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Frequency domain analysis of nonlocal rods embedded in an elastic medium

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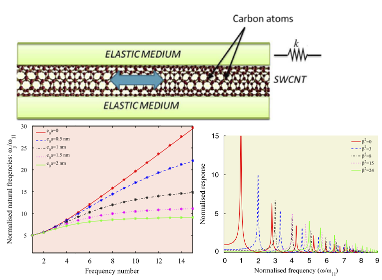
HIGHLIGHTS

- Computational formulation of an elastic medium on nonlocal axial vibration of rods is considered.
- Static and dynamic finite element formulations are proposed.
- The nonlocal parameter impacts both the element mass and stiffness matrices.
- A closed-from exact expression is derived for the upper cut-off natural frequency.
- Nonlocal parameter and the elastic stiffness respectively decrease and increase the natural frequencies.

GRAPHICAL ABSTRACT

Variation of natural frequencies and frequency response function of single walled carbon nanotube embedded in an elastic medium.



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ABSTRACT

A novel dynamic finite element method is carried out for a small-scale nonlocal rod which is embedded in an elastic medium and undergoing axial vibration. Eringen's nonlocal elasticity theory is employed. Natural frequencies are derived for general boundary conditions. An asymptotic analysis is carried out. The stiffness and mass matrices of the embedded nonlocal rod are obtained using the proposed finite element method. Nonlocal rods embedded in an elastic medium have an upper cut-off natural frequency which is independent of the boundary conditions and the length of the rod. Dynamic response for the damped case has been obtained using the conventional finite element and dynamic finite element approaches. The present study would be helpful for developing nonlocal finite element models and study of embedded carbon nanotubes for future nanocomposite materials.

1. Introduction

Recently, classical continuum mechanics is becoming popular for modelling, understanding and predicting the physical behaviour of nanostructures such as bending, vibration and buckling, etc. One reason for employing continuum mechanics is that the experiments at the nanoscale are challenging; and atomistic computational methods such as molecular dynamic (MD) simulations are computationally expensive for nanostructures with large numbers of atoms. At the nanoscale, scale-effects due to atoms, molecules, forces are important and cannot be ignored. Thus classical continuum mechanics requires upgrading for accurate predictions of behaviour of nanostructures. Further, the discrete nature of structures at nanoscale needs to be accounted for in continuum based modelling. To address the discreteness and size-dependency [1–6], continuum mechanics based methods [7–9] are gaining in popularity in the modelling of small sized structures. This approach offers much faster solutions than molecular dynamic simulations for various nano engineering problems.

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One popularly used size-dependant theory is the nonlocal elasticity theory pioneered by Eringen [10]. The theory of nonlocal elasticity (nonlocal continuum mechanics) is being increasingly used [11] for efficient analysis of nanostructures such as carbon nanotubes and graphene. According to nonlocal elasticity, the stress at point is not only dependent on the strain at the point but also on strains at all other points in the domain.

For complex structures and loading, mere analytical modelling is not sufficient for understanding the vibration phenomenon of nanostructures such as CNT or graphene. One popular approach is finite element (FE) modelling. Some works on finite elements and nonlocal elasticity are reported for nanorods; however the research is in its initial stage. Application of nonlocal elasticity and finite elements is reported for the small scale effects on axial free vibration of non-uniform and nonhomogeneous nanorods [12]. Using dynamic nonlocal finite element analysis, Adhikari et al. [13] studied free and forced axial vibrations of damped nonlocal rods. Recently the concept of nonlocal elasticity is applied for the development of a spectral finite element (SFE) for analysis of nanorods [14].

Literature shows various works via nonlocal elasticity theory on study of carbon nanotubes embedded in an elastic medium. Analysis has been carried out both on the phenomenon of axial [15,16] and transverse vibration [17–20] of carbon nanotubes embedded in elastic medium. Single-walled [18,20,21] as well as double-walled carbon nanotubes [22] being embedded were studied. The carbon nanotubes may be with or without fluid flowing [23] through it besides being embedded in an elastic medium. Two types of elastic mediums are generally considered for the study. The elastic medium models are based on one-parameter (Winkler type) [15,21] as well as two-parameter (Pasternak type) [18] elastic medium.

From the brief literature survey, we can see that significant research effort has taken place in the nonlocal analysis of nanostructures embedded in an elastic medium including nanorods. However, not much work has been done on the study of nanorods in an elastic medium and considering finite element in details. The majority of the reported works on nonlocal finite element analysis study free vibration studies where the effect of non-locality on the undamped eigensolutions is studied. Damped nonlocal systems and forced vibration response analysis have received little attention. This type of study is useful for the design and analysis of future generation of nano composite materials and structures. Thus in this paper we develop a novel finite element method based on nonlocal elasticity for axially vibrating nanorods in an elastic medium. Free and forced axial vibration of damped nonlocal embedded rods are investigated. We consider both the damped and undamped cases of vibration. The present work on finite element for nanorods in embedded elastic medium is expected to provide the general framework for improved design methods.

2. Nonlocal rod embedded in an elastic medium

We consider a damped nanorod embedded in an elastic medium [15]. In Fig. 1 a single-walled carbon nanotube (SWCNT) embedded in an elastic medium is shown for example. The equation of motion for the axial vibration can be expressed as

$$EA\frac{\partial^2 U(x,t)}{\partial x^2} - kU(x,t) + (e_0a)^2 k\frac{\partial^2 U(x,t)}{\partial x^2} + \hat{c}_1\frac{\partial^3 U(x,t)}{\partial x^2 \partial t} = \hat{c}_2\frac{\partial U(x,t)}{\partial t} + \left(1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right)\left\{m\frac{\partial^2 U(x,t)}{\partial t^2} + F(x,t)\right\} \quad (1)$$

The mass per unit length is denoted by m, the stiffness of the elastic medium is denoted by k and the axial rigidity is denoted by EA. The constant $\hat{c}_1$ is the strain-rate-dependent viscous damping coefficient and $\hat{c}_2$ is the velocity-dependent viscous damping coefficient. The nonlocal parameter [10], denoted by e_0a, influences the inertial term as well as the stiffness of the elastic medium. The case without the elastic medium was considered in Ref. [13]. Here we aim to understand the impact of the elastic medium on the nonlocal axial vibration of damped nanorods.

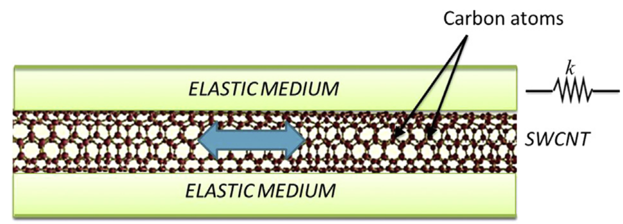

Fig. 1. A single-walled carbon nanotube (SWCNT) embedded in within an elastic medium with stiffness k. Axial vibration is considered in this study.

The equation of motion (1) can be solved by using the separation of variable approach [24]. We express the time-depended axial motion by

$$U(x,t) = u(x)\exp[\mathrm{i}\omega t] \quad (2)$$

Considering the forcing is zero (i.e., free vibration) and substituting this in Eq. (1) one obtains

$$(EA + (e_0a)^2 k + \mathrm{i}\omega\hat{c}_1 - (e_0a)^2 m\omega^2)\frac{d^2u}{dx^2} + (m\omega^2 - k - \mathrm{i}\omega\hat{c}_2)u(x) = 0 \quad (3)$$

For analytical convenience, the damping is expressed as proportional to mass and stiffness by introducing the following two damping factors:

$$\hat{c}_1 = \zeta_1 EA \quad \text{and} \quad \hat{c}_2 = \zeta_2 m \quad (4)$$

Note that ζ_1 and ζ_2 are stiffness and mass proportional damping factors respectively. Eq. (3) can be reorganised as

$$\frac{d^2u}{dx^2} + \frac{(m\omega^2 - k - \mathrm{i}\omega\zeta_2 m)}{(EA + (e_0a)^2 k + \mathrm{i}\omega\zeta_1 EA - (e_0a)^2 m\omega^2)}u(x) = 0 \quad (5)$$

This can be concisely expressed as

$$\frac{d^2u}{dx^2} + \alpha^2 u = 0 \quad (6)$$

with

$$\alpha^2 = \frac{(\omega^2 - \omega_s^2) - \mathrm{i}\zeta_2\omega}{c^2(1 + \mathrm{i}\omega\zeta_1) - (e_0a)^2(\omega^2 - \omega_s^2)} \quad (7)$$

and

$$c^2 = \frac{EA}{m}, \quad \omega_s^2 = \frac{k}{m} \quad (8)$$

We call ω_s is the elastic medium natural frequency. For the special case of the undamped rod embedded in the elastic medium, we set the damping coefficients ζ_1 and ζ_1 to zero. With this α^2 in Eq. (7) reduces to

$$\alpha^2 = \frac{\Omega^2 - \Omega_s^2}{1 - (e_0a)^2(\Omega^2 - \Omega_s^2)} \quad (9)$$

This is a real function of $\Omega = \omega/c$ and $\Omega_s = \omega_s/c = \sqrt{k/EA}$. For the general damped case, α^2 is a complex function of the frequency parameter ω.

Natural frequencies of the system depend on the boundary conditions. We adopt a general approach by which different boundary conditions can be considered in an unified manner.

We define the parameter σ_j as

$$\sigma_j = \frac{j\pi}{L} \quad \text{for clamped–clamped boundary conditions} \tag{10}$$

$$\text{and} \quad \sigma_j = \frac{(2j-1)\pi}{2L} \quad \text{for clamped–free boundary conditions} \tag{11}$$

The length of the rod is denoted by L. The clamped–clamped boundary condition is obtained by imposing zero displacement at both ends of the bar, that is

$$u(0)=0 \quad \text{and} \quad u(L)=0 \tag{12}$$

The clamped–free boundary condition requires some careful implementation. Assuming undamped vibration, the boundary conditions at both ends can be expressed as

$$u(0)=0 \tag{13}$$

and at $x=L$, the nonlocal stress is zero. This can be expressed as

$$EA\frac{\partial U(x,t)}{\partial x} + (e_0a)^2 k\frac{\partial U(x)}{\partial x, t} + (e_0a)^2 m\frac{\partial^3 U(x,t)}{\partial x \partial t^2} = 0 \tag{14}$$

Using the harmonic representation in Eq. (2), the boundary condition in the frequency domain can be expressed in terms of $u(x)$ as

$$\left\{EA + (e_0a)^2 k - (e_0a)^2 m\omega^2\right\}\frac{\partial u(x)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u(x)}{\partial x} = 0 \tag{15}$$

The natural frequencies can be obtained [24,25] by equating σ_j in Eqs. (10) or (11) with α for different values of j as

$$\alpha_j = \sigma_j \tag{16}$$

Replacing ω by ω_j is the expression of α in Eq. (7) one has

$$\{(\omega_j^2-\omega_s^2) - \mathrm{i}\zeta_2\omega_j\} = \sigma_j^2\{c^2(1+\mathrm{i}\omega_j\zeta_1) - (e_0a)^2(\omega_j^2-\omega_s^2)\} \tag{17}$$

or

$$\omega_j^2(1+\sigma_j^2(e_0a)^2) - \mathrm{i}\omega_j(\zeta_2+\zeta_1\sigma_j^2c^2) - [\sigma_j^2c^2 + \omega_s^2(1+\sigma_j^2(e_0a)^2)] = 0 \tag{18}$$

The solution of this quadratic equation for ω_j completely defines the natural frequencies. The natural frequencies are in general complex quantities due to the presence of damping. Considering special cases when the damping is assumed to be zero, we can identify the following expressions:

- *Undamped local systems without elastic medium:* This case can be obtained by substituting $\omega_s=0$, ζ_1, $\zeta_2=0$ and $e_0a=0$. From Eq. (18) we therefore obtain

  $$\omega_j = \sigma_j c \tag{19}$$

  which is the classical expression [24].
- *Undamped nonlocal systems without elastic medium:* This case can be obtained by substituting $\omega_s=0$, $\zeta_1=0$ and $\zeta_2=0$. Solving Eq. (18) we therefore obtain

  $$\omega_j = \frac{\sigma_j c}{\sqrt{1+\sigma_j^2(e_0a)^2}} \tag{20}$$

  which is obtained in Ref. [25].
- *Undamped local systems:* This case can be obtained by substituting $\zeta_1=\zeta_2=0$ and $e_0a=0$. From Eq. (18) we therefore obtain

  $$\omega_j = \sigma_j c\sqrt{1+(\omega_s/\sigma_j c)^2} \tag{21}$$

- *Undamped nonlocal systems:* This case can be obtained by substituting $\zeta_1=0$ and $\zeta_2=0$. From Eq. (18) we therefore obtain

  $$\omega_j = \sigma_j c\sqrt{\frac{1}{1+\sigma_j^2(e_0a)^2} + (\omega_s/\sigma_j c)^2} \tag{22}$$

By using the two expressions of σ_j in Eqs. (10) and (11), the natural frequencies for the clamped–clamped and clamped–free boundary conditions can be obtained.

For the axial vibration of nonlocal rods, it was shown that [13] there exist a maximum cutoff frequency given by $\sqrt{EA/m}/e_0a$. This is important as the nature of the dynamic response depends on the proximity of the excitation frequency to the cutoff frequency. To obtain the cutoff frequency of the rod within the elastic medium, we rewrite the frequency equation in Eq. (22) and take the limit $k\to\infty$ to obtain

$$\lim_{j\to\infty}\omega_j = \lim_{j\to\infty} c\sqrt{\frac{\sigma_j^2}{1+\sigma_j^2(e_0a)^2} + (\omega_s/c)^2} = \lim_{j\to\infty} c\sqrt{\frac{1}{1/\sigma_j^2+(e_0a)^2} + k/EA}$$
$$= c\sqrt{1/(e_0a)^2 + k/EA} = \sqrt{\frac{EA}{m(e_0a)^2} + \frac{k}{m}} \tag{23}$$

This expression also shows that there is a constant difference between the asymptotic frequency of the system with and without the elastic medium.

3. Finite element approach for damped rods embedded in an elastic medium

The finite element approach is useful to consider general cases when the analytical approach outlined in the last section may not be applicable. One example is nanotube reinforced composite materials where one needs to consider nanorods embedded in an elastic polymer matrix. Mechanical and dynamic analysis such nano-composite materials need to be investigated using a finite element analysis. In the last section it was shown that Eq. (18) completely defines the natural frequencies of the system. Finite element approach can be used to obtain dynamic response of damped systems in an efficient manner. In this section we develop new finite element matrices of nanorods embedded in an elastic medium.

Nonlocal finite element approach for nanorods have been considered in Refs. [13,26]. We use the linear shape functions [27] for the two-noded element shown in Fig. 2 as

$$\mathbf{N}(x) = [N_1(x), N_2(x)]^T = [1-x/L, x/L]^T \tag{24}$$

Note that this element has two degrees of freedom and therefore two shape functions are used. Using the shape functions, the nonlocal element stiffness matrix can be obtained as

$$\mathbf{K}_e = EA\int_0^L \frac{d\mathbf{N}(x)}{dx}\frac{d\mathbf{N}^T(x)}{dx}dx + k\int_0^L \mathbf{N}(x)\mathbf{N}^T(x)\,dx + k(e_0a)^2\int_0^L \frac{d\mathbf{N}(x)}{dx}\frac{d\mathbf{N}^T(x)}{dx}dx = \frac{(EA+k(e_0a)^2)}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{kL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \tag{25}$$

The element mass matrix can be derived as

$$\mathbf{M}_e = m\int_0^L \mathbf{N}(x)\mathbf{N}^T(x)\,dx + m(e_0a)^2\int_0^L \frac{d\mathbf{N}(x)}{dx}\frac{d\mathbf{N}^T(x)}{dx}dx = \frac{mL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + mL(e_0a/L)^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = mL\begin{bmatrix}1/3+(e_0a/L)^2 & 1/6-(e_0a/L)^2\\ 1/6-(e_0a/L)^2 & 1/3+(e_0a/L)^2\end{bmatrix} \tag{26}$$

For the special case when there is no elastic medium, substitution of $k=0$ reduces the element matrices to the ones derived in Refs. [13,26]. These element matrices can be assembled to the global matrices and the dynamic response can be obtained using the conventional modal analysis [24].

The derivation of dynamic finite element matrices requires the use of frequency dependent shape functions. This is a sharp difference compared to the conventional finite element approach where linear static shape functions are used. Following the approach outlined in Refs. [13,28,29], the vector containing two shape functions can be expressed as

$$\mathbf{N}(x,\omega)=\begin{bmatrix}-\cot(\alpha L)\sin(\alpha x)+\cos(\alpha x)\\ \operatorname{cosec}(\alpha L)\sin(\alpha x)\end{bmatrix} \tag{27}$$

where α is defined in Eq. (7). The shape functions are obtained using unit axial displacement boundary condition as $u_e(x=0,\omega)=1$ and $u_e(x=L,\omega)=1$. Note that α is frequency dependent and in general

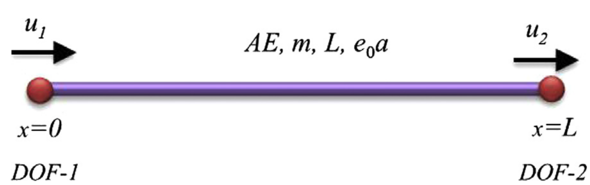

Fig. 2. An element for the damped axially vibrating rod within an elastic medium. The axial rigidity AE, elastic medium stiffness k and mass per unit length m and e_0a is the nonlocal parameter.

complex in nature. Consequently, unlike the classical shape functions in Eq. (24), the shape functions in Eq. (27) are in general complex and frequency dependent. Using the shape function, the displacement within the element can be expressed in terms of the nodal displacements as

$$u_e(x,\omega)=\mathbf{N}^T(x,\omega)\hat{\mathbf{u}}_e(\omega) \tag{28}$$

where $\hat{\mathbf{u}}_e(\omega)\in\mathbb{C}^2$ is the nodal displacement vector. The dynamic stiffness matrix for the rod element is defined as

$$\mathbf{D}_e(\omega)=\mathbf{K}_e(\omega)-\omega^2\mathbf{M}_e(\omega) \tag{29}$$

Here the frequency-dependent stiffness and mass matrices are expressed as

$$\mathbf{K}_e(\omega)=EA\int_0^L\frac{d\mathbf{N}(x,\omega)}{dx}\frac{d\mathbf{N}^T(x,\omega)}{dx}dx \quad \text{and} \quad \mathbf{M}_e(\omega)=m\int_0^L\mathbf{N}(x,\omega)\mathbf{N}^T(x,\omega)\,dx \tag{30}$$

Using the expressions of the shape functions in Eq. (27), the dynamic stiffness matrix can be expressed as

$$\mathbf{D}_e(\omega)=EA\alpha\begin{bmatrix}\cot(\alpha L) & -\operatorname{cosec}(\alpha L)\\ \operatorname{cosec}(\alpha L) & \cot(\alpha L)\end{bmatrix} \tag{31}$$

The equation of dynamic equilibrium is therefore given by

$$\mathbf{D}_e(\omega)\hat{\mathbf{u}}_e(\omega)=\hat{\mathbf{f}}(\omega) \tag{32}$$

where $\hat{\mathbf{u}}_e(\omega)$ and $\hat{\mathbf{f}}(\omega)$ are respectively the frequency-dependent nodal displacement and forcing vectors. The dynamic finite element approach leads to an exact solution for the harmonically driven

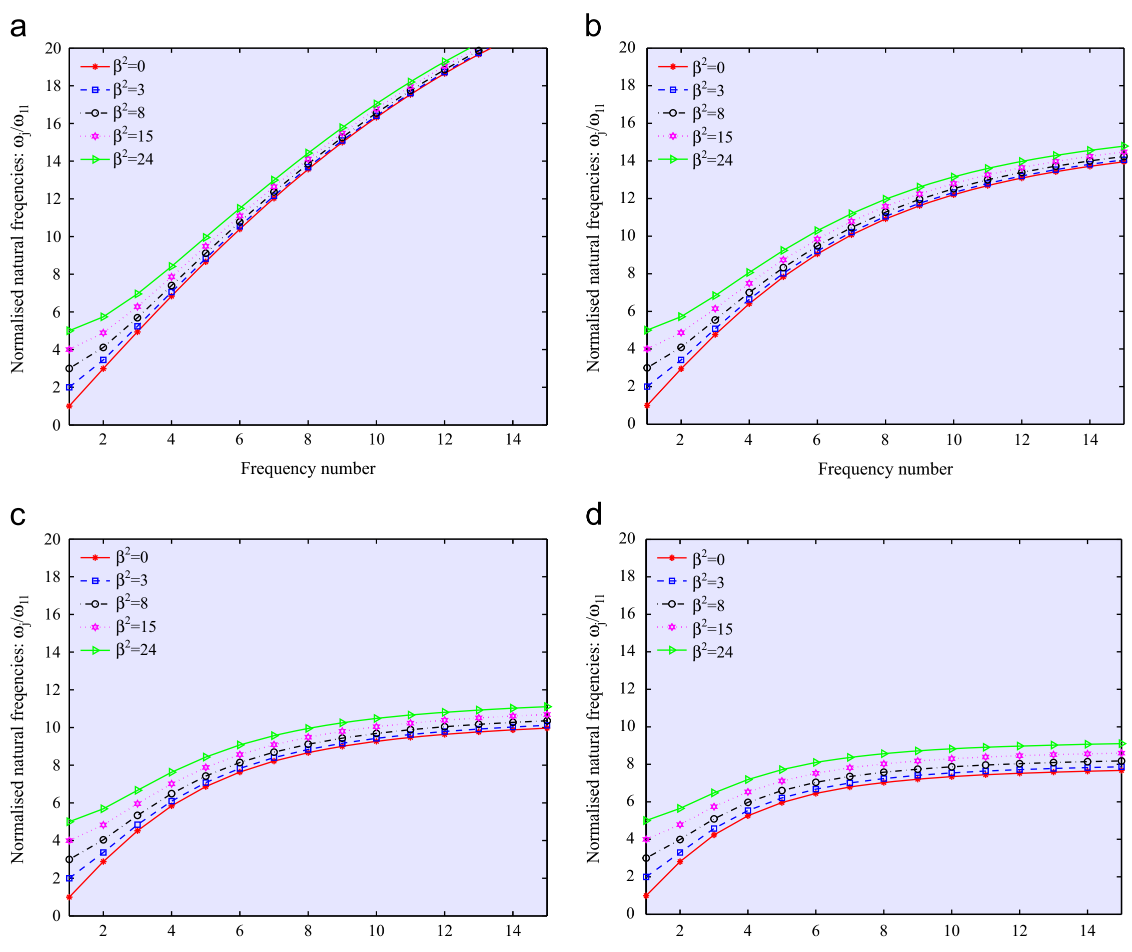

Fig. 3. Normalised natural frequency (ω_j/ω_{1l}) for different values of β for four different values of e_0a. The case of $\beta^2=0$ denotes that there is no elastic medium. (a) $e_0a=0.5$ nm. (b) $e_0a=1.0$ nm. (c) $e_0a=1.5$ nm and (d) $e_0a=2.0$ nm.

forced vibration problem. However, this approach is exact when the properties of the elastic medium and the rod do not change along the length. On the other hand, the nonlocal finite element method introduced earlier is approximate, but it can be used for the case when the system properties are varying along the length.

4. Axial vibration of a SWCNT embedded in an elastic medium

A clamped–free SWCNT embedded in an elastic medium is used for illustration. An armchair (5, 5) SWCNT with Young's modulus $E=6.85$ TPa, $L=25$ nm, density $\rho=9.517\times10^3$ kg/m^3 and thickness $t=0.08$ nm is considered as in Ref. [30]. Only mass proportional damping is considered so that the damping factor $\zeta_2=0.04$ and $\zeta_1=0$. We consider a range of values of e_0a within 0–2 nm to understand its role in the dynamic response. In order to understand the role of the elastic medium, we express ω_s, the elastic medium natural frequency, in terms of the first natural frequency in the local undamped system. The first natural frequency of the local system without the elastic medium is given by Eq. (19) as

$$\omega_{1l}=\frac{\pi}{2L}c \tag{33}$$

We define the non-dimensional elastic medium stiffness as

$$\beta^2=\frac{\omega_s^2}{\omega_{1l}^2}=\frac{4}{\pi^2}\frac{k}{(EA/L^2)} \tag{34}$$

This non-dimensional elastic medium stiffness is used as a variable quantity in the numerical studies to quantify its impact on the dynamics of the system.

4.1. Dynamic characteristics

Undamped natural frequencies and damped frequency response function of the SWCNT are considered. For convenience, the natural frequencies calculated are normalised by the first natural frequency of the local system without the elastic medium as given by Eq. (33). In Fig. 3, the first 15 normalised natural frequencies of the SWCNT are shown for five different values of β for four different values of e_0a. Note that $\beta^2=0$ implies the case when there is no elastic medium. The frequencies are higher for higher values of the elastic medium stiffness. The first natural frequency for $\beta^2=24$ is five times compared to the system without the elastic medium. However, the difference reduces for higher frequencies. When the value of e_0a is small, the frequency values increase sharply with the number of modes. The difference between frequencies with and without the elastic medium also reduces quickly compared the case when the values of e_0a are larger. For larger values of e_0a, as in Fig. 3(d), the frequency curves flatten significantly. This implies that the natural frequencies reach a terminal value as shown by the asymptotic analysis in Section 2. Using Eq. (23), for large values of k, the normalised natural frequency plotted in Fig. 3 would

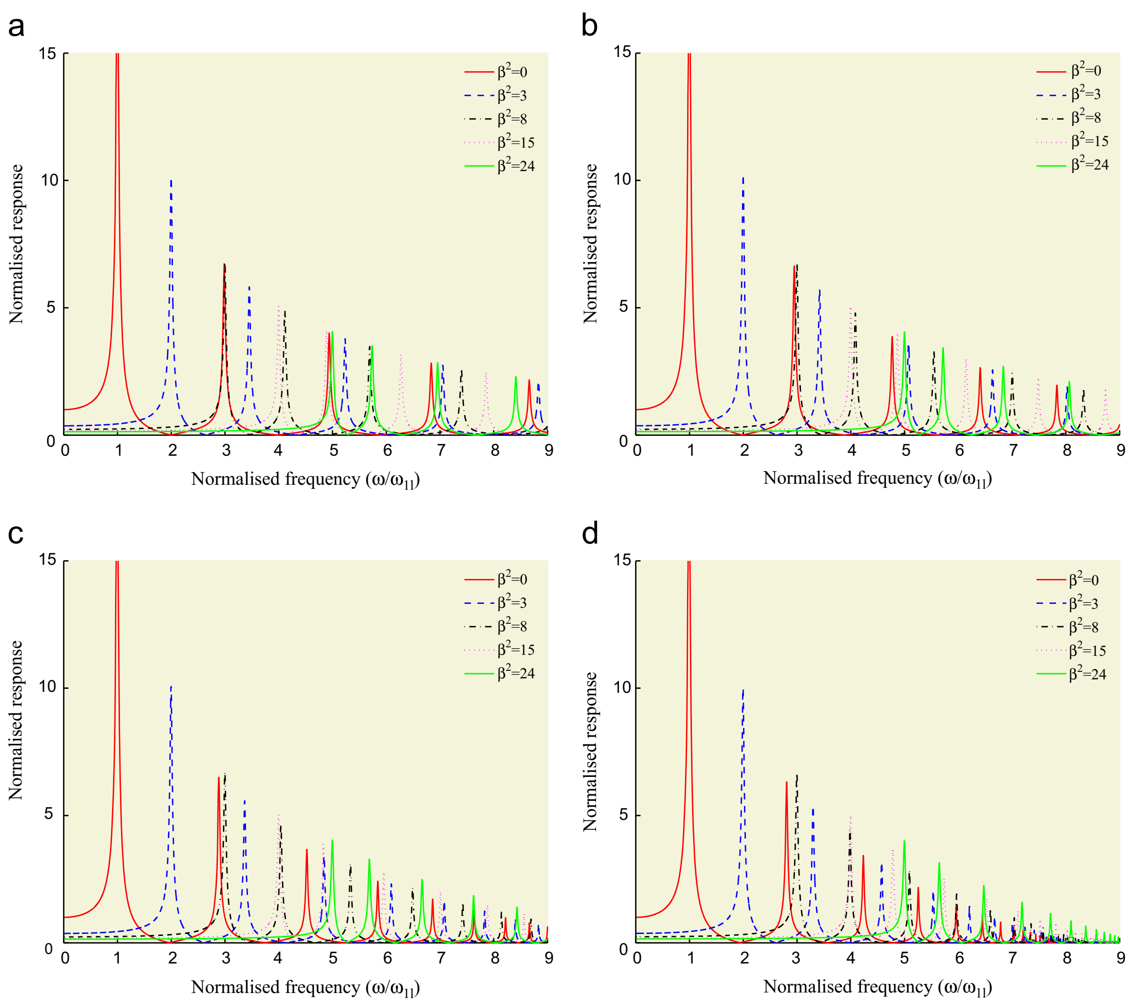

Fig. 4. Amplitude of the normalised dynamic frequency response at the tip for different values of β for four different values of e_0a. The case of $\beta^2=0$ denotes that there is no elastic medium. (a) $e_0a=0.5$ nm, (b) $e_0a=1.0$ nm, (c) $e_0a=1.5$ nm and (d) $e_0a=2.0$ nm.

approach to

$$\lim_{j\to\infty}\frac{\omega_j}{\omega_{1l}}=\frac{c}{\pi c/2L}\sqrt{1/(e_0a)^2+k/EA}=\sqrt{\left(\frac{2/\pi}{(e_0a/L)}\right)^2+\beta^2} \tag{35}$$

Clearly, the smaller the value of e_0a and larger the value of β^2, the larger this upper limit becomes. When $L=25$ nm, for $e_0a=2$ nm, and $\beta^2=24$ we have $\omega_{j_{\max}}\rightarrow\sqrt{(25/\pi)^2+24}=9.3448$. This value can be seen in top line in Fig. 3(d). All the other asymptotic normalised natural frequencies can be obtained from Eq. (35).

Dynamic frequency response is another quantity of interest focused in this paper. We assume that the input force has a unit harmonic excitation at the tip. Given this force we are interested in the dynamic response at the same point. Using the dynamic stiffness matrix, the equation of motion (32) is therefore

$$EA\alpha\begin{bmatrix}\cot(\alpha L) & -\operatorname{cosec}(\alpha L)\\ \operatorname{cosec}(\alpha L) & \cot(\alpha L)\end{bmatrix}\begin{Bmatrix}\widehat{u}_1(\omega)\\ \widehat{u}_2(\omega)\end{Bmatrix}=\begin{Bmatrix}0\\1\end{Bmatrix} \tag{36}$$

Since the left end is fixed, $\widehat{u}_1(\omega)=0$. Therefore solving the above equation we have $\widehat{u}_2(\omega)=1/EA\alpha\cot(\alpha L)=\tan(\alpha L)/EA\alpha$. The normalised displacement is defined as $\delta(\omega)=\widehat{u}_2(\omega)/u_{\text{static}}$, where u_{static} is the static response at the free edge given by $u_{\text{static}}=L/EA$. The normalised displacement can therefore be simplified as

$$\delta(\omega)=\frac{\tan(\alpha L)}{\alpha L} \tag{37}$$

In Fig. 4, the amplitude of this quantity is plotted for different values of β for four different values of e_0a. Note that $\beta^2=0$ implies the case when there is no elastic medium. The main impact of the elastic medium is the right shift in the resonance frequency and an increase in damping. For smaller values of e_0a and β, the asymptotic frequency is higher and consequently there are well separated peaks as seen in Fig. 4(a). The nature of the frequency response function changes significantly for higher values e_0a. In particular, when combined with higher values of β, we observe a remarkable clustering of resonance peaks in Fig. 4(d). This implies that natural frequencies of a nonlocal system are very closely spaced near the asymptotic frequency. For $e_0a=2$ nm, and $\beta^2=24$, the normalised frequency is 9.34, which is close the right end of Fig. 4(d). Another district feature that can be observed in Fig. 4 is the increase of the modal damping with increasing modes. The higher modes are effectively more damped.

4.2. Dynamic finite element versus nonlocal finite element analysis

We compare the results from the dynamic finite element and the nonlocal finite element method. The natural frequencies are obtained using the nonlocal finite element method developed in Section 3. By assembling the element stiffness and mass matrices given by Eqs. (25) and (26) and solving the resulting matrix eigenvalue problem $\mathbf{K}\boldsymbol{\phi}_j=\omega_j^2\mathbf{M}\boldsymbol{\phi}_j, j=1,2,\ldots$ one can obtain the both the eigenvalues and eigenvectors (denoted by $\boldsymbol{\phi}_j$ here). For the numerical calculation we used 200 elements. In Fig. 5, the first 15 natural frequencies obtained from the nonlocal finite element method are compared with the analytical expression in Eq. (22) for

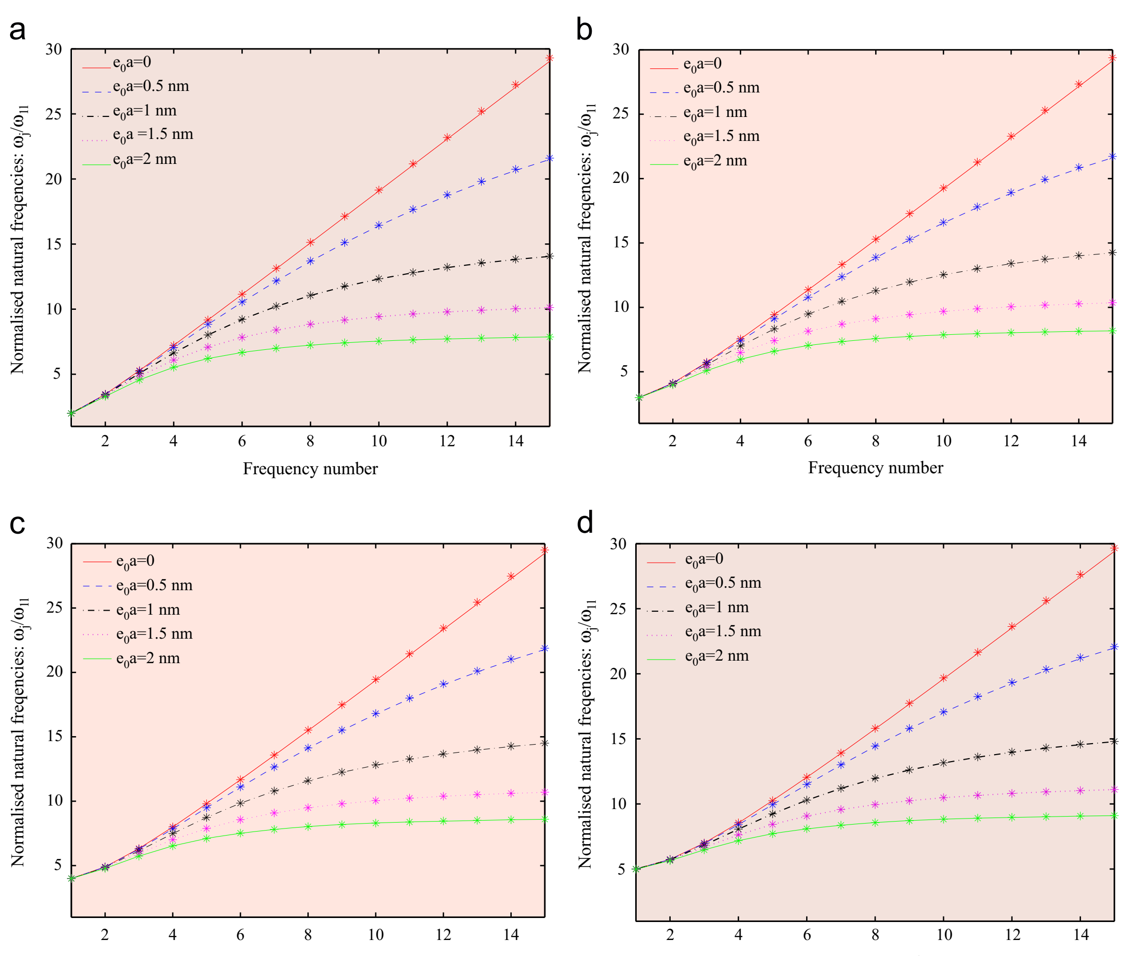

Fig. 5. Normalised natural frequency (ω_j/ω_{1l}) for different values of e_0a. Analytical results are compared with the nonlocal finite element (with 200 elements) results. The lines denote the analytical results and the star marks denote the finite element results. (a) $\beta^2=3$, (b) $\beta^2=8$, (c) $\beta^2=15$ and (d) $\beta^2=24$.

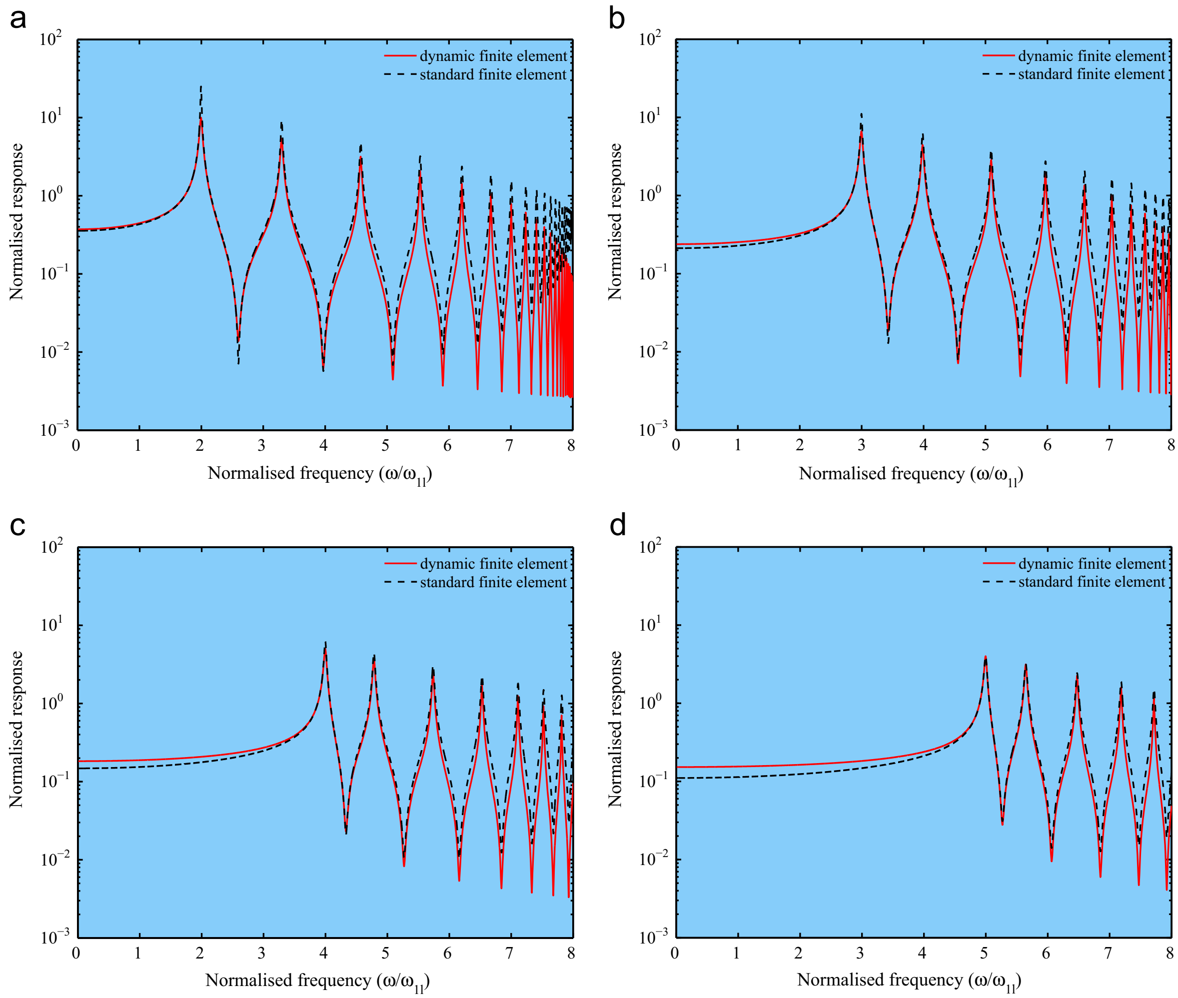

Fig. 6. Amplitude of the normalised dynamic frequency response at the tip for different values of β for a fixed value of $e_0a = 2.0$ nm. Dynamic finite element results (with one element) are compared with the conventional finite element results (with 200 elements). (a) $\beta^2 = 3$, (b) $\beta^2 = 8$, (c) $\beta^2 = 15$ and (d) $\beta^2 = 24$.

five values of e_0a within the range of 0.0–2.0 nm and four values of β^2 within the range of 3–24 nm. Note that $e_0a = 0$ implies the classical case of local elasticity model. The lines are obtained from the analytical expressions and the star marks are obtained from the finite element method. Both of the finite element approach proposed in this paper agree well with each other. However, the results become quite different for the dynamic response as shown in Fig. 6. Here four different values of β are selected while taking the highest value of $e_0a = 2.0$ nm. The highest value of e_0a is selected because this where we expect maximum discrepancy between the two methods [13]. In the numerical calculations, 10^5 points are used in the frequency axis. The frequency response functions from the nonlocal finite element were obtained using the classical modal series method [24]. For higher values of β, the results from the dynamic finite element and nonlocal finite element method agree well over the chosen range of normalised frequency of 0–8. This can be seen in Fig. 6(c,d). The discrepancies between the methods increase for lower values of β as seen in Fig. 6(a). From Eq. (35), the asymptotic frequency for this case can be obtained as $\omega_{j_{\max}} \to \sqrt{(25/\pi)^2 + 3} = 8.1441$. The reason for the discrepancy in Fig. 6(a) is that the excitation frequency is close to this asymptotic frequency. The results from the dynamic finite element approach are exact as it does not suffer from error arising due to finite element discretisation. In this case increasing numbers of natural frequencies lie within a given frequency range. As a result a very fine mesh is necessary to capture the high number of modes. The proposed dynamic finite element is effective in these situations as it does not suffer from discretisation errors as in the conventional finite element method. The results in Fig. 6 show that the nonlocal finite element can be used with confidence when the excitation frequency is below the asymptotic frequency.

5. Conclusions

In this paper a novel dynamic finite element approach for axial vibration of damped nonlocal rods embedded in an elastic medium is proposed. Strain rate dependent viscous damping and velocity dependent viscous damping are considered. Damped and undamped natural frequencies for general boundary conditions are derived. An asymptotic analysis is used to understand the behaviour of the frequencies in the high-frequency limit as function of the nonlocal parameter and elastic medium stiffness. The dynamic response in the frequency domain can be obtained by inverting the dynamic stiffness matrix derived in the paper. The stiffness and mass matrices of the nonlocal rod were also obtained using the nonlocal finite element method. In the special case when the nonlocal parameter becomes zero, the expressions of the mass and stiffness matrices reduce to the classical case. The proposed method is numerically applied to the axial vibration of a (5,5) carbon nanotube. Some of the key findings in this study are:

- Nonlocal rods embedded in an elastic medium have an upper cut-off natural frequency given by $\sqrt{EA/m(e_0a)^2+k/m}$. This is independent on the boundary conditions and the length of the rod.
- In the context of finite element method, in contrast to the system without embedded in an elastic medium, the nonlocal parameter impacts both the mass and stiffness matrices.
- Compared to a local system without embedded in an elastic medium, the natural frequency increases with increasing elastic medium stiffness and decreases with the nonlocal parameter.
- For both local and nonlocal systems, the difference in the natural frequencies arising due to the elastic medium decreases with increasing mode numbers.

The natural frequencies and the dynamic response obtained using the nonlocal finite element approach were compared with the results obtained using the dynamic finite method. Good agreement between the two methods was found for large values of the elastic medium stiffness. Away from the upper cut off frequency, the proposed nonlocal finite element approach is applicable, while the dynamic finite element method is applicable for the complete range of excitation frequency.

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