Analysis of Asymmetric Nonviscously Damped Linear Dynamic Systems

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Multiple-degree-of-freedom linear asymmetric nonviscously damped systems are considered. It is assumed that the nonviscous damping forces depend on the past history of velocities via convolution integrals over exponentially decaying kernel functions. An extended state-space approach involving a single asymmetric matrix is proposed. The nature of the eigensolutions in the extended state space has been explored. Some useful results relating the modal matrix in the extended state space and the modal matrix in the original space has been derived. Numerical examples are provided to illustrate the results.
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1 Introduction

Viscous damping is the most common model for the modeling of vibration damping. This model assumes that the instantaneous generalized velocities are the only relevant variables that determine damping. Viscous damping models are used widely for their simplicity and mathematical convenience even though the true damping behavior is expected to be nonviscous. Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities will be called nonviscous damping models. Of many nonviscous damping models, the convolution integral model ([1–3]) is possibly the most general model within the scope of linear analysis. In this paper we consider that the damping model consists of viscous and nonviscous damping. The equations of motion of a N-degree-of-freedom linear system with such damping can be expressed by

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{D}\dot{\mathbf{u}}(t)+\int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{u}}(\tau)d\tau+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t), \tag{1}$$

where $\mathbf{u}(t)\in\mathbb{R}^N$ is the vector of generalized coordinates, $\mathbf{M}\in\mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K}\in\mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{D}\in\mathbb{R}^{N\times N}$ is the viscous damping matrix, and $\mathbf{f}(t)\in\mathbb{R}^N$ is the forcing vector. The matrix of the damping functions, $\boldsymbol{\mathcal{G}}(t-\tau)$, can have various mathematical forms. For example, when $\boldsymbol{\mathcal{G}}(t-\tau)=\mathbf{D}\delta(t-\tau)$, where $\delta(t)$ is the Dirac delta function, the kernel function reduces to the case of viscous damping. Among many other mathematically possible damping functions, the exponential damping model is physically most meaningful ([4]). For this damping model the kernel function matrix has the special form

$$\boldsymbol{\mathcal{G}}(t-\tau)=\sum_{k=1}^{n}\mu_k e^{-\mu_k(t-\tau)}\mathbf{C}_k, \tag{2}$$

where $\mu_k\in\mathbb{R}^+$ are known as the relaxation parameters, $\mathbf{C}_k\in\mathbb{R}^{N\times N}$ are the damping coefficient matrices, and n denotes the number relaxation parameters used to describe the damping behavior.

A brief review of literature on dynamics of nonviscously damped systems may be found in Ref. [3]. Muravyouv [5,6] and Muravyov and Hutton [7,8] have considered this kind of system where the exponential kernel function is associated with the stiffness matrix. Recently Wagner and Adhikari [9] have proposed an exact state-space method for the analysis of linear systems with exponential damping. Their approach was based on representing Eq. (1) in terms of two *symmetric* matrices in an augmented state space. In this paper an alternative approach based on only one asymmetric matrix is suggested and the relationships between the eigenvectors in the state space and the eigenvectors in the original space have been derived.

It is assumed that $\mathbf{M}^{-1}$ exists, that is, systems with a singular mass matrix is not considered in the present work. For the sake of generality the usual symmetry and non-negative definiteness properties of the system matrices are not assumed. Further, it is also considered that in general the system is neither symmetrizable ([10,11]), nor simultaneously diagonalizable by any linear transformations ([12]). In Sec. 2, the eigenvalue problem associated with Eq. (1) is briefly reviewed. The state-space approach based on internal variables is formulated in Sec. 3. The eigenvalue problem in the state space and some properties of the eigensolutions are discussed in Sec. 4. In Sec. 5, the proposed results are illustrated by a numerical example.

2 Background: The Eigenvalue Problem

Free vibration characteristics of the system is governed by the eigenvalue problem associated with the equations of motion (1). Assuming the initial conditions

$$\mathbf{u}(t=0)=\boldsymbol{u}_0\in\mathbb{R}^N \quad\text{and}\quad \dot{\mathbf{u}}(t=0)=\dot{\boldsymbol{u}}_0\in\mathbb{R}^N \tag{3}$$

and taking the Laplace transform of Eq. (1) one obtains

$$s^2\mathbf{M}\overline{\mathbf{u}}(s)-s\mathbf{M}\boldsymbol{u}_0-\mathbf{M}\dot{\boldsymbol{u}}_0+s[\mathbf{D}+\mathbf{G}(s)]\overline{\mathbf{u}}(s)$$
$$-[\mathbf{D}+\mathbf{G}(s)]\boldsymbol{u}_0+\mathbf{K}\overline{\boldsymbol{u}}(s)=\overline{\mathbf{f}}(s) \tag{4}$$

$$\text{or}\quad s^2\mathbf{M}\overline{\mathbf{u}}(s)+s[\mathbf{D}+\mathbf{G}(s)]\overline{\boldsymbol{u}}(s)+\mathbf{K}\overline{\boldsymbol{u}}(s)=\overline{\mathbf{p}}(s). \tag{5}$$

Here

$$\overline{\mathbf{p}}(s)=\overline{\mathbf{f}}(s)+\mathbf{M}\dot{\boldsymbol{u}}_0+[s\mathbf{M}+\mathbf{D}+\mathbf{G}(s)]\boldsymbol{u}_0 \tag{6}$$

is the equivalent forcing function in the Laplace domain. In the above equations $\overline{\boldsymbol{u}}(s)=\mathcal{L}[\mathbf{u}(t)]\in\mathbb{C}^N$, $\overline{\mathbf{f}}(s)=\mathcal{L}[\mathbf{f}(t)]\in\mathbb{C}^N$ and $\mathcal{L}[\bullet]$ denotes the Laplace transform. The matrix $\mathbf{G}(s)\in\mathbb{C}^{N\times N}$ is the Laplace transform of $\boldsymbol{\mathcal{G}}(t)$ and can be obtained from Eq. (2) as

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$$\mathbf{G}(s)=\sum_{k=1}^{n}\frac{\mu_k}{s+\mu_k}\mathbf{C}_k. \tag{7}$$

In the context of structural dynamics, the Laplace parameter $s=i\omega$, where $i=\sqrt{-1}$ and $\omega\in\mathbb{R}^+$ denotes the frequency. Considering the free vibration from Eq. (5) the right-eigenvalue problem can be represented by

$$\left[\lambda_j^2\mathbf{M}+\lambda_j\mathbf{D}+\lambda_j\sum_{k=1}^{n}\frac{\mu_k}{\lambda_j+\mu_k}\mathbf{C}_k+\mathbf{K}\right]\boldsymbol{u}_j=\mathbf{0}. \tag{8}$$

Similarly the left-eigenvalue problem can be expressed as

$$\boldsymbol{v}_j^T\left[\lambda_j^2\mathbf{M}+\lambda_j\mathbf{D}+\lambda_j\sum_{k=1}^{n}\frac{\mu_k}{\lambda_j+\mu_k}\mathbf{C}_k+\mathbf{K}\right]=\mathbf{0}^T. \tag{9}$$

Here $\lambda_j\in\mathbb{C}$ is the jth eigenvalue and $\boldsymbol{u}_j\in\mathbb{C}^N$ is the jth eigenvector. Suppose the number of eigenvalues is m. The methods for solving this kind of problem follow mainly two routes, (a) the extended state-space method [13–16,7,6,9], and (b) the methods in the configuration space or ''N'' space ([2,3]). In the next section an extended state-space method based on a set of internal variables is proposed.

For lightly damped systems, among the m eigenvalues $2N$ appear in complex conjugate pairs and the rest are purely real and negative ([3,17]). We emphasize that these results are simply observations and a detailed mathematical proof of the conditions under which such results are valid are not yet available. A physical explanation, however, can be given. The N pairs of complex conjugate eigenvalues can be related to the N (complex) modes of structural vibration. These modes are therefore called *elastic modes* ([3]). The other $(m-2N)$ purely dissipative modes appear due to nonviscous nature of the damping model and therefore called *nonviscous modes* ([3]). Nonviscous modes, or similar to these, are known by different names in the literature of different subjects, for example, ''wet modes'' in the context of ship dynamics ([18]) and ''damping modes'' in the context of viscoelastic structures ([15]). For convenience we construct and partition the following matrices:

$$\boldsymbol{\Lambda}=\mathrm{diag}[\lambda_1,\lambda_2,\ldots,\lambda_m]\in\mathbb{C}^{m\times m}=\mathrm{diag}[\boldsymbol{\Lambda}_e,\boldsymbol{\Lambda}_e^*\boldsymbol{\Lambda}_{nv}], \tag{10}$$

$$\mathbf{U}=[\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_m]\in\mathbb{C}^{n\times m}=[\mathbf{U}_e,\mathbf{U}_e^*,\mathbf{U}_{nv}], \tag{11}$$

$$\text{and}\quad \mathbf{V}=[\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_m]\in\mathbb{C}^{n\times m}=[\mathbf{V}_e,\mathbf{V}_e^*,\mathbf{V}_{nv}]. \tag{12}$$

Here $(\bullet)^*$ denotes complex conjugation, the subscript $(\bullet)_e$ corresponds to elastic modes, and the subscript $(\bullet)_{nv}$ corresponds to nonviscous modes.

3 State-Space Formulation

For viscously damped systems, the state-space method based on one asymmetric system matrix has been used extensively in literature (see Newland [19,20]). Here this approach will be extended to system (1) by using a set of internal variables. In what follows next, two physically realistic cases, namely, (a) when all $\mathbf{C}_k$ matrices are of full rank and, (b) all $\mathbf{C}_k$ matrices are rank deficient, have been considered in details. A third possible case, when only some $\mathbf{C}_k$ matrices are rank deficient, can be easily derived from the two preceding cases.

3.1 Case A: All $\mathbf{C}_k$ Matrices are of Full Rank.

Here it is assumed that

$$\mathrm{rank}\;(\mathbf{C}_k)=N,\quad \forall k=1,\ldots,n. \tag{13}$$

We introduce a set of internal variables $\mathbf{y}_k(t)\in\mathbb{R}^N$, $\forall k=1,\ldots,n$ by

$$\mathbf{y}_k(t)=\int_0^t\mu_k e^{-\mu_k(t-\tau)}\dot{\mathbf{u}}(\tau)d\tau. \tag{14}$$

Applying Leibniz's rule for differentiation of an integral to Eq. (14) one obtains

$$\dot{\mathbf{y}}_k(t)=\int_0^t-\mu_k^2 e^{-\mu_k(t-\tau)}\dot{\mathbf{u}}(\tau)d\tau+\mu_k\dot{\mathbf{u}}(t). \tag{15}$$

Multiplying Eq. (14) by the relaxation parameter μ_k, then adding it to Eq. (15) results in

$$\dot{\mathbf{y}}_k(t)+\mu_k\mathbf{y}_k(t)=\mu_k\dot{\mathbf{u}}(t). \tag{16}$$

Now, taking account of the kernel function matrix (2), Eq. (1) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{D}\dot{\mathbf{u}}(t)+\sum_{k=1}^{n}\mathbf{C}_k\left\{\int_0^t\mu_k e^{-\mu_k(t-\tau)}\dot{\mathbf{u}}(\tau)d\tau\right\}+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t). \tag{17}$$

Substituting Eq. (14) into Eq. (17) leads to

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{D}\dot{\mathbf{u}}(t)+\sum_{k=1}^{n}\mathbf{C}_k\mathbf{y}_k(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t). \tag{18}$$

Using additional state variables

$$\mathbf{v}(t)=\dot{\mathbf{u}}(t) \tag{19}$$

and assuming that $\mathbf{M}^{-1}$ exists, Eq. (17) can be rewritten as

$$\dot{\mathbf{v}}(t)+\mathbf{M}^{-1}\mathbf{D}\dot{\mathbf{u}}(t)+\sum_{k=1}^{n}\mathbf{M}^{-1}\mathbf{C}_k\mathbf{y}_k(t)+\mathbf{M}^{-1}\mathbf{K}\mathbf{u}(t)=\mathbf{M}^{-1}\mathbf{f}(t). \tag{20}$$

Rearranging Eqs. (16), (19), and (20) one obtains

$$\dot{\mathbf{u}}(t)=\mathbf{v}(t), \tag{21}$$

$$\dot{\mathbf{v}}(t)=-\mathbf{M}^{-1}\mathbf{D}\mathbf{v}(t)-\sum_{k=1}^{n}\mathbf{M}^{-1}\mathbf{C}_k\mathbf{y}_k(t)-\mathbf{M}^{-1}\mathbf{K}\mathbf{u}(t)+\mathbf{M}^{-1}\mathbf{f}(t), \tag{22}$$

$$\dot{\mathbf{y}}_k(t)=\mu_k\mathbf{I}\mathbf{v}(t)-\mu_k\mathbf{I}\mathbf{y}_k(t),\quad \forall k=1,\ldots,n \tag{23}$$

or in the matrix form

$$\dot{\mathbf{z}}(t)=\mathbf{A}\mathbf{z}(t)+\mathbf{r}(t), \tag{24}$$

where

$$\mathbf{A}=\begin{bmatrix}\mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} & -\mathbf{M}^{-1}\mathbf{C}_1 & -\mathbf{M}^{-1}\mathbf{C}_2 & \cdots & -\mathbf{M}^{-1}\mathbf{C}_n\\ \mathbf{O} & \mu_1\mathbf{I} & -\mu_1\mathbf{I} & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \mu_2\mathbf{I} & \mathbf{O} & -\mu_2\mathbf{I} & \cdots & \mathbf{O}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{O} & \mu_n\mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mu_n\mathbf{I}\end{bmatrix}\in\mathbb{R}^{m\times m}, \tag{25}$$

$$\mathbf{r}(t)=\begin{Bmatrix}\mathbf{O}\\ \mathbf{M}^{-1}\mathbf{f}(t)\\ \mathbf{O}\\ \mathbf{O}\\ \vdots\\ \mathbf{O}\end{Bmatrix}\in\mathbb{R}^m, \tag{26}$$

$$\text{and}\quad \mathbf{z}(t)=\begin{Bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\\ \mathbf{y}_1(t)\\ \mathbf{y}_2(t)\\ \vdots\\ \mathbf{y}_n(t)\end{Bmatrix}\in\mathbb{R}^m. \tag{27}$$

In the above equations $\mathbf{z}(t)$ is the extended state vector, $\mathbf{A}$ is the system matrix in the extended state space, $\mathbf{r}(t)$ is the force vector in the extended state space, $\mathbf{O}\in\mathbb{R}^{N\times N}$ is a null matrix, and $\mathbf{I}\in\mathbb{R}^{N\times N}$ is an identity matrix. It is clear that the order of the system m is

$$m=2N+nN. \tag{28}$$

In the viscous damping limit, *all* the internal variables can be disregarded, that is, all $n\times N$ equations after the first $2N$ rows in Eq. (24) can be deleted from the formulation. Under these conditions it is easy to see that the equations of motion (24) reduce to the standard form ([19,20]) for viscously damped systems with

$$\mathbf{A}=\begin{bmatrix}\mathbf{O} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D}\end{bmatrix},\quad \mathbf{r}(t)=\begin{Bmatrix}\mathbf{O}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{Bmatrix},$$

and

$$\mathbf{z}(t)=\begin{Bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\end{Bmatrix}. \tag{29}$$

This shows that the representation of the equations of motion by Eq. (24) is a natural generalization of the standard state-space formulation for viscously damped systems.

3.2 Case B: All $\mathbf{C}_k$ Matrices are Rank Deficient. In a large engineering structure it is possible to have different damping in different parts of a structure. For example, different members of a space frame may have different damping properties, each associated with its relaxation parameter μ_k. In this case the associated coefficient matrix $\mathbf{C}_k$ will be rank deficient because it will have nonzero blocks corresponding to the associated elements only. In this section we assume that in general

$$r_k=\text{rank}\ (\mathbf{C}_k)\leq N,\quad \forall k=1,\ldots,n. \tag{30}$$

This implies that the number of nonzero eigenvalues of $\mathbf{C}_k$ is r_k. Therefore there exist two matrices $\tilde{\mathbf{U}}_k\in\mathbb{R}^{N\times N}$ and $\tilde{\mathbf{V}}_k\in\mathbb{R}^{N\times N}$ such that

$$\tilde{\mathbf{V}}_k^T\mathbf{C}_k\tilde{\mathbf{U}}_k=\begin{bmatrix}\mathbf{d}_k & \mathbf{O}_{1k}\\ \mathbf{O}_{1k}^T & \mathbf{O}_{2k}\end{bmatrix}. \tag{31}$$

In the above equation $\mathbf{d}_k\in\mathbb{R}^{r_k\times r_k}$ is a diagonal matrix consisting of only the nonzero eigenvalues of $\mathbf{C}_k$, $\mathbf{O}_{1k}$ and $\mathbf{O}_{2k}$ are null matrices of orders $r_k\times(N-r_k)$ and $(N-r_k)\times(N-r_k)$, respectively. For convenience partition $\tilde{\mathbf{U}}_k$ and $\tilde{\mathbf{V}}_k$ as

$$\tilde{\mathbf{U}}_k=[\tilde{\mathbf{U}}_{1k}|\tilde{\mathbf{U}}_{2k}] \tag{32}$$

$$\text{and}\quad \tilde{\mathbf{V}}_k=[\tilde{\mathbf{V}}_{1k}|\tilde{\mathbf{V}}_{2k}]. \tag{33}$$

The columns of matrices $\tilde{\mathbf{U}}_{1k}\in\mathbb{R}^{N\times r_k}$ and $\tilde{\mathbf{V}}_{1k}\in\mathbb{R}^{N\times r_k}$ are the eigenvectors of $\mathbf{C}_k$ and $\mathbf{C}_k^T$ corresponding to the nonzero block $\mathbf{d}_k$ and the columns of matrices $\tilde{\mathbf{U}}_{2k}\in\mathbb{R}^{N\times(N-r_k)}$ and $\tilde{\mathbf{V}}_{2k}\in\mathbb{R}^{N\times(N-r_k)}$ are the eigenvectors of $\mathbf{C}_k$ and $\mathbf{C}_k^T$ corresponding to the other $(N-r_k)$ numbers of zero eigenvalues. Now defining the rectangular transformation matrices

$$\mathbf{R}_k=\tilde{\mathbf{U}}_{1k}\in\mathbb{R}^{N\times r_k} \tag{34}$$

$$\text{and}\quad \mathbf{L}_k=\tilde{\mathbf{V}}_{1k}\in\mathbb{R}^{N\times r_k} \tag{35}$$

it is easy to show that

$$\mathbf{L}_k^T\mathbf{C}_k\mathbf{R}_k=\mathbf{d}_k. \tag{36}$$

Thus the matrices $\mathbf{R}_k$ and $\mathbf{L}_k$ in Eqs. (34) and (35) transform the originally rank deficient matrix $\mathbf{C}_k$ to a full-rank matrix with rank r_k.

Now we define a set of internal variables of reduced dimension $\tilde{\mathbf{y}}_k(t)\in\mathbb{R}^{r_k}$ using the rectangular transformation matrix $\mathbf{R}_k$ as

$$\mathbf{y}_k(t)=\mathbf{R}_k\tilde{\mathbf{y}}_k(t). \tag{37}$$

From this equation it immediately follows that

$$\dot{\mathbf{y}}_k(t)=\mathbf{R}_k\dot{\tilde{\mathbf{y}}}_k(t), \tag{38}$$

where $\mathbf{y}_k(t)$ is defined in Eq. (14). Using these relationships, Eqs. (22) and (23) can be expressed as

$$\dot{\mathbf{v}}(t)=-\mathbf{M}^{-1}\mathbf{D}\mathbf{v}(t)-\sum_{k=1}^{n}\mathbf{M}^{-1}\mathbf{C}_k\mathbf{R}_k\tilde{\mathbf{y}}_k(t)-\mathbf{M}^{-1}\mathbf{K}\mathbf{u}(t)+\mathbf{M}^{-1}\mathbf{f}(t) \tag{39}$$

$$\text{and}\quad \mathbf{R}_k\dot{\tilde{\mathbf{y}}}_k(t)=\mu_k\mathbf{v}(t)-\mu_k\mathbf{R}_k\tilde{\mathbf{y}}_k(t). \tag{40}$$

Because Eq. (40) still represents a set of N equations, we premultiply this by $\mathbf{L}_k^T$ to obtain a reduced set of r_k equations:

$$[\mathbf{L}_k^T\mathbf{R}_k]\dot{\tilde{\mathbf{y}}}_k(t)=\mu_k\mathbf{L}_k^T\mathbf{v}(t)-\mu_k[\mathbf{L}_k^T\mathbf{R}_k]\tilde{\mathbf{y}}_k(t). \tag{41}$$

Taking the inverse of $[\mathbf{L}_k^T\mathbf{R}_k]$, the preceding equation may be rewritten as

$$\dot{\tilde{\mathbf{y}}}_k(t)=\mu_k\mathbf{T}_k\mathbf{v}(t)-\mu_k\mathbf{I}_{r_k}\tilde{\mathbf{y}}_k(t), \tag{42}$$

$$\text{where}\quad \mathbf{T}_k=[\mathbf{L}_k^T\mathbf{R}_k]^{-1}\mathbf{L}_k^T\in\mathbb{R}^{r_k\times N},\quad \forall k=1,\ldots,n. \tag{43}$$

Now Eqs. (21), (39), and (42) can be combined into the first-order form as

$$\dot{\tilde{\mathbf{z}}}(t)=\tilde{\mathbf{A}}\tilde{\mathbf{z}}(t)+\tilde{\mathbf{r}}(t), \tag{44}$$

where

$$\tilde{\mathbf{A}}=\begin{bmatrix}\mathbf{O} & \mathbf{I}_N & \mathbf{O}_{N,r_1} & \mathbf{O}_{N,r_2} & \cdots & \mathbf{O}_{N,r_n}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} & -\mathbf{M}^{-1}\mathbf{C}_1\mathbf{R}_1 & -\mathbf{M}^{-1}\mathbf{C}_2\mathbf{R}_2 & \cdots & -\mathbf{M}^{-1}\mathbf{C}_n\mathbf{R}_n\\ \mathbf{O}_{r_1,N} & \mu_1\mathbf{T}_1 & -\mu_1\mathbf{I}_{r_1} & \mathbf{O}_{r_1,r_2} & \cdots & \mathbf{O}_{r_1,r_n}\\ \mathbf{O}_{r_2,N} & \mu_2\mathbf{T}_2 & \mathbf{O}_{r_2,r_1} & -\mu_2\mathbf{I}_{r_2} & \cdots & \mathbf{O}_{r_2,r_n}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{O}_{r_n,N} & \mu_n\mathbf{T}_n & \mathbf{O}_{r_n,r_1} & \mathbf{O}_{r_n,r_2} & \cdots & -\mu_n\mathbf{I}_{r_n}\end{bmatrix}\in\mathbb{R}^{m\times m}, \tag{45}$$

$$\tilde{\mathbf{r}}(t)=\begin{Bmatrix}\mathbf{0}_N\\ \mathbf{M}^{-1}\mathbf{f}(t)\\ \mathbf{0}_{r_1}\\ \mathbf{0}_{r_2}\\ \vdots\\ \mathbf{0}_{r_n}\end{Bmatrix}\in\mathbb{R}^m, \tag{46}$$

$$\text{and}\quad \tilde{\mathbf{z}}(t)=\begin{Bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\\ \tilde{\mathbf{y}}_1(t)\\ \tilde{\mathbf{y}}_2(t)\\ \vdots\\ \tilde{\mathbf{y}}_n(t)\end{Bmatrix}\in\mathbb{R}^m. \tag{47}$$

In the above equations

$$m=2N+\sum_{k=1}^{n} r_k \tag{48}$$

is the order of the system, $\mathbf{O}_{ij}$ are $i\times j$ null matrices, $\mathbf{I}_j$ are $j\times j$ identity matrices, and $\mathbf{0}_j$ are vectors of j zeros. The terms $\tilde{(\cdot)}$ are corresponding to terms $(\cdot)$ defined in Eq. (24). When all $\mathbf{C}_k$ matrices are of full rank, that is, when $r_k=N,\forall k$, then one can choose all $\mathbf{R}_k$ and $\mathbf{L}_k$ matrices as the identity matrices and Eq. (44) reduces to Eq. (24).

4 The Eigenvalue Problem

4.1 Case A: All $\mathbf{C}_k$ Matrices are of Full Rank. The right and the left eigenvalue problems associated with Eq. (24) can be expressed as

$$\mathbf{A}\boldsymbol{\phi}_j=\lambda_j\boldsymbol{\phi}_j \tag{49}$$

$$\text{and}\quad \boldsymbol{\psi}_j^T\mathbf{A}=\lambda_j\boldsymbol{\psi}_j^T\quad j=1,2,\ldots,(2+n)N, \tag{50}$$

where $\lambda_j\in\mathbb{C}$ is the jth eigenvalue, $\boldsymbol{\phi}_j\in\mathbb{C}^{(2+n)N}$ and $\boldsymbol{\psi}_j\in\mathbb{C}^{(2+n)N}$ are respectively the jth right and left eigenvectors. Because $\mathbf{A}$ is a real matrix the eigenvalues can only be real or, if complex, then must appear in complex conjugate pairs. Construct the "modal matrices:"

$$\boldsymbol{\Phi}=[\phi_1,\phi_2,\ldots,\phi_{(2+n)N}]\in\mathbb{C}^{(2+n)N\times(2+n)N} \tag{51}$$

$$\text{and}\quad \boldsymbol{\Psi}=[\psi_1,\psi_2,\ldots,\psi_{(2+n)N}]\in\mathbb{C}^{(2+n)N\times(2+n)N}. \tag{52}$$

It can be easily shown that the right eigenvectors and the left eigenvectors satisfy a biorthogonality relationship, i.e., $\boldsymbol{\psi}_k^T\mathbf{A}\boldsymbol{\phi}_j=0,\ \forall k\neq j$. We also normalize the eigenvectors such that

$$\boldsymbol{\Psi}^T\boldsymbol{\Phi}=\mathbf{I}_{(2+n)N} \tag{53}$$

$$\text{and}\quad \boldsymbol{\Psi}^T\mathbf{A}\boldsymbol{\Phi}=\boldsymbol{\Lambda}. \tag{54}$$

However, it should be noted that the above normalization is not sufficient to define $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ uniquely.

4.1.1 The Structure of the Modal Matrices. From the definition of $\mathbf{z}(t)$ in Eq. (27), the right eigenvectors in the extended state space can be related to the right eigenvectors in the original space (8) by

$$\boldsymbol{\phi}_j=\begin{Bmatrix}\boldsymbol{u}_j\\ \lambda_j\boldsymbol{u}_j\\ \boldsymbol{y}_{1j}\\ \boldsymbol{y}_{2j}\\ \vdots\\ \boldsymbol{y}_{nj}\end{Bmatrix}, \tag{55}$$

where $\boldsymbol{y}_{1j},\boldsymbol{y}_{2j},\ldots,\boldsymbol{y}_{nj}$ are components of the jth eigenvector corresponding to the internal variables $\mathbf{y}_1(t),\mathbf{y}_2(t),\ldots,\mathbf{y}_n(t)$. The vectors $\boldsymbol{y}_{kj}\in\mathbb{C}^N,\ \forall k=1,2,\ldots,n$ can be related to $\boldsymbol{u}_j$ using Eq. (16). Taking the Laplace transform of Eq. (16) results in

$$s\overline{\mathbf{y}}_k+\mu_k\overline{\mathbf{y}}_k=s\mu_k\overline{\mathbf{u}}, \tag{56}$$

where $\overline{\mathbf{y}}_{\mathbf{k}}$ is the Laplace transform of $\mathbf{y}_k(t)$. For the jth eigenvalue one obtains

$$\lambda_j\boldsymbol{y}_{kj}+\mu_k\boldsymbol{y}_{kj}=\lambda_j\mu_k\boldsymbol{u}_j\quad\text{or}\quad(\lambda_j+\mu_k)\boldsymbol{y}_{kj}=\lambda_j\mu_k\boldsymbol{u}_j. \tag{57}$$

Provided $\lambda_j\neq-\mu_k$, from the preceding equation,

$$\boldsymbol{y}_{kj}=\frac{\lambda_j\mu_k}{\lambda_j+\mu_k}\boldsymbol{u}_j,\quad \forall k=1,2,\ldots,n;\ \forall j=1,2,\ldots,(2+n)N. \tag{58}$$

Using Eqs. (55) and (58) the right eigenvectors in the state space $\boldsymbol{\phi}_j$ can be related to the right eigenvectors in the original space $\boldsymbol{u}_j$. It is useful to represent this relationship in a matrix form. Define a matrix

$$\boldsymbol{Y}_k=[\boldsymbol{y}_{k1},\boldsymbol{y}_{k_2},\ldots,\boldsymbol{y}_{k_{(2+n)N}}]\in\mathbb{C}^{N\times(2+n)N}. \tag{59}$$

For $j=1,2,\ldots,(2+n)N$, Eq. (57) can be written in a matrix form as

$$\boldsymbol{Y}_k\boldsymbol{\Lambda}+\mu_k\boldsymbol{Y}_k=\mu_k\mathbf{U}\boldsymbol{\Lambda}. \tag{60}$$

Dividing this equation by μ_k one obtains

$$\boldsymbol{Y}_k=\mathbf{U}\boldsymbol{\Lambda}[\boldsymbol{\Lambda}/\mu_k+\mathbf{I}_{(2+n)N}]^{-1}. \tag{61}$$

Using this expression the matrix of right eigenvectors in the state space, given by Eq. (51), can be expressed as

$$\boldsymbol{\Phi}=\begin{bmatrix}\mathbf{U}\\ \mathbf{U}\boldsymbol{\Lambda}\\ \mathbf{U}\boldsymbol{\Lambda}[\boldsymbol{\Lambda}/\mu_1+\mathbf{I}_{(2+n)N}]^{-1}\\ \mathbf{U}\boldsymbol{\Lambda}[\boldsymbol{\Lambda}/\mu_2+\mathbf{I}_{(2+n)N}]^{-1}\\ \vdots\\ \mathbf{U}\boldsymbol{\Lambda}[\boldsymbol{\Lambda}/\mu_n+\mathbf{I}_{(2+n)N}]^{-1}\end{bmatrix}. \tag{62}$$

In view of the partitions shown in Eqs. (10) and (11), the preceding equation can be conveniently partitioned as

$$\mathbf{\Phi}=\begin{bmatrix} \mathbf{U}_e & \mathbf{U}_e^* & \mathbf{U}_{nv} \\ \mathbf{U}_e\mathbf{\Lambda}_e & \mathbf{U}_e^*\mathbf{\Lambda}_e^* & \mathbf{U}_{nv}\mathbf{\Lambda}_{nv} \\ \mathbf{U}_e\mathbf{\Lambda}_e[\mathbf{\Lambda}_e/\mu_1+\mathbf{I}_N]^{-1} & \mathbf{U}_e^*\mathbf{\Lambda}_e^*[\mathbf{\Lambda}_e^*/\mu_1+\mathbf{I}_N]^{-1} & \mathbf{U}_{nv}\mathbf{\Lambda}_{nv}[\mathbf{\Lambda}_{nv}/\mu_1+\mathbf{I}_{nN}]^{-1} \\ \mathbf{U}_e\mathbf{\Lambda}_e[\mathbf{\Lambda}_e/\mu_2+\mathbf{I}_N]^{-1} & \mathbf{U}_e^*\mathbf{\Lambda}_e^*[\mathbf{\Lambda}_e^*/\mu_2+\mathbf{I}_N]^{-1} & \mathbf{U}_{nv}\mathbf{\Lambda}_{nv}[\mathbf{\Lambda}_{nv}/\mu_2+\mathbf{I}_{nN}]^{-1} \\ \vdots & \vdots & \vdots \\ \mathbf{U}_e\mathbf{\Lambda}_e[\mathbf{\Lambda}_e/\mu_n+\mathbf{I}_N]^{-1} & \mathbf{U}_e^*\mathbf{\Lambda}_e^*[\mathbf{\Lambda}_e^*/\mu_n+\mathbf{I}_N]^{-1} & \mathbf{U}_{nv}\mathbf{\Lambda}_{nv}[\mathbf{\Lambda}_{nv}/\mu_n+\mathbf{I}_{nN}]^{-1} \end{bmatrix}. \quad (63)$$

This equation completely defines the structure of the right-modal matrix in the state space. Note that the right-modal matrix of a viscously damped system consists of only a $2N\times 2N$ block in the top left corner of this expression.

Now consider the left eigenvectors. Suppose

$$\psi_j=\begin{Bmatrix} \boldsymbol{p}_{1j} \\ \boldsymbol{p}_{2j} \\ \boldsymbol{x}_{1j} \\ \boldsymbol{x}_{2j} \\ \vdots \\ \boldsymbol{x}_{nj} \end{Bmatrix}. \quad (64)$$

Expanding Eq. (50) we get the following equations:

$$-\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{K}=\lambda_j\boldsymbol{p}_{1j}^T, \quad (65)$$

$$\boldsymbol{p}_{1j}^T-\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{D}+\sum_{k=1}^{n}\mu_k\boldsymbol{x}_{kj}^T=\lambda_j\boldsymbol{p}_{2j}^T, \quad (66)$$

$$\text{and } -\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{C}_k-\mu_k\boldsymbol{x}_{kj}^T=\lambda_j\boldsymbol{x}_{kj}^T, \quad \forall k=1,\ldots,n. \quad (67)$$

Multiplying Eq. (66) by λ_j and using Eq. (65) results in

$$-\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{K}-\lambda_j\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{D}+\lambda_j\sum_{k=1}^{n}\mu_k\boldsymbol{x}_{kj}^T=\lambda_j^2\boldsymbol{p}_{2j}^T. \quad (68)$$

Provided $\lambda_j\neq-\mu_k$, from Eq. (67) we further obtain

$$\boldsymbol{x}_{kj}^T=-\frac{1}{\mu_k+\lambda_j}\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{C}_k. \quad (69)$$

Substituting $\boldsymbol{x}_{kj}^T$ from Eq. (69), Eq. (68) results in

$$\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{K}+\lambda_j\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{D}+\lambda_j\sum_{k=1}^{n}\frac{\mu_k}{\mu_k+\lambda_j}\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\mathbf{C}_k+\lambda_j^2\boldsymbol{p}_{2j}^T=\mathbf{0}^T \quad (70)$$

$$\text{or } \boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\left[\lambda_j^2\mathbf{M}+\lambda_j\mathbf{D}+\lambda_j\sum_{k=1}^{n}\frac{\mu_k}{\lambda_j+\mu_k}\mathbf{C}_k+\mathbf{K}\right]=\mathbf{0}^T. \quad (71)$$

Comparing Eq. (71) with Eq. (9) immediately results in

$$\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}=\boldsymbol{v}_j^T \quad (72)$$

$$\text{or } \boldsymbol{p}_{2j}= \ =\mathbf{M}^T\boldsymbol{v}_j. \quad (73)$$

Using Eqs. (72) and (65) one obtains

$$\boldsymbol{p}_{1j}=-\mathbf{K}^T\boldsymbol{v}_j/\lambda_j. \quad (74)$$

Similarly, using Eqs. (72) and (69) results in

$$\boldsymbol{x}_{kj}=-\frac{1}{\mu_k+\lambda_j}\mathbf{C}_k^T\boldsymbol{v}_j. \quad (75)$$

From Eqs. (73)–(75), the left eigenvectors in the state space given by Eq. (64) can be expressed as

$$\psi_j=\begin{Bmatrix} -\mathbf{K}^T\boldsymbol{v}_j/\lambda_j \\ \mathbf{M}^T\boldsymbol{v}_j \\ -\mathbf{C}_1^T\boldsymbol{v}_j/(\mu_1+\lambda_j) \\ -\mathbf{C}_2^T\boldsymbol{v}_j/(\mu_2+\lambda_j) \\ \vdots \\ -\mathbf{C}_n^T\boldsymbol{v}_j/(\mu_n+\lambda_j) \end{Bmatrix}. \quad (76)$$

Recalling the partitions in Eqs. (10) and (12), for $j=1,2,\ldots,(2+n)N$, the matrix of left eigenvectors can be expressed as

$$\mathbf{\Psi}=\begin{bmatrix} -\mathbf{K}^T\mathbf{V}_e\mathbf{\Lambda}_e^{-1} & -\mathbf{K}^T\mathbf{V}_e^*\mathbf{\Lambda}_e^{*-1} & -\mathbf{K}^T\mathbf{V}_{nv}\mathbf{\Lambda}_{nv}^{-1} \\ \mathbf{M}^T\mathbf{V}_e & \mathbf{M}^T\mathbf{V}_e^* & \mathbf{M}^T\mathbf{V}_{nv} \\ -\mathbf{C}_1^T\mathbf{V}_e[\mathbf{\Lambda}_e+\mu_1\mathbf{I}_N]^{-1} & -\mathbf{C}_1^T\mathbf{V}_e^*[\mathbf{\Lambda}_e^*+\mu_1\mathbf{I}_N]^{-1} & -\mathbf{C}_1^T\mathbf{V}_{nv}[\mathbf{\Lambda}_{nv}+\mu_1\mathbf{I}_{nN}]^{-1} \\ -\mathbf{C}_2^T\mathbf{V}_e[\mathbf{\Lambda}_e+\mu_2\mathbf{I}_N]^{-1} & -\mathbf{C}_2^T\mathbf{V}_e^*[\mathbf{\Lambda}_e^*+\mu_2\mathbf{I}_N]^{-1} & -\mathbf{C}_2^T\mathbf{V}_{nv}[\mathbf{\Lambda}_{nv}+\mu_2\mathbf{I}_{nN}]^{-1} \\ \vdots & \vdots & \vdots \\ -\mathbf{C}_n^T\mathbf{V}_e[\mathbf{\Lambda}_e+\mu_n\mathbf{I}_N]^{-1} & -\mathbf{C}_n^T\mathbf{V}_e^*[\mathbf{\Lambda}_e^*+\mu_n\mathbf{I}_N]^{-1} & -\mathbf{C}_n^T\mathbf{V}_{nv}[\mathbf{\Lambda}_{nv}+\mu_n\mathbf{I}_{nN}]^{-1} \end{bmatrix}. \quad (77)$$

This equation completely defines the structure of the left-modal matrix in the extended state space. For viscously damped systems, the left-modal matrix consists of only $2N\times 2N$ block in the top left corner of this expression.

4.2 Case B: All $\mathbf{C}_k$ Matrices are Rank Deficient. The right and the left eigenvalue problems associated with Eq. (44) can be expressed as

$$\tilde{\mathbf{A}}\tilde{\phi}_j=\lambda_j\tilde{\phi}_j \quad (78)$$

$$\text{and } \tilde{\psi}_j^T\tilde{\mathbf{A}}=\lambda_j\tilde{\psi}_j^T \quad j=1,2,\ldots,m, \quad (79)$$

where $\tilde{\boldsymbol{\phi}}_j\in\mathbb{C}^m$ and $\tilde{\boldsymbol{\psi}}_j\in\mathbb{C}^m$ are respectively the jth right and left eigenvectors and the order of the system m is defined in Eq. (48). Again we construct the modal matrices

$$\tilde{\mathbf{\Phi}}=[\tilde{\phi}_1,\tilde{\phi}_2,\ldots,\tilde{\phi}_m]\in\mathbb{C}^{m\times m} \quad (80)$$

$$\text{and } \tilde{\mathbf{\Psi}}=[\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_m]\in\mathbb{C}^{m\times m}. \quad (81)$$

These modal matrices also satisfy the biorthogonality property defined in Eqs. (53) and (54).

4.2.1 The Structure of the Modal Matrices. From the definition of $\mathbf{z}(t)$ in Eq. (47), the right eigenvectors in the extended state space can be related to the right eigenvectors in the original space (9) by

$$\phi_j=\begin{Bmatrix}\boldsymbol{u}_j\\ \lambda_j\boldsymbol{u}_j\\ \tilde{\boldsymbol{y}}_{1_j}\\ \tilde{\boldsymbol{y}}_{2_j}\\ \vdots\\ \tilde{\boldsymbol{y}}_{n_j}\end{Bmatrix}, \tag{82}$$

where $\tilde{\boldsymbol{y}}_{1_j}\in\mathbb{C}^{r_1}$, $\tilde{\boldsymbol{y}}_{2_j}\in\mathbb{C}^{r_2},\ldots,\tilde{\boldsymbol{y}}_{n_j}\in\mathbb{C}^{r_n}$ are components of the jth eigenvector corresponding to the internal variables $\tilde{\mathbf{y}}_1(t),\tilde{\mathbf{y}}_2(t),\ldots,\tilde{\mathbf{y}}_n(t)$. From Eq. (37) we may obtain

$$\boldsymbol{y}_{k_j}=\boldsymbol{R}_k\tilde{\boldsymbol{y}}_{k_j}. \tag{83}$$

Premultiplying by $\mathbf{L}_k^T$ yields

$$\tilde{\boldsymbol{y}}_{k_j}=\mathbf{T}_k\boldsymbol{y}_{k_j}, \tag{84}$$

where $\mathbf{T}_k$ is defined in Eq. (43) and $\boldsymbol{y}_{k_j}$ is defined in Eq. (58). Substituting $\tilde{\boldsymbol{y}}_{k_j}$ in Eq. (82) for $j=1,2,\ldots,m$, the matrix of right eigenvectors in the extended state space can be obtained as

$$\tilde{\boldsymbol{\Phi}}=\begin{bmatrix}\mathbf{U}_e & \mathbf{U}_e^* & \mathbf{U}_{nv}\\ \mathbf{U}_e\boldsymbol{\Lambda}_e & \mathbf{U}_e^*\boldsymbol{\Lambda}_e^* & \mathbf{U}_{nv}\boldsymbol{\Lambda}_{nv}\\ \mathbf{T}_1\mathbf{U}_e\boldsymbol{\Lambda}_e[\boldsymbol{\Lambda}_e/\mu_1+\mathbf{I}_N]^{-1} & \mathbf{T}_1\mathbf{U}_e^*\boldsymbol{\Lambda}_e^*[\boldsymbol{\Lambda}_e^*/\mu_1+\mathbf{I}_N]^{-1} & \mathbf{T}_1\mathbf{U}_{nv}\boldsymbol{\Lambda}_{nv}[\boldsymbol{\Lambda}_{nv}/\mu_1+\mathbf{I}_{nN}]^{-1}\\ \mathbf{T}_2\mathbf{U}_e\boldsymbol{\Lambda}_e[\boldsymbol{\Lambda}_e/\mu_2+\mathbf{I}_N]^{-1} & \mathbf{T}_2\mathbf{U}_e^*\boldsymbol{\Lambda}_e^*[\boldsymbol{\Lambda}_e^*/\mu_2+\mathbf{I}_N]^{-1} & \mathbf{T}_2\mathbf{U}_{nv}\boldsymbol{\Lambda}_{nv}[\boldsymbol{\Lambda}_{nv}/\mu_2+\mathbf{I}_{nN}]^{-1}\\ \vdots & \vdots & \vdots\\ \mathbf{T}_n\mathbf{U}_e\boldsymbol{\Lambda}_e[\boldsymbol{\Lambda}_e/\mu_n+\mathbf{I}_N]^{-1} & \mathbf{T}_n\mathbf{U}_e^*\boldsymbol{\Lambda}_e^*[\boldsymbol{\Lambda}_e^*/\mu_n+\mathbf{I}_N]^{-1} & \mathbf{T}_n\mathbf{U}_{nv}\boldsymbol{\Lambda}_{nv}[\boldsymbol{\Lambda}_{nv}/\mu_n+\mathbf{I}_{nN}]^{-1}\end{bmatrix}. \tag{85}$$

Now consider the left eigenvectors. Suppose

$$\psi_j=\begin{Bmatrix}\mathbf{p}_{1_j}\\ \mathbf{p}_{2_j}\\ \tilde{\mathbf{x}}_{1_j}\\ \tilde{\mathbf{x}}_{2_j}\\ \vdots\\ \tilde{\mathbf{x}}_{n_j}\end{Bmatrix}. \tag{86}$$

Following the procedure outlined in the previous section, it can be shown that $\boldsymbol{p}_{2_j}$ and $\boldsymbol{p}_{1_j}$ are again given by Eqs. (73) and (74) while $\tilde{\boldsymbol{x}}_{k_j}$ is given by

$$\tilde{\mathbf{x}}_{k_j}=-\frac{1}{\mu_k+\lambda_j}\mathbf{R}_k^T\mathbf{C}_k^T\mathbf{v}_j. \tag{87}$$

Substituting $\tilde{\boldsymbol{x}}_{k_j}$ in Eq. (86) for $j=1,2,\ldots,m$, the matrix of left eigenvectors in the extended state space can be expressed as

$$\tilde{\boldsymbol{\Psi}}=\begin{bmatrix}-\mathbf{K}^T\mathbf{V}_e\boldsymbol{\Lambda}_e^{-1} & -\mathbf{K}^T\mathbf{V}_e^*\boldsymbol{\Lambda}_e^{*^{-1}} & -\mathbf{K}^T\mathbf{V}_{nv}\boldsymbol{\Lambda}_{nv}^{-1}\\ \mathbf{M}^T\mathbf{V}_e & \mathbf{M}^T\mathbf{V}_e^* & \mathbf{M}^T\mathbf{V}_{nv}\\ -\mathbf{R}_1^T\mathbf{C}_1^T\mathbf{V}_e[\boldsymbol{\Lambda}_e+\mu_1\mathbf{I}_N]^{-1} & -\mathbf{R}_1^T\mathbf{C}_1^T\mathbf{V}_e^*[\boldsymbol{\Lambda}_e^*+\mu_1\mathbf{I}_N]^{-1} & -\mathbf{R}_1^T\mathbf{C}_1^T\mathbf{V}_{nv}[\boldsymbol{\Lambda}_{nv}+\mu_1\mathbf{I}_{nN}]^{-1}\\ -\mathbf{R}_2^T\mathbf{C}_2^T\mathbf{V}_e[\boldsymbol{\Lambda}_e+\mu_2\mathbf{I}_N]^{-1} & -\mathbf{R}_2^T\mathbf{C}_2^T\mathbf{V}_e^*[\boldsymbol{\Lambda}_e^*+\mu_2\mathbf{I}_N]^{-1} & -\mathbf{R}_2^T\mathbf{C}_2^T\mathbf{V}_{nv}[\boldsymbol{\Lambda}_{nv}+\mu_2\mathbf{I}_{nN}]^{-1}\\ \vdots & \vdots & \vdots\\ -\mathbf{R}_n^T\mathbf{C}_n^T\mathbf{V}_e[\boldsymbol{\Lambda}_e+\mu_n\mathbf{I}_N]^{-1} & -\mathbf{R}_n^T\mathbf{C}_n^T\mathbf{V}_e^*[\boldsymbol{\Lambda}_e^*+\mu_n\mathbf{I}_N]^{-1} & -\mathbf{R}_n^T\mathbf{C}_n^T\mathbf{V}_{nv}[\boldsymbol{\Lambda}_{nv}+\mu_n\mathbf{I}_{nN}]^{-1}\end{bmatrix}. \tag{88}$$

The analysis presented here clarifies the structure of the modal matrices in the extended state space. The response to the system subjected to dynamic forces and initial conditions can be easily obtained by utilizing the biorthogonality of the left and the right eigenvectors (see the Appendix). In the next section the results derived here are illustrated by a numerical example.

5 Numerical Example

We consider a three-degree-of-freedom system with asymmetric coefficient matrices. The purpose of this example is to verify some of the mathematical expressions derived in this paper. The damping of the system is expressed as a sum of two exponential kernels. For this special case the equations of motion (1) reads

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\int_0^t[\mu_1e^{-\mu_1(t-\tau)}\mathbf{C}_1+\mu_2e^{-\mu_2(t-\tau)}\mathbf{C}_2]\dot{\mathbf{u}}(\tau)\mathrm{d}\tau+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t). \tag{89}$$

The mass and the stiffness matrices of the system are defined by

$$\mathbf{M}=\begin{bmatrix} 0.5740 & 1.3858 & 1.3858 \\ 0.7070 & 0.7070 & -0.7070 \\ 0.4620 & -0.1914 & -0.1914 \end{bmatrix} \tag{90}$$

and

$$\mathbf{K}=\begin{bmatrix} 1.3748 & 10.9440 & 25.2975 \\ 1.2625 & 2.8770 & -17.4195 \\ 0.7455 & -4.1244 & 0.8625 \end{bmatrix}. \tag{91}$$

Numerical values for the entries of **M** and **K** matrices are taken from Adhikari [21]. Note that these matrices are asymmetric and not positive definite. The damping coefficient matrices are given by

$$\mathbf{C}_1=\begin{bmatrix} 0.3588 & -1.3747 & -1.1471 \\ -0.3574 & 2.6618 & -2.1707 \\ 0.0210 & -1.4199 & 3.3674 \end{bmatrix} \tag{92}$$

and

$$\mathbf{C}_2=\begin{bmatrix} 1.1198 & 1.1915 & 1.1495 \\ 1.7641 & 1.8770 & 1.8109 \\ 0.6881 & 0.7321 & 0.7063 \end{bmatrix}. \tag{93}$$

Numerical values for the relaxation parameters are assumed to be

$$\mu_1=1.5 \quad \text{and} \quad \mu_2=0.1. \tag{94}$$

Both of the damping coefficient matrices have rank deficiencies because one can easily verify that

$$\tau_1=\operatorname{rank}(\mathbf{C}_1)=2\leqslant 3 \tag{95}$$

$$\text{and} \quad \tau_2=\operatorname{rank}(\mathbf{C}_2)=1\leqslant 3. \tag{96}$$

The order of the system matrix in the extended state space can be obtained from Eq. (48) as $m=2\times 3+(2+1)=9$ and the matrix itself can be obtained using the procedure described in Sec. 3.2. The transformation matrices $\mathbf{R}_k$ and $\mathbf{L}_k$ for $k=1,2$ given by Eqs. (34) and (35) are obtained as

$$\mathbf{R}_1=\begin{bmatrix} 0.0397 & -0.8274 \\ -0.7128 & 0.4356 \\ 0.7002 & 0.3545 \end{bmatrix}, \quad \mathbf{L}_1=\begin{bmatrix} 0.0497 & -0.1887 \\ -0.5719 & 0.6828 \\ 0.8188 & 0.7058 \end{bmatrix} \tag{97}$$

$$\text{and} \quad \mathbf{R}_2=\begin{bmatrix} 0.5091 \\ 0.8019 \\ 0.3128 \end{bmatrix}, \quad \mathbf{L}_2=\begin{bmatrix} 0.5603 \\ 0.5961 \\ 0.5751 \end{bmatrix}. \tag{98}$$

Using these, the system matrix in the extended state space in Eq. (45) is given by

$$\tilde{\mathbf{A}}=\begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ -1.7281 & 4.8272 & -8.0485 & 0 & 0 & 0 & -6.2763 & -0.6986 & -2.6209 \\ -0.1670 & -9.3966 & 8.8830 & 0 & 0 & 0 & 6.7959 & 0.4790 & -0.9270 \\ -0.1093 & -0.5001 & -23.8041 & 0 & 0 & 0 & -4.3340 & 0.7501 & 0.6523 \\ 0 & 0 & 0 & 0.0759 & -0.8728 & 1.2494 & -1.5000 & 0 & 0 \\ 0 & 0 & 0 & -0.4021 & 1.4554 & 1.5044 & 0 & -1.5000 & 0 \\ 0 & 0 & 0 & 0.0594 & 0.0632 & 0.0610 & 0 & 0 & -0.1000 \end{bmatrix} \tag{99}$$

The nine eigenvalues of the system are arranged according to Eq. (10). The diagonal matrices containing the eigenvalues are given by

$$\begin{aligned}\mathbf{\Lambda}_e=\operatorname{diag}[&0.0217+1.4060i,\ \ 0.0371+3.1929i,\\ &-0.2359+5.6940i]\end{aligned} \tag{100}$$

$$\text{and} \quad \mathbf{\Lambda}_{nv}=\operatorname{diag}[-0.0905,\ \ -0.8755,\ \ -1.7798]. \tag{101}$$

Note that the eigenvalues corresponding to the nonviscous modes are purely real and negative. The right and the left eigenvector matrices corresponding to these eigenvalues are obtained as

$$\tilde{\mathbf{\Phi}}=\begin{bmatrix} 0.0088-0.5690i & 0.0070+0.1364i & 0.0020-0.0866i & 0.0088+0.5690i & 0.0070-0.1364i & 0.0020+0.0866i & 0.8662 & 0.3905 & -0.1046 \\ -0.0029+0.0147i & 0.0026-0.2245i & -0.0017+0.1015i & -0.0029-0.0147i & 0.0026+0.2245i & -0.0017-0.1015i & 0.0171 & -0.2630 & 0.0428 \\ -0.0082+0.0102i & 0.0209-0.0531i & -0.0043-0.1030i & -0.0082-0.0102i & 0.0209+0.0531i & -0.0043+0.1030i & -0.0168 & 0.1207 & 0.0320 \\ 0.8002+0.0000i & -0.4353+0.0275i & 0.4925+0.0317i & 0.8002-0.0000i & -0.4353-0.0275i & 0.4925-0.0317i & -0.0784 & -0.3419 & 0.1861 \\ -0.0207-0.0037i & 0.7169+0.0000i & -0.5774-0.0338i & -0.0207+0.0037i & 0.7169-0.0000i & -0.5774+0.0338i & -0.0015 & 0.2302 & -0.0762 \\ -0.0146-0.0114i & 0.1702+0.0646i & 0.5872+0.0000i & -0.0146+0.0114i & 0.1702-0.0646i & 0.5872-0.0000i & 0.0015 & -0.1057 & -0.0570 \\ 0.0179-0.0238i & -0.0335+0.1236i & 0.0527-0.2122i & 0.0179+0.0238i & -0.0335-0.1236i & 0.0527+0.2122i & -0.0019 & -0.5748 & -0.0338 \\ -0.1399+0.1145i & 0.2024-0.3644i & -0.0161+0.0236i & -0.1399-0.1145i & 0.2024+0.3644i & -0.0161-0.0236i & 0.0224 & 0.5021 & 0.9702 \\ 0.0021-0.0321i & 0.0021-0.0093i & -0.0002-0.0050i & 0.0021+0.0321i & 0.0021+0.0093i & -0.0002+0.0050i & -0.4923 & 0.0157 & -0.0016 \end{bmatrix} \tag{102}$$

and

$$
\tilde{\boldsymbol{\Psi}}=\begin{bmatrix}
0.0001+0.8070i & -0.0033+0.0672i & 0.0020+0.0129i & 0.0001-0.8070i & -0.0033-0.0672i & 0.0020-0.0129i & 0.1088 & 0.0407 & 0.1185\\
0.0193+0.5788i & 0.1252+1.7383i & 0.0302-0.2378i & 0.0193-0.5788i & 0.1252-1.7383i & 0.0302+0.2378i & 0.1130 & -0.8364 & -0.7962\\
0.4716-0.7060i & -0.1955+2.4606i & 0.3681+3.1361i & 0.4716+0.7060i & -0.1955-2.4606i & 0.3681-3.1361i & 0.1211 & 1.4596 & -2.3903\\
0.6231-0.0078i & 0.0344+0.0088i & 0.0159+0.0013i & 0.6231+0.0078i & 0.0344-0.0088i & 0.0159-0.0013i & 0.0053 & 0.0257 & 0.1441\\
0.4119-0.0068i & 0.5793-0.0450i & -0.1716-0.0202i & 0.4119+0.0068i & 0.5793+0.0450i & -0.1716+0.0202i & 0.0038 & -0.0658 & -0.0634\\
-0.0991-0.0271i & 0.5349+0.0026i & 0.6844-0.0649i & -0.0991+0.0271i & 0.5349-0.0026i & 0.6844+0.0649i & 0.0001 & 0.0204 & -0.2511\\
-0.2023+0.2660i & 0.0770-0.4024i & -0.1344+0.7134i & -0.2023-02660i & 0.0770+0.4024i & -0.1344-0.7134i & -0.0056 & -1.1162 & 0.8826\\
-0.1167+0.0959i & 0.0736-0.1696i & 0.0057-0.0725i & -0.1167-0.0959i & 0.0736+0.1696i & 0.0057+0.0725i & -0.0013 & -0.0547 & 1.1414\\
-0.1207+1.4685i & 0.0026+0.0872i & -0.0071-0.0988i & -0.1207-1.4685i & 0.0026-0.0872i & -0.0071+0.0988i & -1.8409 & -0.0090 & 0.2873
\end{bmatrix}. \tag{103}
$$

The eigenvectors are normalized so that $\tilde{\boldsymbol{\Psi}}^T\tilde{\boldsymbol{\Phi}}$ is an identity matrix. However, it should be noted that this normalization does not make the eigenvectors unique.

In view of Eq. (85), the right-eigenvector matrix corresponding to the elastic modes in the space of the original variables, $\mathbf{U}_e$, can be obtained directly by taking a 3×3 block in the top left corner of Eq. (102):

$$
\mathbf{U}_e=\begin{bmatrix}
0.0088-0.5690i & 0.0070+0.1364i & 0.0020-0.0866i\\
-0.0029+0.0147i & 0.0026-0.2245i & -0.0017+0.1015i\\
-0.0082+0.0102i & 0.0209-0.0531i & -0.0043-0.1030i
\end{bmatrix}. \tag{104}
$$

Similarly $\mathbf{U}_{nv}$ can be obtained by taking first three rows and last three columns of Eq. (102):

$$
\mathbf{U}_{nv}=\begin{bmatrix}
0.8662 & 0.3905 & -0.1046\\
0.0171 & -0.2630 & 0.0428\\
-0.0168 & 0.1207 & 0.0320
\end{bmatrix}. \tag{105}
$$

Now consider the left eigenvectors. Because it is assumed that $\mathbf{M}^{-1}$ exists, from the blocks (2,1) and (2,3) of Eq. (88), $\mathbf{V}_e$ and $\mathbf{V}_{nv}$ can be obtained. So, from the corresponding blocks in Eq. (103) we obtain

$$
\mathbf{V}_e=\begin{bmatrix}
0.1901-0.0150i & 0.3462-0.0047i & 0.2712-0.0316i\\
0.3614+0.0144i & 0.0314-0.0337i & -0.6054+0.0317i\\
0.5594-0.0202i & -0.4039+0.0765i & 0.6239-0.0065i
\end{bmatrix} \tag{106}
$$

$$
\text{and}\quad \mathbf{V}_{nv}=\begin{bmatrix}
0.0021 & 0.0036 & -0.0840\\
0.0026 & -0.0610 & 0.1327\\
0.0049 & 0.1445 & 0.2131
\end{bmatrix}. \tag{107}
$$

As mentioned before, $\mathbf{U}_{nv}$ and $\mathbf{V}_{nv}$ turned out to be real matrices. This is expected because the eigenvalues corresponding to these modes (the nonviscous modes) are purely real.

Using these numerical values one can easily verify Eqs. (85) and (88). A typical case for block (3,2) in Eq. (88) is considered here. Using the numerical values given by Eqs. (92), (94), (97), (100), and (106) one obtains

$$
-\mathbf{R}_1^T\mathbf{C}_1^T\mathbf{V}_e^*[\boldsymbol{\Lambda}_e^*+\mu_1\mathbf{I}_3]^{-1}=\begin{bmatrix}
-0.2023-0.2660i & 0.0770+0.4024i & -0.1344-0.7134i\\
-0.1167-0.0959i & 0.0736+0.1696i & 0.0057+0.0725i
\end{bmatrix}. \tag{108}
$$

These values can be exactly identified in $\tilde{\Psi}_{ij}$ given by Eq. (103) for $i=7,8$ and $j=4,5,6$, which corresponds to block (3,2) in Eq. (88). This illustrates the relationship between the modal matrices in the extended state space and the modal matrices in the original N space.

6 Conclusions

Linear vibration of multiple-degree-of-freedom damped systems with combined viscous damping and exponentially fading damping memory kernels has been considered. It has been assumed that in general, the mass, the stiffness and the damping coefficient matrices are neither symmetric nor positive definite. An extended state-space method based on a set of internal variables has been proposed. Two physically realistic cases, namely (a) when all the damping coefficient matrices are of full rank, and (b) when the damping coefficient matrices are rank deficient, have been presented. It was shown that for both the cases the equation of motion in the extended state space can be represented in terms of a single asymmetric matrix of higher dimension. The dimension of this matrix depends on the rank of the damping coefficient matrices. The eigenvalues and the corresponding eigenvectors of the system were obtained by solving the standard eigenvalue problem in the state space.

Closed-form exact relationships relating the modal matrices in the extended state space and the modal matrices in the original space have been derived. All the entries of the modal matrices in the extended state space can be represented in terms of the eigenvalues, the systems matrices, and the modal matrices in the original space. It is expected that these results will be useful to understand the nature of the eigensolutions of nonviscously damped systems.

Appendix A: Dynamic Response of Asymmetric Nonviscously Damped Systems

In this section the dynamic response of the system will be obtained by using the mode superposition method, commonly employed for undamped or proportionally damped systems. We begin by assuming that all the initial conditions are zero. Taking the Laplace transform of Eqs. (24) or (44) one obtains

$$s\overline{\mathbf{z}}(s)=\boldsymbol{\mathcal{A}}\overline{\mathbf{z}}(s)+\overline{\mathbf{r}}(s), \tag{109}$$

where the system matrix $\boldsymbol{\mathcal{A}}$ is given by Eqs. (25) or (45), depending on the ranks of the $\mathbf{C}_k$ matrices. Similarly $\overline{\mathbf{z}}(s)$ and $\overline{\mathbf{r}}(s)$ are the Laplace transforms of $\mathbf{z}(t)$ and $\mathbf{r}(t)$ or $\tilde{\mathbf{z}}(t)$ and $\tilde{\mathbf{r}}(t)$, respectively. Using the modal transformation

$$\overline{\mathbf{z}}(s)=\mathbf{\Phi}\overline{\mathbf{q}}(s) \tag{110}$$

and the orthogonality relationships (53) and (54), we obtain

$$(s\mathbf{I}-\mathbf{\Lambda})\overline{\mathbf{q}}(s)=\mathbf{\Psi}^T\overline{\mathbf{r}}(s) \quad \text{or} \quad \overline{\mathbf{q}}(s)=(s\mathbf{I}-\mathbf{\Lambda})^{-1}\mathbf{\Psi}^T\overline{\mathbf{r}}(s). \tag{111}$$

Substitution of $\overline{\mathbf{q}}(s)$ from the preceding equation in Eq. (110) results in

$$\overline{\mathbf{z}}(s)=\mathbf{\Phi}(s\mathbf{I}-\mathbf{\Lambda})^{-1}\mathbf{\Psi}^T\overline{\mathbf{r}}(s)=\sum_{j=1}^{m}\frac{\phi_j\psi_j^T}{s-\lambda_j}\overline{\mathbf{r}}(s). \tag{112}$$

Using the expression of $\boldsymbol{\psi}_j$ in Eq. (64) and recalling that only $N+1$ to $2N$ rows of $\overline{\mathbf{r}}(s)$ is nonzero, one has

$$\boldsymbol{\psi}_j^T\overline{\mathbf{r}}(s)=\boldsymbol{p}_{2j}^T\mathbf{M}^{-1}\overline{\mathbf{f}}(s). \tag{113}$$

Substituting p_{2j} from Eq. (73), Eq. (112) can be rewritten as

$$\overline{\mathbf{z}}(s)=\sum_{j=1}^{m}\frac{\boldsymbol{v}_j^T\overline{\mathbf{f}}(s)}{s-\lambda_j}\boldsymbol{\phi}_j. \tag{114}$$

Taking only the first N rows of Eq. (114) and using Eq. (55) one obtains the displacement response

$$\overline{\mathbf{u}}(s)=\sum_{j=1}^{m}\frac{\boldsymbol{v}_j^T\overline{\mathbf{f}}(s)}{s-\lambda_j}\boldsymbol{u}_j. \tag{115}$$

In view of the partitions (10)–(12), the preceding expression can be conveniently expressed as

$$\overline{\mathbf{u}}(s)=\sum_{j=1}^{N}\left[\frac{\boldsymbol{v}_j^T\overline{\mathbf{f}}(s)}{s-\lambda_j}\boldsymbol{u}_j+\frac{\boldsymbol{v}_j^{*T}\overline{\mathbf{f}}(s)}{s-\lambda_j^*}\boldsymbol{u}_j^*\right]+\sum_{j=2N+1}^{m}\frac{\boldsymbol{v}_{nv_j}^T\overline{\mathbf{f}}(s)}{s-\lambda_{nv_j}}\boldsymbol{u}_{nv_j}. \tag{116}$$

This expression of the dynamic response is in terms of the left and the right eigenvectors of the system in N space. The second part of Eq. (116) is the contribution of the nonviscous modes to the global dynamic response. This part is not present for viscously damped systems and consequently the response of a viscously damped system can be obtained only form the first part of the right-hand side of Eq. (116). In the presence of nonzero initial conditions the vector $\overline{\mathbf{f}}(s)$ needs to be replaced by $\overline{\mathbf{p}}(s)$ defined in Eq. (6). Therefore the dynamic response with nonzero initial conditions can be expressed as

$$\overline{\mathbf{u}}(s)=\sum_{j=1}^{m}\frac{1}{s-\lambda_j}\left(\boldsymbol{v}_j^T\overline{\mathbf{f}}(s)+\boldsymbol{v}_j^T\mathbf{M}\dot{\mathbf{u}}_0+\boldsymbol{v}_j^T\mathbf{D}\mathbf{u}_0+s\boldsymbol{v}_j^T\mathbf{M}\mathbf{u}_0+\sum_{k=1}^{n}\mu_k\frac{\boldsymbol{v}_j^T\mathbf{C}_k\mathbf{u}_0}{s+\mu_k}\right)\boldsymbol{u}_j. \tag{117}$$

In the time domain the response can be obtained by taking the inverse Laplace transform of Eq. (117):

$$\begin{aligned}\mathbf{u}(t)&=\mathcal{L}^{-1}[\overline{\mathbf{u}}(s)]\\&=\mathcal{L}^{-1}\left[\sum_{j=1}^{m}\frac{1}{s-\lambda_j}\left(\boldsymbol{v}_j^T\overline{\mathbf{f}}(s)+\sum_{k=1}^{n}\frac{1}{s+\mu_k}\mu_k(\boldsymbol{v}_j^T\mathbf{C}_k\mathbf{u}_0)\right)+\frac{\boldsymbol{v}_j^T\mathbf{M}\dot{\mathbf{u}}_0+\boldsymbol{v}_j^T\mathbf{D}\mathbf{u}_0}{s-\lambda_j}+\left(\frac{s}{s-\lambda_j}\right)\boldsymbol{v}_j^T\mathbf{M}\mathbf{u}_0\right]\boldsymbol{u}_j\\&=\sum_{j=1}^{m}\left(\int_0^t e^{\lambda_j(t-\tau)}\boldsymbol{v}_j^T\mathbf{f}(\tau)d\tau+\sum_{k=1}^{n}\int_0^t\mu_k e^{\lambda_j t-(\lambda_j+\mu_k)\tau}\boldsymbol{v}_j^T\mathbf{C}_k\mathbf{u}_0 d\tau\right)\boldsymbol{u}_j+e^{\lambda_j t}(\boldsymbol{v}_j^T\mathbf{M}\dot{\mathbf{u}}_0\\&\quad+\boldsymbol{v}_j^T\mathbf{D}\mathbf{u}_0)\boldsymbol{u}_j+\{\lambda_j e^{\lambda_j t}+\delta(t)\}(\boldsymbol{v}_j^T\mathbf{M}\mathbf{u}_0)\boldsymbol{u}_j.\end{aligned} \tag{118}$$

For $t>0$ the preceding equation may be rewritten as

$$\mathbf{u}(t)=\sum_{j=1}^{m}\left\{\int_0^t e^{\lambda_j(t-\tau)}\boldsymbol{v}_j^T\mathbf{f}(\tau)d\tau+a_j(t)\right\}\boldsymbol{u}_j, \tag{119}$$

$$\text{where}\quad a_j(t)=e^{\lambda_j t}(\boldsymbol{v}_j^T\mathbf{M}\dot{\mathbf{u}}_0+\boldsymbol{v}_j^T\mathbf{D}\mathbf{u}_0)+\lambda_j e^{\lambda_j t}(\boldsymbol{v}_j^T\mathbf{M}\mathbf{u}_0)+\sum_{k=1}^{n}\mu_k\frac{(e^{\lambda_j t}-e^{-\mu_k t})}{\lambda_j+\mu_k}\boldsymbol{v}_j^T\mathbf{C}_k\mathbf{u}_0. \tag{120}$$

It is interesting to note that the expression of $a_j(t)$ is independent of the ranks of the $\mathbf{C}_k$ matrices. The ranks of the $\mathbf{C}_k$ matrices only effect the number of terms (m) to be added in Eq. (119) to obtain $\mathbf{u}(t)$.

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