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ORIGINAL PAPER

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Flexoelectric effect on vibration responses of piezoelectric nanobeams embedded in viscoelastic medium based on nonlocal elasticity theory

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Abstract In this study, vibration characteristics of a piezoelectric nanobeam embedded in a viscoelastic medium are investigated based on nonlocal Euler–Bernoulli beam theory. In doing this, the governing equations of motion and boundary conditions for vibration analysis are first derived using Hamilton's principle, where nonlocal effect, piezoelectric effect, flexoelectric effect, and viscoelastic medium are considered simultaneously. Subsequently, the transfer function method is employed to obtain the natural frequencies and corresponding mode shapes in closed form for the embedded piezoelectric nanobeam with arbitrary boundary conditions. The proposed mechanics model is validated by comparing the obtained results with those available in the literature, where good agreement is achieved. The effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient, and viscoelastic medium on vibration responses are also examined carefully for the embedded nanobeam. The results demonstrate the efficiency and robustness of the developed model for vibration analysis of a complicated multi-physics system comprising piezoelectric nanobeam with flexoelectric effect, viscoelastic medium, and electrical loadings.

1 Introduction

Piezoelectric nanobeams have various practical applications in smart devices and systems due to their exceptional electromechanical coupling effect [1,2], such as nanotransducers, nanoresonators, nanosensors, and nanogenerators [3–6]. It is thus essential to quantify and understand the vibration behaviors of piezoelectric nanobeams. As a universal electromechanical mechanism in all piezoelectric materials, flexoelectricity has been reported to have strong influence on the vibration responses of piezoelectric nanobeams [7–9]. As a result, the study of the flexoelectric effect on the vibration responses of embedded piezoelectric nanobeams may provide valuable information for the above-mentioned potential applications of piezoelectric nanobeams.

Numerous studies have been performed so far by researchers to examine the mechanical properties of piezoelectric nanobeams with flexoelectric effect. Based on the flexoelectricity theory and strain gradient theory, a size-dependent bending model was developed by Qi et al. [10] to investigate the static bending of an electro-elastic bilayer nanobeam. The influence of flexoelectric effect on the static bending and free vibration was examined by Yan and Jiang [11] for a simply supported piezoelectric nanobeam. In this study, the governing equations of motion were derived using Hamilton's principle and the explicit expressions of deflection and natural frequencies were also obtained. A modified couple stress theory and Euler–Bernoulli beam theory

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were proposed by Li and Luo [6] to study the effects of couple stress, flexoelectricity and piezoelectricity on vibration characteristics of piezoelectric microbeams. In this study, the results showed that the effective bending rigidity of the piezoelectric microbeam was hardened due to enhanced flexoelectric effect. Liang et al. [9] examined the effects of surface and flexoelectricity on static bending of piezoelectric nanobeams based on Euler–Bernoulli beam theory and variational principle. The study pointed out that the flexoelectric effect had a momentous influence on the bending rigidity of the piezoelectric nanobeam. A microscale Timoshenko beam model was developed by Yue et al. [12] to investigate the static bending and free vibration problems for a piezoelectric nanobeam with simply supported boundary conditions. Here, the governing equations of motion and related boundary conditions were derived by using the variational principle and Hamilton's principle. It is noted that in the above studies the size effect was taken into account by considering the flexoelectric effect of piezoelectric nanobeams based on strain gradient theory.

However, some researchers point out that nonlocal elasticity theory should be incorporated to the strain gradient theory for the more accurate prediction of a mechanical behavior of nanostructures [13–15]. On this basis, to include the nonlocal effect in vibration analysis of piezoelectric nanobeams, Ebrahimi and Barati [8] investigated the vibration characteristics of a flexoelectric nanobeam resting on Winkler–Pasternak elastic foundation based on nonlocal elasticity theory. In this study, Hamilton's principle was used to derive the governing equations of motion and a Galerkin-based method was applied to obtain the natural frequencies. Based on nonlocal Timoshenko beam theory, Ke and Wang [16] derived the governing equations and boundary conditions for vibration analysis of piezoelectric nanobeams by utilizing Hamilton's principle and computed the natural frequencies for various boundary conditions by using differential quadrature method. Nonlocal Timoshenko beam theory was also used by Ke et al. [17] to investigate the nonlinear vibration of piezoelectric nanobeams, which were subjected to an applied voltage and a uniform temperature change. Thermo-electro-mechanical vibration was examined by Ansari et al. [18] for a postbuckled piezoelectric Timoshenko nanobeam based on nonlocal elasticity theory. In the study, the governing equations of motion were derived using Hamilton's principle and then solved via generalized differential quadrature (GDQ) method.

It is noted that piezoelectric nanobeams are often embedded in a medium in many of their nanotechnology applications, such as nanosensors and nanogenerators [5,8,19]. In particular, some media employed in the nanotechnology applications normally exhibit viscoelastic behavior. To the best of the present authors' knowledge, the vibration responses of piezoelectric nanobeams with both flexoelectric effect and nonlocal effect have not been reported in the literature when the surrounding viscoelastic medium is taken into account. The information, however, is essential for the engineering applications of piezoelectric nanobeams in nanotechnology. Hence, the objective of the present work is to study the vibration responses of a piezoelectric nanobeam with flexoelectric and nonlocal effects, which is embedded in viscoelastic medium and subjected to electrical loadings. Here, the nonlocal Euler–Bernoulli beam model is employed to derive the governing equations and a numerical approach named the transfer function method (TFM) is proposed to calculate the natural frequencies of nanobeams with arbitrary boundary conditions. Subsequently, a detailed parametric study is conducted to investigate the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient, and viscoelastic medium on the vibration responses of piezoelectric nanobeams.

2 Mathematical modeling

Here let us consider a piezoelectric nanobeam with flexoelectric and nonlocal effects, which is embedded in a viscoelastic medium and subjected to electrical loadings, as shown in Fig. 1. The piezoelectric nanobeam is modeled as nonlocal Euler–Bernoulli beam with length L, thickness h and width b. The viscoelastic medium is described by a visco-Pasternak foundation model, whose Winkler's modulus parameter is k_w, Pasternak's modulus parameter is k_G and damping parameter is c_t. A Cartesian coordinate system $oxyz$ is also defined, in which the x-, y- and z-axes are taken along the length, width, and thickness directions of the nanobeam, respectively. To account for the flexoelectric effect, the electric Gibbs free energy density G_b can be written as [6]

$$G_b = -\frac{1}{2}\kappa_{ij} E_i E_j - \frac{1}{2} b_{ijkl} E_{i,j} E_{k,l} + \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - e_{ijk} E_i \varepsilon_{jk} - \mu_{ijkl} \left(E_k \varepsilon_{ij,l} - \varepsilon_{ij} E_{k,l} \right), \tag{1}$$

where κ_{ij}, b_{ijkl}, c_{ijkl}, e_{ijk} and μ_{ijkl} are dielectric constant tensor, nonlocal electrical coupling coefficient tensor, elastic stiffness tensor, piezoelectric coefficient tensor, and flexoelectric coefficient tensor, respectively. In addition, E_i and E_j are electric field vectors, ε_{ij}, and ε_{kl} are strain tensors, and $\varepsilon_{ij,l}$ and $E_{k,l}$ are the gradients of strain and electric field, respectively.

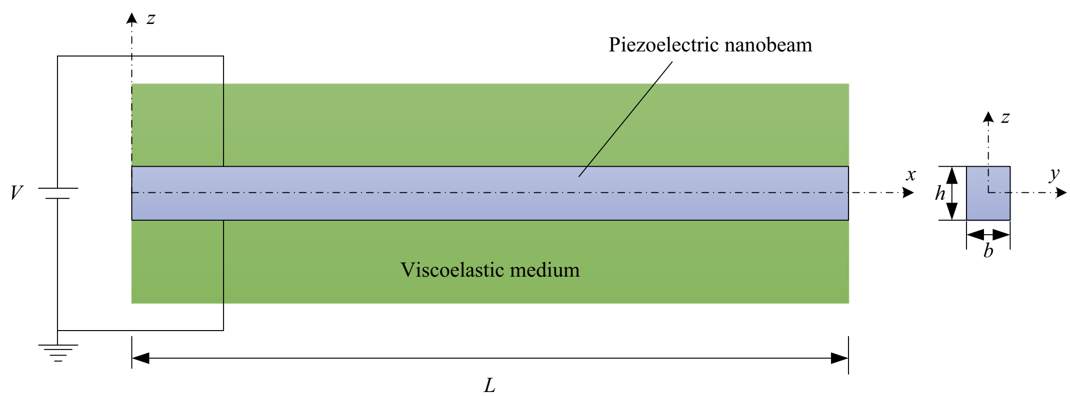

Fig. 1 Schematic of a piezoelectric nanobeam embedded in viscoelastic medium under electrical field

For an Euler–Bernoulli beam, the displacement field can be expressed as

$$u = -z\frac{\partial w}{\partial x},\ v = 0,\ w = w(x, t), \tag{2}$$

where u, v, and w are the displacement components along the x-, y-, and z-directions on the cross section. Accordingly, the components of strains and strain gradients are given as

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2},\ \varepsilon_{xx,z} = -\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

Based on nonlocal Euler–Bernoulli beam theory [20,21], the constitutive equations for piezoelectric nanobeams with flexoelectric effect can be derived in terms of the electric Gibbs free energy density as

$$\left(1 - (e_0a)^2\nabla^2\right)\sigma_{xx} = c_{11}\varepsilon_{xx} - e_{31}E_z + \mu_{31}E_{z,z}, \tag{4}$$

$$\left(1 - (e_0a)^2\nabla^2\right)\tau_{xxz} = -\mu_{31}E_z, \tag{5}$$

$$\left(1 - (e_0a)^2\nabla^2\right)D_z = \kappa_{33}E_z + e_{31}\varepsilon_{xx} + \mu_{31}\varepsilon_{xx,z}, \tag{6}$$

$$\left(1 - (e_0a)^2\nabla^2\right)Q_{zz} = b_{33}E_{z,z} - \mu_{31}\varepsilon_{xx}, \tag{7}$$

where σ_{xx}, τ_{xxz}, D_z and Q_{zz} denote the nonlocal stress, higher-order stress, electric displacement, and electric quadrupole, respectively. Moreover, e_0a is the nonlocal parameter and ∇ denotes the Hamilton arithmetic operator. In addition, the poling direction of the piezoelectric material is assumed to coincide with the z-direction, and only the electric field in the z-direction is considered [12], i.e.,

$$E_z = -\frac{\partial \Phi}{\partial z}, \tag{8}$$

where Φ is the electric potential. In the absence of free electric charges, Gauss's law requires

$$-\frac{\partial^2 Q_{zz}}{\partial z^2} + \frac{\partial D_z}{\partial z} = 0. \tag{9}$$

Substituting Eqs. (6) and (7) into Eq. (9) yields

$$\kappa_{33}E_{z,z} + e_{31}\varepsilon_{xx,z} + 2\mu_{31}\varepsilon_{xx,zz} - b_{33}E_{z,zzz} = 0. \tag{10}$$

In addition, the boundary conditions of electric potential Φ are given by

$$\Phi\left(-\frac{h}{2}\right) = 0,\quad \Phi\left(\frac{h}{2}\right) = V. \tag{11}$$

From Eqs. (3), (10), and (11), the expression of electric potential Φ can be determined as

$$\Phi = -\frac{e_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z^2 - \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{V}{h}z + \frac{e_{31}h^2}{8\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + \frac{V}{2} + \frac{e_{31}}{\eta^2\kappa_{33}}\frac{e^{\eta z}+e^{-\eta z}}{e^{\eta z/2}+e^{-\eta z/2}}\frac{\partial^2 w}{\partial x^2} + \frac{\mu_{31}h}{4\eta^2\kappa_{33}}\frac{e^{\eta z}-e^{-\eta z}}{e^{\eta z/2}-e^{-\eta z/2}}\frac{e^{\eta z}+e^{-\eta z}}{e^{\eta z/2}+e^{-\eta z/2}}, \tag{12}$$

where $\eta = \sqrt{\kappa_{33}/b_{33}}$. For simplification of analysis, the high-order nonlocal electrical coupling effect in Eq. (12) is neglected by taking $b_{33} = 0$, and we have [6]

$$\Phi = -\frac{e_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z^2 - \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{V}{h}z + \frac{e_{31}h^2}{8\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + \frac{V}{2}. \tag{13}$$

Substituting Eq. (13) into Eqs. (4)–(8) leads to

$$E_z = \frac{e_{31}}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2} - \frac{V}{h}, \tag{14}$$

$$E_{z,z} = \frac{e_{31}}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2}, \tag{15}$$

$$\left(1-(e_0a)^2\nabla^2\right)\sigma_{xx} = -c_{11}\frac{\partial^2 w}{\partial x^2}z - \frac{e_{31}^2}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{e_{31}\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + e_{31}\frac{V}{h}, \tag{16}$$

$$\left(1-(e_0a)^2\nabla^2\right)\tau_{xxz} = -\frac{e_{31}\mu_{31}}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z - \frac{\mu_{31}^2}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + \mu_{31}\frac{V}{h}, \tag{17}$$

$$\left(1-(e_0a)^2\nabla^2\right)D_z = -\frac{\mu_{31}}{2}\frac{\partial^2 w}{\partial x^2} - \kappa_{33}\frac{V}{h}. \tag{18}$$

The governing equations and boundary conditions for vibration analysis of piezoelectric nanobeams can be obtained by using Hamilton's principle, i.e.,

$$\int_0^t (\delta\Pi_{\mathrm{k}} + \delta\Pi_{\mathrm{F}} - \delta\Pi_{\mathrm{s}})\,\mathrm{d}t = 0, \tag{19}$$

where the strain energy Π_{s}, kinetic energy Π_{k} and external work Π_{F} can be calculated by the following equations

$$\Pi_{\mathrm{s}} = \frac{1}{2}\int_0^L\int_A \left(\sigma_{xx}\varepsilon_{xx} + \tau_{xxz}\varepsilon_{xx,z}\right)\mathrm{d}A\mathrm{d}x = -\frac{1}{2}\int_0^L\left(M_{xx}\frac{\partial^2 w}{\partial x^2} + P_{xxz}\frac{\partial^2 w}{\partial x^2}\right)\mathrm{d}x, \tag{20}$$

$$\Pi_{\mathrm{k}} = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2 dx, \tag{21}$$

$$\Pi_{\mathrm{F}} = -\frac{1}{2}\int_0^L\left[bN_Q w + N_{xx}\left(\frac{\partial w}{\partial x}\right)^2\right]dx. \tag{22}$$

Here M_{xx} is the internal bending moment, P_{xxz} is the higher-order axial couple, N_{xx} is the internal stress resultant and N_Q is the reaction of the viscoelastic medium, which can be expressed as

$$M_{xx} = \int_A \sigma_{xx}z\mathrm{d}A, \quad P_{xxz} = \int_A \tau_{xxz}\mathrm{d}A, \quad N_{xx} = \int_A \sigma_{xx}\mathrm{d}A, \tag{23}$$

$$N_Q = k_w w - k_G\nabla^2 w + c_t\frac{\partial w}{\partial t}. \tag{24}$$

Substituting Eqs. (20)–(22) into Hamilton's principle (19), the governing equation of embedded piezoelectric nanobeams with flexoelectric effect can be derived by integrating it by parts and setting the coefficients in front of δw to zero:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 P_{xxz}}{\partial x^2} + N_{xx}\frac{\partial^2 w}{\partial x^2} - b\left(k_w w - k_G\nabla^2 w + c_t\frac{\partial w}{\partial t}\right) = \rho A\frac{\partial^2 w}{\partial t^2}. \tag{25}$$

The corresponding boundary conditions also can be obtained as follows:

$$w = 0 \text{ or } \frac{\partial M_{xx}}{\partial x} + \frac{\partial P_{xxz}}{\partial x} + N_{xx}\frac{\partial w}{\partial x} = 0, \tag{26}$$

$$\frac{\partial w}{\partial x} = 0 \text{ or } M_{xx} + P_{xxz} = 0. \tag{27}$$

From Eqs. (3), (16), and (17), we have

$$\left(1 - (e_0 a)^2 \nabla^2\right) M_{xx} = -\left(c_{11} + \frac{e_{31}^2}{\kappa_{33}}\right) \frac{bh^3}{12} \frac{\partial^2 w}{\partial x^2}, \tag{28}$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) P_{xxz} = -\frac{\mu_{31}^2 bh}{2\kappa_{33}} \frac{\partial^2 w}{\partial x^2} + \mu_{31} bV, \tag{29}$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) N_{xx} = e_{31} bV. \tag{30}$$

Inserting Eqs. (28)–(30) into the governing equation (25) leads to

$$\begin{aligned} &-\left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \bar{N}_{xx} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right)\left(\bar{k}_w \bar{w} - \bar{k}_G \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \bar{c}_t \frac{\partial \bar{w}}{\partial \iota}\right) \\ &= \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \eta \frac{\partial^2 \bar{w}}{\partial \iota^2}, \end{aligned} \tag{31}$$

in which the dimensionless terms are defined for the sake of convenience and generality as follows:

$$\begin{aligned} &\bar{x} = \frac{x}{L}, \bar{w} = \frac{w}{L}, \alpha = \frac{e_0 a}{L}, \bar{c}_{11} = c_{11} L, \eta = \frac{h}{L}, \bar{k}_w = \frac{k_w L^2}{\bar{c}_{11}}, \bar{k}_G = \frac{k_G}{\bar{c}_{11}}, \bar{c}_t = \frac{c_t L}{\sqrt{\rho L \bar{c}_{11}}}, \\ &\iota = \frac{t}{L}\sqrt{\frac{\bar{c}_{11}}{\rho L}}, \bar{P}_{xxz} = \frac{\mu_{31}^2 \eta}{2\kappa_{33} \bar{c}_{11} L}, \bar{N}_{xx} = \frac{e_{31} V}{\bar{c}_{11}}, \bar{M}_{xx} = \left(1 + \frac{e_{31}^2 L}{\kappa_{33} \bar{c}_{11}}\right) \frac{\eta^3}{12}. \end{aligned} \tag{32}$$

Using these dimensionless terms, the boundary conditions can be rewritten as

$$\bar{w} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} - \bar{N}_{xx} \frac{\partial \bar{w}}{\partial \bar{x}} = 0, \tag{33}$$

$$\frac{\partial \bar{w}}{\partial \bar{x}} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \frac{\mu_{31} V}{\bar{c}_{11} L} = 0. \tag{34}$$

The governing equation (31) satisfying the boundary conditions in Eqs. (33) and (34) can be solved by assuming the solutions in the form

$$\bar{w}(\bar{x}) = \bar{W}(\bar{x})\, e^{i\Omega\iota}, \tag{35}$$

where Ω is the dimensionless angular frequency, and $\bar{W}$ is the corresponding mode shape. Also, the dimensionless angular frequency Ω can be given by

$$\Omega = \omega L \sqrt{\frac{\rho L}{\bar{c}_{11}}}. \tag{36}$$

Substituting Eq. (35) into Eqs. (31), (33) and (34), one has

$$\begin{aligned} &-\left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^4 \bar{W}}{\partial \bar{x}^4} + \bar{N}_{xx} \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} - \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right)\left(\bar{k}_w \bar{W} - \bar{k}_G \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} + i\Omega \bar{c}_t \bar{W}\right) \\ &= -\Omega^2 \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \eta \bar{W}, \end{aligned} \tag{37}$$

$$\bar{W} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^3 \bar{W}}{\partial \bar{x}^3} - \bar{N}_{xx} \frac{\partial \bar{W}}{\partial \bar{x}} = 0, \tag{38}$$

$$\frac{\partial \bar{W}}{\partial \bar{x}} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} - \frac{\mu_{31} V}{\bar{c}_{11} L} = 0. \tag{39}$$

It should be noted the proposed mechanics model above is a simplified model, which is only available for vibration analysis of piezoelectric nanobeams as the electric field and strain gradient are homogeneous in the direction of width b. In addition, the governing Eq. (37) is a fourth-order ordinary differential equation for $\bar{W}$, and the coefficients in front of $\bar{W}$ and its derivatives are the functions of Ω. In the next section, the natural frequencies and corresponding mode shapes in closed form are calculated for the embedded piezoelectric nanobeam with arbitrary boundary conditions by using the TFM.

3 Transfer function method

To achieve the eigenvalues and frequency response functions, the state vector $\boldsymbol{\eta}(\bar{x}, \Omega)$ is defined as

$$\boldsymbol{\eta}(\bar{x}, \Omega) = \left[\bar{W}, \frac{\mathrm{d}\bar{W}}{\mathrm{d}\bar{x}}, \frac{\mathrm{d}^2\bar{W}}{\mathrm{d}\bar{x}^2}, \frac{\mathrm{d}^3\bar{W}}{\mathrm{d}\bar{x}^3}\right]^{\mathrm{T}}, \tag{40}$$

where the superscript “T” denotes the matrix transpose. Then the governing equation (37) can be rewritten in matrix form as

$$\frac{\mathrm{d}\boldsymbol{\eta}(\bar{x}, \Omega)}{\mathrm{d}\bar{x}} = \boldsymbol{F}(\Omega)\, \boldsymbol{\eta}(\bar{x}, \Omega), \tag{41}$$

where

$$\boldsymbol{F}(\Omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_1 & 0 & f_2 & 0 \end{bmatrix} \tag{42}$$

and

$$f_1 = \frac{\Omega^2 \eta - \bar{k}_w - \mathrm{i}\Omega \bar{c}_t}{\bar{M}_{xx} + \bar{P}_{xxz} + \alpha^2 \bar{k}_w}, \quad f_2 = \frac{\bar{N}_{xx} + \bar{k}_G + \alpha^2 \bar{k}_w + \alpha^2 \mathrm{i}\Omega \bar{c}_t - \alpha^2 \Omega^2 \eta}{\bar{M}_{xx} + \bar{P}_{xxz} + \alpha^2 \bar{k}_w}. \tag{43}$$

Also, the boundary conditions can be expressed in matrix form as

$$\boldsymbol{M}(\Omega)\, \boldsymbol{\eta}(0, \Omega) + \boldsymbol{N}(\Omega)\, \eta(1, \Omega) = 0, \tag{44}$$

where $\boldsymbol{M}(\Omega)$ and $\boldsymbol{N}(\Omega)$ are the boundary condition set matrices at the left and the right ends of the nanobeam, respectively. To demonstrate the approach, several typical boundary condition set matrices are given as examples in the following.

1. For clamped-clamped (C–C) boundary conditions, we have

$$\boldsymbol{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{45}$$

2. For simply supported–simply supported (S–S) boundary conditions, one has

$$\boldsymbol{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \end{bmatrix}, \tag{46}$$

where

$$M_1 = \frac{\mu_{31} V}{\bar{c}_{11} L \left(\bar{M}_{xx} + \bar{P}_{xxz}\right)}. \tag{47}$$

3. For clamped-free (C–F) boundary conditions, $\boldsymbol{M}(\varOmega)$ is given by Eq. (45), and $\boldsymbol{N}(\varOmega)$ can be expressed as

$$\boldsymbol{N}(\varOmega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & -\bar{N}_{xx} & 0 & \bar{M}_{xx} + \bar{P}_{xxz} \end{bmatrix}. \tag{48}$$

The solution of Eq. (41) can be calculated as

$$\boldsymbol{\eta}(\bar{x}, \varOmega) = e^{\boldsymbol{F}(\varOmega)\bar{x}} \boldsymbol{\eta}(0, \varOmega). \tag{49}$$

Substituting Eqs. (49) into (44), one has

$$\left[\boldsymbol{M}(\varOmega) + \boldsymbol{N}(\varOmega) e^{\boldsymbol{F}(\varOmega)}\right] \boldsymbol{\eta}(0, \varOmega) = 0. \tag{50}$$

Thus the dimensionless frequencies $\varOmega$ of the nanobeam can be obtained by solving the following transcendental characteristic equation

$$\det\left[\boldsymbol{M}(\varOmega) + \boldsymbol{N}(\varOmega) e^{\boldsymbol{F}(\varOmega)}\right] = 0. \tag{51}$$

In addition, the mode shape corresponding to the dimensionless frequency $\varOmega_j$ can be expressed as

$$\boldsymbol{\eta}\left(\bar{x}, \varOmega_j\right) = e^{\boldsymbol{F}(\varOmega_j)\bar{x}} \boldsymbol{\eta}\left(0, \varOmega_j\right). \tag{52}$$

According to Eq. (36), the natural frequencies ω of the embedded piezoelectric nanobeam with flexoelectric effect can be calculated from

$$\omega = \frac{\varOmega}{L}\sqrt{\frac{\bar{c}_{11}}{\rho L}}. \tag{53}$$

4 Numerical results and discussion

In this section, the formulation achieved above is first validated by comparing the obtained results with those available in the literature. This is followed by a detailed parametric study of the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient and viscoelastic medium on the vibration responses of nanobeams. In doing this, the material of the piezoelectric nanobeam with flexoelectric effect is assumed to be $BaTiO_3$. Unless otherwise stated, the values of some parameters used for numerical calculations are given as follows: The length of the nanobeam $L = 40\,\text{nm}$, thickness $h = 2\,\text{nm}$, width $b = h\,\text{nm}$, mass density $\rho = 7500\,\text{kg/m}^3$, elastic stiffness $c_{11} = 131\,\text{GPa}$, flexoelectric coefficient $\mu_{31} = 1\times10^{-6}\,\text{C/m}$, dielectric constant $\kappa_{33} = 12.56 \times 10^{-9}\,\text{C/V m}$ and piezoelectric coefficient $e_{31} = -4.35\,\text{C/m}^2$. These geometrical and material properties of the piezoelectric nanobeam are adopted from the papers [6,11,12].

The dimensionless fundamental frequencies of piezoelectric nanobeams with various boundary conditions and slenderness ratios L/h in comparison with those of Ref. [17] are listed in Table 1. In this calculation, the values of basic parameters are the same as those in the paper [17], and the effects of flexoelectricity and viscoelastic medium are omitted by taking $\mu_{31} = 0$, $k_w = 0$, $k_G = 0$ and $c_t = 0$. Table 1 shows that the results of this study are in good agreement with those in the literature, which verifies the accuracy and efficiency of the proposed method for vibration analysis of piezoelectric nanobeams.

For future comparisons with other researchers, Table 2 presents the first three dimensionless frequencies of piezoelectric nanobeams in the absence of medium (i.e., $k_w = 0$, $k_G = 0$ and $c_t = 0$) and in the presence of viscoelastic medium ($k_w = 1\,\text{GPa/nm}$, $k_G = 0.25\,\text{GPa/nm}$ and $c_t = 10^{-4}\,\text{GPa ns/nm}$) under typical boundary conditions, including C–F, S–S and C–C. From Table 2, we can observe that the imaginary parts related to damping ratios appear in the natural frequencies of nanobeams when the viscoelastic medium is considered.

Table 1 Dimensionless fundamental frequencies of piezoelectric nanobeams with various boundary conditions and slenderness ratios L/h in comparison with those of Ref. [17]

BCs	L/h					
	6	8	10	16	20	30
S–S						
Present	0.4571	0.3428	0.2743	0.1714	0.1371	0.0914
Ke et al. [17]	0.4570	0.3428	0.2742	0.1714	0.1371	0.0914
C-S						
Present	0.7079	0.5309	0.4248	0.2655	0.2124	0.1416
Ke et al. [17]	0.7077	0.5310	0.4250	0.2658	0.2127	0.1420

Table 2 First three dimensionless frequencies of piezoelectric nanobeams with different boundary conditions and nonlocal parameter α

BCs	In the absence of medium			In the presence of viscoelastic medium		
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$
C–F	1.5331	1.5399	1.5607	2.9086 + 0.0319i	2.9121 + 0.0319i	2.9232 + 0.0319i
	9.6082	8.6976	7.0049	9.9219 + 0.0319i	9.0429 + 0.0319i	7.4294 + 0.0319i
	26.9033	21.1570	14.4468	27.0176 + 0.0319i	21.3022 + 0.0319i	14.6586 + 0.0319i
S–S	4.3036	4.1058	3.6440	4.9835 + 0.0319i	4.7930 + 0.0319i	4.4039 + 0.0319i
	17.2145	14.5761	10.7191	17.3921 + 0.0319i	14.7853 + 0.0319i	11.0019 + 0.0319i
	38.7327	28.1868	18.1521	38.8125 + 0.0319i	28.2964 + 0.0319i	18.3218 + 0.0319i
C–C	9.7557	8.8189	7.0870	10.0648 + 0.0319i	9.1597 + 0.0319i	7.5069 + 0.0319i
	26.8912	21.1150	14.4430	27.0067 + 0.0319i	21.2952 + 0.0319i	14.6549 + 0.0319i
	52.7197	35.4709	21.8228	52.7787 + 0.0319i	35.5585 + 0.0319i	21.9649 + 0.0319i

The reason for this is that the damping effect of the viscoelastic medium is introduced into the system. It also can be seen that boundary conditions have strong influence on the real parts of natural frequencies but have no effect on the imaginary parts. Furthermore, nonlocal parameter α has no effect on the imaginary parts of natural frequencies. This is because that the imaginary parts of natural frequencies are only related to the damping of the viscoelastic medium, in which nonlocal effect is not taken into account.

Since nonlocal parameter α has no effect on the imaginary parts of natural frequencies for embedded piezoelectric nanobeams, only the effect of nonlocal parameter α on the real parts of the first three natural frequencies are presented in Figs. 2, 3 and 4. Here the frequency ratios $Re(\omega_{\mathrm{NL}}/\omega_{\mathrm{L}})$ in the vertical axis are used to denote the real part of the ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ between the two frequencies ω_{NL} and ω_{L}, which are the damped frequencies obtained based on the nonlocal and local (classical) mechanics theories, respectively. Figure 2 shows that the real parts of the first frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ decrease significantly with rising nonlocal parameter α for both S–S and C–C nanobeams but increase slightly for C–F nanobeams. This suggests that the rigidity of the embedded nanobeam is reduced due to enhanced nonlocal effect for S–S and C–C boundary conditions but hardened for C–F boundary conditions. A similar phenomenon was also described by Lei et al. [20] and Lu et al. [22]. Furthermore, such an effect of nonlocal parameter α turns out to be less pronounced when the viscoelastic medium is considered. For example, as α increases from 0 to 0.2 the real parts of the first frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ for S–S nanobeams decrease about 15.33% in the absence of viscoelastic medium but about 10.99% in the presence of viscoelastic medium. The strong influence of boundary conditions on vibration responses of nanobeams also is observed in Fig. 2. The values of the first frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ follow the order: C–C < S–S < C–F, which implies that the nonlocal effect on the natural frequencies is more substantial when the stronger constrains are imposed on the boundaries. The effect of nonlocal parameter α on the higher frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ is also investigated in Figs. 3 and 4. It can be seen from the figures that the real parts of the higher frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ decrease significantly with α no matter what boundary conditions are imposed on the boundaries. In addition, the nonlocal effect on the higher frequency ratios turns out to be much larger than its effect on the fundamental ones. For instance, as α changes from 0 to 0.2 the real parts of the second frequency ratios $\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$ for S–S nanobeams with and without viscoelastic medium decrease about 52.79 and 53.14%, respectively, in Fig. 3 much larger than the values shown above for the first frequency ratios. As expected, the effect of viscoelastic medium on the frequency ratios becomes less pronounced as the mode number increases.

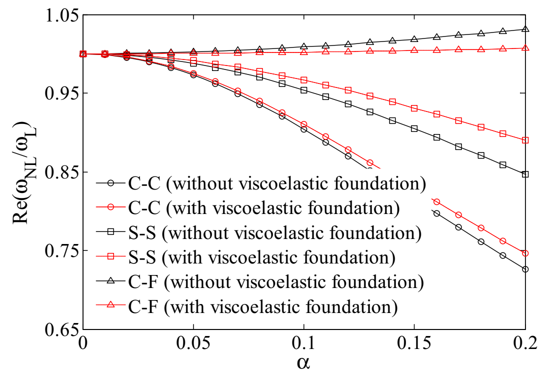

Fig. 2 Effect of nonlocal parameter α on the real parts of the first frequency ratios ω_{NL}/ω_L for piezoelectric nanobeams with various boundary conditions

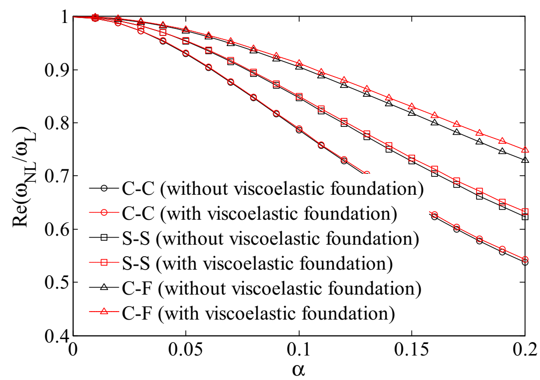

Fig. 3 Effect of nonlocal parameter α on the real parts of the second frequency ratios ω_{NL}/ω_L for piezoelectric nanobeams with various boundary conditions

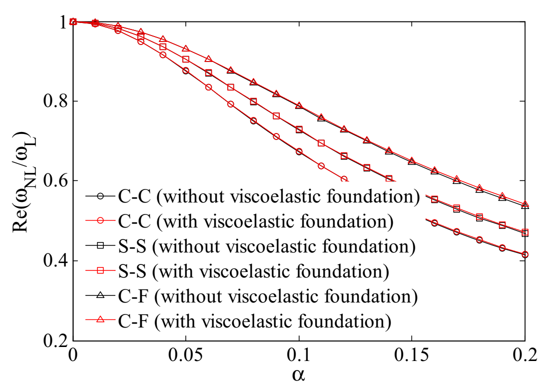

Fig. 4 Effect of nonlocal parameter α on the real parts of the third frequency ratios ω_{NL}/ω_L for piezoelectric nanobeams with various boundary conditions

Figure 5 depicts the effect of slenderness ratio L/h on the imaginary parts of the first dimensionless frequencies for embedded S–S piezoelectric nanobeams with various flexoelectric coefficient μ_{31}. In this case, the thickness of the nanobeam is taken as $h = 2$ nm and the length L changes to satisfy different slenderness ratio L/h. Figure 5 shows that the imaginary parts of the first dimensionless frequencies increase almost linearly with an increase in slenderness ratio L/h. The possible reason for this is that the damping effect of viscoelastic medium introduced into the system becomes more pronounced as the length L increases. Both flexoelectric coefficient μ_{31} and nonlocal parameter α have no effect on the imaginary parts of natural frequencies. Hence, the effect of slenderness ratio L/h on the real parts of the first three natural frequencies is mainly examined in

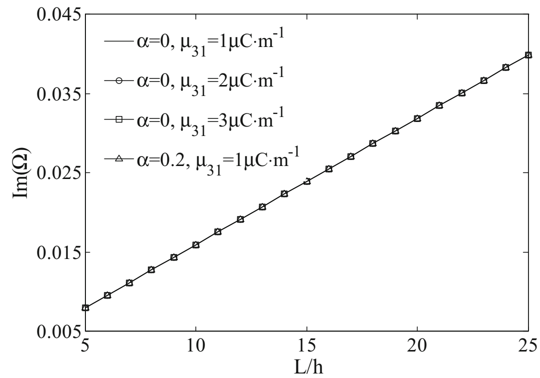

Fig. 5 Effect of slenderness ratio L/h on the imaginary parts of the first dimensionless frequencies for S–S piezoelectric nanobeams with various flexoelectric coefficient μ_{31}

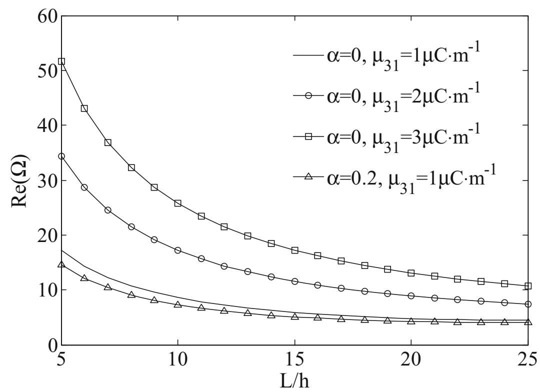

Fig. 6 Effect of slenderness ratio L/h on the real parts of the first dimensionless frequencies for S–S piezoelectric nanobeams with various flexoelectric coefficient μ_{31}

Figs. 6, 7, and 8. From the figures, we can find that the real parts of the first three natural frequencies decrease significantly as slenderness ratio L/h increases for various flexoelectric coefficient μ_{31}. This is because the rigidity of the nanobeam is reduced with rising slenderness ratio L/h. In addition, the effect of slenderness ratio L/h turns out to be more substantial as flexoelectric coefficient μ_{31} and mode number increase. For example, as L/h increases from 5 to 25 the first and the second natural frequencies decrease about 12.73 and 54.93 at $\mu_{31} = 1$ μ C/m but about 40.91 and 165.17 at $\mu_{31} = 3$ μC/m. The significant effect of the flexoelectric coefficient μ_{31} is also observed in Figs. 6, 7, and 8, where the real parts of the first three natural frequencies increase significantly with rising μ_{31}. This effect of μ_{31} is found to be less pronounced as slenderness ratio L/h increases. For instance, as μ_{31} changes from 1 to 3 μC/m, the real parts of the first natural frequencies increase about 34.43 at $L/h = 5$ but only about 6.25 at $L/h = 25$.

Figures 9, 10, and 11 present the effect of flexoelectric coefficient μ_{31} on the real parts of the first three dimensionless frequencies for embedded S–S piezoelectric nanobeams with various nonlocal parameter α and electric voltage V. From the figures, it can be observed that flexoelectric coefficient μ_{31} has a strong influence on the natural frequencies of piezoelectric nanobeams. The real parts of the first three dimensionless frequencies increase almost linearly as flexoelectric coefficient μ_{31} increases. This implies that the rigidity of the embedded nanobeam is hardened due to enhanced flexoelectric effect. In addition, the effect of flexoelectric coefficient μ_{31} turns out to be more substantial as nonlocal parameter α decreases or mode number increases. For example, as μ_{31} increases from 1 to 2 μC/m the first and the third natural frequencies with $V = 10$ V increase, respectively, about 3.35 and 18.01 at $\alpha = 0.2$ but about 4.06 and 38.57 at $\alpha = 0$. Compared with the nonlocal parameter α, the effect of the electric voltage V on the sensibility of natural frequencies to the flexoelectric coefficient μ_{31} is small.

As the final numerical example, the effect of viscoelastic medium on the vibration responses of embedded piezoelectric nanobeams is examined. To this end, Figs. 12 and 13 show the variations of the first two complex natural frequencies versus damping parameter c_t for S–S piezoelectric nanobeams. Figures 12 and 13 show that

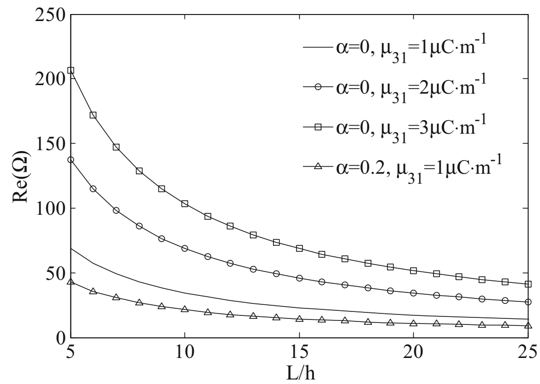

Fig. 7 Effect of slenderness ratio L/h on the real parts of the second dimensionless frequencies for S–S piezoelectric nanobeams with various flexoelectric coefficient μ_{31}

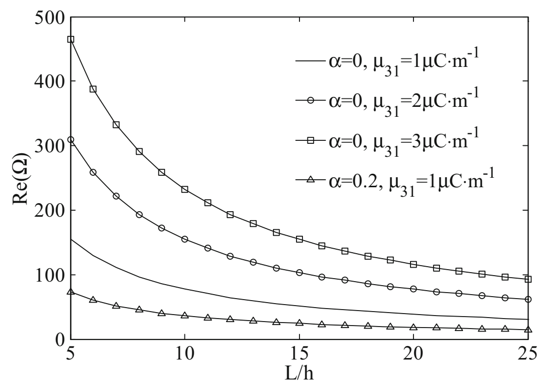

Fig. 8 Effect of slenderness ratio L/h on the real parts of the third dimensionless frequencies for S–S piezoelectric nanobeams with various flexoelectric coefficient μ_{31}

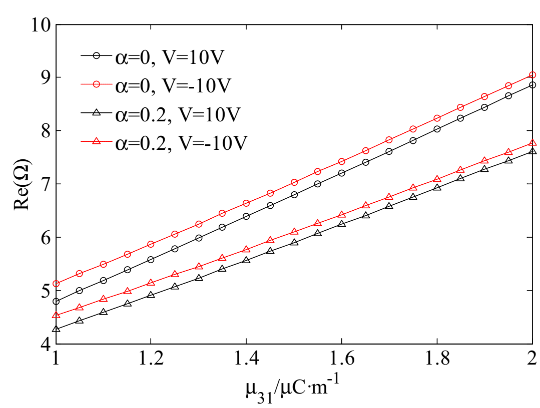

Fig. 9 Effect of flexoelectric coefficient μ_{31} on the real parts of the first dimensionless frequencies for S–S piezoelectric nanobeams with various nonlocal parameter α and electric voltage V

the real parts of the first two natural frequencies remain zero as damping parameter c_t is larger than a certain value, which is denoted by $(c_t)_{\text{crit}}$ to represent the critical value of c_t for nonoscillatory eigenfrequencies. Accordingly, a sharp change also can be observed in the imaginary parts of the first two natural frequencies. In addition, the value of $(c_t)_{\text{crit}}$ increases significantly with an increase in Winkler's modulus parameter k_w, flexoelectric coefficient μ_{31} and mode number or a decrease in nonlocal parameter α and electric voltage V. It also can be seen from the figures that as damping parameter c_t is smaller than $(c_t)_{\text{crit}}$, the real parts of the

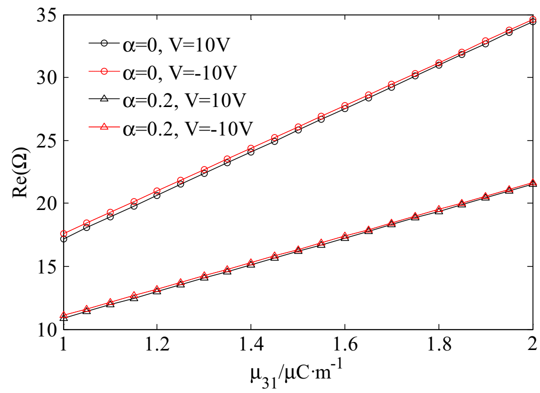

Fig. 10 Effect of flexoelectric coefficient μ_{31} on the real parts of the second dimensionless frequencies for S–S piezoelectric nanobeams with various nonlocal parameter α and electric voltage V

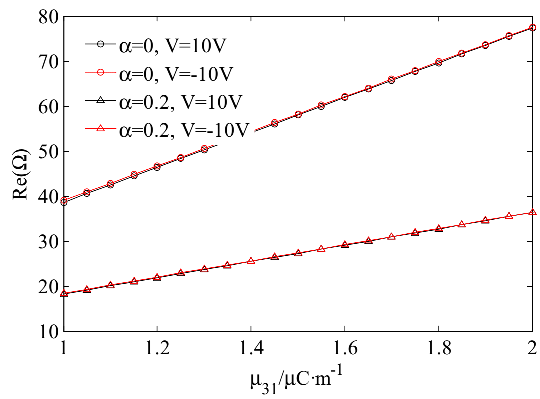

Fig. 11 Effect of flexoelectric coefficient μ_{31} on the real parts of the third dimensionless frequencies for S–S piezoelectric nanobeams with various nonlocal parameter α and electric voltage V

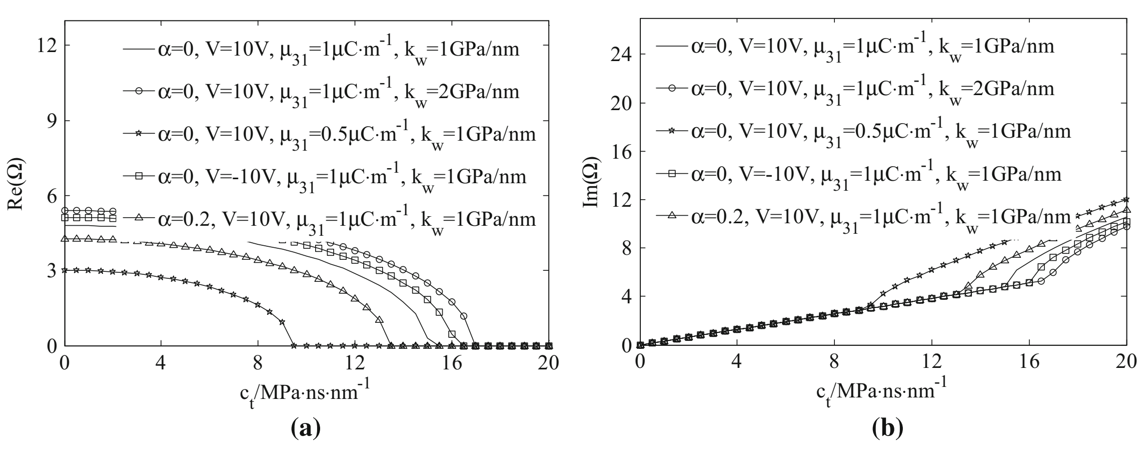

Fig. 12 Variation of the first dimensionless frequencies for S–S piezoelectric nanobeams with damping parameter c_t. **a** The real parts of dimensionless frequencies, **b** the imaginary parts of dimensionless frequencies

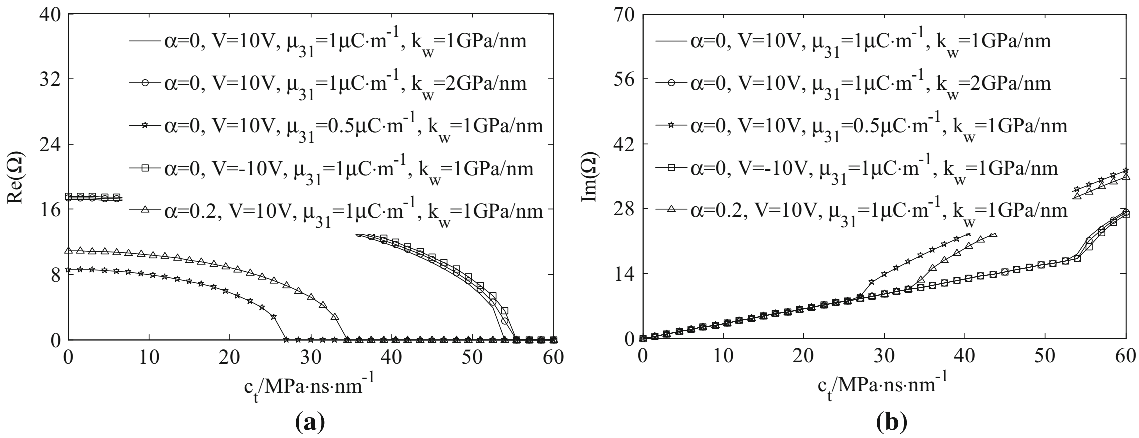

Fig. 13 Variation of the second dimensionless frequencies for S–S piezoelectric nanobeams with damping parameter c_t. **a** The real parts of dimensionless frequencies, **b** the imaginary parts of dimensionless frequencies

first two natural frequencies decrease nonlinearly with rising damping parameter c_t, but the imaginary parts increase almost linearly.

5 Conclusions

Based on nonlocal Euler–Bernoulli beam theory, the transverse vibration was investigated for a piezoelectric nanobeam embedded in viscoelastic medium with various boundary conditions. Considering nonlocal effect, piezoelectric effect, and flexoelectric effect simultaneously, the governing equations of motion and boundary conditions for vibration analysis are first derived by using Hamilton's principle. Subsequently, the natural frequencies and corresponding mode shapes in closed form for embedded piezoelectric nanobeams with arbitrary boundary conditions were obtained by utilizing the transfer function method (TFM). In benchmark cases, the proposed model was validated by comparing the obtained results with those available in the literature, where good agreement has been achieved. In addition, a detailed parametric study was also conduced to investigate the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient, and viscoelastic medium on the vibration responses of nanobeams. Some of the key contributions made in this study include:

- The novelty of this work includes the simultaneous consideration of nonlocal effect, piezoelectric effect, flexoelectric effect, viscoelastic surrounding medium and electrical loadings for nanobeam dynamics.
- An the increase in the nonlocal parameter α leads to a significant decrease in the fundamental frequencies of both C–C and S–S nanobeams, but an increase in those of C–F nanobeams. This effect of α becomes more substantial with rising frequency modes and boundary condition stiffness.
- The flexoelectric coefficient μ_{31} has a strong influence on the vibration responses of nanobeams. The natural frequencies of the embedded nanobeams increase significantly as μ_{31} increases. This effect of μ_{31} turns out to be less pronounced as the nonlocal parameter α and the slenderness ratio L/h increase or the mode number decreases.
- The nonlocal parameter α, boundary conditions, flexoelectric coefficient μ_{31} and mode numbers have no effect on the imaginary parts of the natural frequencies, which are only related to the damping of the viscoelastic surrounding medium.
- The critical values of the damping parameter c_t to obtain nonoscillatory eigenfrequencies also can be determined by using the proposed method. Moreover, the value of $(c_t)_{\text{crit}}$ increases significantly as k_w, μ_{31} and the mode number increase or α and V decrease.

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References

1. Liu, C., Ke, L.L., Wang, Y.S., Yang, J., Kitipornchai, S.: Thermo-electro-mechanical vibration of piezoelectric nanoplates based on the nonlocal theory. Compos. Struct. **106**, 167–174 (2013)
2. Zhang, S.J., Xia, R., Lebrun, L., Anderson, D., Shrout, T.R.: Piezoelectric materials for high power, high temperature applications. Mater. Lett. **59**, 3471–3475 (2005)
3. Asemi, S.R., Farajpour, A., Mohammadi, M.: Nonlinear vibration analysis of piezoelectric nanoelectromechanical resonators based on nonlocal elasticity theory. Compos. Struct. **116**, 703–712 (2014)
4. Wang, Z.L., Song, J.H.: Piezoelectric nanogenerators based on zinc oxide nanowire arrays. Science **312**(5771), 242–246 (2006)
5. Song, S., Hou, Y., Guo, M., Wang, L., Tong, X., Wu, J.: An investigation on the aggregate-shape embedded piezoelectric sensor for civil infrastructure health monitoring. Constr. Build. Mater. **131**, 57–65 (2017)
6. Li, X.J., Luo, Y.: Flexoelectric effect on vibration of piezoelectric microbeams based on a modified couple stress theory. Shock Vib. **2017**, 1–7 (2017)
7. Ma, W., Cross, L.E.: Flexoelectric polarization of barium strontium titanate in the paraelectric state. Appl. Phys. Lett. **81**(18), 3440–3442 (2002)
8. Ebrahimi, F., Barati, M.R.: Surface effects on the vibration vehavior of flexoelectric nanobeams based on nonlocal elasticity theory. Eur. Phys. J. Plus **132**(19), 1–13 (2017)
9. Liang, X., Hu, S.L., Shen, S.P.: Effects of surface and flexoelectricity on a piezoelectric nanobeam. Smart Mater. Struct. **23**, 035020 (2014)
10. Qi, L., Zhou, S.J., Li, A.Q.: Size-dependent bending of an electro-elastic bilayer nanobeam due to flexoelectricity and strain gradient elastic effect. Compos. Struct. **135**, 167–175 (2016)
11. Yan, Z., Jiang, L.Y.: Size-dependent bending and vibration behaviour of piezoelectric nanobeams due to flexoelectricity. J. Phys. D Appl. Phys. **46**, 355502 (2013)
12. Yue, Y.M., Xu, K.Y., Chen, T.: A micor scale Timoshenko beam model for piezoelectricity with flexoelectricity and surface effects. Compos. Struct. **136**, 278–286 (2016)
13. Ebrahimi, F., Barati, M.R., Dabbagh, A.: A nonlocal strain gradient theory for wave propagation analysis in temperature-dependent inhomogeneous nanoplates. Int. J. Eng. Sci. **107**, 169–182 (2016)
14. Li, L., Li, X.B., Hu, Y.J.: Free vibration analysis of nonlocal strain gradient beams made of functionally graded material. Int. J. Eng. Sci. **201**, 77–92 (2016)
15. Li, L., Hu, Y.J., Ling, L.: Wave propagation in viscoelastic single-walled carbon nanotubes with surface effect under magnetic field based on nonlocal strain gradient theory. Physica E **75**, 118–124 (2016)
16. Ke, L.L., Wang, Y.S.: Thermoelectric-mechanical vibration of piezoelectric nanobeams based on the nonlocal theory. Smart Mater. Struct. **21**, 025018 (2012)
17. Ke, L.L., Wang, Y.S., Wang, Z.D.: Nonlinear vibration of the piezoelectric nanobeams based on the nonlocal theory. Compos. Struct. **94**, 2038–2047 (2012)
18. Ansari, R., Oskouie, M.F., Gholami, R., Sadeghi, F.: Thermo-electro-mechanical vibration of postbuckled piezoelectric Timoshenko nanobeams based on the nonlocal elasticity theory. Compos. Part B **89**, 316–327 (2016)
19. Stassi, S., Cauda, V., Ottone, C., Chiodoni, A., Pirri, C.F., Canavese, G.: Flexible piezoelectric energy nanogenerator based on ZnO nanotubes hosted in a polycarbonate membrane. Nano Energy **13**(36), 474–481 (2015)
20. Lei, Y., Murmu, T., Adhikari, S., Friswell, M.I.: Dynamic characteristics of damped viscoelastic nonlocal Euler–Bernoulli beams. Eur. J. Mech. A Solids **42**, 125–136 (2013)
21. Zhang, D.P., Lei, Y., Shen, Z.B.: Vibration analysis of horn-shaped single-walled carbon nanotubes embedded in viscoelastic medium under a longitudinal magnetic field. Int. J. Mech. Sci. **118**, 219–230 (2016)
22. Lu, P., Lee, H.P., Lu, C.: Dynamic properties of flexural beams using a nonlocal elasticity model. J. Appl. Phys. **99**, 073510 (2006)