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Nonlocal effects on the longitudinal vibration of a complex multi-nanorod system subjected to the transverse magnetic field

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Abstract In this communication we examine the free longitudinal vibration of a complex multi-nanorod system (CMNRS) using the nonlocal elasticity theory. Discussion is limited to the cases of two types of boundary conditions, namely, clamped–clamped (C–C) and clamped–free (C–F), where nanorods are coupled in the "Free-Chain" system by an elastic medium. Each nanorod in CMNRS is subjected to the influence of transversal magnetic field. The longitudinal vibration of the system are described by a set of m partial differential equations, derived by using D'Alembert's principle and classical Maxwell's relation, which includes Lorentz magnetic force. Analytical expressions for the nonlocal natural frequencies are obtained in closed-form by using the method of separations of variables and trigonometric method. Results for the nonlocal natural frequencies are compared for the special cases of a single and double-nanorod system with the existing results in the literature. Numerical examples are given in order to examine the effects of nonlocal parameter, stiffness coefficient and transversal magnetic field on nonlocal natural frequencies of axially vibrating CMNRS.

Keywords Natural frequency · Nonlocal effects · Multi-nanorod system · Exact solution · Magnetic field effects

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1 Introduction

In recent years, a growing application of nanodevices in industry, medicine, measurement equipment and elsewhere, enforce the scientists to direct their research effort in the field of nanoscience and nanotechnology [1–3]. Nanomaterials underpinning many nanodevices are often in turn made of complex structures. Mechanical, thermal, electrical, chemical and optical properties [4–6] of such structures are generally not easy to describe. Hence, under certain phenomenological assumptions, development of mathematical models of nanostructure based systems can be important for the future design of new nanodevices. To examine mechanical behavior of one-dimensional nanostructures such as nanorods, nanowires, nanotubes and nanobelts one may use modified continuum theories similar to the application of classical continuum theories in macro structures with the same geometry. These modified theories can be Eringen's nonlocal elasticity, modified coupled stress or gradient theory [7–9]. Using the nonlocal continuum models one is able to introduce a material parameter into the constitutive equation, which describes the forces between atoms and internal length-scale. Thus, Eringen's nonlocal elasticity theory is widely used by numerous of authors to examine mechanical behavior of nanostructures [10–15].

Vibration behavior of structures on macro level such as double and multiple rod, beam, disc and plate systems coupled with elastic, viscoelastic or other type of layers has been investigated by various authors [16–19]. In most of the papers, motion equations of the free vibration of a single sandwich systems are solved analytically using Bernoulli–Fourier method of separation of variables. Often, for the forced vibrations of sandwich structures, modal expansion method is used [20]. In the paper by Kelly and Srinivas [21], a general theory for the determination of mode shapes and natural frequencies of an axially loaded multi-beam system coupled with elastic layers was given. The authors used Euler–Bernoulli beam theory and obtained the exact solution in the general case. As an alternative way to find a solution, they developed a Rayleigh–Ritz approximation method using the mode shapes of the corresponding upstretched beams. One way to find exact analytical solutions, which proved to be very efficient for determination of natural frequencies of complex systems with multiple structures, is the trigonometric method. According to the authors knowledge, first application of this method to the vibration problems of complex systems based on identical multiple structures is in the work by Rasković [22] and later in [23]. Moreover, Stojanović et al. [24] suggested a general procedure for finding the natural frequencies and buckling loads of an elastic multi-beam system coupled with elastic layers and under compressive axial load. For beams, they used Timoshenko and other higher order theories and obtained exact solutions for natural frequencies and critical buckling loads using the trigonometric method. The similar procedure is used to find explicit expressions for complex natural frequencies of viscoelastic orthotropic multi-nanoplate system embedded in viscoelastic medium [25] and natural frequencies of the elastic isotropic multi-nanoplate system [26]. In both cases, asymptotic values of complex and classical natural frequencies are determined in the case when the number of nanoplates tends to infinity. Rossikhin and Shitikova [27] gave a wide overview of papers using fractional calculus for dynamic problems of solid mechanics. In their analysis, two and more degrees of freedom linear and nonlinear systems as well as vibration analysis of linear and nonlinear infinite number degrees of freedom systems such as beams, rods, plates, shells and multilayered systems are considered. In summary, it can be noted that systems with multiple nanorods, nanobeams and nanoplates have not been sufficiently explored by the researchers.

Nanorods and nanowires are one-dimensional structures [28] which can be grown using several methods [29–31]. As reported in the literature, dimensions of nanorods and nanowires are in the range from one nanometer to several micrometers. Nanorods growing from nanowire core can be viewed as multi-nanorod system Fig. 1. Theoretical analysis of such systems can be of great importance for their application and comprehension since experiments on nano-scale level cannot be well controlled. Deterministic numerical methods as molecular dynamics (MD), nondeterministic methods as Monte-Carlo and modern multiscale methods as a mixture of molecular dynamics and continuum methods can be computationally prohibitive when nanoscale systems are composed of multiple structures with large number of molecules and atoms. Hereby, size-dependent nonlocal continuum theory provides numerical solutions much faster than previously mentioned methods and even, for some cases, it is possible to obtain exact closed-form solutions. Value of small-scale parameter or so called nonlocal parameter used in nonlocal continuum models is usually obtained

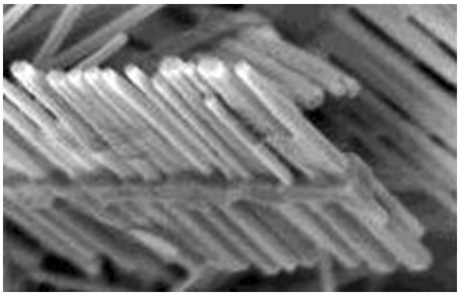

Fig. 1 ZnO nanorods growing on nanowire core (scanning electron microscopy image reproduced from [24])

by fitting the results for nonlocal models of nanostructures with the results from molecular dynamics simulations (MDS). In the paper by Duan et al. [32], nonlocal parameters are calibrated by comparison of the results obtained for clamped–clamped and clamped–free nonlocal Timoshenko model of single walled carbon nanotube (SWCNT) with those of molecular dynamics. Parameters are obtained for different aspect ratios of the model. Arash and Ansari [33] examined the vibration properties of SWCNTs with initial strain using the nonlocal shell model and molecular dynamics model. They calibrated the nonlocal parameter for a large spectrum of aspect ratios by comparing the resonant frequencies of the nonlocal model with those of MDS.

Nonlocal longitudinal vibration of nanorod systems has been the research topic in many papers. In the follow, we will review some of the works where the free or forced vibration of complex systems based on nanorods coupled with elastic, viscoelastic or other type of layers are examined for various boundary conditions, aspect ratios and small-scale parameter. Danesh et al. [34] accounted the small-scale effect in the longitudinal vibration of tapered nanorod using the nonlocal elasticity theory. The authors have used the differential quadrature method to solve motion equations for three different boundary conditions. Carbon nanotube embedded in elastic medium was modeled by Aydogdu [35] as a nanorod surrounded with elastic layers by using the Eringen's nonlocal elasticity theory. The author compared the longitudinal frequencies for the nonlocal and classical continuum models. Narendar and Gopalakrishnan [36] have considered the nonlocal effects in the longitudinal wave propagation within the system of two nanorods coupled with elastic layer. The authors studied the influence of small-scale (nonlocal) parameter and stiffness of the layer on axial wave propagation. Hsu et al. [37] investigated the longitudinal frequencies of cracked nanobeams for different boundary conditions and using the theory of nonlocal elasticity. A wide study of the longitudinal, transversal and torsional vibration and instability has been made by Kiani [38] for a system of SWCNT. Şimşek [39] used a Galerkin approach to obtain the natural frequencies for the longitudinal vibration of axially functionally graded tapered nanorods. The author performed the analysis for nanorods with variable cross section, different tapered ratios, material properties and boundary conditions. Longitudinal vibration of nanorods, which takes the nonlocal long-range interactions into account, was examined by Huang [40]. Chang [41] considered the small-scale effects to investigate the axial vibration of elastic nanorods. The author used the differential quadrature method to solve the model equations. Filiz and Aydogdu [42] analyzed the longitudinal vibration of carbon nanotubes with heterojunctions using the nonlocal elasticity for different lengths, diameters and chirality of heterojunctions. Karličić et al. [43] proposed detailed analyses for the free longitudinal vibrational response of the system of coupled viscoelastic nanorods and investigated the influence of different physical parameters on the complex natural frequency. Murmu and Adhikari [44] analyzed the longitudinal vibration of double-nanorod system (DNRS) coupled with elastic layer. The effects of nonlocal parameter, layers stiffness and aspect ratio on the free longitudinal vibration of DNRS are investigated.

Magnetic field subjected to a nanorod can change its dynamic properties depending of the magnitude of the field and deformation regime. In the literature [45, 46], vibration response of carbon nanotubes (CNTs) or graphene sheets within the magnetic field become a new subject of investigation. The reason for this is that the mechanical properties of nanostructures are changed since electromagnetic force is acting on each element of a nanostructure. Taking into account Maxwell's equations and Lorentz's forces it is possible to connect the forces acting on each particle of the nanostructure with the parameters of magnetic field.

In recent time, several vibration studies of nanostructures under the influence of magnetic field are published. Murmu et al. [47] examined the transversal vibrations of double single-walled carbon nanotube system (DSWNTS) under the influence of longitudinal magnetic field and by using the nonlocal Euler–Bernoulli beam theory. They investigated the influence of nonlocal effects and strength of magnetic field on vibration behavior of DSWNTS. Further, Murmu et al. [48] proposed an analytical approach using nonlocal theory to study the bending vibration of double-walled CNTs under the influence of longitudinal magnetic field. Arani et al. [49] studied the effect of in-plane two-dimensional magnetic field and biaxial preload on vibration behavior of double orthotropic graphene sheets which are coupled by the elastic layer of Pasternak type. Equations are solved using the differential quadrature method (DQM). In addition, authors examined the effects of in-plane preload, nonlocal parameter, different aspect ratios and magnetic field on natural frequencies. Narendar et al. [50] applied the nonlocal continuum model to investigate the influence of longitudinal magnetic field on wave dispersion characteristics of a single-walled carbon nanotube embedded in elastic medium. Kiani [51–53] published several papers on mechanical behavior of CNTs affected by a magnetic field. Recently, Murmu et al. [54] analyzed the effect of transverse magnetic field on axial vibration of nanorods by applying the nonlocal continuum theory.

In spite of these recent developments, the model which considers the nonlocal longitudinal vibration of multi-nanorod system coupled with elastic layers and under the influence of transversal magnetic field has not been considered in the literature yet. Systems like this can play significant role in the development of future generation of nano-electro mechanical devices. Understanding the vibration behavior of such systems, particularly complex multi-nanorod system could be a key step for the application in electromechanical resonators where magnetic field is used to shift a spectrum of resonant frequencies. In addition, the resonant frequencies of nanorods are in gigahertz range, which allows us to use them for the high-frequency systems. The exact solutions for natural frequencies are derived by using the method of separation of variables and trigonometric method. Moreover, in the case when the number of nanorods or mode number tends to the infinity, we derived the asymptotic values of the natural frequencies of the CMNRS. Numerical results are presented in order to show the effects of nonlocal parameter, magnetic field parameter and stiffness of the elastic layers on natural frequencies of CMNRS for different number of nanorods and boundary conditions.

2 Theoretical model

2.1 Introduction to constitutive relations

As opposed to the classical theory of elasticity, where the stress is a function of the strain at a same point, nonlocal theory of elasticity assumes that the stress at a point is a function of strains at all points of an elastic body. Further, nonlocal elasticity theory takes into account the size-effects via single material parameter called small-scale or nonlocal parameter. Based on Eringen's work [55], fundamental equations of nonlocal theory of elasticity for a three-dimensional elastic body can be written as

$$\sigma_{ij}(x) = \int \alpha(|x - x'|, \tau) C_{ijkl} \varepsilon_{kl}(x') dV(x'), \quad \forall x \in V, \tag{1a}$$

$$\sigma_{ij,j} = 0, \tag{1b}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1c}$$

where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij} and ε_{ij} are the stress and strain tensors, respectively and u_i denotes displacement vector. With $\alpha(|x - x'|, \tau)$ we denote the nonlocal modulus or attenuation function which incorporates nonlocal effects into the constitutive equation at the reference point x produced by a local strain at the source x', where $|x - x'|$ denotes the Euclidean metric. For the parameter $\tau = (e_0 a)/l$, l is the external characteristic length (wave length), a describes internal characteristic length (lattice parameter, granular size or distance between C–C bounds) and e_0 is a constant appropriate to each material that can be identified from atomistic simulations. Integral constitutive relation (1a) is reformulated into a differential form in order to make it more convenient for use in nonlocal elasticity problems. According to the Eringen [56], differential form of nonlocal elasticity constitutive relation for the one dimensional case is of the form

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{2}$$

where E is elastic modulus of the beam; $\mu = (e_0 a)^2$ is the nonlocal parameter; σ_{xx} is normal nonlocal stress and $\varepsilon_{xx} = \partial u/\partial x$ is axial deformation. As the exact value of nonlocal parameter is scattered and depends on various parameters, in the present study, the free vibration analysis of CMNRS is carried out assuming for $e_0 a$ to be from 0 to 2 nm. If $e_0 a = 0$, i.e. there is no influence of nonlocal parameter (same as in macro-scale modeling) we get back to the classical elasticity stress–strain relation.

2.2 Classical Maxwell's relation

The classical Maxwell's relation for electromagnetic theory [47–54] gives the relationships between the current density **J**, distributing vector of magnetic field **h**, strength vectors of the electric fields **e** and magnetic field permeability η in differential form as

$$\mathbf{J} = \nabla \times \mathbf{h}, \quad \nabla \times \mathbf{e} = -\boldsymbol{\eta}\frac{\partial \mathbf{h}}{\partial t}, \quad \nabla \cdot \mathbf{h} = 0, \tag{3}$$

where vectors of distributing magnetic field **h** and the electric field **e** are defined as

$$\mathbf{h} = \nabla \times (\mathbf{U} \times \mathbf{H}), \quad \mathbf{e} = -\eta\left(\frac{\partial \mathbf{U}}{\partial t} \times \mathbf{H}\right). \tag{4}$$

in which equation $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ is the Hamilton operator, $\mathbf{U} = (u, v, w)$ is the displacement vector and $\mathbf{H} = \left(0, H_y, H_z\right)$ is the vector of the transversal magnetic field. Now we assume that the transversal magnetic field acts on CMNRS in the y and z direction, where the vector of distributing magnetic field is given in the following form

$$\mathbf{h} = 0\mathbf{i} - H_y\frac{\partial u}{\partial x}\mathbf{j} - H_z\frac{\partial u}{\partial x}\mathbf{k}. \tag{5}$$

In Eq. (5) displacements v and w along y and z directions are neglected. By substituting Eq. (5) into the first expressions of Eq. (3) we obtain

$$\mathbf{J} = \nabla \times \mathbf{h} = 0\mathbf{i} + H_z\frac{\partial^2 u}{\partial x^2}\mathbf{j} - H_y\frac{\partial^2 u}{\partial x^2}\mathbf{k}. \tag{6}$$

Further, using Eq. (6) into the expressions for the Lorentz force induced by the longitudinal magnetic field, yields

$$\mathbf{f}\left(f_x, f_y, f_z\right) = \eta(\mathbf{J} \times \mathbf{H}) = \eta\left[\left(H_y^2 + H_z^2\right)\frac{\partial^2 u}{\partial x^2}\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}\right]. \tag{7}$$

Now we assume that the axial displacement of the ith nanorod $u_i(x, t)$ and the Lorentz force acts only in x- direction which can be written as

$$f_{x,i} = \eta\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2}. \tag{8}$$

Finally, it is possible to obtain force per unit length of the ith nanorod in the following form

$$\tilde{q}_i(x,t) = \int_A f_{x,i}dA = \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2}. \tag{9}$$

2.3 Governing equations of complex multi-nanorod system (CMNRS)

In Fig. 2, an Illustrative example of nonlocal nanorod model representing the carbon nanotube subjected to transverse magnetic field is shown. Further, in Fig. 3 we show CMNRS with multiple coupled nanorods where each of them is subjected to the same transverse magnetic field in y and z direction. The free longitudinal vibration of CMNRS are considered for two types of boundary conditions, clamped–clamped (C–C) and clamped–free (C–F), as shown in Fig. 3. Distinction between these two systems is in the way of coupling of the left and the right side of nanorods in CMNRS. The CMNRS shown in Figs. 3a, b is the so called "Free-Chain" system, which main characteristic is that the first and the last nanorod in the system are not connected with a fixed base ($K_0 = 0$ and $K_m = 0$). Both models presented in Fig. 3 are consist of a set of identical, parallel and uniform nanorods connected by an axially distributed elastic layers with stiffness per length denoted as $K_1 = K_2 = \ldots = K_i = \ldots = K_{m-1} = K$. The set of m nanorods of CMNRS are referred to as nanorod 1, nanorod 2 and so on until the m-th nanorod. All nanorods are having the elastic modulus E, mass density ρ, uniform cross-section area A and length L.

Let us assume the ith nanorod and a differential element of the length dx. The cross sectional area A of differential elements of nanorods is constant and the longitudinal displacement of elements in the x-direction is given by $u_i(x, t)$. We presume that a nanorod is under a dynamically-varying stress field $\sigma_{xx}(x, t)$, so that adjacent sections are subjected to varying stresses. The external forces $F_i(x, t)$ and $F_{i\text{-}1}(x, t)$ per unit length and transversal magnetic field H_y and H_z are also considered. The equation of motion in the x-direction then becomes

$$-N_i + \left(N_i + \frac{\partial N_i}{\partial x}dx\right) + F_i dx - F_{i-1}dx + \tilde{q}_i(x,t)dx = \ddot{u}_i dm, \tag{10}$$

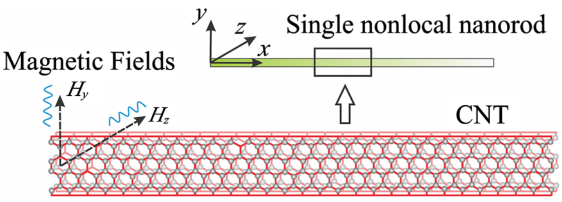

Fig. 2 Single nonlocal nanorod model of CNT subjected to transverse magnetic field

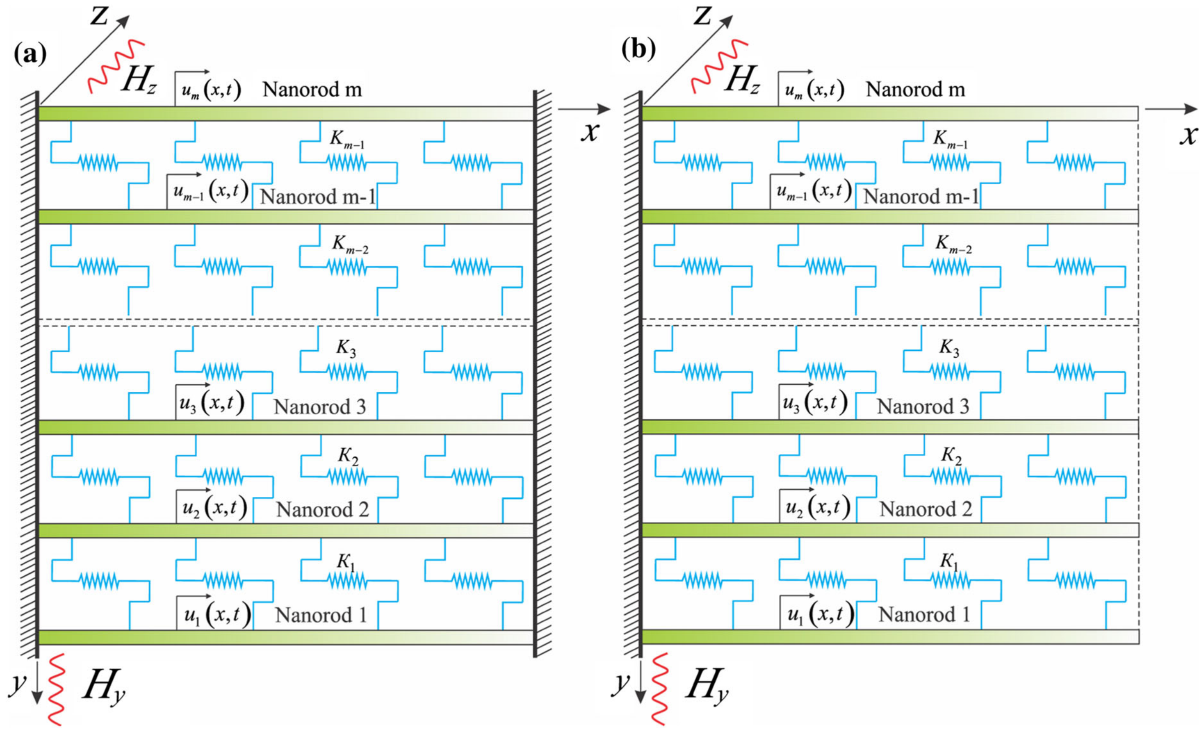

Fig. 3 CMNRS for different boundary conditions: **a** Free-Chain system with C–C boundary conditions, **b** Free-Chain system with C–F boundary conditions

where $dm = \rho A dx$ is the mass of the infinitesimal element; $N_i(x,t)$ is the stress resultant of an axial stress σ_{xx} acting internally on cross section area A; F_i and F_{i-1} are external forces which results from elastic layers and $\tilde{q}_i(x,t)$ is the external transversal magnetic field effect given in Eq. (9).Introducing $dm = \rho A dx$ into the Eq. (10), gives

$$\frac{\partial N_i}{\partial x} = -F_i + F_{i-1} - \tilde{q}_i(x,t) + \rho A \ddot{u}_i, \tag{11}$$

where N_i is the stress resultant defined as

$$N_i(x,t) = \int_A \sigma_{xx}(x,t) dA = \sigma_{xx}(x,t)A, \tag{12}$$

and

$$F_i = K_i(u_{i+1} - u_i), \quad F_{i-1} = K_{i-1}(u_i - u_{i-1}), \\ \tilde{q}_i(x,t) = \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2}, \tag{13}$$

are external forces from elastic layers and transversal magnetic field. Introducing Eq. (12) into Eq. (2), the stress resultant for the nonlocal theory is obtained as

$$N_i - \mu\frac{\partial^2 N_i}{\partial x^2} = EA\frac{\partial u_i}{\partial x}. \tag{14}$$

By substituting Eq. (11) and (13) into the Eq. (14) we get to the following equation of motion expressed in terms of the displacement $u_i(x, t)$ for nonlocal elastic constitutive relation

$$\begin{aligned} &\bar{m}\ddot{u}_i + K_i(u_i - u_{i+1}) + K_{i-1}(u_i - u_{i-1}) \\ &\quad - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2} - \bar{e}\frac{\partial u_i}{\partial x^2} \\ &\quad = \mu\frac{\partial^2}{\partial x^2}[\bar{m}\ddot{u}_i + K_i(u_i - u_{i+1}) + K_{i-1}(u_i - u_{i-1}) \\ &\quad\quad -\eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2}\Big], \quad i = 1,2,\ldots,m, \end{aligned} \tag{15}$$

and

$$EA = \bar{e} = constant, \tag{16a}$$

$$\rho A = \bar{m} = constant, \tag{16b}$$

where e and m denotes axial rigidity and mass per unit length, respectively.From Eq. (15), we can obtain equations of motion of the "Free-Chain" system with conditions $K_0 = K_m = 0$ and $u_0 = u_{m+1} = 0$ (see Fig. 3) in the following form

$$\bar{m}\ddot{u}_1 - \bar{e}\frac{\partial^2 u_1}{\partial x^2} + K_1(u_1 - u_2) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_1}{\partial x^2} = \mu\frac{\partial^2}{\partial x^2}\left[\bar{m}\ddot{u}_1 + K_1(u_1 - u_2) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_1}{\partial x^2}\right], \quad (17a)$$

$$\bar{m}\ddot{u}_i - \bar{e}\frac{\partial^2 u_i}{\partial x^2} + K_i(u_i - u_{i+1}) + K_{i-1}(u_i - u_{i-1}) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2} = \mu\frac{\partial^2}{\partial x^2}\left[\bar{m}\ddot{u}_i + K_i(u_i - u_{i+1}) + K_{i-1}(u_i - u_{i-1}) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_i}{\partial x^2}\right],$$
$$i = 2, 3, \ldots, m-1, \quad (17b)$$

$$\bar{m}\ddot{u}_m - \bar{e}\frac{\partial^2 u_m}{\partial x^2} + K_{m-1}(u_m - u_{m-1}) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_m}{\partial x^2} = \mu\frac{\partial^2}{\partial x^2}\left[\bar{m}\ddot{u}_m + K_{m-1}(u_m - u_{m-1}) - \eta A\left(H_y^2 + H_z^2\right)\frac{\partial^2 u_m}{\partial x^2}\right]. \quad (17c)$$

3 Analytical solution

Suppose that ends of a set of m nanorods are clamped–clamped Fig. 3a and clamped–free Fig. 3b, the boundary conditions are given by
Clamped–clamped:

$$u_i(0,t) = u_i(L,t) = 0, \quad (18)$$

Clamped–free:

$$u_i(0,t) = N_i(L,t) = 0, \quad (19)$$

where $N_i(L,t), (i = 1, 2, \ldots, m)$ are stress resultants on the right side of a set of m nanorods.

The equations of motion (15) with boundary conditions (18) and (19), can be solved using the method of separations of variables and assuming the solutions of the following form

$$u_i(x,t) = \sum_{n=1}^{\infty} U_{ni} \sin \alpha_n x\, e^{j\omega_n t}, \quad (20)$$

where for the *clamped–clamped* boundary conditions, we have

$$\alpha_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots, \infty, \quad (21)$$

for *clamped–free* boundary conditions, we have

$$\alpha_n = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, \ldots, \infty, \quad (22)$$

where $j = \sqrt{-1}$, U_{ni} is the amplitude and ω_n is the natural frequency in n-th mode of vibration.

Introducing the assumed solutions (20) into Eq. (15), we obtain the system of m algebraic equations as

$$-v_{ni-1}U_{ni-1} + S_{ni}U_{ni} - v_{ni}U_{ni+1} = 0, \quad i = 1, 2, 3, \ldots, m, \quad (23)$$

where

$$S_{ni} = -\bar{m}\omega_n^2\left(1 + \mu\alpha_n^2\right) + K_i\left(1 + \mu\alpha_n^2\right) + K_{i-1}\left(1 + \mu\alpha_n^2\right) + \eta A\alpha_n^2\left(H_y^2 + H_z^2\right)\left(1 + \mu\alpha_n^2\right) + \alpha_n^2\bar{e} \quad (24a)$$

$$v_{ni} = K_i\left(1 + \mu\alpha_n^2\right), \quad (24b)$$

$$v_{ni-1} = K_{i-1}\left(1 + \mu\alpha_n^2\right). \quad (24c)$$

3.1 Free-chain system

In this case, we assume that the stiffness $K_0 = K$ and $K_m = K$ are equal to zero, so there is no coupling of the first and the last nanorod with a fixed base. It is also assumed that all nanorods have the same physical characteristics and are joined by elastic layers of the same stiffness. Introducing the assumed solutions (20) into the set of m partial differential Eqs. (17a, 17b, 17c), we obtained the system of algebraic equations in the matrix form as

$$\begin{bmatrix} S_n - v_n & -v_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v_n & S_n & -v_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & S_n & -v_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_n & S_n & -v_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -v_n & S_n & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & -v_n & S_n & -v_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_n & S_n - v_n \end{bmatrix} \begin{Bmatrix} U_{n1} \\ U_{n2} \\ U_{n3} \\ \dots \\ U_{ni-1} \\ U_{ni} \\ U_{ni+1} \\ \dots \\ U_{nm-2} \\ U_{nm-1} \\ U_{nm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{25}$$

where expressions for S_n and v_n are given as

$$S_n = -\bar{m}\omega_n^2\left(1+\mu\alpha_n^2\right) + 2v_n + \eta A\alpha_n^2\left(H_y^2 + H_z^2\right)\left(1+\mu\alpha_n^2\right) + \alpha_n^2\bar{e}, \tag{26a}$$

$$v_n = K\left(1+\mu\alpha_n^2\right). \tag{26b}$$

The closed form solution of the system of algebraic equations is possible only for the case when the CMNRS is consist of identical nanorods and the identical coupling elastic layers. For a homogenous system of algebraic equations Eq. (25), analytical expressions for natural frequencies of the system of m elastically coupled nanorods can be determined using the trigonometric method [22–26]. According to the Rašković [22] we assumed the solution of the ith algebraic equations as

$$U_{ni} = N cos\left(i\varphi_{fc}\right) + M sin\left(i\varphi_{fc}\right), \quad i = 1,2,3,\dots,m. \tag{27}$$

Introducing Eq. (27) into the ith algebraic equation of the system (25), we get two trigonometric equations, assuming that the constants M and N are not simultaneously equal to zero

$$N\left\{-v_n cos\left[(i-1)\varphi_{fc}\right] + S_n cos\left(i\varphi_{fc}\right) - v_n cos\left[(i+1)\varphi_{fc}\right]\right\} = 0, \quad i = 2,3,\dots,m-1, \tag{28a}$$

$$M\left\{-v_n\, sin\left[(i-1)\varphi_{fc}\right] + S_n\, sin\left(i\varphi_{fc}\right) - v_n\, sin\left[(i+1)\varphi_{fc}\right]\right\} = 0, \quad i = 2,3,\dots,m-1, \tag{28b}$$

and after some transformation, this can be written as

$$\left(S_n - 2v_n cos\varphi_{fc}\right) N cos\left(i\varphi_{fc}\right) = 0, \tag{29a}$$

$$\left(S_n - 2v_n cos\varphi_{fc}\right) M sin\left(i\varphi_{fc}\right) = 0. \tag{29b}$$

From Eqs. (29a, 29b), we can conclude that conditions must use this form $N \neq 0$ and $cos\left(i\varphi_{fc}\right) \neq 0$ or $M \neq 0$ and $sin\left(i\varphi_{fc}\right) \neq 0$ in order to consider the oscillatory behavior of the system, for $i = 2, 3, \dots, m-1$. Now, we get to the frequency equation in the following form

$$S_n = 2v_n\, cos\varphi_{fc}. \tag{30}$$

where unknown φ_{cf} can be determined based on the condition that the assumed solutions (27) must satisfy the first and the last equation of the system (25). Introducing the assumed solutions

$$U_{1n} = N\, cos\varphi_{fc} + M\, sin\varphi_{fc} \text{ and } U_{2n} = N\, cos\left(2\varphi_{fc}\right) + M\, sin\left(2\varphi_{fc}\right) \tag{31a}$$

into the first equation of the system (25) and

$$U_{m-1n} = N\, cos\left[(m-1)\varphi_{fc}\right] + M\, sin\left[(m-1)\varphi_{fc}\right] \text{ and } U_{m-1n} = N\, cos\left(m\varphi_{fc}\right) + M\, sin\left(m\varphi_{fc}\right) \tag{31b}$$

into the last algebraic equation of the system (25), after some algebra we obtain the system of algebraic equations

$$N\left[(S_n - v_n)cos\varphi_{fc} - v_n cos\left(2\varphi_{fc}\right)\right] + M\left[(S_n - v_n)sin\varphi_{fc} - v_n sin\left(2\varphi_{fc}\right)\right] = 0, \tag{32a}$$

$$N\left[(S_n - v_n)cos\left(m\varphi_{fc}\right) - v_n cos\left[(m-1)\varphi_{fc}\right]\right] + M\left[(S_n - v_n)sin\left(m\varphi_{fc}\right) - v_n sin\left[(m-1)\varphi_{fc}\right]\right] = 0. \tag{32b}$$

Non-trivial solutions for the constants N and M can be obtained, which yields a trigonometric equation in the following form

$$\begin{vmatrix} 1-cos\varphi_{fc} & -sin\varphi_{fc} \\ cos[(m+1)\varphi_{cc}]-cos(m\varphi_{fc}) & sin[(m+1)\varphi_{cc}]-sin(m\varphi_{fc}) \end{vmatrix} = 0 \Rightarrow sin(m\varphi_{fc}) = 0, \tag{33}$$

from which we obtain solutions for unknown $\varphi_{fc,s}$ as

$$\varphi_{fc,s} = \frac{s\pi}{m}, \quad s = 0, 1, \ldots, m-1. \tag{34}$$

Introducing expression (26a, 26b) and Eq. (34) into the Eq. (30), the frequency equation is obtained in the following form

$$-\bar{m}\omega_n^2(1+\mu\alpha_n^2) + \alpha_n^2\bar{e} + 2v_n(1-cos\varphi_{fc,s}) + \eta A\alpha_n^2\left(H_y^2+H_z^2\right)(1+\mu\alpha_n^2) = 0. \tag{35}$$

The natural frequency of the "Free-Chain" CMNRS is given as

$$\omega_{nfc,s} = \sqrt{\frac{\alpha_n^2\bar{e} + 2v_n(1-cos\varphi_{fc,s}) + \eta A\alpha_n^2\left(H_y^2+H_z^2\right)(1+\mu\alpha_n^2)}{\bar{m}(1+\mu\alpha_n^2)}}, \quad s = 0, 1, \ldots, m-1. \tag{36}$$

or in dimensionless form as

$$\Omega_{nfc,s} = \sqrt{\frac{\bar{\alpha}_n^2 + 2\bar{K}(1+\bar{\alpha}_n^2\nu^2)(1-\cos\varphi_{fc,s}) + MP\bar{\alpha}_n^2(1+\delta^2)(1+\bar{\alpha}_n^2\nu^2)}{1+\bar{\alpha}_n^2\nu^2}}, \quad s = 0, 1, \ldots, m-1. \tag{37}$$

where

$$\Omega_{nfc,s} = \omega_{nfc,s}L\sqrt{\frac{\rho}{E}}, \quad \bar{\alpha}_n^2 = \alpha_n^2 L, \quad \nu = \frac{\sqrt{\mu}}{L}, \quad \bar{K} = K\frac{L^2}{EA}, \quad H_z = \delta H_y, \quad MP = \frac{\eta H_y^2}{E}, \tag{38}$$

are dimensionless parameters.

3.2 Asymptotic analysis

Assuming that a number of nanorods tend to infinity, i.e. introducing $m \to \infty$ into Eq. (36), we get the asymptotic natural frequency as

$$\omega_{n,m\to\infty} = \sqrt{\frac{\alpha_n^2\bar{e} + \eta A\alpha_n^2\left(H_y^2+H_z^2\right)(1+\mu\alpha_n^2)}{\bar{m}(1+\mu\alpha_n^2)}}. \tag{39}$$

or in dimensionless form as

$$\Omega_{n,m\to\infty} = \sqrt{\frac{\bar{\alpha}_n^2 + MP\bar{\alpha}_n^2(1+\delta^2)(1+\bar{\alpha}_n^2\nu^2)}{1+\bar{\alpha}_n^2\nu^2}}. \tag{40}$$

Expression (39) is representing the lowest natural frequency of the system when the number of nanorods tends to infinity. From the physical point of view this expression represents the natural frequency of a single nanorod without the influence of coupled medium or other nanorods in the system.

4 Results and discussions

In the literature, one can find equivalent nonlocal continuum models of nanorod structures representing the single-walled carbon nanotubes [57] or ZnO nanotube [42]. When complex systems such as double [43, 44] or multiple nanorod systems coupled with certain type of medium are considered, the advantage of the nonlocal theory is evident since much less effort is needed to examine the vibration behavior compared to the atomistic dynamics simulation methods. Validation study is usually performed by comparing the results for natural frequencies obtained with the nonlocal theory and molecular dynamics. As stated in [54, 57], nonlocal rod models are giving good agreement for the values of nonlocal parameter $(e_0 a) = 1$ nm. Therefore, we validate our results with the results from [58] obtained for axial vibration of CNT by using the molecular dynamics. The main point of this study is to examine the influence of transverse magnetic field on axial vibration of CMNRS with arbitrary number of nanorods coupled with elastic medium. The proposed model generalizes the problem of axial vibration of coupled nanorods, where single and double nanorod systems are special cases of our model. Moreover, we obtain the analytical solution for natural frequencies of CMNRS composed of an arbitrary number of identical nanorods subjected to the transverse magnetic field in the y and z directions. To examine the influence of different effects on the longitudinal vibration of CMNRS affected by transversal magnetic field, natural frequencies are analyzed for various values of (1) dimensionless nonlocal parameter (2) number of nanorods, (3) dimensionless stiffness parameter of elastic medium and (4) dimensionless parameter of transversal magnetic field. In the parametric study, we used dimensionless parameters in order to cover a wide range of values.

4.1 Validation of the proposed method

To confirm the accuracy of our results, we consider a special case of CMNRS and reduce it to the double-nanorod system (DNRS) presented in [44], where the influence of magnetic field is not observed. The following values are used in the comparative study: $L = 1$ nm, $\bar{m} = 10^{-9}$ kg/m, $\bar{e} = 1$ nN, $e_0 a = 0 - 2$ nm and stiffness coefficient $K = 8$ N/nm. Table 1 shows the obtained results for natural frequencies of the "Free-Chain" CMNRS for C–F boundary conditions, when $n = 2, m = 2$ and $s = 0, 1$, and results for natural frequencies presented by Murmu and Adhikari [44]. Based on the given results, it can be observed that the natural frequencies obtain by the trigonometric method are in excellent agreement with the corresponding results found in the literature.

In order to validate the results obtained for the case with an arbitrary number of nanords in CMNRS $(m > 2)$, one can compare obtained analytical results with the numerical solutions of the characteristic polynomial of the system of algebraic Eqs. (25). It should be noted that numerical solutions of the system of algebraic Eqs. (25), obtained by using the function NSolve available in software package Wolfram Mathematica, gives almost identical results as analytical solution presented in Eq. (36). Thus, in Table 2 we give only the results for natural frequencies of CMNRS coupled in the "Free-Chain" system with three, five and seven nanorods and for two types of boundary conditions that are obtained from analytical expression Eq. (36). It can be noticed that the first natural frequency of the system is fundamental frequency, and independent from the elastic interaction between nanorods. In the case of the "Free-

Table 1 Comparison of the first four natural frequencies (GHz) of the "Free-Chain" CMNRS for C–F boundary conditions and different values of nonlocal parameter $e_0 a$

C–F		$e_0a = 0$ nm	$e_0a = 0.5$ nm	$e_0a = 1$ nm	$e_0a = 1.5$ nm	$e_0a = 2$ nm
		Murmu and Adhikari [44]				
$K = 8$ N/nm	1	1.5708	1.2353	0.8436	0.6137	0.4764
	2	4.2974	4.1864	4.0880	4.0468	4.0283
	3	4.7124	1.8411	0.9782	0.6601	0.4972
	4	6.1812	4.4033	4.1179	4.0541	4.0308
		Coupled multiple-nanorod "Free-Chain" system (m—number of nanorods)—presented analysis				
Analytical solution Eq. (36)	1	1.570796326794896	1.235335649677712	0.84356360806876	0.61368350567533	0.476445256994343
	2	4.297371417537975	4.186413043091264	4.08798233372626	4.04680212577017	4.028275075378096
$m = 2$	3	4.712388980384690	1.841050517025992	0.97821716853262	0.66009381221733	0.497209065755821
$K = 8$ N/nm	4	6.181149561566284	4.403347250245168	4.11787673793327	4.05409963381853	4.030783652724341

Table 2 The natural frequencies (GHz) of CMNRS with C–F boundary conditions obtained from analytical expression Eq. (36)

Coupled multiple-nanorod system for the "Free-Chain" system			
Number of nanorods		C–F boundary conditions	C–C boundary conditions
		Analytical Solutions Eq. (36)	
$m = 3$	1	0.476445256994343	0.493785246075695
	2	2.868274757221212	2.871205995612651
	3	4.922093059147948	4.923801769897122
$m = 5$	1	0.476445256994343	0.493785246075695
	2	1.811830061819609	1.816466889112729
	3	3.358977251026456	3.361480620090331
	4	4.601225053495162	4.603052875998840
	5	5.401043602204259	5.402600834712962
$m = 7$	1	0.476445256994343	0.493785246075695
	2	1.345919089868963	1.352154570603275
	3	2.500232639810277	2.503594823349477
	4	3.559025869477683	3.561388623267756
	5	4.448295744014041	4.450186379528734
	6	5.118870667701240	5.120513714363215
	7	5.535566996193896	5.537086395901795

m number of nanorods

Chain" system, the fundamental frequency of CMNRS is equivalent to the frequency of a single nanorod. The next lowest frequency is considerably higher than the fundamental one. However, the intervals between frequencies are decreasing with increase of the number of nanorods in CMNRS. Further, it can be observe that the lowest natural frequency for C–C boundary conditions tends to the fundamental one with increase of the number of nanorods. This can be confirmed with proposed the asymptotic analysis, for the case when the number of nanorods tends to the infinity.

To the best of author's knowledge, the longitudinal vibrations study of a system of m coupled nanorods using the molecular dynamic (MD) simulation is not available in the literature. This is to be expected since such systems are cumbersome to apply the atomistic simulation methods due to the large number of atoms in the system. From the obtained analytical results expressed in Eq. (36) and numerical results shown in Table 2, we can see that the first or fundamental natural frequency of CMNRS corresponds to the fundamental natural frequency of one single nanorod. This conclusion is based on the fact that the fundamental frequency does not change with an increase of the number of nanorods in CMNRS. Therefore, the fundamental natural frequency of CMNRS can be compared with the frequency obtained for a single nanorod using the molecular dynamics simulation method presented in [58]. If we neglect the external magnetic field parameter in expression for natural frequencies of CMNRS in Eq. (36) and introduce new form of natural resonant frequency $f = \omega_{nfc,s}/2\pi$ as proposed in [58], we can obtain the comparative values of natural resonant frequency with the results obtained by MD simulation [58]. Based on the material and geometrical parameters given for armchair SWCNT (5, 5) where mass density is $\rho = 9517\ \mathrm{kg/m^3}$, elastic modulus $E = 6.85$ TPa, length $L = 12.2$ nm and mode number $n = 1$, numerical values for natural resonant frequency for different values of nonlocal parameter are given in Table 3. We can notice that in the case when nonlocal effects are neglected i.e. $e_0a = 0$, the fundamental natural frequency is larger comparing to the result obtained via MD simulations. However, taking into account nonlocal effects by increasing the values of e_0a, natural resonant frequencies are decreasing and better agreement is achieved with MD simulation results. It can be concluded that the nonlocal parameter has a dampening effect on the fundamental natural frequency of CMNRS. Since the value of internal length scale parameter a is fixed for C–C bonds in SWCNT, to fit the results for natural frequency it is necessary to

Table 3 Validation of the fundamental natural frequency $f = \omega_{nfc,s}/2\pi$ of CMNRS given in (THz) with the results obtained with MD simulation for armchair SWCNT (5, 5)

Boundary conditions C–F		Nonlocal parameter			MD simulation Ref. [58]
		$e_0a = 0$ nm	$e_0a = 1$ nm	$e_0a = 2$ nm	
$H_y = H_z = 0$ $s = 0$ Equation (36)	f	0.549763	0.545262	0.532395	0.544

optimize the value of parameter e_0 for each problem of boundary conditions and geometrical properties of nanostructures.

4.2 Nonlocal and transversal magnetic field effects on the natural frequency of CMNRS

To analyze of the influence of different physical parameters on dynamic behavior of CMNRS, we use the dimensionless expression for natural frequency Eq. (37). Numerical results for dimensionless natural frequency are determined for $m = 3, 5,$ and 10 nanorods in CMNRS, three different values of magnetic field parameter $MP = 3, 5,$ and 7, ratio $\delta = H_z/H_y = 0.5$ and for values of nonlocal parameter in the range $\nu = 0$ - 1 as shown in Figs. 4, 5, 6 and 7. In this study we investigated CMNRS for both boundary conditions, clamped–clamped and clamped–free, where the system is coupled as "Free-Chain" for the value of parameter s = 1.

In Figs. 4a, b we plotted the values of dimensionless natural frequency for changes of the nonlocal parameter and the number of nanorods in CMNRS for three different values of dimensionless magnetic field parameter and C–C and C–F boundary conditions, respectively. It can be noticed that the values of natural frequency are higher when nonlocal parameter is not taken into account. An increase of nonlocal parameter

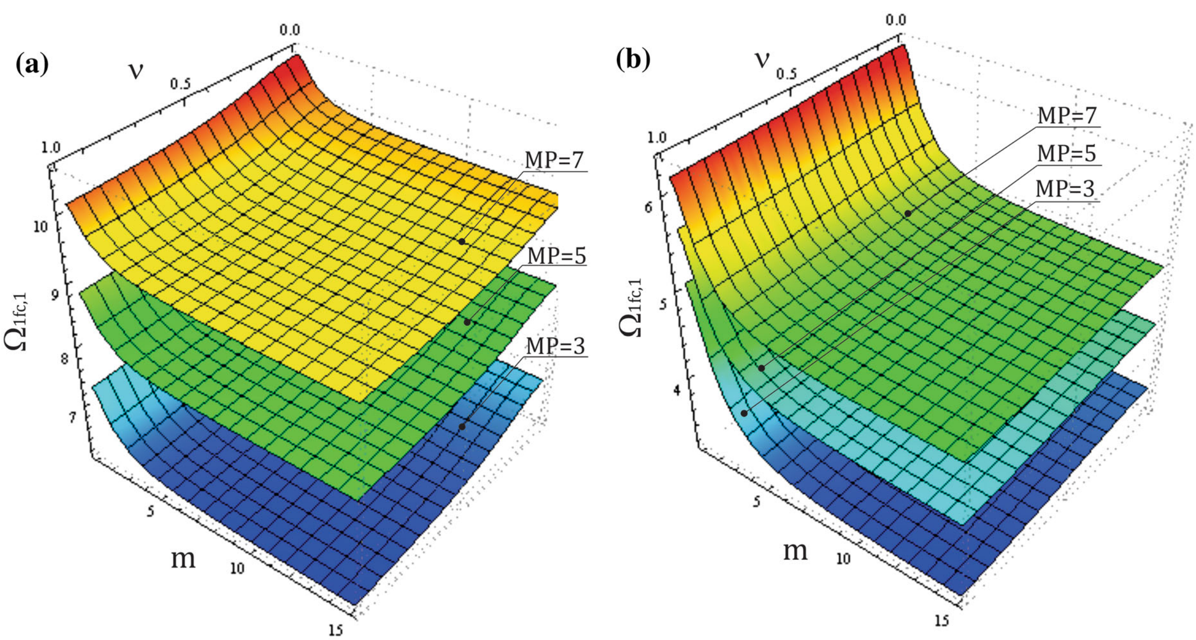

Fig. 4 The effect of number of nanorods (m) and nonlocal parameter (ν) on the natural frequency for two boundary conditions, **a** C–C and **b** C–F

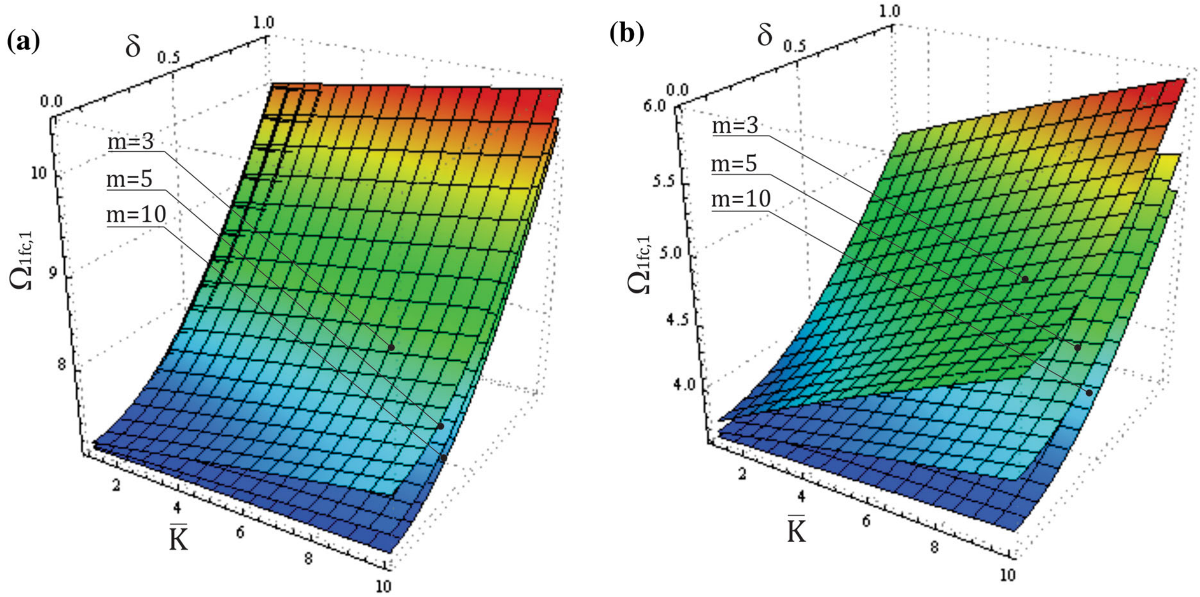

Fig. 5 The effect of ratio of transversal field in two directions δ and stiffness coefficients $\bar{K}$ on the natural frequency for two boundary conditions, **a** C–C and **b** C–F

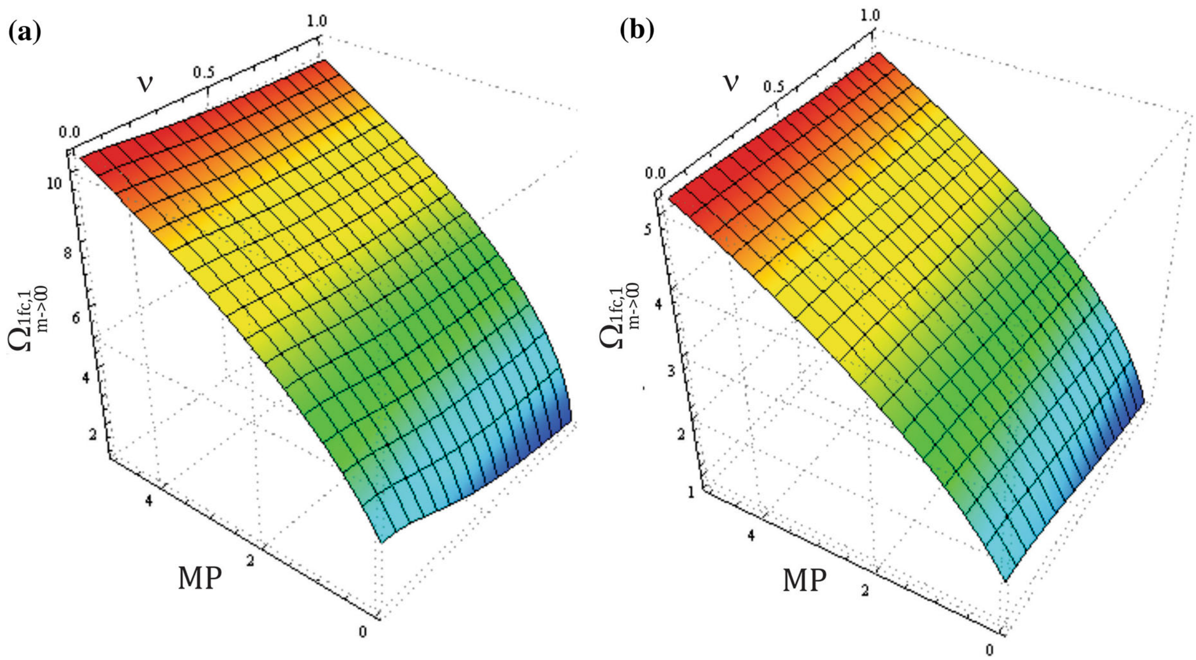

Fig. 6 The effect of magnetic field *MP* and nonlocal parameter *v* on the critical natural frequency for two boundary conditions, **a** C–C and **b** C–F

decreases the values of natural frequency as expected for both boundary conditions. However, this effect is less pronounced for C–F boundary conditions. This behavior is addressed to the nature of nonlocal elasticity models to predict the lower values of frequencies compared to the frequencies obtained

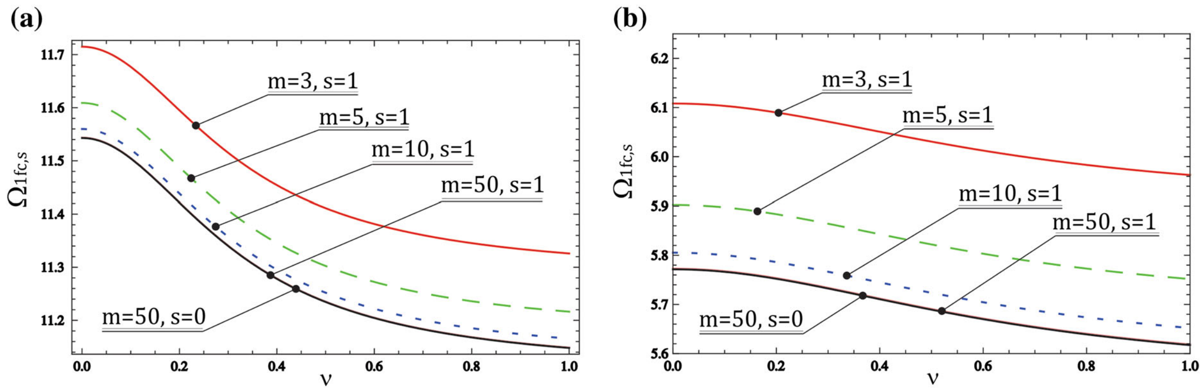

Fig. 7 The effect of nonlocal parameter (ν) with different number of nanorod (m) on the natural frequencies and critical natural frequency for two boundary conditions, **a** C–C and **b** C–F

with classical elasticity models. From the physical point of view this behavior is caused by the softening of the stiffness in axial direction of nanorods at nanoscale [54]. Further, we can notice a decrease of natural frequency for an increase of a number of nanorods in CMNRS. Such behavior can be explained based on the asymptotic analysis [26] where natural frequency tends to the lowest fundamental natural frequency of the system when the number of coupled nanorods (or other nano-structural elements such as nanoplate and nanobeam) tends to infinity. An opposite case can be noticed for an increase of the value of dimensionless magnetic field parameter. In this case, natural frequency increases which is attributed to the coupling effect of transverse magnetic field and axial vibration of nanorods. This effect is more pronounced for C–C boundary conditions.

From Figs. 5a, b one can observe the changes in dimensionless natural frequency for changes of dimensionless stiffness parameter of elastic medium, ratio δ and the number of nanorods in CMNRS for C–C and C–F boundary conditions, respectively. Figure 5 shows that an increase of ratio causes an increase of natural frequency. This behavior is attributed to an increase of overall magnitude of magnetic field. In addition, it can be noticed that change of natural frequency is more pronounced at higher values of ratio than for the lower one. Further, an increase of natural frequency for an increase of the mediums stiffness parameter is much less pronounced than for the change of ratio. The influence of an increase of the number of nanorods in CMNRS is the same as for the previous case and it reflects in a decrease of the natural frequency.

Figure 6 shows the change of dimensionless natural frequency for changes of the magnetic field parameter and nonlocal parameter. Again, it can be observed that natural frequency increases for an increase of magnetic field parameter while it decreases for an increase of nonlocal parameter. However, the effect of nonlocal parameter is weakening for higher values of magnetic field.

From the previous plots it is evident that the magnetic field increases the natural frequency. This consequence is caused by increased rigidity of the system. The similar conclusion is given in [54] for a single nanorod embedded in elastic medium. The authors stated that a magnetic field and vibration in transverse direction with respect to nanorod would result in the increase of corresponding natural frequency. The same is true for the multiple nanorod system with only difference that the natural frequency decreases for an increase of the number of nanorods in the system.

In order to present some results more clearly we give Fig. 7. As we stated earlier in the text, it is obvious that an increase of the number of nanorods causes a decrease of the second lowest frequency towards the fundamental frequency, which value is independent of the number of nanorods in CMNRS. This conclusion is in line with the analytical results obtained for the asymptotic natural frequency in Eq. (40).

In spite molecular simulation results for observed systems are not available in the literature we proved that the fundamental frequency of CMNRS is close to the results from molecular dynamics obtained for a single nanorod. The presented results show the property of magnetic field to change material characteristics of nanostructures and shift their natural frequencies. This could be important for the development of new generation of adaptable nanodeviceses such as nanosensors and nanoresonators by increasing the range of their resonant frequencies only by changing the magnitude of magnetic field.

5 Conclusion

A novel approach for studying the nonlocal longitudinal vibration of complex multi-nanorod system affected by transversal magnetic field is presented. The nonlocal theory applied to CMNRS is a generalized theory and it can be used to model other magnetically influenced vibration of complex multiple nanostructure systems. The mathematical model of CMNRS consists of a set of m homogeneous partial differential equations that have been derived by using the D'Alembert's principle and nonlocal constitutive relationship. The Lorentz magnetic force is introduced by classical Maxwell's equations. Explicit closed-form expressions for natural frequencies are derived successfully by using the method of separations of variables and the trigonometric method, for both boundary conditions, clamped–clamped (C–C) and clamped–free (C–F), and "Free-Chain" system. The obtained analytical results are validated by comparison with the results found in the literature for the special case of CMNRS with two elastically coupled nanorods. Also, the first natural frequency is compared with the results obtained by MD simulation for the single nanorod and excellent agreement is shown. Moreover, the first natural frequency of CMNRS in the case of "Free-Chain" system represent the fundamental natural frequency which is independent of the number of nanorods and the influence of elastic medium. Comparing the obtained values of natural frequencies for two types of boundary conditions, it is found that the natural frequencies for C–C boundary conditions are larger than for the C–F boundary conditions since the system with C–C boundary conditions has stronger constraints. The effects of nonlocal parameter, stiffness coefficient and the number of nanorods in CMNRS on natural frequencies are also analyzed in the parametric study. It is found that an increase of the nonlocal parameter v and a number of nanorods m reduces the natural frequency. Also, it is noted that an increase of the stiffness coefficient and transversal magnetic field causes an increase of natural frequency. Finally, it should be noted that the possibility to change natural frequencies of magnetically affected nano-structures only by changing the magnitude of external magnetic field without changing other physical parameters is very applicable in nanoelectromechanical systems.

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