be different. Such ‘modelling errors’ can have significant effect in the dynamics of light space structures [2]. Moreover, increasing use of modern composite materials and

active control mechanisms in aerospace and automotive industries demands sophisticated treatment of dissipative forces for proper analysis and design. For these reasons there has been an increase in interest in the recent years on non-viscous damping models with an aim to represent dissipative forces in a more general manner compared to the limited scope offered by a viscous damping model. Majority of the linear non-viscous damping models, like fractional derivative models [3–6] and hereditary/viscoelastic/convolution integral models [7,2,8,9], are based on the fact that the dissipative force can depend on quantities other than only the generalized velocities as assumed in the viscous damping model.

Exponential damping model is a special case of non-viscous damping models described by convolution integrals [7,2,8,9]. The equation of motion of a

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$N$ -degree-of-freedom linear system with exponential damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \sum_{k=1}^n \int_0^t \mu_k e^{-\mu_k(t-\tau)} \mathbf{C}_k \dot{\mathbf{u}}(\tau) d\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t), \quad (1)$$

together with the initial conditions

$$\mathbf{u}(t=0) = \mathbf{u}_0 \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_0 \in \mathbb{R}^N. \quad (2)$$

Here  $\mathbf{u}(t) \in \mathbb{R}^N$  is the displacement vector,  $\mathbf{M} \in \mathbb{R}^{N \times N}$  is the mass matrix,  $\mathbf{K} \in \mathbb{R}^{N \times N}$  is the stiffness matrix and  $\mathbf{f}(t) \in \mathbb{R}^N$  is the forcing vector. The constants  $\mu_k \in \mathbb{R}^+$  are known as the relaxation parameters,  $\mathbf{C}_k \in \mathbb{R}^{N \times N}$  are known as the damping coefficient matrices and  $n$  denotes the number relaxation parameters used to describe the damping behavior. A physical justification (using the principles of mechanics and thermodynamics) as to why a general structure should always have this type of damping is hard to provide. From this point of view this damping model is on a similar footing to that of the viscous model. However, based on engineering judgement and intuition several reasons behind the selection of this model could be given:

- In the context of viscoelastic materials, the physical basis for exponential models has been well established—as in the words of Cremer and Heckl [10]: ‘Of the many after-effect functions that are possible in principle, only one—the so-called relaxation function—is physically meaningful.’
- In a large engineering structure it is possible to have different damping in different parts of a structure. For example, various members of a space-frame may have different damping properties, each characterized by a relaxation parameter  $\mu_k$ . Then the associated coefficient matrix  $\mathbf{C}_k$  would have non-zero blocks corresponding to the relevant elements only. One could perform experiments for individual members and use the finite element method to obtain the element damping matrix, say  $\mathbf{C}_k^{(e)}$ . Using standard approach it is possible to assemble all the element matrices associated with relaxation parameter  $\mu_k$  to obtain a global damping matrix  $\mathbf{C}_k$ . This procedure may be repeated for all damping types present in the structure to obtain  $\mu_k$  and  $\mathbf{C}_k$  for all  $k$ .
- In a recent work Adhikari and Woodhouse [11] have proposed a method to identify  $\mu_k$  and  $\mathbf{C}_k$  from vibration measurements when  $n = 1$  in Eq. (1). It was also noted [12] that when the damping is non-viscous, forceful fitting of viscous damping may produce non-physical result (for example, a non-symmetric coefficient matrix). Thus, from parameter estimation point of view damping model in Eq. (1) gives additional flexibility to fit measured data obtained from modal testings.

Because most vibration analysis textbooks, finite element packages and modal analysis softwares only allow viscous damping, it is useful to relate the exponential damping model with the viscous damping model. From Eq. (1) it may be observed that in the limit when  $\mu_k \rightarrow \infty, \forall k$  then the equation of motion reduces to that of a viscously damped system with an equivalent viscous damping matrix

$$\mathbf{C} = \sum_{k=1}^n \mathbf{C}_k. \quad (3)$$

Thus, the exponential damping model is a further generalization of the more familiar viscous damping model. The purpose of this paper is to propose an efficient numerical method to solve Eq. (1) together with the initial conditions in (2).

In recent years some authors have considered systems similar to (1). Muravyov [13,14] and Muravyov and Hutton [15,16] have considered a similar system where the exponential kernel function is associated with the stiffness term. Recently Wagner and Adhikari [17] have proposed a state-space approach for the analysis of linear systems with exponential damping. Their method was based on representing Eq. (1) in terms of two symmetric matrices in an augmented state-space. The dynamic response of the system was obtained by mode superposition method using the state-space eigensolutions. Although the method gives exact results, it requires significant computation because the size of the eigenvalue problem in the extended state-space is usually very large. In this article an alternative approach based on direct integration in the time-domain has been proposed. This approach utilizes the state-space representation proposed in the earlier work [17]. Two physically realistic cases, namely, (a) when all the damping coefficient matrices are of full rank, and (b) when the damping coefficient matrices are rank deficient, have been presented. The proposed method is illustrated by numerical examples.

## 2. Review of the state-space formalism

In a recent paper Wagner and Adhikari [17] have proposed a state-space method for exponentially damped systems. Here we briefly outline the main results.

### 2.1. Case A: All $\mathbf{C}_k$ matrices are of full rank

In this case it is considered that

$$\text{rank}(\mathbf{C}_k) = N, \quad \forall k = 1, \dots, n. \quad (4)$$

We introduce the internal variables  $\mathbf{y}_k(t) \in \mathbb{R}^N, \forall k = 1, \dots, n$  through following relationship:

$$\mathbf{y}_k(t) = \int_0^t \mu_k e^{-\mu_k(t-\tau)} \dot{\mathbf{u}}(\tau) d\tau. \quad (5)$$

Differentiating Eq. (5) one obtains the evolution equation:

$$\dot{\mathbf{y}}_k(t) + \mu_k \mathbf{y}_k(t) = \mu_k \dot{\mathbf{u}}(t). \quad (6)$$

Using additional state-variables

$$\mathbf{v}(t) = \dot{\mathbf{u}}(t), \quad (7)$$

Eq. (2) can be represented in the first-order form as

$$\mathbf{B}\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{r}(t), \quad (8)$$

where

$$\mathbf{B} = \begin{bmatrix} \sum_{k=1}^n \mathbf{C}_k & \mathbf{M} & -\mathbf{C}_1/\mu_1 & \cdots & -\mathbf{C}_n/\mu_n \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{C}_1/\mu_1^2 & \mathbf{O} & \mathbf{O} \\ \vdots & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ -\mathbf{C}_n/\mu_n & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C}_n/\mu_n^2 \end{bmatrix} \in \mathbb{R}^{m \times m}, \quad (9)$$

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{C}_n/\mu_n \end{bmatrix} \in \mathbb{R}^{m \times m}, \quad (10)$$

$$\mathbf{r}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix} \in \mathbb{R}^m \quad \text{and} \quad \mathbf{z}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \\ \mathbf{y}_1(t) \\ \vdots \\ \mathbf{y}_n(t) \end{Bmatrix} \in \mathbb{R}^m. \quad (11)$$

In the above equations  $\mathbf{z}(t)$  is the extended state-vector,  $\mathbf{A}$  and  $\mathbf{B}$  are the system matrices in the extended state-space,  $\mathbf{r}(t)$  is the force vector in the extended state-space, and  $\mathbf{O}$  is a  $N \times N$  null matrix. Clearly, the order of the system,  $m$ , is given by

$$m = 2N + nN. \quad (12)$$

Because it is assumed that  $\mathbf{M}$ ,  $\mathbf{K}$  and  $\mathbf{C}_k$ ,  $\forall k$  are symmetric matrices,  $\mathbf{B}$  is a symmetric matrix and  $\mathbf{A}$  is a block-diagonal, therefore, also a symmetric matrix.

### 2.2. Case B: $\mathbf{C}_k$ matrices are rank deficient

In this section we assume that in general

$$r_k = \text{rank}(\mathbf{C}_k) \leq N, \quad \forall k = 1, \dots, n. \quad (13)$$

This implies that the number of non-zero eigenvalues of  $\mathbf{C}_k$  is  $r_k$ . We introduce a matrix  $\mathbf{R}_k \in \mathbb{R}^{N \times r_k}$  whose columns are the eigenvectors corresponding to the  $r_k$  non-zero eigenvalues of  $\mathbf{C}_k$ . Defining a set of internal variables of reduced dimension  $\tilde{\mathbf{y}}_k(t) \in \mathbb{R}^{r_k}$  using the rectangular transformation matrix  $\mathbf{R}_k$  by

$$\mathbf{y}_k(t) = \mathbf{R}_k \tilde{\mathbf{y}}_k(t), \quad (14)$$

it can be shown that [17] Eq. (1) can be represented in a first-order form as

$$\tilde{\mathbf{B}}\dot{\tilde{\mathbf{z}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{z}}(t) + \tilde{\mathbf{r}}(t). \quad (15)$$

Here

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{k=1}^n \mathbf{C}_k & \mathbf{M} & -\mathbf{C}_1 \mathbf{R}_1 / \mu_1 & \cdots & -\mathbf{C}_n \mathbf{R}_n / \mu_n \\ \mathbf{M} & \mathbf{O}_{N,N} & \mathbf{O}_{N,r_1} & \cdots & \mathbf{O}_{N,r_n} \\ -\mathbf{R}_1^T \mathbf{C}_1 / \mu_1 & \mathbf{O}_{N,r_1}^T & \mathbf{R}_1^T \mathbf{C}_1 \mathbf{R}_1 / \mu_1^2 & \cdots & \mathbf{O}_{r_1,r_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{R}_n^T \mathbf{C}_n / \mu_n & \mathbf{O}_{N,r_n}^T & \mathbf{O}_{r_1,r_n}^T & \cdots & \mathbf{R}_n^T \mathbf{C}_n \mathbf{R}_n / \mu_n^2 \end{bmatrix} \in \mathbb{R}^{\tilde{m} \times \tilde{m}}, \quad (16)$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} -\mathbf{K} & \mathbf{O}_{N,N} & \mathbf{O}_{N,r_1} & \cdots & \mathbf{O}_{N,r_n} \\ \mathbf{O}_{N,N} & \mathbf{M} & \mathbf{O}_{N,r_1} & \cdots & \mathbf{O}_{N,r_n} \\ \mathbf{O}_{N,r_1}^T & \mathbf{O}_{N,r_1}^T & -\mathbf{R}_1^T \mathbf{C}_1 \mathbf{R}_1 / \mu_1 & \cdots & \mathbf{O}_{r_1,r_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{N,r_n}^T & \mathbf{O}_{N,r_n}^T & \mathbf{O}_{r_1,r_n}^T & \cdots & -\mathbf{R}_n^T \mathbf{C}_n \mathbf{R}_n / \mu_n \end{bmatrix} \in \mathbb{R}^{\tilde{m} \times \tilde{m}}, \quad (17)$$

$$\tilde{\mathbf{r}}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0}_N \\ \mathbf{0}_{r_1} \\ \vdots \\ \mathbf{0}_{r_n} \end{Bmatrix} \in \mathbb{R}^{\tilde{m}} \quad \text{and} \quad \tilde{\mathbf{z}}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \\ \tilde{\mathbf{y}}_1(t) \\ \vdots \\ \tilde{\mathbf{y}}_n(t) \end{Bmatrix} \in \mathbb{R}^{\tilde{m}}. \quad (18)$$

In the above equations

$$\tilde{m} = 2N + \sum_{k=1}^n r_k, \quad (19)$$

is the order of the system,  $\mathbf{O}_{ij}$  are  $i \times j$  null matrices and  $\mathbf{0}_j$  are vectors of  $j$  zeros. The terms  $(\bullet)$  are corresponding to terms  $(\bullet)$  defined in Eq. (8). When all  $\mathbf{C}_k$  matrices are of full rank, that is, when  $r_k = N$ ,  $\forall k$ , then one can choose each  $\mathbf{R}_k$  matrix as the identity matrix and Eq. (15) reduces to Eq. (8).

### 3. Direct time-domain approach

First consider the full rank case. The linear approximations

$$\mathbf{u}(t) = \mathbf{u}_j \left(1 - \frac{t}{h}\right) + \mathbf{u}_{j+1} \frac{t}{h} \quad (20)$$

and

$$\mathbf{v}(t) = \mathbf{v}_j \left(1 - \frac{t}{h}\right) + \mathbf{v}_{j+1} \frac{t}{h}, \quad (21)$$

for the displacements  $\mathbf{u}(t)$  and the velocities  $\mathbf{v}(t)$  are used. In addition, the internal variables  $\mathbf{y}_k(t)$  have also been interpolated in a linear manner

$$\mathbf{y}_k(t) = \mathbf{y}_{k_j} \left(1 - \frac{t}{h}\right) + \mathbf{y}_{k_{j+1}} \frac{t}{h}, \quad k = 1, \dots, n. \quad (22)$$

Here the step size  $h = T/N_d$ , where  $T$  is the time up to which the response calculation is required and  $N_d$  is the number of divisions in the time axis. In order to simplify the formulas only constant time-steps throughout the whole time axis are used. The next and the last step concerns the integration of the system (8) with respect to the time interval with  $t/h \in [j, j + 1]$ . This integration, exact where it is possible and by using the linear approximation where it is necessary, leads to a totally discretized one-step formulation which allows  $\mathbf{z}_{j+1}$  to be calculated by means of  $\mathbf{z}_j$

$$\mathbf{B}(\mathbf{z}_{j+1} - \mathbf{z}_j) = \frac{h}{2} \mathbf{A}(\mathbf{z}_{j+1} + \mathbf{z}_j) + \mathbf{i}_r, \quad \mathbf{i}_r = \int_{jh}^{(j+1)h} \mathbf{r}(t) dt \quad (23)$$

or

$$\left[ \mathbf{B} - \frac{h}{2} \mathbf{A} \right] \mathbf{z}_{j+1} = \left[ \mathbf{B} + \frac{h}{2} \mathbf{A} \right] \mathbf{z}_j + \mathbf{i}_r, \quad (24)$$

where

$$\mathbf{z}_0^T = [\mathbf{u}_0^T, \mathbf{v}_0^T, \mathbf{0}^T, \dots, \mathbf{0}^T]. \quad (25)$$

The initial values for the internal variables follow from their definition in Eq. (5). Eq. (24) can now be used to compute  $\mathbf{z}(t)$  at discrete points. The result in Eq. (24) is also identical with the transition process based on a  $\mathbf{P}_{11}$  Padé-expansion of an exponential representation proposed by Ruge and Wagner [18]. A result similar to Eq. (24) can also be obtained for the rank deficient case.

#### 4. Numerical realization

From Eq. (24) note that factorization (LU factorization for example) of the matrix  $\mathbf{S} = \mathbf{B} - \frac{h}{2} \mathbf{A}$  is required to compute each  $\mathbf{z}_j$ . The order of the matrix  $\mathbf{S}$  is often too large to do efficient numerical computations. For example, for a three DOF system with four exponential terms, factorization of a  $18 \times 18$  matrix may be required at each time step. Therefore, although the numerical method outlined the previous section is very simple, it is required to reduce the computational effort in order to put the method into practice. Following Ruge [19], the most important properties required for an effective numerical scheme are as follows:

- The coefficient matrix  $\mathbf{S}$  of the time-stepping scheme  $\mathbf{Sx} = \mathbf{p}$  finally to be solved should contain the original system matrices  $\mathbf{M}$ ,  $\mathbf{C}_k$ ,  $\mathbf{K}$  in their original form without any mass scaling form  $\mathbf{M}^{-1}\mathbf{K}$  or products like  $\mathbf{MK}$ .
- Any special properties of  $\mathbf{M}$ ,  $\mathbf{C}_k$ ,  $\mathbf{K}$  like sparsity or symmetry should be preserved when generating  $\mathbf{S}$ .
- The column  $\mathbf{x}$  of unknowns finally to be solved for should contain nothing but the  $N$  nodal quantities  $\mathbf{u}$ . Thus,  $\mathbf{S}$  should be of order  $N \times N$ .

An essential advantage of the  $\mathbf{P}_{11}$ -scheme employed here is the possibility of satisfying these requirements. In the following section we outline how one can achieve them. It may be noted that in some problems the use of variable time-steps may be desired. Variable time-stepping algorithms have not been considered in this paper and our discussions are confined to fixed time-stepping algorithm only.

##### 4.1. Case A: All $\mathbf{C}_k$ matrices are of full rank

The second row in (24) is used to eliminate the velocities

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (26)$$

All succeeding rows in (24) are employed to express the internal variables in terms of the displacements as

$$\frac{\mathbf{C}_k}{\mu_k} \mathbf{y}_{k,j+1} = \frac{2\mu_k}{2 + \mu_k h} \left[ \frac{\mathbf{C}_k}{\mu_k} (\mathbf{u}_{j+1} - \mathbf{u}_j) + \frac{\mathbf{C}_k}{2\mu_k^2} (2 - \mu_k h) \mathbf{y}_{k,j} \right]. \quad (27)$$

Substituting (26) and (27) into the first row of Eq. (24) leads to a totally discretized one-step formulation of order  $N$  as

$$\begin{aligned} & \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \mathbf{u}_{j+1} \\ & = \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] \mathbf{u}_j + 2\mathbf{M}\mathbf{v}_j \\ & \quad - 2h \sum_{k=1}^n \frac{\mathbf{C}_k}{2 + \mu_k h} \mathbf{y}_{k,j} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt. \end{aligned} \quad (28)$$

Simplifying Eq. (27) we also have

$$\frac{2 + \mu_k h}{2\mu_k^2} \mathbf{C}_k \mathbf{y}_{k,j+1} = \frac{2 - \mu_k h}{2\mu_k^2} \mathbf{C}_k \mathbf{y}_{k,j} + \frac{1}{\mu_k} \mathbf{C}_k [\mathbf{u}_{j+1} - \mathbf{u}_j] \quad (29)$$

or

$$\mathbf{y}_{k,j+1} = \frac{2 - \mu_k h}{2 + \mu_k h} \mathbf{y}_{k,j} + \frac{2\mu_k}{2 + \mu_k h} [\mathbf{u}_{j+1} - \mathbf{u}_j]. \quad (30)$$

Eq. (28) together with Eqs. (26) and (30) allow  $\mathbf{u}_{j+1}$  to be calculated by means of  $\mathbf{u}_j$ ,  $\mathbf{y}_{k,j} \forall k = 1, \dots, n$ . Note that

this scheme meets all the requirements for an effective numerical scheme just mentioned.

4.2. Case B:  $C_k$  matrices are rank deficient

We rewrite Eq. (24) for the rank deficient case as

$$\left[ \tilde{\mathbf{B}} - \frac{h}{2} \tilde{\mathbf{A}} \right] \tilde{\mathbf{z}}_{j+1} = \left[ \tilde{\mathbf{B}} + \frac{h}{2} \tilde{\mathbf{A}} \right] \tilde{\mathbf{z}}_j + \tilde{\mathbf{i}}_r, \quad (31)$$

where

$$\tilde{\mathbf{z}}_0^T = [\mathbf{u}_0^T, \mathbf{v}_0^T, \mathbf{0}_{r_1}^T, \dots, \mathbf{0}_{r_n}^T] \quad (32)$$

and

$$\tilde{\mathbf{i}}_r = \int_{jh}^{(j+1)h} \tilde{\mathbf{r}}(t) dt. \quad (33)$$

From the second row in (31), the velocities can be eliminated by using (26). All succeeding rows in (31) are employed to express the internal variables in terms of the displacements

$$\frac{C_k \mathbf{R}_k}{\mu_k} \tilde{\mathbf{y}}_{k,j+1} = \frac{2\mu_k}{2 + \mu_k h} \left[ \frac{C_k}{\mu_k} (\mathbf{u}_{j+1} - \mathbf{u}_j) + \frac{C_k \mathbf{R}_k}{2\mu_k^2} (2 - \mu_k h) \tilde{\mathbf{y}}_{k,j} \right]. \quad (34)$$

Substituting (26) and (34) into the first row of (31) again leads to a totally discretized one-step formulation of order  $N$ , which allows  $\mathbf{u}_{j+1}$  to be calculated by means of  $\mathbf{u}_j$ ,  $\mathbf{y}_{k,j} \forall k = 1, \dots, n$  as

$$\begin{aligned} & \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \mathbf{u}_{j+1} \\ &= \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] \mathbf{u}_j + 2\mathbf{M}\mathbf{v}_j \\ & - 2h \sum_{k=1}^n \frac{C_k \mathbf{R}_k}{2 + \mu_k h} \tilde{\mathbf{y}}_{k,j} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt, \end{aligned} \quad (35)$$

with

$$\begin{aligned} \frac{2 + \mu_k h}{2\mu_k^2} \mathbf{R}_k^T C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,j+1} &= \frac{2 - \mu_k h}{2\mu_k^2} \mathbf{R}_k^T C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,j} \\ & + \frac{1}{\mu_k} \mathbf{R}_k^T C_k [\mathbf{u}_{j+1} - \mathbf{u}_j], \end{aligned} \quad (36)$$

or

$$\begin{aligned} [\mathbf{R}_k^T C_k \mathbf{R}_k] \tilde{\mathbf{y}}_{k,j+1} &= \frac{2 - \mu_k h}{2 + \mu_k h} [\mathbf{R}_k^T C_k \mathbf{R}_k] \tilde{\mathbf{y}}_{k,j} \\ & + \frac{2\mu_k}{2 + \mu_k h} \mathbf{R}_k^T C_k [\mathbf{u}_{j+1} - \mathbf{u}_j] \end{aligned} \quad (37)$$

and

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (38)$$

It should be noted that at the most  $n$  additional factorizations regarding the internal variables  $\mathbf{R}_k^T C_k \mathbf{R}_k$  needs

to be computed when  $r_k = \text{rank}(C_k) < N$ ,  $\forall k = 1, \dots, n$ . Nevertheless the order of those matrices is always less than  $N$ .

Recall that the computation of each  $\mathbf{R}_k$  matrix requires the solution of an eigenvalue problem involving the matrix  $C_k$ . This is a significant computation for systems with many exponential terms. Now it will be shown that these additional decompositions can be avoided. We start with the first step  $t \in [0, h]$

$$\begin{aligned} & \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \mathbf{u}_1 \\ &= \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] \mathbf{u}_0 + 2\mathbf{M}\mathbf{v}_0 \\ & - 2h \sum_{k=1}^n \frac{C_k \mathbf{R}_k}{2 + \mu_k h} \tilde{\mathbf{y}}_{k,0} + \int_0^h \mathbf{f}(t) dt, \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{2 + \mu_k h}{2\mu_k^2} \mathbf{R}_k^T C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,1} &= \frac{2 - \mu_k h}{2\mu_k^2} \mathbf{R}_k^T C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,0} \\ & + \frac{1}{\mu_k} \mathbf{R}_k^T C_k [\mathbf{u}_1 - \mathbf{u}_0], \end{aligned}$$

alternatively

$$C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,1} = \frac{2 - \mu_k h}{2 + \mu_k h} C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,0} + \frac{2\mu_k}{2 + \mu_k h} C_k [\mathbf{u}_1 - \mathbf{u}_0] \quad (40)$$

and

$$\mathbf{v}_1 = \frac{2}{h} [\mathbf{u}_1 - \mathbf{u}_0] - \mathbf{v}_0. \quad (41)$$

Since  $\mathbf{y}_{k,0} = \mathbf{0} \forall k$ , Eqs. (39) and (40) are simplified to

$$\begin{aligned} & \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \mathbf{u}_1 \\ &= \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] \mathbf{u}_0 \\ & + 2\mathbf{M}\mathbf{v}_0 + \int_0^h \mathbf{f}(t) dt, \end{aligned} \quad (42)$$

and

$$C_k \mathbf{R}_k \tilde{\mathbf{y}}_{k,1} = \frac{2\mu_k}{2 + \mu_k h} C_k [\mathbf{u}_1 - \mathbf{u}_0] =: \mathbf{s}_{k,1}, \quad (43)$$

where auxiliary quantities  $\mathbf{s}_k$  are introduced conveniently. All succeeding steps  $t \in [jh, (j+1)h]$ ,  $j \geq 1$

$$\begin{aligned} & \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \mathbf{u}_{j+1} \\ &= \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] \mathbf{u}_j + 2\mathbf{M}\mathbf{v}_j \\ & - 2h \sum_{k=1}^n \frac{\mathbf{s}_{k,j}}{2 + \mu_k h} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt, \end{aligned} \quad (44)$$

$$\mathbf{s}_{k,j+1} = \frac{2 - \mu_k h}{2 + \mu_k h} \mathbf{s}_{k,j} + \frac{2\mu_k}{2 + \mu_k h} \mathbf{C}_k [\mathbf{u}_{j+1} - \mathbf{u}_j] \quad (45)$$

and

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (46)$$

From this analysis it can be seen that it is not necessary to compute the matrices  $\mathbf{R}_k$  unless one is interested in  $\tilde{\mathbf{y}}_k$  explicitly. If, and only if, that is the case, then additional decompositions  $\mathbf{R}_k^T \mathbf{C}_k \mathbf{R}_k$  must be computed and  $\tilde{\mathbf{y}}_k$  should be obtained from (36). The full rank scenario follows directly from (35) and (36) by substituting  $\mathbf{R}_k = \mathbf{I}$ . Finally, because the modulus of the eigenvalues of the associated amplification matrix is one, the linear interpolation scheme proposed here is unconditionally stable.

## 5. Summary of the method

Following the procedure outlined in the previous sections, the time-domain response of an exponentially damped linear system can be obtained in an efficient manner. Here we briefly summarize the steps to be followed:

- (1) Select a sufficiently small step size  $h = T/N_d$  and construct the symmetric matrices

$$\mathbf{S}_1 = \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k + \frac{h}{2} \mathbf{K} \right] \quad (47)$$

and

$$\mathbf{S}_2 = \left[ \frac{2}{h} \mathbf{M} + \sum_{k=1}^n \frac{2\mu_k}{2 + h\mu_k} \mathbf{C}_k - \frac{h}{2} \mathbf{K} \right] = \mathbf{S}_1 - h\mathbf{K}. \quad (48)$$

- (2) Rank deficient [yes/no]: if yes step [3] else step [6]
- (3) Explicit computation of  $\tilde{\mathbf{y}}_k$  [yes/no]: if yes step [4] else step [5]

- (4) Explicit computation of  $\tilde{\mathbf{y}}_k$ :

- (a) Solve the eigenvalue problem  $\mathbf{C}_k \mathbf{U}_k = \mathbf{U}_k \begin{bmatrix} \mathbf{d}_k & \mathbf{O}_{1k} \\ \mathbf{O}_{1k}^T & \mathbf{O}_{2k} \end{bmatrix}$  for all  $k$ . Normalize the eigenvector matrix  $\mathbf{U}_k$  so that  $\mathbf{U}_k^T \mathbf{U}_k = \mathbf{I}$  and partition  $\mathbf{U}_k = [\mathbf{U}_{1k} | \mathbf{U}_{2k}]$ . Select  $\mathbf{R}_k = \mathbf{U}_{1k} \in \mathbb{R}^{N \times r_k}$ .

- (b) For all  $j$  solve the displacements from

$$\mathbf{S}_1 \mathbf{u}_{j+1} = \mathbf{S}_2 \mathbf{u}_j + 2\mathbf{M} \mathbf{v}_j - 2h \sum_{k=1}^n \frac{\mathbf{C}_k \mathbf{R}_k}{2 + \mu_k h} \tilde{\mathbf{y}}_{k,j} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt. \quad (49)$$

- (c) Velocities from

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (50)$$

- (d) Solve for the internal variables

$$[\mathbf{R}_k^T \mathbf{C}_k \mathbf{R}_k] \tilde{\mathbf{y}}_{k,j+1} = \frac{2 - \mu_k h}{2 + \mu_k h} [\mathbf{R}_k^T \mathbf{C}_k \mathbf{R}_k] \tilde{\mathbf{y}}_{k,j} + \frac{2\mu_k}{2 + \mu_k h} \mathbf{R}_k^T \mathbf{C}_k [\mathbf{u}_{j+1} - \mathbf{u}_j]. \quad (51)$$

- (5) Explicit computation of  $\tilde{\mathbf{y}}_k$  is omitted: for all  $j$ 
  - (a) Solve for displacements

$$\mathbf{S}_1 \mathbf{u}_{j+1} = \mathbf{S}_2 \mathbf{u}_j + 2\mathbf{M} \mathbf{v}_j - 2h \sum_{k=1}^n \frac{\mathbf{s}_{k,j}}{2 + \mu_k h} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt. \quad (52)$$

- (b) Velocities

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (53)$$

- (c) Auxiliary variables

$$\mathbf{s}_{k,j+1} = \frac{2 - \mu_k h}{2 + \mu_k h} \mathbf{s}_{k,j} + \frac{2\mu_k}{2 + \mu_k h} \mathbf{C}_k [\mathbf{u}_{j+1} - \mathbf{u}_j], \quad (54)$$

where  $\mathbf{s}_{k,1}$  are defined in Eq. (43).

- (6) Full rank scenario: for all  $j$

- (a) Solve for displacements

$$\mathbf{S}_1 \mathbf{u}_{j+1} = \mathbf{S}_2 \mathbf{u}_j + 2\mathbf{M} \mathbf{v}_j - 2h \sum_{k=1}^n \frac{\mathbf{C}_k \mathbf{y}_{k,j}}{2 + \mu_k h} + \int_{jh}^{(j+1)h} \mathbf{f}(t) dt. \quad (55)$$

- (b) Velocities

$$\mathbf{v}_{j+1} = \frac{2}{h} [\mathbf{u}_{j+1} - \mathbf{u}_j] - \mathbf{v}_j. \quad (56)$$

- (c) Solve for the internal variables

$$\mathbf{y}_{k,j+1} = \frac{2 - \mu_k h}{2 + \mu_k h} \mathbf{y}_{k,j} + \frac{2\mu_k}{2 + \mu_k h} [\mathbf{u}_{j+1} - \mathbf{u}_j]. \quad (57)$$

Numerical efficiency is mainly governed by the special properties of  $\mathbf{S}_1 \in \mathbb{R}^{N \times N}$ . If  $\mathbf{S}_1$  is symmetric positive definite a Cholesky decomposition  $\mathbf{S}_1 = \mathbf{L}\mathbf{L}^T$  is used. It is a factor of two faster than alternative methods for solving linear equations. Once  $\mathbf{S}_1$  is decomposed, the triangular factor  $\mathbf{L}$  can be used to solve (49), (52) and (55) by backsubstitution. However, if time-step adaptation is considered the decomposition  $\mathbf{S}_1$  has to be done at each time when  $h$  is modified. Finally, recall that because the viscous damping model is a special case of the exponential damping model, the proposed algorithm is also applicable to linear systems with non-proportional viscous damping.

## 6. Numerical examples

### 6.1. Example 1

A three DOF system with double exponential model is considered [17]. The equations of motion of this model



system can be represented by Eq. (1) with  $n = 2$ . The mass and the stiffness matrices of the system are given by

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix}. \quad (58)$$

The damping coefficient matrices are given by

$$\mathbf{C}_1 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix}. \quad (59)$$

For the numerical calculation we have assumed  $m_u = 3.0 \text{ kg}$ ,  $k_u = 2.0 \text{ N/m}$ ,  $c_1 = 0.6 \text{ Ns/m}$ ,  $c_2 = 0.2 \text{ Ns/m}$ ,  $\mu_1 = 1.0 \text{ s}^{-1}$  and  $\mu_2 = 5.0 \text{ s}^{-1}$ . Both the damping coefficient matrices have rank deficiency because  $r_1 = \text{rank}(\mathbf{C}_1) = 2 \leq 3$  and  $r_2 = \text{rank}(\mathbf{C}_2) = 1 \leq 3$ . The order of the system matrices in the state-space, expressed by Eq. (19), can be obtained as  $m = 2 \times 3 + (2 + 1) = 9$ .

The attention is focused on the dynamic response of the system subjected to an unit initial displacement at the first DOF, that is,  $\bar{\mathbf{f}}(i\omega) = \mathbf{0}$ ,  $\mathbf{u}_0 = \{1, 0, 0\}^T$  and  $\dot{\mathbf{u}}_0 = \mathbf{0}$ . The responses of the three masses are obtained using the direct time-domain approach outlined in Section 4 with time step  $h = 0.02 \text{ s}$ . Fig. 1 shows the response of the first DOF as a function of time.

To verify the numerical accuracy of the proposed method, the same quantity is obtained using the exact state-space mode superposition method [17] and shown in the same figure. It may be noted that the results obtained using both the approaches match with excellent accuracy. For numerical calculation a total of 100 steps with step-size  $h = 0.5 \text{ s}$  have been used. The time-domain method proposed here, however, is more efficient be-

cause the computation of state-space eigensolutions is avoided.

### 6.2. Example 2

We consider a fixed-free axially vibrating rod with double-exponential damping. A finite element model of the rod, shown in Fig. 2, with  $N$  linear elements is used.

The element stiffness matrix and the corresponding (consistent) element mass matrix for a rod element are well known

$$\mathbf{M}_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{K}_e = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (60)$$

Here  $\rho$  is the mass density,  $A$  is the area,  $E$  is the Young's modulus, and  $l_e = L/N$  is the length of an element. Numerical values of these quantities are assumed to be  $A = 6.25 \times 10^{-4} \text{ m}^2$ ,  $E = 2.1 \times 10^{11} \text{ N/m}^2$ ,  $\rho = 7.8 \times 10^3 \text{ kg/m}^3$  and the length of the rod  $L = 4 \text{ m}$ . The global stiffness matrix,  $\mathbf{K}$  and the global mass matrix,  $\mathbf{M}$  can be obtained using the usual finite-element assembly procedure. For numerical calculations 80 elements have been used so that the degrees-of-freedom of the system  $N = 80$ . Thus the response vector  $\mathbf{u}(t) = \{u_1(t), u_2(t), \dots, u_{80}(t)\}^T$  and the (discretized) equation of motion of this rod with double-exponential damping can be expressed as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t [\mu_1 e^{-\mu_1(t-\tau)} \mathbf{C}_1 + \mu_2 e^{-\mu_2(t-\tau)} \mathbf{C}_2] \dot{\mathbf{u}}(\tau) d\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t). \quad (61)$$

Here numerical values of the relaxation parameters  $\mu_1$  and  $\mu_2$  are conveniently selected as

$$\mu_1 = \frac{1}{\gamma_1 T_{\min}} \quad \text{and} \quad \mu_2 = \frac{1}{\gamma_2 T_{\min}}, \quad (62)$$

with  $\gamma_1 = 1$ ,  $\gamma_2 = 2$  and  $T_{\min}$  is the minimum time period given by  $T_{\min} = 2\pi/\omega_{\max}$ . For an axially vibrating rod, the highest undamped natural frequency  $\omega_{\max} = \sqrt{\frac{E}{\rho}} \frac{2N-1}{2L} \pi$ . The global damping coefficient matrices  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are assumed to be proportional to the mass and stiffness matrices respectively

$$\mathbf{C}_1 = \alpha \mathbf{M} \quad \text{and} \quad \mathbf{C}_2 = \beta \mathbf{K}. \quad (63)$$

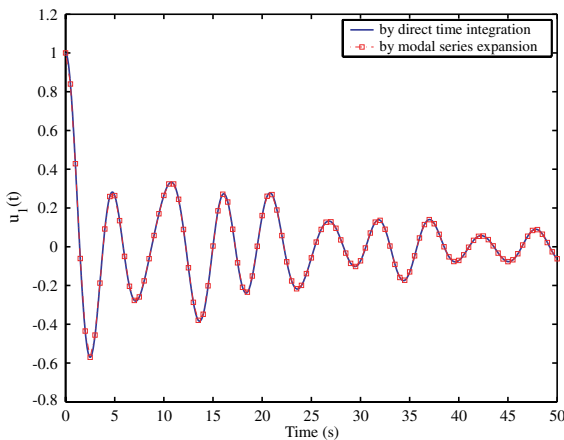


Fig. 1. Displacement at the first DOF.

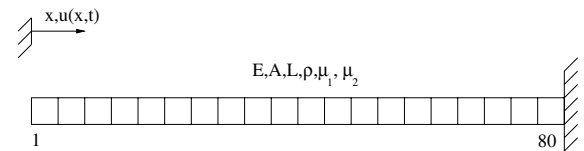


Fig. 2. Axially vibrating free-fixed rod with double-exponential damping.

The proportionality constants  $\alpha$  and  $\beta$  are assumed to be

$$\alpha = 2\xi \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \quad \text{and} \quad \beta = 2\xi \frac{1}{\omega_1 + \omega_2}, \quad (64)$$

where the damping factor  $\xi = 0.05$  (i.e., 5% damping) and  $j$ th undamped natural frequency  $\omega_j = \sqrt{\frac{E}{\rho} \frac{2j-1}{2L}} \pi$ .

The interest here is to obtain the time-domain response of the free end of the rod subjected to an unit initial velocity at the same end. For this problem the forcing vectors  $\mathbf{f}(t) = \mathbf{0}$  and the initial conditions  $\mathbf{u}_0 = \mathbf{0}$ ,  $\mathbf{v}_0 = \mathbf{e}_1$  (unit vector). The problem is solved by the time-stepping procedure developed in this paper. Because we have assumed proportional damping and  $\mathbf{M}$ ,  $\mathbf{K}$  are symmetric positive definite, the matrices  $\mathbf{C}_1$  and  $\mathbf{C}_2$ , are of full rank. Figs. 3 and 4 show the tip displacement  $u(x=0, t) = u_1(t)$  and the tip velocity  $\dot{u}(x=0, t) = v_1(t)$  as a function of time.

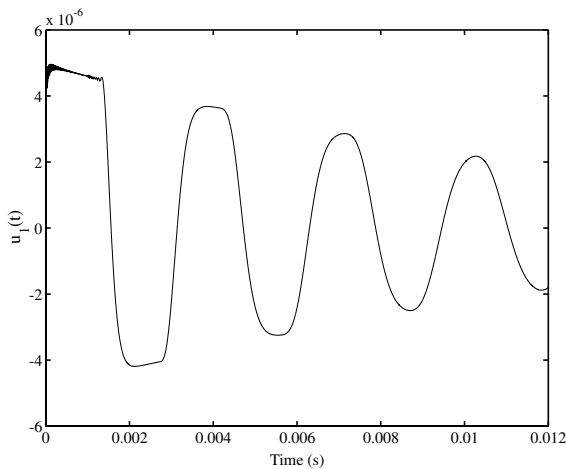


Fig. 3. Tip displacement of the axially vibrating rod.

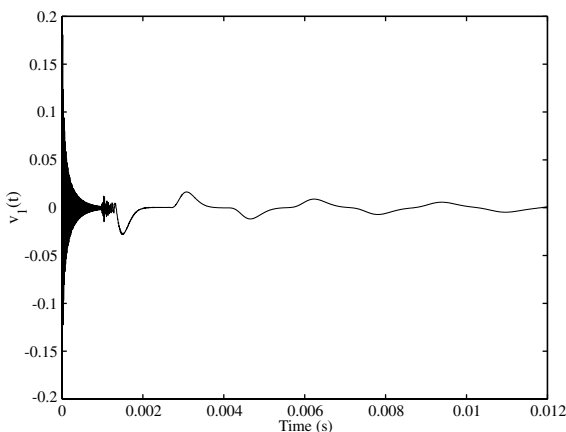


Fig. 4. Tip velocity of the axially vibrating rod.

For numerical computations a total of 8000 steps with step-size  $h = 1.5 \times 10^{-6}$ s have been used.

## 7. Conclusions

Time-domain analysis of linear systems with exponentially decaying damping memory kernels has been considered. The proposed method is based on an extended state-space representation of the equations of motion. The novelty of the proposed method is that the coefficient matrix of the time-stepping scheme finally to be solved contain nothing but a linear combination of the original system matrices  $\mathbf{M}$ ,  $\mathbf{C}_k$  and  $\mathbf{K}$ . This is useful because we can benefit from the special properties of  $\mathbf{M}$ ,  $\mathbf{C}_k$  and  $\mathbf{K}$ , like sparsity or symmetry. Thus, although the problem is originally formulated in the extended state-space, in effect the actual solution is carried out in the original reduced space. This implies that the extra effort needed to handel non-viscous damping model like the one considered here is small compared to a viscous damping model. The proposed algorithm also extends to the physically realistic case when some or all damping coefficient matrices  $\mathbf{C}_k$  are rank-deficient. If the interest is limited to the evolution of displacements and velocities only (not the internal variables), the proposed algorithm allows to bypass the expensive process of factorization of the  $\mathbf{C}_k$  matrices for the rank-deficient case. The proposed algorithm currently allows only uniform time-steps. In principle it can be extended to variable time-steps but at additional computational cost. Research is underway to develop efficient numerical algorithms for variable time stepping.

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