Dynamics of multiple viscoelastic carbon nanotube based nanocomposites with axial magnetic field

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Nanocomposites and magnetic field effects on nanostructures have received great attention in recent years. A large amount of research work was focused on developing the proper theoretical framework for describing many physical effects appearing in structures on nanoscale level. Great step in this direction was successful application of nonlocal continuum field theory of Eringen. In the present paper, the free transverse vibration analysis is carried out for the system composed of multiple single walled carbon nanotubes (MSWCNT) embedded in a polymer matrix and under the influence of an axial magnetic field. Equivalent nonlocal model of MSWCNT is adopted as viscoelastically coupled multi-nanobeam system (MNBS) under the influence of longitudinal magnetic field. Governing equations of motion are derived using the Newton second low and nonlocal Rayleigh beam theory, which take into account small-scale effects, the effect of nanobeam angular acceleration, internal damping and Maxwell relation. Explicit expressions for complex natural frequency are derived based on the method of separation of variables and trigonometric method for the "Clamped-Chain" system. In addition, an analytical method is proposed in order to obtain asymptotic damped natural frequency and the critical damping ratio, which are independent of boundary conditions and a number of nanobeams in MNBS. The validity of obtained results is confirmed by comparing the results obtained for complex frequencies via trigonometric method with the results obtained by using numerical methods. The influence of the longitudinal magnetic field on the free vibration response of viscoelastically coupled MNBS is discussed in detail. In addition, numerical results are presented to point out the effects of the nonlocal parameter, internal damping, and parameters of viscoelastic medium on complex natural frequencies of the system. The results demonstrate the efficiency of the suggested methodology to find the closed form solutions for the free vibration response of multiple nanostructure systems under the influence of magnetic field.
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I. INTRODUCTION

The consideration of multiphysics phenomena such as heat conduction, fluid flow, magnetic fields and others, in the theoretical analysis of nano-structures is of great importance for the fabrication and exploitation of micro-electromechanical (MEMS) and nano-electromechanical (NEMS) devices. The role of natural frequency is important for understanding the working mechanism of some MEMS and NEMS devices. Two principal components at all scales that are common in electromechanical systems are: mechanical elements and transducers.[1] Response of mechanical elements (structures) to an applied force is either a deflection or vibration, which can be used to sense static or time-varying forces. On the other hand, transducers are used to convert mechanical energy into optical or electrical signals and vice versa. In some cases, input transducers are measuring a response only when perturbed steady vibration signals appears rather than the input signals itself. These perturbations can measure changes in system damping by pressure variations, change of nanoresonators mass due to attached nanoparticles or changes of elasticity properties caused by variations in temperature. The last two cases results in change of natural frequency as a unique property of every structure or a mechanical system. Further, decrease of a dimension of structures and mechanical systems towards nano-scale results in an increase of natural frequency towards GHZ range that makes new possibilities to build high-frequency resonators and mechanical devices. Thus, prediction of natural frequencies can be substantial in design procedures of NEMS and MEMS devices.

Experimental investigation of different physical phenomena on nano-scale level is generally difficult to perform due to the weak control of parameters. Using atomistic simulation methods[2–4] to simulate physical processes in nano-structures with large number of atoms may be computationally prohibitive. This brings us to the consideration of continuum field theories. Nevertheless, classical continuum theories are not always able to predict the behavior of nanostructures well and nonlocal continuum field theory of Eringen needs to be considered.[5–7] This theory can take into account the small-scale effects and it is convenient to introduce different physical phenomena. First by Pedisson *et al.*[8] and later by many other authors, this theory has been extensively applied in numerous works to examine vibration and other properties of nanostructures.[9–22]

Nanocomposites are significant materials for the application in nanoengineering. Nano-structures such as nanotubes, nanorods, nanobeams, nanoplates and others are grown from carbon, zinc-oxide, gold, silver, and boron-nitride by different fabrication techniques. Single and multi-walled carbon nanotubes embedded in certain polymers improve their mechanical, electrical and thermal properties, which gives them advantageous characteristics for application in engineering practice. In the review paper by Moniruzzaman and Winey[23] various properties of polymer based nanocomposites containing carbon nanotubes are discussed. Various nanostructures based nanocomposites often with enhanced certain physical properties are reported in many scientific works.[23–33]

Behavior and properties of nanotubes and other nanostructures within a magnetic filled have become significant subject of investigation among the scientific community.[34–44] In paper by Wang *et al.*,[45] effect of longitudinal magnetic field on wave propagation in carbon nanotubes (CNT) embedded in elastic matrix is investigated. The authors derived dynamic equations of CNTs with Lorentz magnetic forces considered. Narendar *et al.*[46] analyzed the effect of longitudinal magnetic field on wave dispersion in single-walled carbon nanotubes (SWCNT) embedded in elastic medium. They modeled SWNCT using the nonlocal Euler-Bernoulli beam theory and elastic medium with Pasternak foundation model. Lorentz magnetic force is incorporated into the governing equations of the system. Murmu *et al.*[47] used nonlocal elasticity model of nanobeams in order to investigate the influence of magnetic field on bending vibration of double-walled carbon nanotube (DWCNT) system. It was reported that longitudinal magnetic field applied to DWCNT increases natural frequencies of the systems. In addition, synchronous and asynchronous vibration phases and effect of longitudinal magnetic field on higher natural frequencies were examined. In other paper,[48] the same authors considered the system where the longitudinal magnetic field is exerted on the double SWCNT system coupled with elastic medium. They investigated nonlocal effects, effects of longitudinal magnetic field and synchronous and asynchronous phase on bending vibration of the system. In several papers,[49–51] Kiani has analyzed influence of axial magnetic field on wave propagation and magneto-thermo-elastic fields in CNTs and nanowires, respectively. The similar problem on vibration properties of graphene sheets in magnetic field have been considered in several papers. Murmu *et al.*[52] considered the effect of in-plane magnetic field on transverse vibration of single-layer graphene sheet embedded in elastic medium. The nonlocal elastic plate theory was employed to model a graphene sheet and governing equations are derived by taking into account Lorentz force. It was also reported that natural frequencies increases when single-layer graphene sheet is under the influence of in-plane magnetic field. Arani *et al.*[53] observed the effects of two-dimensional magnetic field exerted on the double bonded graphene sheet system under the biaxial in-plane pre-load. The authors conducted the detail parametric study and analyzed the effects of aspect ratio, small-scale parameter, tensile and compressive in-plane pre-load and different magnitude and angle of magnetic field. Recently, Kiani[54] analyzed vibration and instability of SWCNT exposed to the three-dimensional magnetic field. Nonlocal Rayleigh beam theory for SWCNT and Maxwell's equations were used to obtain governing equations for the free vibration analysis of the system. In addition, the author obtained critical value of the transverse magnetic field affiliated with buckling of SWCNT.

It is known that CNTs dispersed in dielectric host, i.e., polymer matrix, can be used as electromagnetic shielding material,[55–57] in optoelectronic and optomechanical devices and actuators.[58–62] In addition, magnetic field can be used in polymer composites to align CNTs in the fabrication process[63] as well as to change mechanical properties of CNTs[64] in nanocomposites. In this communication we investigate change of mechanical, i.e., vibration properties of a system with multiple single walled carbon nanotubes (MSWCNT) embeded in a polymer matrix, which is modeled as multi-nanobeam system (MNBS) coupled with viscoelastic layers, under the influence of longitudinal magnetic field. Nonlocal Rayleigh beam theory is used in order to represent small-scale and rotary inertia effects. Exact closed form solutions for complex natural frequencies are derived using the trigonometric method for simply supported nanobeams and "Clamped-Chain" system. In addition, asymptotic damped natural frequency and the critical damping ratio are determined analytically. Results are validated by comparison of the numerical solutions of a system of algebraic equations against the trigonometric solution. The effect of longitudinal magnetic field on the free vibration response of the viscoelastically coupled MNBS is discussed together with the effects of nonlocal parameter, internal damping, and parameters of viscoelastic medium on complex natural frequencies of the system.

It should be noted that the limitations of the present study lies in the certain assumptions introduced into the proposed model of carbon nanotube/polymer nanocomposite, that is, assumed identical properties of carbon nanotubes, ideally parallel assembly of nanotubes embedded in polymer matrix, and neglected presence of catalyst nanoparticles. In paper by Shitikova,[65] the similar procedure of modeling was proposed for multi-beam system but with fractional order viscoelastic properties. As stated in there, the proposed way of modeling is valid when no slip or welded contact is considered between structures and medium. In our case, the proposed methodology is partially justified since a weak bonding between polymer matrix and straight nanotubes was reported.[66] Further, it is possible to reach some kind of carbon nanotubes alignment by using different methods as stated previously. Finally, CNTs contains catalyst nanoparticles remaining from the production process. These nanoparticles can significantly change mechanical properties of CNTs and needs to be considered in the modeling process. Nevertheless, some procedures were proposed to remove catalyst impurities from carbon nanotube materials[67] and even their presence cannot be avoided it can be significantly reduced. In the future, model used in this work can be closer to the real world nano-systems due to the constant improvements in techics of making nanocomposites and it can be used as a starting point to construct future continuum based models.

II. PROBLEM FORMULATION

A. Nonlocal viscoelastic constitutive relation

We consider the fundamental equations of the nonlocal elasticity and viscoelasticity theory. The key assumption in the nonlocal elasticity theory is that the stress at a point is a function of the strains at all points of the elastic body. Based on the experimental observations, Eringen derived a constitutive relation in an integral form for nonlocal stress tensor at a point $\mathbf{x}$. The form of the nonlocal elastic relation for a three-dimensional linear, homogeneous isotropic body is given as follows:

$$\sigma_{ij}(x) = \int \alpha(|x - x'|, \tau) C_{ijkl}\varepsilon_{kl}(x')dV(x'), \forall x \in V, \tag{1a}$$

$$\sigma_{ij,j} = 0, \tag{1b}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1c}$$

where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij} and ε_{ij} are the stress and the strain tensors, respectively, and u_i is the displacement vector. With $\alpha(|x - x'|, \tau)$ we denote the nonlocal modulus or attenuation function, which incorporates nonlocal effects into the constitutive equation at a reference point x produced by the local strain at a source x'. The above absolute value of the difference $|x - x'|$ denotes the Euclidean metric. The parameter $\tau = (e_0 a)/l$ where l is the external characteristic length (crack length, wave length), a describes the internal characteristic length (lattice parameter, granular size and distance between C-C bounds), and e_0 is a constant appropriate to each material that can be identified from atomistic simulations or by using the dispersive curve of the Born-Karman model of lattice dynamics. Since it is difficult to use the constitutive relations in integral form for solving practical problems, a simplified constitutive relation in differential form is developed. Based on works by Eringen[5] and Eringen and Edelen,[6] constitutive relations in differential form for the one-dimensional case are

$$\sigma_{xx} - \mu\frac{d^2\sigma_{xx}}{dx^2} = E\varepsilon_{xx}, \tag{2a}$$

$$\sigma_{xz} - \mu\frac{d^2\sigma_{xz}}{dx^2} = G\gamma_{xz}, \tag{2b}$$

where E and G are elastic modulus and shear modulus of the beam, respectively; $\mu = (e_0 a)^2$ is the nonlocal parameter (length scales); σ_{xx}, σ_{xz} are the normal and the shear nonlocal stresses, respectively, and $\varepsilon_{xx} = \partial u/\partial x$ is the axial deformation. Nano-size structures such as CNTs, ZnO nanotubes, and other two dimensional structures are modeled as nanobeams and nanorods by using the nonlocal theory, where internal characteristic lengths $(e_0 a)$ is often assumed to be in the range 0–2 [nm]. More details and discussions on different values of parameter e_0 one can find in Ref. 68. When $e_0 a = 0$, nonlocal constitutive relation is reduced to the classical constitutive relation of the elastic body. The nonlocal viscoelastic constitutive relation for Kelvin-Voigt viscoelastic model proposed by Lei *et al.*[69] is a combination of nonlocal elasticity and viscoelasticity theory. Therefore, for one-dimensional nonlocal viscoelastic solids constitutive relations are given by

$$\sigma_{xx} - \mu\frac{d^2\sigma_{xx}}{dx^2} = E(\varepsilon_{xx} + \tau_d\dot{\varepsilon}_{xx}), \tag{3a}$$

$$\sigma_{xz} - \mu\frac{d^2\sigma_{xz}}{dx^2} = G(\gamma_{xz} + \tau_d\dot{\gamma}_{xz}), \tag{3b}$$

where τ_d is the internal damping coefficient of nanobeam. If $\tau_d = 0$, i.e., there is no influence of internal viscosity we get back to the constitutive relation for nonlocal elasticity. In the following, we use constitutive relation for nonlocal viscoelasticity to derive governing equations of motion.

B. Maxwell's relation

Based on the classical electromagnetic theory,[47,48] the relationships between the current density $\mathbf{J}$, distributing vector of magnetic field $\mathbf{h}$, strength vectors of the electric fields $\mathbf{e}$, and magnetic field permeability η are represented by Maxwell's equations in differential form and can be retrieved as

$$\mathbf{J} = \nabla \times \mathbf{h}, \quad \nabla \times \mathbf{e} = -\boldsymbol{\eta}\frac{\partial \mathbf{h}}{\partial t}, \quad \nabla \cdot \mathbf{h} = 0. \tag{4}$$

Here vectors of distributing magnetic field $\mathbf{h}$ and the electric field $\mathbf{e}$ are defined as

$$\mathbf{h} = \nabla \times (\mathbf{U} \times \mathbf{H}), \quad \mathbf{e} = -\eta\left(\frac{\partial \mathbf{U}}{\partial t} \times \mathbf{H}\right). \tag{5}$$

In the above equation, $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ is the Hamilton operator, $\mathbf{U} = (x, y, z)$ is the displacement vector, and $\mathbf{H} = (H_x, 0, 0)$ is the vector of the longitudinal magnetic field. It is assumed that the longitudinal magnetic field acts on MNBS in the axial direction of each nanobeam. We can write the vector of the distributing magnetic field in the following form:

$$\mathbf{h} = -H_x\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\mathbf{i} + H_x\frac{\partial v}{\partial x}\mathbf{j} + H_x\frac{\partial w}{\partial x}\mathbf{k}. \tag{6}$$

Substituting Eq. (6) into the first expressions of Eq. (4), one has

$$\begin{aligned}\mathbf{J} = \nabla \times \mathbf{h} = H_x&\left(-\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y}\right)\mathbf{i} \\ &- H_x\left(\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right)\mathbf{j} \\ &+ H_x\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y}\right)\mathbf{k}.\end{aligned} \tag{7}$$

Further, using Eq. (7) into the expressions for the Lorentz force induced by the longitudinal magnetic field yields

$$\mathbf{f}(f_x,f_y,f_z) = \eta(\mathbf{J}\times\mathbf{H}) = \eta\left[0\mathbf{i} + H_x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right)\mathbf{j} + H_x^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right)\mathbf{k}\right]. \tag{8}$$

Here f_x, f_y, and f_z express the Lorentz force along the x, y, and z directions, as follows:

$$f_x = 0, \tag{9a}$$

$$f_y = \eta H_x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right), \tag{9b}$$

$$f_z = \eta H_x^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 v}{\partial z\partial y}\right). \tag{9c}$$

In this study, we assume that the displacement of the i-th nanobeam $w_i(x,t)$ and the Lorentz force acts only in z direction which can be written as

$$f_{z,i} = \eta H_x^2 \frac{\partial^2 w_i}{\partial x^2}. \tag{10}$$

Finally, it is possible to obtain force per unit length of the i-th nanobeam in the following form:

$$\tilde{q}_i(x,t) = \int_A f_{z,i}dA = \eta A H_x^2 \frac{\partial^2 w_i}{\partial x^2}. \tag{11}$$

C. Mathematical model of viscoelastic MNBS

Suppose that the system of m slender single walled carbon nanotubes (SWCNT) is embedded in the polymer matrix, this system can be modeled as MNBS embedded in the viscoelastic medium. As shown in Fig. 1(a), a vertically distributed springs and dampers represent this medium as nanobeams coupling layers. The stiffness k_i of the spring and viscosity b_i of the damper are material parameters of the viscoelastic medium. CNTs are modeled using nonlocal Rayleigh beam theory with internal damping and rotary inertia effects included. It is assumed that CNTs are with same material properties: Young's modulus E, mass density ρ, cross section area A, moment of inertia I, nonlocal parameter μ, and internal damping τ_d. Each CNT is under the influence of Lorenz magnetic force induced by the longitudinal magnetic field. The CNTs in the system are denoted as CNT1, CNT 2 and so on until the last CNT m. Their transversal displacements are labeled as $w_1(x,t), w_2(x,t), \ldots w_{m-1}(x,t)$ and $w_m(x,t)$. Here, we consider that the system of multiple CNTs is coupled with the fixed base in the so called "Clamped-Chain" system (Fig. 1(b)), i.e., it is assumed that the first and the last CNT in the system are continuously joined with a fixed base by the viscoelastic layers of stiffness k_0 and k_m and viscosity b_0 and b_m. Where $w_0(x,t) = 0$ and $w_{m+1}(x,t) = 0$ are boundary conditions for "Clamped-Chain" system. The viscoelastic layers with the same stiffness $k_1 = k_2 = \cdots = k_i = \cdots = k_{m-1} = k_m = k$ and viscosity $b_1 = b_2 = \cdots = b_i = \cdots = b_{m-1} = b_m = b$ also couples other CNTs in the system.

Let us consider $i-th$ CNT under the influence of longitudinal magnetic field and with external load from viscoelastic medium as shown in Fig. 1(c). Now, we apply the Rayleigh beam theory on the differential element of i-*th* nanobeam where the rotatory inertia of the cross-section would be incorporated into the moment equation. According to this theory and using Newton's second law, the equilibrium equations of the *i-th* differential element can be expressed as

$$\frac{\partial F_T}{\partial x} + q_i - q_{i-1} + \tilde{q}_i = \rho A \frac{\partial^2 w_i}{\partial t^2}, \tag{12a}$$

$$F_T = \frac{\partial M_f}{\partial x} + \rho I \frac{\partial^3 w_i}{\partial x \partial t^2}, \quad i = 1,2,3,\ldots m, \tag{12b}$$

where M_f is the stress resultant defined as

$$M_f = \int_0^A z\sigma_{xx} dA. \tag{13a}$$

The external loads from the viscoelastic interaction are

$$q_i = k_i(w_{i+1} - w_i) + b_i(\dot{w}_{i+1} - \dot{w}_i), \quad q_{i-1} = k_{i-1}(w_i - w_{i-1}) + b_{i-1}(\dot{w}_i - \dot{w}_{i-1}), \tag{13b}$$

where $\tilde{q}_i$ is the magnetic force per unit length defined in Eq. (11) and $\partial w_i/\partial x$ is the angle of rotation of the nanobeam element. Governing equations of motion of a coupled system of m viscoelastic CNTs in terms of transversal displacement $w_i(x,t)$ are obtained by introducing Eqs. (12a), (12b), and (13a), (13b) into Eq. (3a) in the following form:

$$\rho A \frac{\partial^2 w_i}{\partial t^2} - \rho I \frac{\partial^4 w_i}{\partial t^2 \partial x^2} + k_i(w_i - w_{i+1}) + b_i(\dot{w}_i - \dot{w}_{i+1}) + k_{i-1}(w_i - w_{i-1}) + b_{i-1}(\dot{w}_i - \dot{w}_{i-1}) - \eta A H_x^2 \frac{\partial^2 w_i}{\partial x^2} + EI\left(1 + \tau_d \frac{\partial}{\partial t}\right)\frac{\partial^4 w_i}{\partial x^4} = \mu \frac{\partial^2}{\partial x^2}\left[\rho A \frac{\partial^2 w_i}{\partial t^2} - \rho I \frac{\partial^4 w_i}{\partial t^2 \partial x^2} + k_i(w_i - w_{i+1}) + b_i(\dot{w}_i - \dot{w}_{i+1}) + k_{i-1}(w_i - w_{i-1}) + b_{i-1}(\dot{w}_i - \dot{w}_{i-1}) - \eta A H_x^2 \frac{\partial^2 w_i}{\partial x^2}\right], \quad i = 1,2,3,\ldots m \tag{14a}$$

or in the dimensionless form

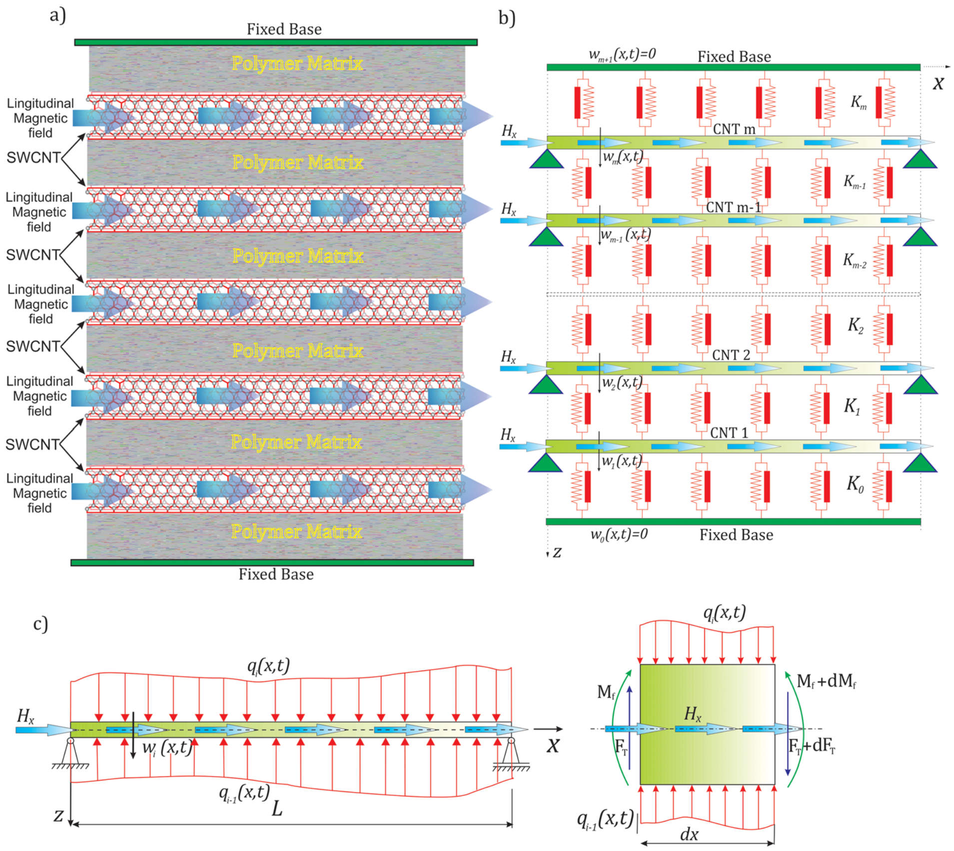

FIG. 1. (a) Axial magnetic field exerted on multiple SWCNTs embedded in polymer matrix; (b) Equivalent nonlocal model of MNBS coupled with viscoelastic layers in axial magnetic field; (c) Single simply supported nanobeam with its elementary part in the axial magnetic field.

$$
\begin{aligned}
&\frac{\partial^2 \bar{w}_i}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_i}{\partial \tau^2 \partial \xi^2} + K_i(\bar{w}_i - \bar{w}_{i+1}) + B_i(\dot{\bar{w}}_i - \dot{\bar{w}}_{i+1}) \\
&+ K_{i-1}(\bar{w}_i - \bar{w}_{i-1}) + B_{i-1}(\dot{\bar{w}}_i - \dot{\bar{w}}_{i-1}) - MP \frac{\partial^2 \bar{w}_i}{\partial \xi^2} \\
&+ \left(1 + T_d \frac{\partial}{\partial \tau}\right) \frac{\partial^4 \bar{w}_i}{\partial \xi^4} \\
&= \nu^2 \frac{\partial^2}{\partial \xi^2}\left[\frac{\partial^2 \bar{w}_i}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_i}{\partial \tau^2 \partial \xi^2} + K_i(\bar{w}_i - \bar{w}_{i+1})\right. \\
&\quad + B_i(\dot{\bar{w}}_i - \dot{\bar{w}}_{i+1}) + K_{i-1}(\bar{w}_i - \bar{w}_{i-1}) \\
&\quad \left. + B_{i-1}(\dot{\bar{w}}_i - \dot{\bar{w}}_{i-1}) - MP \frac{\partial^2 \bar{w}_i}{\partial \xi^2}\right], \quad i = 1, 2, 3, \ldots m, \qquad (14b)
\end{aligned}
$$

where

$$
\begin{aligned}
&\bar{S}_n = S_n \frac{L^4}{EI}, \quad \bar{v}_n = v_n \frac{L^4}{EI}, \quad \delta = \frac{I}{AL^2}, \quad K = k\frac{L^4}{EI}, \\
&B = b\sqrt{\frac{L^4}{\rho A EI}}, \quad \mathrm{T}_d = \frac{\tau_d}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad MP = \frac{\eta A H_x^2}{EI} L^2, \\
&\nu^2 = \frac{\mu}{L^2}, \quad \bar{w}_i = \frac{w_i}{L}, \quad \xi = \frac{x}{L}, \quad \tau = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}}, \\
&\dot{\bar{w}}_i = \frac{\partial \bar{w}_i}{\partial \tau}. \qquad (14c)
\end{aligned}
$$

In the above equations Ω_n, K, B, T_d, MP, and ν are dimensionless complex natural frequency, viscoelastic layer stiffness and damping coefficients, internal damping parameter, magnetic field parameter, and nonlocal parameter, respectively. Introducing the boundary conditions for the "Clamped-Chain" system into Eq. (14b) and assuming that all viscoelastic layers are with same material properties, we obtain equations of motion as

$$
\begin{aligned}
&\frac{\partial^2 \bar{w}_1}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_1}{\partial \tau^2 \partial \xi^2} + K(\bar{w}_1 - \bar{w}_2) + B(\dot{\bar{w}}_1 - \dot{\bar{w}}_2) \\
&+ K\bar{w}_1 + B\dot{\bar{w}}_1 - MP \frac{\partial^2 \bar{w}_1}{\partial \xi^2} + \left(1 + T_d \frac{\partial}{\partial \tau}\right) \frac{\partial^4 \bar{w}_1}{\partial \xi^4} \\
&= \nu^2 \frac{\partial^2}{\partial \xi^2}\left[\frac{\partial^2 \bar{w}_1}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_1}{\partial \tau^2 \partial \xi^2} + K(\bar{w}_1 - \bar{w}_2)\right. \\
&\quad \left. + B(\dot{\bar{w}}_1 - \dot{\bar{w}}_2) + K\bar{w}_1 + B\dot{\bar{w}}_1 - MP \frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right], \qquad (15a)
\end{aligned}
$$

$$
\begin{aligned}
&\rho \frac{\partial^2 \bar{w}_i}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_i}{\partial \tau^2 \partial \xi^2} + K(\bar{w}_i - \bar{w}_{i+1}) + B(\dot{\bar{w}}_i - \dot{\bar{w}}_{i+1}) \\
&+ K(\bar{w}_i - \bar{w}_{i-1}) + B(\dot{\bar{w}}_i - \dot{\bar{w}}_{i-1}) - MP \frac{\partial^2 \bar{w}_i}{\partial \xi^2} \\
&+ \left(1 + T_d \frac{\partial}{\partial \tau}\right) \frac{\partial^4 \bar{w}_i}{\partial \xi^4} \\
&= \nu^2 \frac{\partial^2}{\partial \xi^2}\left[\frac{\partial^2 \bar{w}_i}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_i}{\partial \tau^2 \partial \xi^2} + K(\bar{w}_i - \bar{w}_{i+1})\right. \\
&\quad + B(\dot{\bar{w}}_i - \dot{\bar{w}}_{i+1}) + K(\bar{w}_i - \bar{w}_{i-1}) + B(\dot{\bar{w}}_i - \dot{\bar{w}}_{i-1}) \\
&\quad \left. - MP \frac{\partial^2 \bar{w}_i}{\partial \xi^2}\right], \quad i = 2, 3, \ldots m - 1. \qquad (15b)
\end{aligned}
$$

$$\frac{\partial^2 \bar{w}_m}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_m}{\partial \tau^2 \partial \xi^2} + K\bar{w}_m + B\dot{\bar{w}}_m + K(\bar{w}_m - \bar{w}_{m-1}) + B(\dot{\bar{w}}_m - \dot{\bar{w}}_{m-1}) - MP\frac{\partial^2 \bar{w}_m}{\partial \xi^2} + \left(1 + T_d \frac{\partial}{\partial \tau}\right)\frac{\partial^4 \bar{w}_m}{\partial \xi^4} = \nu^2 \frac{\partial^2}{\partial \xi^2}\left[\frac{\partial^2 \bar{w}_m}{\partial \tau^2} - \delta \frac{\partial^4 \bar{w}_m}{\partial \tau^2 \partial \xi^2} + K\bar{w}_m + B\dot{\bar{w}}_m + K(\bar{w}_m - \bar{w}_{m-1}) + B(\dot{\bar{w}}_m - \dot{\bar{w}}_{m-1}) - MP\frac{\partial^2 \bar{w}_m}{\partial \xi^2}\right]. \tag{15c}$$

The initial conditions in general form and boundary conditions in dimensionless form for simply supported nonlocal viscoelastic Rayleigh's nanobeams of the same length can be written as

$$\bar{w}_i(\xi, 0) = \bar{w}_{i0}(\xi), \quad \frac{\partial \bar{w}_i(\xi, 0)}{\partial t} = \bar{v}_{i0}(\xi), \tag{16a}$$

$$\bar{w}_i(0,\tau) = \bar{w}_i(1,\tau) = 0, \quad \bar{M}_{fi}(0,\tau) = \bar{M}_{fi}(1,\tau) = 0, \quad i = 1,2,3,..,m. \tag{16b}$$

III. EXACT SOLUTION FOR COMPLEX NATURAL FREQUENCY

First, we suggest the analytical solution of the governing equations for the free transverse vibration and boundary conditions of the viscoelastically coupled MNBS. Assuming the solution of Eqs. (15) and (16) as the expansions of the generalized displacement $w_i(x,t)$ in the following form:

$$\bar{w}_i(\xi,\tau) = \sum_{n=1}^{\infty} \bar{W}_{in} \sin n\pi\xi e^{i\Omega_n \tau}, \quad i = 1,2,3,..,m, \quad n = 1,2,3,\ldots \tag{17}$$

where $\bar{W}_{in}$ and $\Omega_n = \omega_n L^2 \sqrt{\frac{\rho A}{EI}}$, $(i = 1,2,3,..,m)$ are amplitudes and complex natural frequencies, respectively and $i = \sqrt{-1}$. By substituting expression Eq. (17) for the assumed solution into Eq. (15), the system of m partial differential equations is reduced to the system of m algebraic equations as

$$\bar{S}_n \bar{W}_{1n} - \bar{v}_n \bar{W}_{2n} = 0, \quad i = 1, \tag{18a}$$

$$-\bar{v}_n \bar{W}_{i-1n} + \bar{S}_n \bar{W}_{in} - \bar{v}_n \bar{W}_{i+1n} = 0, \quad i = 2,3,\ldots,m-1, \tag{18b}$$

$$-\bar{v}_n \bar{W}_{m-1n} + \bar{S}_n \bar{W}_{mn} = 0, \quad i = m \tag{18c}$$

in which

$$\bar{S}_n = -\Omega_n^2(1 + \delta n^2\pi^2)\bar{\lambda}_n + 2\bar{v}_n + n^2\pi^2 MP\bar{\lambda}_n + n^4\pi^4(1 + i\Omega_n T_d), \tag{19a}$$

$$\bar{v}_n = (K + i\Omega_n B)\bar{\lambda}_n, \tag{19b}$$

$$\bar{\lambda}_n = (1 + \nu^2 n^2 \pi^2). \tag{19c}$$

The system of algebraic equations (18) with dimensionless term can be written in the matrix form as

$$\begin{bmatrix} \bar{S}_n & -\bar{v}_n & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -\bar{v}_n & \bar{S}_n & -\bar{v}_n & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \bar{S}_n & -\bar{v}_n & & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & -\bar{v}_n & \bar{S}_n & -\bar{v}_n & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -\bar{v}_n & \bar{S}_n & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \bar{S}_n & -\bar{v}_n \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -\bar{v}_n & \bar{S}_n \end{bmatrix} \begin{Bmatrix} \bar{W}_{1n} \\ \bar{W}_{2n} \\ \bar{W}_{3n} \\ \ldots \\ \bar{W}_{i-1n} \\ \bar{W}_{in} \\ \bar{W}_{i+1n} \\ \ldots \\ \bar{W}_{m-2n} \\ \bar{W}_{m-1n} \\ \bar{W}_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{20}$$

From the system of homogeneous algebraic equations (20), it is possible to obtain dimensionless complex natural frequency in two ways. The first way is related to some of the available numerical methods for solving determinant of the system of algebraic equations, i.e., solving high order polynomial equation. The second way is so-called trigonometric method by which we can find solution of i-th algebraic equation in the trigonometric form. We proposed this method in order to obtain closed form solutions for complex natural frequencies. It should be noted that the analytical solution of the system of algebraic equations (20) is possible only for the case when the system is composed of identical CNTs embedded in homogeneous medium. Based on works by Rašković,[70] Stojanović *et al.*[71] and Karličić *et al.*[72] we propose the solution of $\boldsymbol{i-th}$ algebraic equation in the following form:

$$\bar{W}_{in} = N\cos(i\,\varphi_{cc}) + M\sin(i\,\varphi_{cc}), \; i = 1,2,3,\ldots,m. \tag{21}$$

Substituting Eq. (21) into the $i-th$ algebraic equation of the system (20), and after some algebra, we obtain two trigonometric equations with the assumption that constants M and N are not simultaneously equal to zero

$$N\{-\bar{v}_n\cos[(i-1)\varphi_{cc}]+\bar{S}_n\cos(i\,\varphi_{cc})-\bar{v}_n\cos[(i+1)\varphi_{cc}]\}=0,\quad i=2,3,\ldots,m-1, \tag{22a}$$

$$M\{-\bar{v}_n\sin[(i-1)\varphi_{cc}]+\bar{S}_n\sin(i\,\varphi_{cc})-\bar{v}_n\sin[(i+1)\varphi_{cc}]\}=0,\quad i=2,3,\ldots,m-1, \tag{22b}$$

so that

$$(\bar{S}_n-2\bar{v}_n\cos\varphi_{cc})N\cos(i\,\varphi_{cc})=0, \tag{23a}$$

$$(\bar{S}_n-2\bar{v}_n\cos\varphi_{cc})M\sin(i\,\varphi_{cc})=0. \tag{23b}$$

To have an oscillatory behavior of our system, it is necessary that $N\neq 0$ and $\cos(i\,\varphi_{cc})\neq 0$ or $M\neq 0$ and $\sin(i\,\varphi_{cc})\neq 0$ for $i=2,3,\ldots,m-1$. From Eq. (23), we determine the frequency equation as follows:

$$\bar{S}_n=2\bar{v}_n\cos\varphi_{cc}. \tag{24}$$

The closed form solutions for complex natural frequencies are obtained from frequency equation (24), where unknown φ_{cc} is determined from the first and the last equation of the system (20). By substituting expressions $\bar{W}_{1n}=N\cos\varphi_{cc}+M\sin\varphi_{cc}$ and $\bar{W}_{2n}=N\cos(2\varphi_{cc})+M\sin(2\varphi_{cc})$ into the first equation and $\bar{W}_{m-1n}=N\cos[(m-1)\varphi_{cc}]+M\sin[(m-1)\varphi_{cc}]$ and $\bar{W}_{mn}=N\cos(m\varphi_{cc})+M\sin(m\varphi_{cc})$ into the last equation of the system (20), we obtain the following system of algebraic equations:

$$N[\bar{S}_n\cos\varphi_{cc}-\bar{v}_n\cos(2\varphi_{cc})]+M[\bar{S}_n\sin\varphi_{cc}-\bar{v}_n\sin(2\varphi_{cc})]=0, \tag{25a}$$

$$N[\bar{S}_n\cos(m\varphi_{cc})-\bar{v}_n\cos[(m-1)\varphi_{cc}]]+M[\bar{S}_n\sin(m\varphi_{cc})-\bar{v}_n\sin[(m-1)\varphi_{cc}]]=0. \tag{25b}$$

Here, we obtain the non-trivial solutions for constants N and M from the following trigonometric equation:

$$\begin{vmatrix}1 & 0\\ \cos[(m+1)\varphi_{cc}] & \sin[(m+1)\varphi_{cc}]\end{vmatrix}=0 \;\Rightarrow\; \sin[(m+1)\varphi_{cc}]=0. \tag{26}$$

From the above trigonometric equation, we determine the solutions for unknown φ_{cc} as

$$\varphi_{cc,s}=\frac{s\pi}{m+1},\quad s=1,2,\ldots,m. \tag{27}$$

Introducing expression for $\varphi_{cc,s}$ and Eqs. (19) into frequency equation (24) yields

$$-\Omega_n^2(1+\delta n^2\pi^2)\bar{\lambda}_n+2(K+i\Omega_nB)\bar{\lambda}_n(1-\cos\varphi_{cc,s})+n^2\pi^2MP\bar{\lambda}_n+n^4\pi^4(1+i\Omega_nT_d)=0. \tag{28}$$

Expression for the dimensionless complex natural frequency is obtained by solving the quadratic equation (28) which gives

$$\Omega_{ncc,s}=\pm\sqrt{\frac{4(1+\delta n^2\pi^2)\bar{\lambda}_n\left[2K\bar{\lambda}_n(1-\cos\varphi_{cc,s})+n^2\pi^2MP\bar{\lambda}_n+n^4\pi^4\right]-\left[2B\bar{\lambda}_n(1-\cos\varphi_{cc,s})+T_dn^4\pi^4\right]^2}{4(1+\delta n^2\pi^2)^2\bar{\lambda}_n^2}}+i\frac{2B\bar{\lambda}_n(1-\cos\varphi_{cc,s})+T_dn^4\pi^4}{2(1+\delta n^2\pi^2)\bar{\lambda}_n},\quad s=1,2,\ldots m,\quad n=1,2,\ldots. \tag{29}$$

This expression is complex natural frequency and it is composed of the real and imaginary part. The real part represents damped natural frequency and the imaginary part represents damping ratio of the complex frequency.

IV. ASYMPTOTIC DAMPED FREQUENCY AND CRITICAL DAMPING

In order to obtain critical values of damped natural frequency and damping ratio, i.e., asymptotic values of complex natural frequency, we assume that the number of CNTs or a vibration mode number tends to infinity, i.e., we introduce $m\to\infty$ or $n\to\infty$ into Eq. (29). According to this assumption, we get critical complex natural frequencies for $m\to\infty$ as follows:

$$\Omega_{\substack{n\\ m\to\infty}}=\pm\sqrt{\frac{4(1+\delta n^2\pi^2)\bar{\lambda}_n(n^2\pi^2MP\bar{\lambda}_n+n^4\pi^4)-(T_dn^4\pi^4)^2}{4(1+\delta n^2\pi^2)^2\bar{\lambda}_n^2}}+i\frac{T_dn^4\pi^4}{2(1+\delta n^2\pi^2)\bar{\lambda}_n}, \tag{30}$$

where real parts of complex natural frequencies represent the lowest damped natural frequency of MNBS when the number of nanobeams tends to infinity. When the real part of critical complex frequency vanishes, we can determine the critical values of internal damping as

$$Re\left[\Omega_{\substack{n\\ m\to\infty}}\right]=0,$$

$$\Rightarrow (T_d)_{\substack{cr \\ m\to\infty}} = \frac{\sqrt{4(1+\delta n^2\pi^2)\bar{\lambda}_n\left(n^2\pi^2 MP\bar{\lambda}_n + n^4\pi^4\right)}}{n^4\pi^4}. \quad (31)$$

It should be noted that the both critical values expressed in Eqs. (30) and (31) are independent of the influence of viscoelastic medium. The expression for the critical complex natural frequency (30) can be reduced to the fundamental frequency of in-phase mode of vibration for double nanobeam system proposed by Murmu and Adhikari[47] when the following system parameters are vanishes $\delta = 0, T_d = 0$ and $B = 0$. Then, we obtain

$$\bar{\Omega}_n = \pm\sqrt{\frac{n^2\pi^2 MP\bar{\lambda}_n + n^4\pi^4}{\bar{\lambda}_n}}. \quad (32)$$

Accordingly, we can conclude that the fundamental natural frequency of the system is independent of the number of nanobeams in the system and influence of the viscoelastic medium. Let us now consider the case when the vibration mod tends to infinity, i.e., introducing $n \to \infty$ into Eq. (29)

$$\Omega_{n\to\infty} = \pm\frac{\sqrt{4\delta\nu^2(1+\nu^2 MP) - (T_d)^2}}{2\delta\nu^2} + i\frac{T_d}{2\delta\nu^2}. \quad (33)$$

The critical values of internal damping are obtained by setting the oscillation frequency to zero in case when the vibration mode tends to infinity

$$Re[\Omega_{n\to\infty}] = 0, \quad \Rightarrow (T_d)_{\substack{cr \\ n\to\infty}} = 2\nu\sqrt{\delta(1+\nu^2 MP)}. \quad (34)$$

For the last case, we can conclude that the both parts of critical complex natural frequency are functions of nanobeam material parameters and are independent of a number of nanobeams, boundary conditions and influences of the viscoelastic medium.

V. RESULTS AND DISCUSSIONS

In this section, several characteristic effects appearing in viscoelastically coupled MNBS under the influence of longitudinal magnetic field are presented using the nonlocal model of the system. It is assumed that all nanobeams in MNBS are with same dimensions and material properties that allows as to use suggested trigonometric method to find exact solutions for complex natural frequencies. Following dimensions and values of parameters are used in following simulations: $K = 50, B = 15, T_d = 0.05, \nu = 0.5, \delta = 0.5$, and $MP = 25$. To corroborate results for complex natural frequencies find by trigonometric method from system of algebraic equations, we compare them with the numerical solutions of the same system of equations as shown in Table I. It is obvious that by using the proposed analytical method to find exact solutions for complex frequencies we achieved excellent agreement with the numerical results obtained by solving the system of algebraic equations (20) for the first vibration mode. We can notice that increase of a number of nanobeams in MNBS leads to decrease of both parts of complex natural frequency. As concluded in asymptotic analysis, increase in a number of nanobeams to infinity results in decrease of complex natural frequency towards fundamental frequency, which is a frequency of a system, composed of single nanobeam. From a physical point of view, this means that the viscoelastic medium has the reduced influence on the complex natural frequencies for the system with a larger number of nanotubes, which is in agreement with the expression (30).

In the following, we give a detailed analysis of asymptotic values of the complex natural frequency, when the number of nanobeams and vibration mode tends to infinity.

TABLE I. Numerical validation of dimensionless complex natural frequencies of a nonlocal viscoelastically coupled MNBS in "Clamped-Chain" system and for various numbers of nanobeams.

$n=1$		Numerical method Eq. (20)	Trigonometric method Eq. (29)
$m=3$	1	7.106803207007234 + 0.858616467032355i	7.106803207007234 + 0.8586164670323561i
	2	7.444570457727209 + 4.432990594218853i	7.444570457727200 + 4.4329905942188590i
	3	7.493876043935568 + 2.645803530625613i	7.493876043935575 + 2.6458035306256074i
$m=5$	1	6.953943128363628 + 0.456955340272326i	6.953943128363753 + 0.4569553402723789i
	2	7.267974436531825 + 1.382071438710205i	7.267974436531376 + 1.3820714387099455i
	3	7.374070686560821 + 4.834651720978800i	7.374070686561144 + 4.8346517209788360i
	4	7.493876043934701 + 2.645803530625339i	7.493876043935575 + 2.6458035306256074i
	5	7.503257047003444 + 3.909535622541435i	7.503257047002612 + 3.9095356225412687i
$m=10$	1	6.851460422768405 + 0.220719405037133i	6.851460422839132 + 0.2207194051771522i
	2	6.979511814332560 + 0.519565357800634i	6.979511814312691 + 0.5195653568732285i
	3	7.151418559430514 + 0.990666475118477i	7.151418559043098 + 0.9906664801814060i
	4	7.322006580188391 + 5.070887654979121i	7.322006580644942 + 5.0708876560740626i
	5	7.322025077236097 + 1.595856969829816i	7.322025078960496 + 1.5958569638375157i
	6	7.386541385971538 + 4.772041706313426i	7.386541383798568 + 4.7720417043779860i
	7	7.452126829862110 + 2.286107864740458i	7.452126825875755 + 2.2861078740621320i
	8	7.462882253737417 + 4.300940581996953i	7.462882258592229 + 4.3009405810698090i
	9	7.516616897306076 + 3.695750090978755i	7.516616890636129 + 3.6957500974136980i
	10	7.518204576847910 + 3.005499200270248i	7.518204583815520 + 3.0054991871890824i

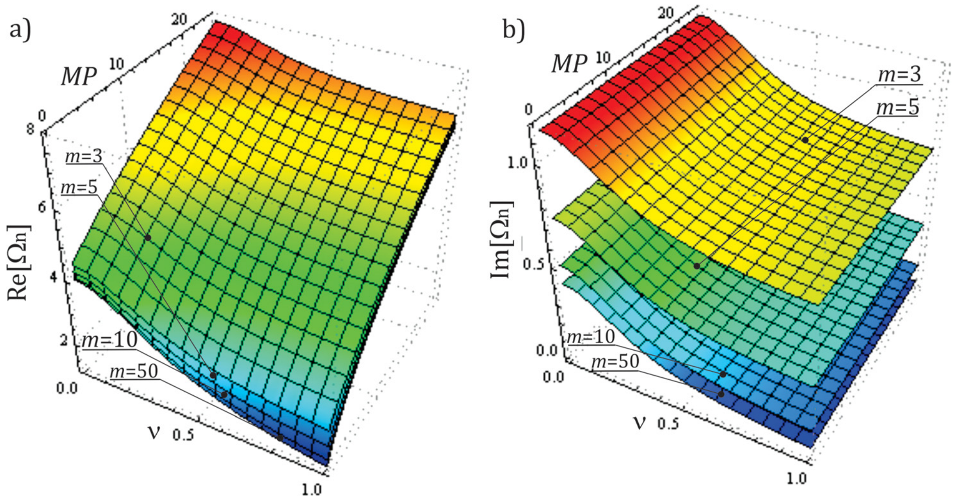

FIG. 2. The effect of magnetic field and nonlocal parameter on complex natural frequency.

First, we should note that all numerical results in Figs. 2–5 are performed in the first mode of vibration. Figs. 2(a) and 2(b) show the real and imaginary parts, respectively of the complex natural frequency of viscoelastically coupled MNBS for increase of a magnitude of magnetic field in the range 0–25 and nonlocal parameter in the range 0–1. The real part of complex natural frequency increases nearly linearly with the increase of the magnitude of magnetic field. On the other side, change of the magnetic field has no effect on the imaginary part. It is obvious that without the presence of magnetic field damped natural frequencies are much lower and they are only affected by a small-scale parameter. As expected, influence of nonlocal parameter is nonlinear and its increase decreases both parts of complex natural frequency. However, it can be seen that this decrease is more pronounced in the real part of the complex frequency than the imaginary one. Generally, it is noticeable that increase of the nonlocal parameter decreases the influence of a magnetic field on damped natural frequency. Further, it can be concluded that the magnetic field has a reinforced effects by increasing the stiffness of the system while the nonlocal parameter has a damping effect so that the system of coupled SWCNTs becomes softer. By selecting these two parameters, we can obtain nanocomposites with very wide range of dynamic characteristics. In addition, it can be noticed that both parts of complex frequency are decreasing for increase of a number of nanobeams in MNBS. This effect is common for all following plots.

Figs. 3(a) and 3(b) show the effect of change of internal damping coefficient and nonlocal parameter on complex natural frequency. In this case, internal damping coefficient has negligible influence on the real part of complex natural frequency while its effect on imaginary part is linear and its increase causes an increase of damping ratio. Further, it can be observed that the influence of internal damping on the damping ratio is reduced with the increase of nonlocal parameter.

In Figs. 4 and 5, the effects of magnetic field on complex natural frequency for variations in internal damping coefficient and damping coefficient of viscoelastic layer are shown. It is noticeable that there are no coupling effects between both damping coefficients and magnetic field. In both cases, damped natural frequencies are increasing almost linearly for an increase of the magnitude of magnetic field and the level of change is not affected by an increase of both damping coefficients. Damping ratios remain constant for any changes in the magnitude of magnetic field while it linearly increases for an increase of both damping coefficient.

The effects of magnetic field and nonlocal parameter on critical values of internal damping are shown in Figs. 6(a)

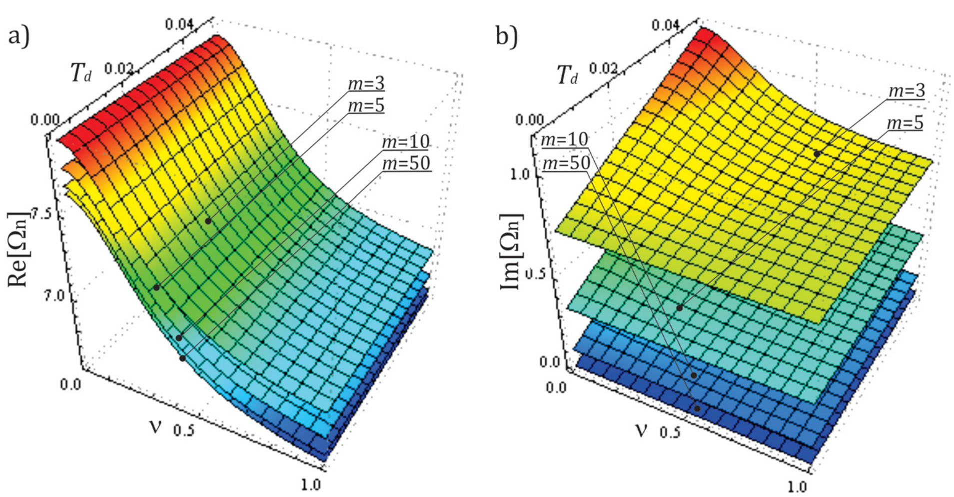

FIG. 3. The effect of the internal damping coefficient and nonlocal parameter on complex natural frequency.

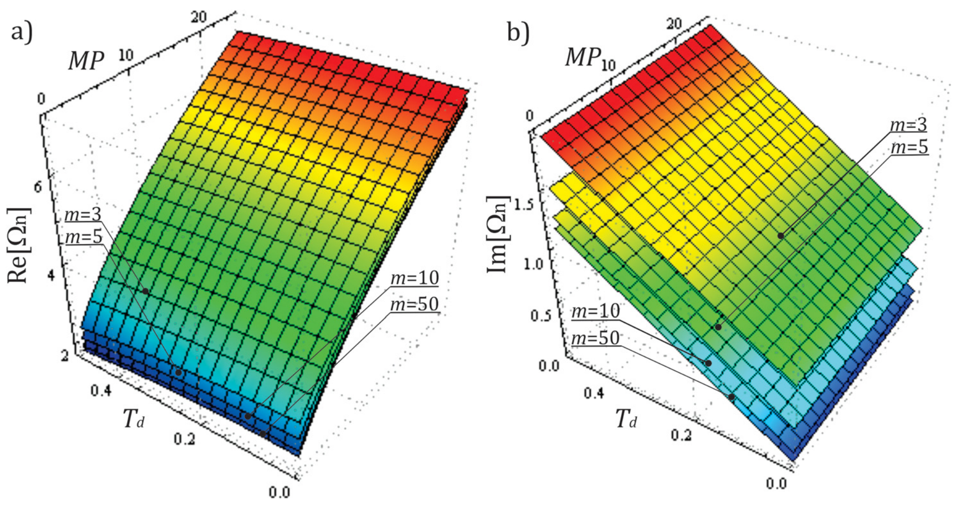

FIG. 4. The effect of magnetic field and internal damping on complex natural frequency.

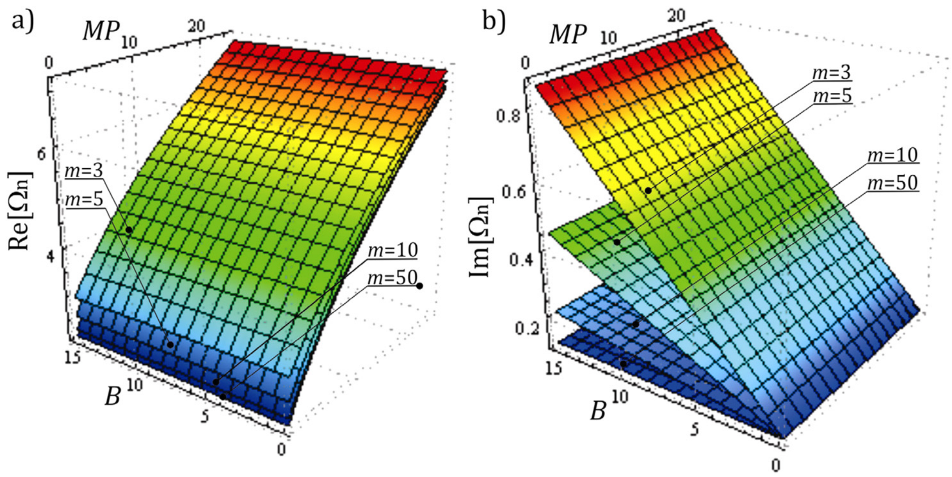

FIG. 5. The effect of magnetic field and damping coefficient of viscoelastic layer on complex natural frequency.

and 6(b). One can notice that an increase of the magnitude of magnetic field has a weak influence on critical damping for low values of nonlocal parameter. However, this influence increases for an increase of the nonlocal parameter. For higher values of small-scall parameter, the critical damping becomes very sensitive to changes in the magnitude of magnetic field. In that case, for an increase of the magnitude of a magnetic field we have a considerable increase of the critical value of internal damping.

Figs. 7(a) and 7(b) show the real and imaginary parts of complex natural frequency for an increase from lower to higher vibration modes of viscoelastically coupled MNBS with different numbers of nanobeams in the system. It can be noticed that for lower modes there is an upsurge of the

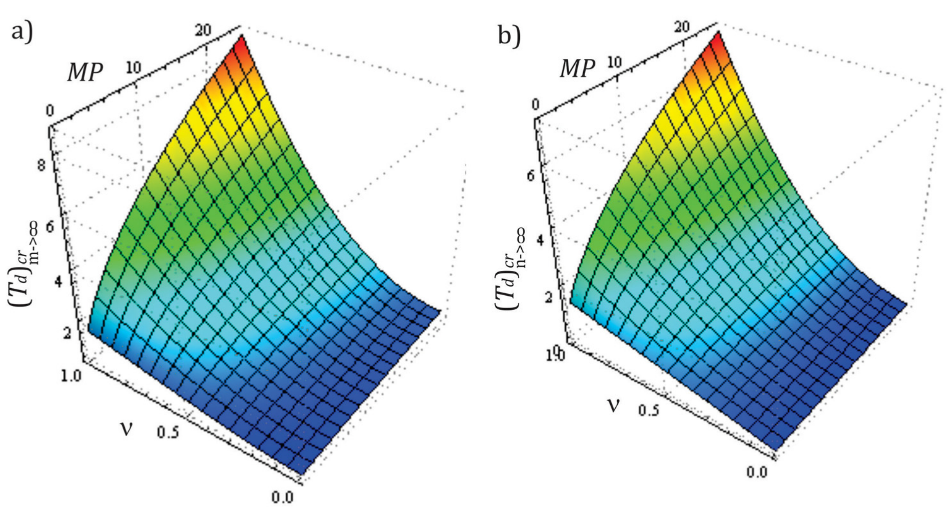

FIG. 6. The effect of magnetic field and nonlocal parameter on the critical values of internal damping; (a) Number of nanobeams tends to infinity, (b) vibration mode tends to infinity.

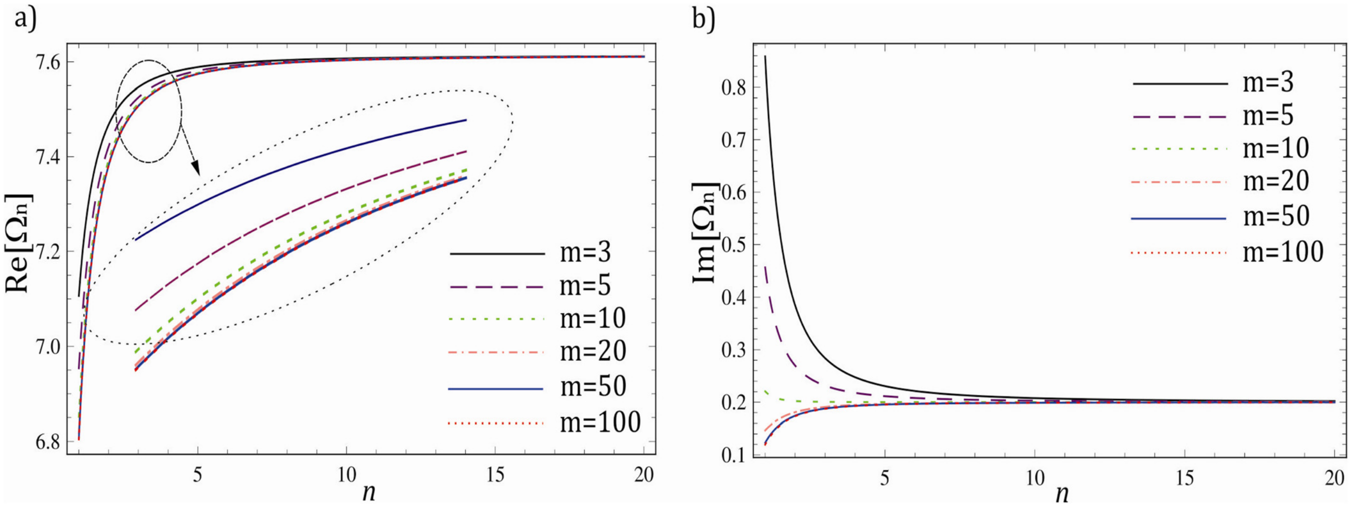

FIG. 7. The effect of vibration modes n and number of nanobeams m on complex natural frequency.

damped natural frequency for an increase of a mode number. This jump is noticeable until the certain higher value of mode number where the real part of complex frequency tends to the critical value of damped natural frequency as discussed in the asymptotic analysis. In addition, damping ratio is higher for the system with lower number of nanobeams while it decreases as the number of nanobeams increases. The damping ratio decreases only until the certain number of nanobeams in MNBS. When the damping ratio reaches its critical value, it is than independent of further increase of a number of nanobeams. However, it is interesting to notice that for higher number of nanobeams in MNBS, e.g., more than 20 nanobeams, the value of damping ratio is lower than the critical value but its value asymptotically approaches to the value of critical damping ratio as the mode number increases.

In Figs. 8(a) and 8(b), it can be seen how the viscoelastically coupled MNBS respond to changes in inertia parameter δ for different number of nanobeams in the system. It can be noticed that in general, both part of complex natural frequency decreases for an increase of the parameter δ for MNBS with any number of nanobeams. Nevertheless, in imaginary part of complex frequency the value of damping ratio is much higher for MNBS with lower number of nanobeams than those with the higher number of nanobeams. Anyways, as the number of nanobeams in MNBS increases value of complex natural frequency tends to its critical value.

From the presented results, it is apparent that longitudinal magnetic field exerted on the viscoelastically coupled MNBS affects only the real part of complex frequency. According to the best of authors' knowledge, so far no experimental work has been reported on the vibration behavior of complex nanobeam systems in the presence of magnetic field, but some conclusion can be made after summarizing the obtained numerical results. It is obvious that an increase of the magnitude of magnetic field leads to an increase of the damped natural frequency. However, from a physical point of view this means that the axial magnetic field is acting on the system in such a manner that causes an increase of the total stiffness of carbon nanotubes and consequently of the whole system. This means that we can change properties, i.e., natural frequencies of nano-mechanical systems by applying the external field without any change of other material or geometrical parameters of the system. Such externally applied

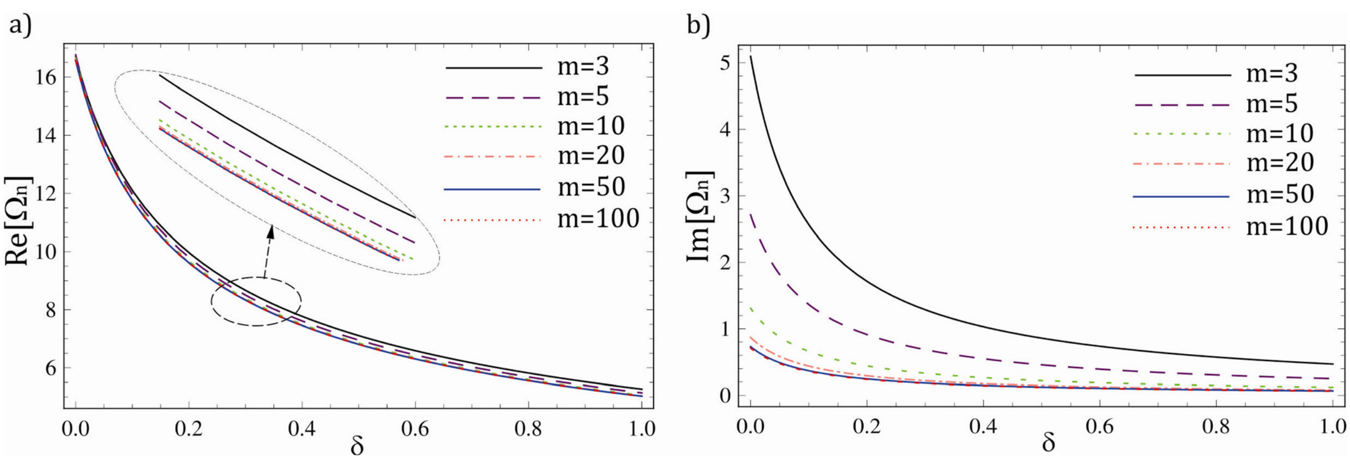

FIG. 8. The effect of inertia parameter δ and number of nanobeams m on complex natural frequency.

magnetic field enables the practical application of complex nano-scale systems as nano-resonators or nanocomposits[73] with enchased mechanical, damping and thermal properties of polymers due to the presence of carbon nanotubes with changeable stiffness. Further, by selecting the appropriate magnitude of axial magnetic field, corresponding values of damped natural frequencies of the system are obtained. This allows us to capture a certain range of excitation resonant frequencies or avoid them in order to prevent resonant states.

Since magnetic field has no effect on imaginary part of complex frequency and by neglecting the internal damping, we come to the results obtained by Murmu *et al.*[48] for a double SWCNT elastically coupled system under the influence of magnetic field. As reported in there, longitudinal magnetic field increases natural frequency of the system similar to the effect of Pasternak foundation. More precise, the effect of magnetic field parameter is equivalent to the effect of Pasternak shear modulus. This is in line with our results since in the viscoelastically coupled MNBS the damped natural frequency also increases nearly linearly with an increase of the magnitude of a magnetic field.

The results of this study may be useful for the design of nanocomposites and NEMS with CNTs. Complex natural frequency determined by the given analytical expression is a function of small-scale parameter, damping coefficient of the medium, internal damping parameter of a nanostructure and magnetic parameter. Such determined approximate frequencies for multi SWCNT embedded in the viscoelastic polymer matrix may be substantial for a production of NEMS devices due to the lack of experimental or molecular dynamics results.

VI. CONCLUSION

In this paper, dynamic behavior of multiple viscoelastic SWCNTs embedded in viscoelastic medium and under the influence of axial magnetic field is investigated. The system of m partial differential equation of motion and corresponding characteristic equation for the complex natural frequency are derived for "Clamped-Chain" system and solved analytically. Nonlocal Rayleigh beam theory is used in order to take into account rotary inertia effects and to carry out the asympthotic analysis in which we have determined the critical compex frequencies and critical internal damping in an exact manner. The effect of longitudinal magnetic field is introduced into dynamic equations of the system by considering the Lorentz magnetic force obtained from Maxwell's relation. It is observed that the influence of magnetic field on vibration properties of MSWCNT system is significant. This system has an ability to change the complex natural frequency by variation of the magnitude of external magnetic field without changing other physical parameters of the system. This can be important property for practical applications in nano-resonators and nanocomposites. In addition, it is shown that the nonlocal parameter has damping effects on the complex natural frequency i.e. it accounts for the small-scale effects that obviously decreases natural frequencies which are over determined by use of classical continuum theories. Some of the main theoretical contributions made in this work include:

(1) Exact solution for the complex natural frequencies of MSWCNT system in the case of an arbitrary number of nanotubes in the system is presented in Eq. (29).
(2) Asymptotic values of complex natural frequencies of MSWCNT system are obtained in Eq. (30) for the case when the number of carbon nanotubes (m) tends to infinity. From the physical viewpoint, this can be the complex natural frequency of a single simply supported SWCNT independent of the influence of external viscoelastic medium.
(3) Asymptotic values of complex natural frequency of MSWCNT system are obtained in Eq. (33) for the case when mode number (n) tends to infinity. This asymptotic value represents the maximal value of a complex natural frequency of the system.
(4) We determined critical values of internal damping for two cases: (i) when the number of nanotubes (m) tends to the infinity (Eq. (31)) and (ii) when the mode number (n) tends to infinity (Eq. (34)). These critical values of internal damping imply aperiodic behavior of MSCNT system.

The results of this research could be used as a starting point for extended models of nanocomposites with more physical effects included. Additional effects that could be interested for future theoretical research are thermal elasticity problem due to different thermal expansion coefficients of carbon nanotubes and polymer matrix and consideration of catalyst nanoparticle impurities.

ACKNOWLEDGMENTS

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