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The 'damping effect' in the dynamic response of stochastic oscillators

Sondipon Adhikari *, Blanca Pascual

College of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, United Kingdom

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ABSTRACT

We consider the dynamics of linear damped oscillators with stochastically perturbed natural frequencies. When average dynamic response is considered, it is observed that stochastic perturbation in the natural frequency manifests as an increase of the effective damping of the system. Assuming uniform distribution of the natural frequency, a closed-from expression of equivalent damping for the mean response has been derived to explain the 'increasing damping' behaviour. In addition to this qualitative analysis, a comprehensive quantitative analysis is proposed to calculate the statistics of frequency response functions from the probability density functions of the natural frequencies. Firstly, single-degree-of-freedom-systems are considered and closed-form analytical expressions for the mean and variance are obtained using a hybrid Laplace's method. Several probability density functions, including gamma, normal and lognormal distributions, are considered for the derivation of the analytical expressions. The method is extended to calculate the mean and the variance of the frequency response function of multiple-degrees-of-freedom dynamic systems. Proportional damping is assumed and the eigenvalues are considered to be independent. Results are derived for several probability density functions and damping factors. The accuracy of the approach for both single and multiple-degrees-of-freedom systems is examined using the direct Monte Carlo simulation.

1. Introduction

Damped linear oscillators have been used to model a range of physical problems across different length and time scales, and disciplines including engineering, biology and nanotechnology. Examples include nanoscale oscillators used as ultra sensitive sensors [1], vibration of buildings and bridges under earthquake loads, vibration of automobiles and aircrafts. The equation of motion of a damped oscillator can be expressed as

$$m\ddot{u}(t) + c\dot{u}(\tau) + ku(t) = f(t) \quad (1)$$

where t, $u(t)$, m, c, k are respectively the time, displacement, mass, damping, stiffness and applied forcing. Diving by m, this equation can be expressed as

$$\ddot{u}(t) + 2\zeta_n\omega_n\dot{u}(\tau) + \omega_n^2u(t) = f(t)/m \quad (2)$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency and $\zeta_n = c/2\sqrt{km}$ is the damping ratio. A rich body of literature on random vibration [2,3] is available for the case when the forcing function is random in nature. We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.

Uncertainty in the natural frequency can arise in uncertainties in the stiffness or inertia properties of the structure. These can be attributed to stochastic parametric variation in the Young's modulus, Poisson's ratio, density, or geometry of the system. In general, stochastic finite element based methods (for example, [4–8]) are well suited to deal with problems with random (distributed) parameters. For a single degree of freedom (SDOF) system, the dynamic response due to uncertainties in the natural frequency can be easily obtained using Monte Carlo simulation. Such an approach, however, may not shed light into the nature of the response statistics to be discussed in the paper. The use of reduced computational methods such as perturbation method or polynomial chaos [9] works well in general except when response near the resonance frequency is considered [10]. From an engineering point of view, this is exactly where a reliable estimate of dynamic response is necessary as this is crucial to safe design of dynamic structures.

This paper gives an explanation as to why mean based analytical approximations (e.g., perturbation, polynomial chaos) fail to provide accurate statistical description of the dynamics response near the resonance frequency of a damped system. In Section 2 some simulation results are provided as the motivation of this study. Based on this, few key observations are made and an explanation based on the mean response for the case of uniform distribution of the natural frequency is provided in Section 3. A

* Corresponding author. Fax: + 44 1792 295676.
E-mail address: S.Adhikari@swansea.ac.uk (S. Adhikari).
URL: http://engweb.swan.ac.uk/~adhikaris (S. Adhikari).

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quantitative analytical approach for dynamic response statistics of single-degree-of-freedom (SDOF) systems is presented in Section 4. The calculation of the probability density function (pdf) of the response is outlined in Section 4.1 and the expressions for the mean and standard deviation are derived in Section 4.2. These expressions depend on the calculation of three integrals, which are evaluated through Laplace's method and through a proposed modified Laplace's method in Sections 4.2.1 and 4.2.2. Exact expressions of the mean and standard deviation are obtained for the uniform distribution of eigenvalues in Section 5.1. Laplace's method and modified Laplace's method are developed for normal, gamma and lognormal distributions respectively in Sections 5.2, 5.3 and 5.4. The method is extended to obtain mean and standard deviation of the response for multiple-degree-of-freedom (MDOF) systems in Section 6. A numerical example for a MDOF system is shown in Section 6.4, where the proposed methods are compared to MCS. The main results and the key conclusions arising from this study are discussed in Sections 7 and 8.

2. Dynamic response of damped stochastic oscillators

2.1. Uncertainty model

Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 x$, where ω_{n_0} is the deterministic frequency and x is a random variable with a given probability distribution function. We assume that the mean of x is 1 and the standard deviation is σ. Stochastic perturbation of this kind can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency. Of course in the special case when the standard deviation of the random variable is close to zero, the stochastic oscillator approaches the classical deterministic oscillator. For initial simulation results, three different types of random variables, namely uniform, normal and lognormal, are considered as shown in Fig. 1.

Note that normal random variable is not a good choice for a positive quantity as the squared natural frequency. It is kept here only for comparing the results later.

2.2. Dynamic response in the time and frequency domain

Dynamic response of a SDOF system with initial displacement u_0 and initial velocity v_0 can be obtained [11] using

$$u(t) = A\, e^{\zeta_n \omega_n t} \sin(\omega_d t + \phi) \tag{3}$$

where the $\omega_d = \omega_n \sqrt{1 - \zeta_n^2}$ is the damped natural frequency and the amplitude and the phase of the response are

$$A = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta_n \omega_n u_0}{\omega_d} \right)^2} \quad \text{and} \quad \phi = \tan^{-1} \frac{u_0 \omega_d}{v_0 + \zeta_n \omega_n u_0}. \tag{4}$$

In Fig. 2 we show the deterministic and mean response of the oscillator due to an initial displacement. The time axis is scaled with the deterministic time period $T_{n0} = 2\pi/\omega_{n0}$ so that the results become general. A representative damping factor of 5%, three types of random variables and two values of standard deviations are used for illustration. Deterministic response, sample responses of the random system (with uniform distribution) and mean response due to the three cases with random natural frequencies are shown in the figure. The mean response is significantly 'damped' compared to the deterministic response. Additionally, the 'damping effect' is almost independent to the nature of the statistical distribution of the natural frequencies.

The normalised steady-state response amplitude in the frequency domain of an SDOF oscillator can be expressed as

$$\left| \frac{u}{u_{st}} \right| = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta_n r)^2}}. \tag{5}$$

Here the static deformation $u_{st} = F/k$ where F is the amplitude of the harmonic excitation and the frequency ratio $r = \omega/\omega_{n0}$. In Fig. 3, the dynamic response of the deterministic system and the mean responses due to three cases with random natural frequencies are shown. The frequency axis is scaled with the deterministic frequency ω_{n0} for generality. Like the time-domain response, we observe that the mean response is significantly more damped compared to the deterministic response. Although the mean responses for different pdfs of ${\omega_n}^2$ are slightly different, the predominant feature (i.e., the 'damping effect') is mainly depended on the standard deviation of the random variable. The observations in these results can be summarised as:

- The mean response of a SDOF oscillator with random natural frequency is more damped compared to the underlying deterministic response.
- The higher the randomness (standard deviation), the higher the 'effective damping'.

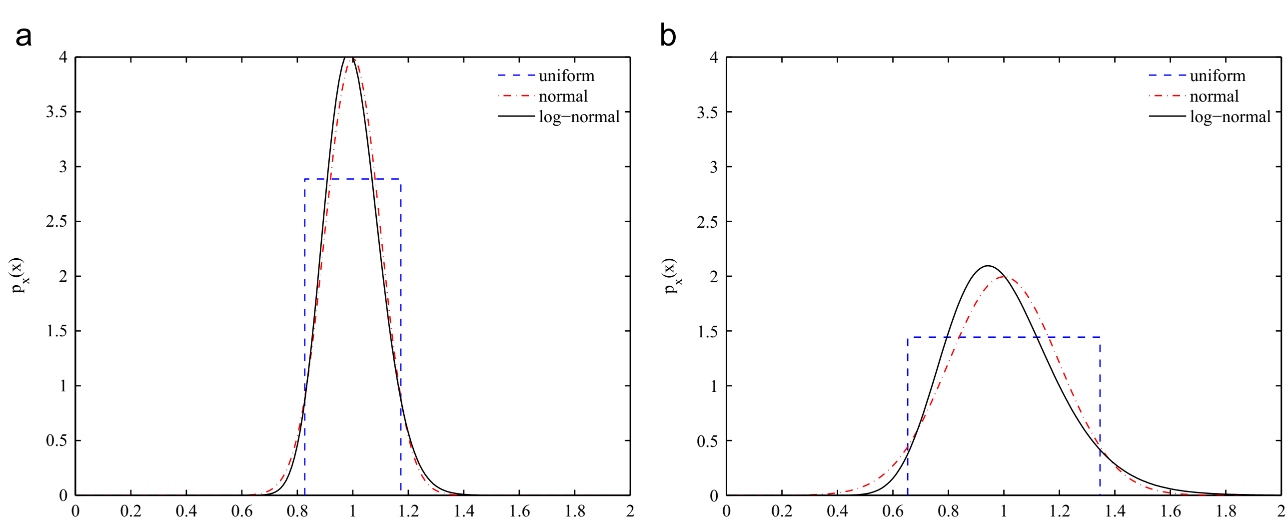

Fig. 1. Assumed probability density functions of the squared natural frequency $\omega_n^2 = \omega_{n0}^2 x$. We consider that the mean of x is 1 and the standard deviation is σ_a. (a) Pdf: $\sigma_a = 0.1$. (b) Pdf: $\sigma_a = 0.2$.

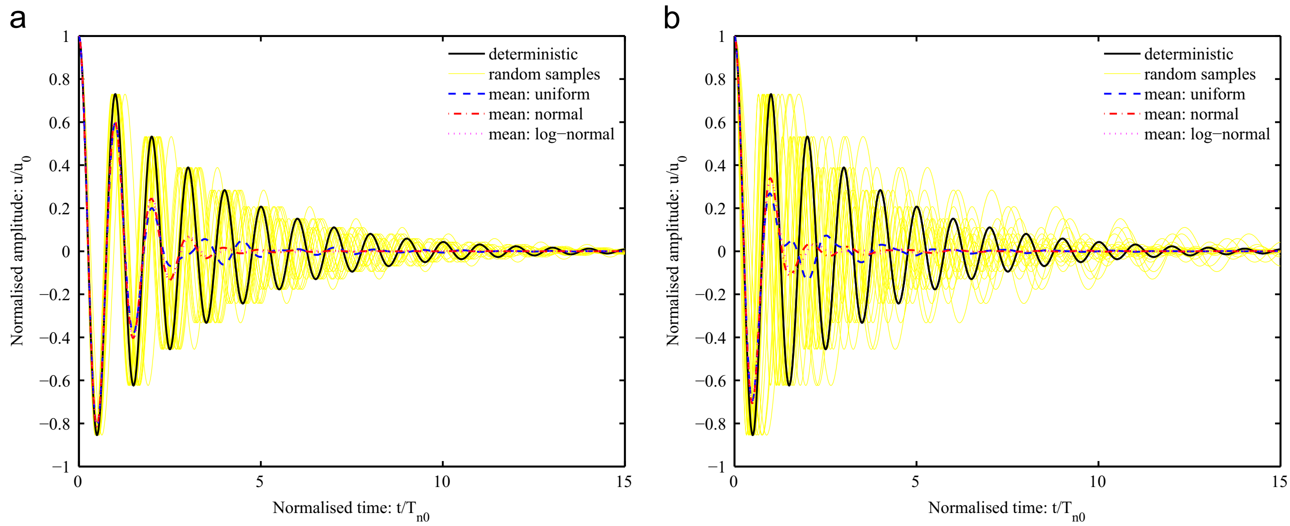

Fig. 2. Response in the time domain due to initial displacement u_0 with 5% damping ($v_0 = 0$). (a) Response: $\sigma_a = 0.1$. (b) Response: $\sigma_a = 0.2$.

- The qualitative features are almost independent of the distribution the random natural frequency.

Assuming uniform random variable for ${\omega_n}^2$, we aim to explain these observations in the next section.

3. Equivalent damping for the mean response

To explain the main observations reported in the previous section, we consider the amplitude-square of the normalised frequency response function. This is employed to simplify the analytical calculations by avoiding the square-root function in the denominator. Without any loss of generality, assume that the random natural frequencies are expressed as

$$\omega_n^2 = \omega_{n_0}^2(1 + \epsilon x) \tag{6}$$

where x has zero mean and unit standard deviation. The normalised harmonic response in the frequency domain can be expressed as

$$\frac{u}{f/k} = \frac{k/m}{[-\omega^2 + \omega_{n_0}^2(1 + \epsilon x)] + 2\mathrm{i}\zeta_n\omega\omega_{n_0}\sqrt{1 + \epsilon x}} \tag{7}$$

Considering $\omega_{n_0} = \sqrt{k/m}$ and the frequency ratio $r = \omega/\omega_{n_0}$ we have

$$\frac{u}{f/k} = \frac{1}{[(1 + \epsilon x) - r^2] + 2\mathrm{i}\zeta_n r\sqrt{1 + \epsilon x}}. \tag{8}$$

The squared-amplitude of the normalised dynamic response at $\omega = \omega_{n_0}$ (that is $r=1$) can be obtained as

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 = \frac{1}{\epsilon^2x^2 + 4\zeta_n^2(1 + \epsilon x)}. \tag{9}$$

Since x is an uniform random variable with zero mean and unit standard deviation, its pdf is given by

$$p_x(x) = 1/2\sqrt{3}, \quad -\sqrt{3} \le x \le \sqrt{3}. \tag{10}$$

The use of uniform random variable is justified due to the fact that the qualitative feature of the mean response is independent of the distribution of the natural frequency (refer Figs. 2 and 3). The mean response is obtained using the following integral:

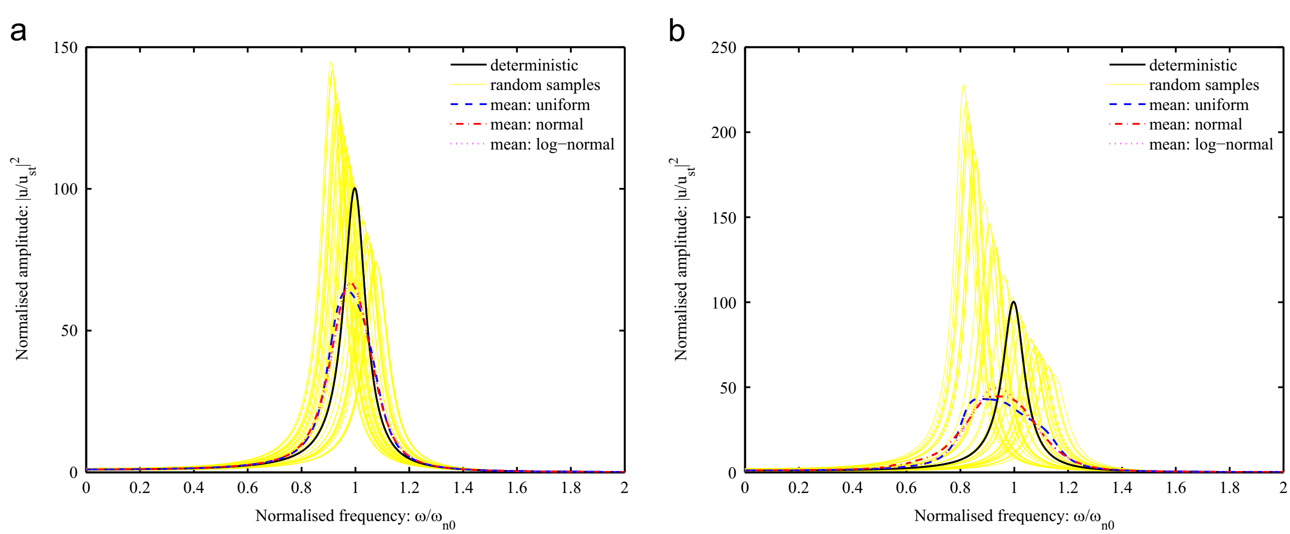

Fig. 3. Amplitude square of the normalised frequency response function with 5% damping. (a) Response: $\sigma_a = 0.1$. (b) Response: $\sigma_a = 0.2$.

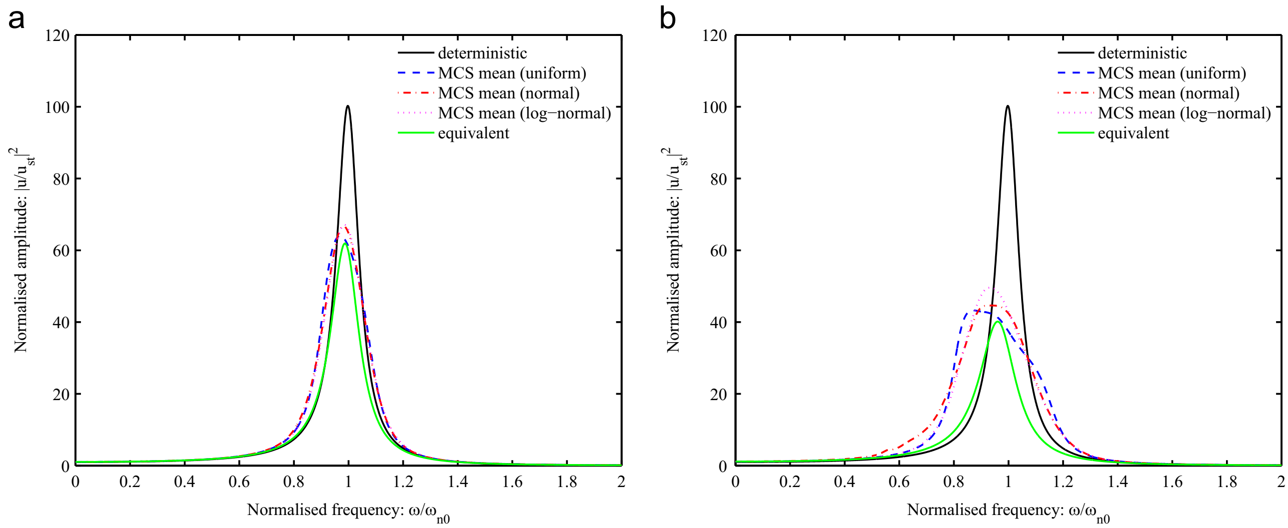

Fig. 4. Normalised frequency response function with equivalent damping ($\zeta_e = 0.05$ in the ensembles) when $\omega_n{}^2$ has the uniform distribution. For the two cases $\zeta_e = 0.0643$ and $\zeta_e = 0.0819$. (a) Response: $\sigma_a = 0.1$. (b) Response: $\sigma_a = 0.2$.

$$\begin{aligned}\mathrm{E}[\hat{U}] &= \int \frac{1}{\epsilon^2 x^2 + 4\zeta_n^2(1+\epsilon x)} p_x(x)\,\mathrm{d}x \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{2\sqrt{3}} \frac{1}{\epsilon^2 x^2 + 4\zeta_n^2(1+\epsilon x)}\,\mathrm{d}x = \frac{1}{4\sqrt{3}\,\epsilon\zeta_n\sqrt{1-\zeta_n^2}} \\ &\quad \tan^{-1}\left(\frac{\sqrt{3}\,\epsilon}{2\zeta_n\sqrt{1-\zeta_n^2}} - \frac{\zeta_n}{\sqrt{1-\zeta_n^2}}\right) \\ &\quad + \frac{1}{4\sqrt{3}\,\epsilon\zeta_n\sqrt{1-\zeta_n^2}} \tan^{-1}\left(\frac{\sqrt{3}\,\epsilon}{2\zeta_n\sqrt{1-\zeta_n^2}} + \frac{\zeta_n}{\sqrt{1-\zeta_n^2}}\right).\end{aligned} \tag{11}$$

Using the Taylor's series expansion of $\tan^{-1}(\bullet)$ function we can establish that

$$\tfrac{1}{2}\{\tan^{-1}(a+\delta) + \tan^{-1}(a-\delta)\} = \tan^{-1}(a) + O(\delta^2). \tag{12}$$

This implies that the linear term in δ disappears in the above expression. Therefore, provided there is a small δ, the above approximation can be exploited to simplify the expression of the mean as

$$\mathrm{E}[\hat{U}] \approx \frac{1}{2\sqrt{3}\,\epsilon\zeta_n\sqrt{1-\zeta_n^2}} \tan^{-1}\left(\frac{\sqrt{3}\,\epsilon}{2\zeta_n\sqrt{1-\zeta_n^2}}\right) + O(\zeta_n^2). \tag{13}$$

Considering light damping (that is, $\zeta^2 \lll 1$), the validity of this approximation relies on the following inequality:

$$\frac{\sqrt{3}\,\epsilon}{2\zeta_n} \ggg \zeta_n^2 \quad \text{or} \quad \epsilon \ggg \frac{2}{\sqrt{3}}\zeta_n^3. \tag{14}$$

Since damping is usually quite small ($\zeta_n < 0.2$), the above inequality will normally hold even for systems with very small uncertainty. To give an example, for $\zeta_n = 0.2$, which is quite high damping in practice, from the above equation we get $\epsilon_{\min} = 0.0092$, which is less than 0.1% randomness. In practice we will be interested in randomness of more than 0.1% and consequently the criteria in Eq. (14) will be met. Therefore, the conditions for the proposed approximation lie within the usual engineering limits of practical applications.

Considering light damping, Eq. (13) can be further simplified as

$$\mathrm{E}[\hat{U}] \approx \frac{\tan^{-1}(\sqrt{3}\,\epsilon/2\zeta_n)}{2\sqrt{3}\,\epsilon\zeta_n}. \tag{15}$$

Now consider that we want to 'replace' the actual oscillator with an oscillator with equivalent damping factor ζ_e such that its mean response at the deterministic natural frequency matches with the analytical expression derived here. For small damping, the maximum deterministic amplitude at $\omega = \omega_{n_0}$ is $1/4\zeta_e^2$. Therefore, the equivalent damping for the mean response is given by

$$(2\zeta_e)^2 = \frac{2\sqrt{3}\,\epsilon\zeta_n\sqrt{1-\zeta_n^2}}{\tan^{-1}(\sqrt{3}\,\epsilon/2\zeta_n\sqrt{1-\zeta_n^2})} \approx \frac{2\sqrt{3}\,\epsilon\zeta_n}{\tan^{-1}(\sqrt{3}\,\epsilon/2\zeta_n)}. \tag{16}$$

This expression of $(2\zeta_e)^2$ can be expanded by a Taylor series in ζ_n as

$$(2\zeta_e)^2 = 4\frac{\zeta_n\epsilon\sqrt{3}}{\pi} + 16\frac{\zeta_n^2}{\pi^2} + 2\left(-\frac{\epsilon}{\pi} + \frac{32}{3}\frac{1}{\epsilon\pi^3}\right)\sqrt{3}\,\zeta_n^3 + O(\zeta_n^4). \tag{17}$$

For the case when damping is small compared to the degree of randomness, neglecting all terms $O(\zeta_n^2)$, we have

$$\zeta_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\zeta_n}. \tag{18}$$

This implies that, under the approximations sated before, the effective damping for the mean response of a stochastic oscillatory system is approximately proportional to the square root of the damping of the baseline model. As we assumed $\zeta_n < 1$, this implies that effective damping increases as observed in the results presented in the previous section. The analytical expression shows that the 'effective damping' increases with increasing randomness of the natural frequency, an observation also made in the numerical results in the previous section.

In order to verify the expression of the equivalent damping derived here, we look into the results in the previous section again. In Fig. 4, the mean obtained using MCS is compared with the response of an equivalent oscillator with damping ζ_e in Eq. (16). It is observed that the simple closed-form expression in Eq. (16) captures the essential feature of the mean dynamic response.

The analysis carried out in this section qualitatively explain the nature of dynamic response of damped stochastic oscillators. As the 'effective damping' of the mean response is significantly more compared to the baseline model, this also explains as to why approximate numerical methods based on mean properties fail near the resonance. Based on this qualitative analysis, in the rest of the paper a semi-analytical quantitative approach for dynamic response statistics is proposed.

4. Analytical approach for dynamic response of single-degree-of-freedom (SDOF) systems

Single-degree-of-freedom (SDOF) system is the simplest way to model a structural dynamic system. The study of SDOF systems is often undertaken prior to the study of multiple-degrees-of-freedom (MDOF) systems due to physical insights and analytical conveniences. Solutions of an SDOF problem are easily obtained in comparison to MDOF problems. They are also useful when investigating the response of general MDOF problems, as MDOF problems with proportional damping reduce to a linear combination of SDOF problems. From the results in the last two sections, it is known that approximate analysis (e.g, perturbation, polynomial-chaos) based on the properties of underlying deterministic system will not work very well for dynamic response near the resonance frequency. This is due to the fact that effective damping of the mean response is significantly lower that the baseline damping. Based on this insight, an analytical approach based on asymptotic integral is proposed to obtain dynamic response statistics.

For an SDOF system, the frequency response function (also known as the transfer function) is given by

$$h(\mathrm{i}\omega) = \frac{1}{(-\omega^2 + \mathrm{i}2\omega_n\zeta_n\omega + \omega_n^2)} \tag{19}$$

where ω_n and ζ_n are respectively the natural frequency and the damping factor, and ω is the frequency. The transfer function is a complex number for every nonzero ω. Its moments can be related to the moments of its real and imaginary parts and its absolute value. The real and imaginary parts of the transfer function and its absolute value are respectively given by

$$\Re\big(h(\omega, \omega_n)\big) = \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2}, \tag{20}$$

$$\Im\big(h(\omega, \omega_n)\big) = \frac{-2\zeta_n\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2} \tag{21}$$

$$\text{and} \quad |h(\omega, \omega_n)| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta_n^2\omega_n^2\omega^2}}. \tag{22}$$

In this set of equations, the squared natural frequency of the system ${\omega_n}^2$ is assumed to be a random variable.

4.1. The probability density function of the response

Probability density functions of the real part, the imaginary part and the absolute value of the response can be found analytically if these quantities are considered as functions of one random variable, i.e. $x = \omega_n^2$. These functions (denoted by z) have to satisfy that its domain includes the range of ${\omega_n}^2$, z are Borel functions and $z(x, \omega) = \pm\infty$ has zero probability. All these conditions are ratified if $\zeta_n \neq 0$ and $\omega_n^2 = 0$ has zero probability. This is a physically realistic situation as it points to a damped system with positive natural frequency. If the new random variables $\Re(h(x, \omega))$, $\Im(h(x, \omega))$ and $|h(x, \omega)|$ are denoted by z, their pdf (denoted by f_z) can be derived using the transformation of random variables (see for example [12])

$$z = z(x_1) = \cdots = z(x_n) = \cdots \tag{23}$$

$$f_z(z) = \frac{f_x(x_1)}{|z'(x_1)|} + \cdots + \frac{f_x(x_n)}{|z'(x_n)|} + \cdots \tag{24}$$

Here x_n are the real roots of $z = z(x)$, and $z'(x_n)$ is the derivative of $z(x)$ evaluated at x_n. The pdf of the real and imaginary parts of the transfer function and its absolute value are derived below.

To obtain the pdf or the real part of the transfer function from (24), the roots of (23) with $z = \Re(h(x, \omega))$ have to be obtained. The real part of the transfer function and its derivative with respect to x are given by

$$z = \frac{x - \omega^2}{(x - \omega^2)^2 + 4\zeta_n^2 x\omega^2}, \quad z'(x) = \frac{-(x - \omega^2)^2 + 4\zeta_n^2 x\omega^2}{((x - \omega^2)^2 + 4\zeta_n^2 x\omega^2)^2}. \tag{25}$$

The roots of $z = \Re(h(x, \omega))$ are the roots of a second order polynomial, and are given by

$$x_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \quad x_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{26}$$

with

$$A = z \tag{27}$$

$$B = z2\omega^2(2\zeta_n^2 - 1) - 1 \tag{28}$$

$$C = z\omega^4 + \omega^2. \tag{29}$$

The pdf of the real part of the transfer function is obtained by introducing $z'(x_i)$ and $f(x_i)$ in (24), that is $f_{\Re}(z) = (f_x(x_1))/|z'(x_1)| + (f_x(x_2))/|z'(x_2)|$.

For the imaginary part of response ($z = \Im(h(x, \omega))$) and Eq. (23) reduces to

$$z = \frac{-2\zeta_n\sqrt{x}\omega}{(x - \omega^2)^2 + 4\zeta_n^2 x\omega^2}. \tag{30}$$

From this equation, it can be derived that x_n are the roots of a third order polynomial

$$x_n^3 z^2 + x_n^2 4z^2\omega^2(2\zeta_n^2 - 1) + x_n 6\omega^4 z^2 + (4z^2\omega^6(2\zeta_n^2 - 1) + 4\zeta_n^2\omega^2) = 0. \tag{31}$$

The first derivative of z with respect to x is given by

$$z'(x) = -2\zeta_n\omega\frac{\frac{1}{2}x^{-1/2}((x - \omega^2)^2 + 4\zeta_n^2 x\omega^2) - (2(x - \omega^2) + 4\zeta_n^2\omega^2)x^{1/2}}{((x - \omega^2)^2 + 4\zeta_n^2 x\omega^2)^2} \tag{32}$$

The pdf of the imaginary part of the transfer function is obtained by introducing $z'(x_n)$ and $f(x_n)$ into (24).

The absolute value of response and its derivative with respect to x are given by

$$z = \frac{1}{\sqrt{(x - \omega^2)^2 + 4\zeta_n^2 x\omega^2}} \tag{33}$$

$$z'(x) = -\frac{(2(x - \omega^2) + 4\zeta_n^2\omega^2)((x - \omega^2)^2 + 4\zeta_n^2 x\omega^2)^{-1/2}}{(x - \omega^2)^2 + 4\zeta_n^2 x\omega^2} \tag{34}$$

and the roots of $z = |h(x, \omega)|$ are the roots of a second order polynomial, and are given by

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{35}$$

with

$$A = z \tag{36}$$

$$B = z^2 2\omega^2(2\zeta_n^2 - 1) \tag{37}$$

$$C = -z^2\omega^4 - 1. \tag{38}$$

The pdf of the absolute value of the transfer function is obtained by introducing $z'(x)$ and $f(x_n)$ into (24).

The derivation of the pdf of the response is not straightforward even for single-degree-of-freedom system. For multiple-degrees-of-freedom systems, the problem becomes even more difficult, and, generally, not analytically solvable.

4.2. Response statistics

It has been seen that the calculation of the pdf of the quantities of interest (i.e. real and imaginary parts of the FRF and its absolute value) is not straightforward. Therefore, we focus on the calculation of their moments. As formerly, a probability density function $f_x(x)$ is assumed for the random variable $x = \omega_n^2$, where $f_x(x \le 0) = 0$. The first moment (mean) of the real part, imaginary part and the absolute value of the transfer function, derived from Eqs. (20)–(22), are given by

$$\mu_{\Re(h)} = \mathrm{E}\left[\Re(h)\right] = \int_{\mathcal{D}} \frac{x f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx - \omega^2 \int_{\mathcal{D}} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx \tag{39}$$

$$\mu_{\Im(h)} = \mathrm{E}\left[\Im(h)\right] = -2\zeta_n\omega \int_{\mathcal{D}} \frac{\sqrt{x} f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx \tag{40}$$

$$\mathrm{E}\left[|h|\right] = \sigma_h^2 + \mu_h^2 = \int_{\mathcal{D}} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx \tag{41}$$

here $\mathcal{D}$ is the domain of x, $\mu_h = \mu_{\Re(h)} + \mathrm{i}\mu_{\Im(h)}$ is the mean of the transfer function and σ_h is its standard deviation. The square of the mean is obtained by multiplying the mean by its complex conjugate, so that

$$\mu_h^2 = \mu_{\Re(h)}^2 + \mu_{\Im(h)}^2. \tag{42}$$

In the next section, integrals appearing in Eqs. (39)–(41) are approximated with Laplace's method.

4.2.1. Laplace's method

Integrals appearing in Eqs. (39)–(41) can be related to Laplace's integral (see, for example, [13])

$$I(w) = \int_{x_1}^{x_2} g(x) e^{wy(x)}\,dx. \tag{43}$$

It is known that if w is a large positive number, the major contribution to the integral comes from the vicinity of points at which $y(x)$ assumes its largest value. If g is continuous, y is twice continuously differentiable and $y'(\theta) = 0$, $y''(\theta) < 0$, this integral can be approximated with Laplace's method as

$$I(w) \backsim g(\theta) e^{wy(\theta)} \left[\frac{-2\pi}{wy''(\theta)}\right]^{1/2}. \tag{44}$$

We will assume $g(x) = 1$ and $w = 1$. Then

$$I(w) \backsim \sqrt{2\pi} e^{y(\theta)} \left[-y''(\theta)\right]^{-1/2}. \tag{45}$$

A general function $y(x, a)$, obtained from integrals appearing in Eqs. (39)–(41), can be given by

$$y(x, a) = \ln(f_x(x)) + a\ln(x) - \ln(|h(\omega, x)|^2) \tag{46}$$

where

$$a = 0 \quad \text{for} \int_{\mathcal{D}} \frac{f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx \tag{47}$$

$$a = \frac{1}{2} \quad \text{for} \int_{\mathcal{D}} \frac{\sqrt{x} f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx \tag{48}$$

$$a = 1 \quad \text{for} \int_{\mathcal{D}} \frac{x f_x(x)}{(x-\omega^2)^2 + 4\zeta_n^2\omega^2 x}\,dx. \tag{49}$$

Function $y(x, a)$ is maximum at $x = \theta_a$. Therefore, θ_a is the solution of

$$\frac{f_x'(\theta_a)}{f_x(\theta_a)} + \frac{a}{\theta_a} - \frac{2(\theta_a - \omega^2) + 4\omega^2\zeta_n^2}{(\theta_a - \omega^2)^2 + 4\zeta_n^2\omega^2\theta_a} = 0 \tag{50}$$

for which $y(\theta_a, a)$ is the absolute maximum. The second derivative of $y(x, a)$ is

$$y''(x, a) = \bar{f}(x) - \frac{a}{x^2} - \bar{h}(x) \tag{51}$$

where

$$\bar{h}(x, \omega) = \frac{2}{(-\omega^2 + x)^2 + 4\zeta_n^2\omega^2 x} - \left(\frac{2(-\omega^2 + x) + 4\zeta_n^2\omega^2}{(-\omega^2 + x)^2 + 4\zeta_n^2\omega^2 x}\right)^2 \tag{52}$$

and $\bar{f}(x)$ depends on the pdf of the random variable x. Laplace's method will be applied for different pdfs in the remaining of the paper.

4.2.2. Hybrid Laplace-numerical integration and modified Laplace

Laplace's method approximates (46) with a second-degree polynomial given by the first two terms of Taylor expansion of $y(x)$ around its maximum θ_a. The method works well if the behaviour of $y(x)$ in the vicinity of its absolute maximum point θ_a is well represented by the approximation used. From (46) it may be observed that $y(x)$ is obtained through the addition of three functions. The first one, $\ln(f_x(x))$, is related to the pdf of the distribution of the random variable x. This pdf is chosen such that it has only one maximum, situated at $x = \mu$, with μ being the mean of the distribution. It is expected that $\ln(f_x(x))$ will have its maximum at μ. The second function added to obtain $y(x)$ is $a\ln(x)$, an increasing function. The last function, $-\ln(|h(\omega, x)|^2)$, is related to the transfer function and therefore has its maximum at $\omega^2(1 - 2\zeta_n^2)$. The function $-\ln(|h(\omega, x)|^2)$ has three points (x_{h1}, x_{h2} and x_{h3}) where its third derivative is zero. These three points are

$$x_{h1} = \omega^2(1 - 2\zeta_n^2) - 2\sqrt{3}\omega^2\zeta_n\sqrt{1 - \zeta_n^2} \tag{53}$$

$$x_{h2} = \omega^2(1 - 2\zeta_n^2) \tag{54}$$

$$x_{h3} = \omega^2(1 - 2\zeta_n^2) + 2\sqrt{3}\omega^2\zeta_n\sqrt{1 - \zeta_n^2}. \tag{55}$$

Therefore, $y(x)$ can have up to two local maximums, the first one, $\theta_{a\mu}$, is close to μ and the second one ($\theta_{a\omega^2}$) is close to $\omega^2(1-2\zeta_n^2)$. That is, a Newton iteration [14] with these starting points should converge to the solution in few steps if the two maximums exist. Three roots of $y'''(x)=0$ can appear close to x_{h1}, x_{h2} and x_{h3}, respectively, $x_1 \approx x_{h1}$, $x_2 \approx x_{h2}$ and $x_3 \approx x_{h3}$. As formerly, Newton iteration is a procedure leading to the solution in few steps if the three points, x_{h1}, x_{h2} and x_{h3}, exist. Function $y(x)$ can have different shapes, and depending on it, a different method to calculate the integral is applied

- The function has two maximums and $y(\theta_{a\,\omega^2}) < y(\theta_{a\mu})$, then, Laplace's method is used.
- The function has two maximums and $y(\theta_{a\mu}) \le y(\theta_{a\omega^2})$. Laplace's approximation does not work well, in this case, a numerical integration (i.e. Gaussian quadrature, trapezoidal method) is a good alternative to approximate the integral.
- The function has only one maximum, $\theta_{a\mu}$, and x_1, x_2 and x_3 do not exist or are situated at the same side of $\theta_{a\mu}$, that is, $x_3 < \theta_{a\mu}$ or $\theta_{a\mu} < x_1$. Then, Laplace's method is used.
- The function has only one maximum, $\theta_{a\omega^2}$, and x_1, x_2 and x_3 exist and are such that $x_1 < \theta_{a\omega^2} < x_3$. In those situations, Laplace's approximation can work well if no discontinuity has appeared both in $\theta(a\omega)$ and $y''(\theta(a\omega))$. Generally, Laplace's approximation will provide a value smaller than the exact one. A different second order polynomial, having its maximum at θ_a and going through x_1 if $\theta_a > \mu$ and through θ_3 if $\theta_a \le \mu$, can be used instead of the Taylor expansion used in Laplace's method. Then

$$I(\omega) = e^{y(\theta_a)}\sqrt{\frac{-\pi(\theta_a - x_1)^2}{y(x_1) - y(\theta_a)}} \quad \text{if } \theta_a > \mu \tag{56}$$

$$I(\omega) = e^{y(\theta_a)}\sqrt{\frac{-\pi(\theta_a - x_3)^2}{y(x_3) - y(\theta_a)}} \quad \text{if } \theta_a \le \mu. \tag{57}$$

This second approximation to the integral provides, mostly, a larger value than the exact result. This approximation will be referred as modified Laplace's method.

If Laplace's method is applied when leading to a good approximation and numerical integration is applied in the remaining frequencies, the resulting method is called Hybrid Laplace-numerical integration. If Laplace's method or modified Laplace's method are applied when leading to a good approximation and numerical integration is performed for the remaining frequencies, the resulting method is referred to as modified Laplace.

5. FRF statistics for different probability density functions of the natural frequency

The pdfs of $x=\omega_n^2$ have to satisfy that $f_x(x \le 0) = 0$, as it is not physically possible to have a negative or zero squared natural frequency. It is observed that $x=\omega_n^2$ is the solution to the random eigenvalue problem $\mathbf{K} - \omega_n^2\mathbf{M} = 0$, such that its pdf would be directly obtained when solving the random eigenvalue problem. The purpose of this section is to understand how different possible pdfs of $\omega_n{}^2$ influence the dynamic response. The pdfs studied can be obtained with the maximum entropy principle, if some information on the eigenvalues is available [15]. That is, the pdfs are the functions maximizing the following entropy equation:

$$S(f_x) = -\int f_x(x)\ln(f_x(x))\,dx - \gamma_0\left(\int f_x(x)\,dx - 1\right) - \sum_{i=1}^{M}\gamma_i g_i(x) \tag{58}$$

where γ_0, γ_i for $i=1,\dots,M$ are Lagrange multipliers and g_i are functions related to the constraints imposed on the pdf f_x. Numerical examples are provided, where $\mu_x = 9$, $\sigma_x = 1$ and the damping factor has values $\zeta_n = 0.1$ or $\zeta_n = 0.01$. The chosen pdfs, other than the uniform distribution, are assumed to be unimodal with maximum at $x=\mu$. The number of samples in MCS is 10,000. A summary of results based on SDOF and MDF systems is given later in the paper.

5.1. Uniform distribution

The pdf of an uniform distribution $U(u_1, u_2)$ is defined by a constant α_u over the interval $x \in [u_1, u_2]$. Parameters α_u, u_1 and u_2 of the distribution can be expressed through its mean (μ_x) and variance (σ_x)

$$u_1 = \mu_x - \sqrt{3\sigma_x} \tag{59}$$

$$u_2 = \mu_x + \sqrt{3\sigma_x} \tag{60}$$

$$\alpha_u = \frac{1}{2\sqrt{3\sigma_x}}. \tag{61}$$

This pdf can be obtained with the maximum entropy principle, where the constraint g_i imposed on the entropy equation is that the random variable x belongs to a given interval. For this pdf, all integrals appearing in Eqs. (47)–(49) can be calculated exactly. For example if $a=0$

$$I(\omega) = \hat{I}_0(u_2, \omega) - \hat{I}_0(u_1, \omega) \quad \text{where} \tag{62}$$

$$\hat{I}_0(x, \omega) = \frac{\alpha_u}{2\omega^2\zeta_n\sqrt{1-\zeta_n^2}}\arctan\frac{x - \omega^2(1-2\zeta_n^2)}{2\omega^2\zeta_n\sqrt{1-\zeta_n^2}} \tag{63}$$

For $a=1$ we have

$$I(\omega) = \hat{I}_1(u_2, \omega) - \hat{I}_1(u_1, \omega) \quad \text{where} \tag{64}$$

$$\begin{aligned}\hat{I}_1(x, \omega) = {}& \alpha_u\frac{\log(\omega^2(\omega^2 - 2x(1-2\zeta_n^2)) + x^2)}{2} \\ & - \alpha_u\frac{\sqrt{1-4\zeta_n^2(1-\zeta_n^2)}}{2\zeta_n\sqrt{1-\zeta_n^2}} \\ & \times \arctan\frac{(\omega^2(2\zeta_n^2 - 1) + 2x)\sqrt{1-4\zeta_n^2(1-\zeta_n^2)}}{\omega^2(2\zeta_n^2-1)2\zeta_n\sqrt{1-\zeta_n^2}}\end{aligned} \tag{65}$$

An expression for $a=0.5$ can also be calculated analytically. In Figs. 5 and 6 we compared the results of the analytical expressions and MCS for different ζ_n and same uniform distribution $U(9-\sqrt{3}, 9+\sqrt{3})$ for x. Note the significant difference between the mean and the deterministic results for the smaller value of the damping factor in Fig. 6 compared to that for a higher value of the damping factor in Fig. 5.

5.2. Normal distribution

The pdf $f_x(x)$ and $f_x'(x)/f_x(x)$ of the normal distribution conditional to $x>0$ are given by

$$f_x(x) = \frac{e^{-(x-\mu_x)^2/2\sigma_x^2}}{\sqrt{2\pi\sigma_x^2}\,P(0)}, \quad x \in (0, +\infty), \quad \frac{f_x'(x)}{f_x(x)} = -\frac{x-\mu_x}{\sigma_x^2} \tag{66}$$

where μ_x and $\sigma_x{}^2$ are respectively the mean and the variance of the

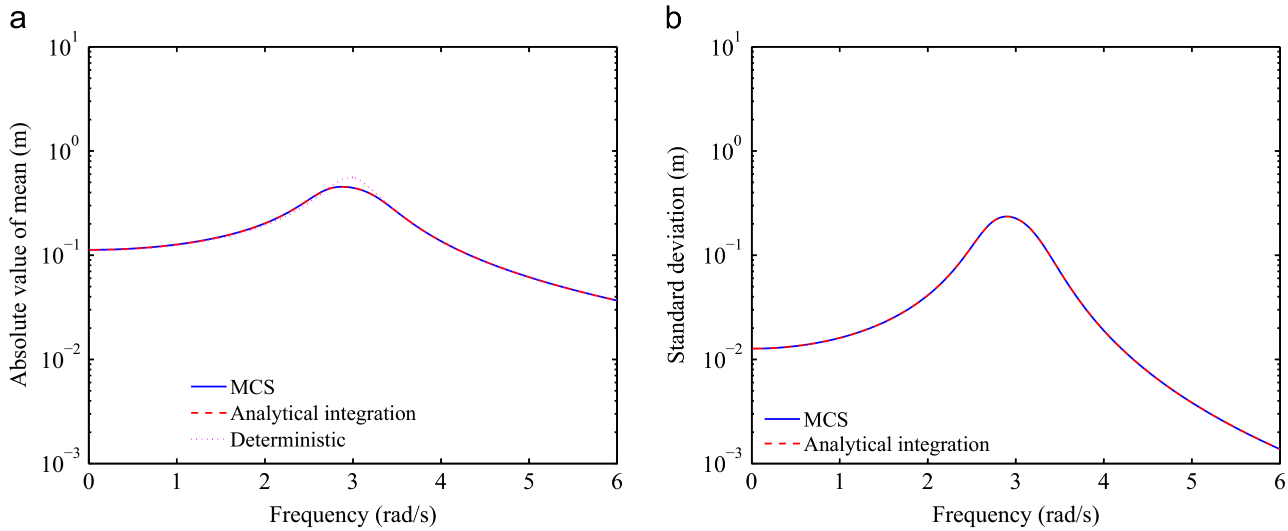

Fig. 5. Mean and standard deviation of the absolute value of the transfer function for uniform distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

distribution and $P(0)$ is the probability of $x \le 0$ for a normal distribution $N(\mu_x\ \sigma_x)$. This pdf can be obtained with the maximum entropy principle, where the constraints g_i imposed on the entropy equation is that the mean and variance of the random variable x are known.

From (50), the parameter θ_a can be identified as the solution of a fourth order polynomial

$$b_{1_a}x^4 + b_{2_a}x^3 + b_{3_a}x^2 + b_{4_a}x + b_{5_a} = 0 \tag{67}$$

with coefficients

$$\begin{aligned} b_{1_b} &= -1 & b_{3_b} &= -\omega^4 + (a-2)\sigma_x^2 + \mu_x\bar{\zeta}_\omega \\ b_{2_b} &= -\bar{\zeta}_\omega + \mu_x & b_{4_b} &= (1+a)\sigma_x^2\bar{\zeta}_\omega + \mu_x\omega^4 \\ & & b_{5_b} &= a\sigma_x^2\omega^4 \end{aligned} \tag{68}$$

and $\bar{\zeta}_\omega = 2\omega^2(2\zeta_n^2 - 1)$. Up to four solutions can be expected, and we assume the existence of a first real solution close to the mean of the distribution, $\theta_{a\mu}$. If a second real solution $\theta_{a\omega^2}$ exists and is not a saddle point, a relative minimum between the two relative maximums exists. The remaining real solution is a spurious value, and likely to be negative or close to zero. Depending on the shape of $y(x)$, and as indicated in Section 4.2.2, Laplace's method may be applied to approximate the integrals appearing in expressions of $\mathrm{E}[h]$ and $\mathrm{E}[|h|^2]$, given by Eqs. (39)–(41). The second derivative of $y(x)$ is given by (51) where $\bar{h}(x, \omega)$ is given by (52) and

$$\bar{f}(x) = -\frac{1}{\sigma_x^2}. \tag{69}$$

Laplace's approximation assuming normal distribution is therefore given by

$$I(\omega, a) \backsim \frac{\theta_a^a\sqrt{2\pi}\,e^{-(\theta_a-\mu_x)^2/2\sigma_x^2}}{P(0)\sqrt{2\pi\sigma_x^2}\left((\theta_a-\omega^2)^2 + 4\zeta_n^2\omega^2\theta_a\right)^{a/2}}\left[\frac{1}{\sigma_x^2} - \frac{a}{\theta_a^2} + \bar{h}(\theta_a)\right]^{-1/2} \tag{70}$$

where a is given in Eqs. (47)–(49). Parameters μ_x and σ_x are the mean and the standard deviation of $x = \omega_n^2$. Another approximation of the integral can be obtained from Eqs. (56) and (57). Plots of approximations for an SDOF with $\mu_x = 9$ and $\sigma_x = 1$ are displayed, in Fig. 7 for $\zeta_n = 0.1$ and in Fig. 8 for $\zeta_n = 0.01$. For the small damping case one can again observe the significant difference between the deterministic response and the mean response. Additionally, it can be noted that hybrid modified Laplace method also produced some discrepancies for the low damping case.

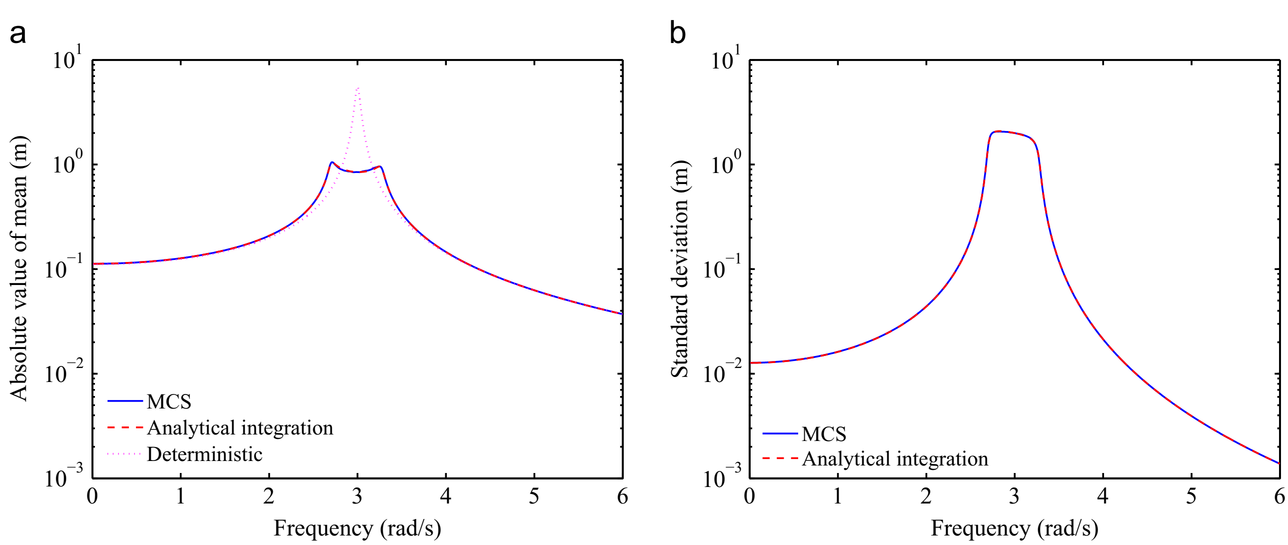

Fig. 6. Mean and standard deviation of the absolute value of the transfer function for uniform distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

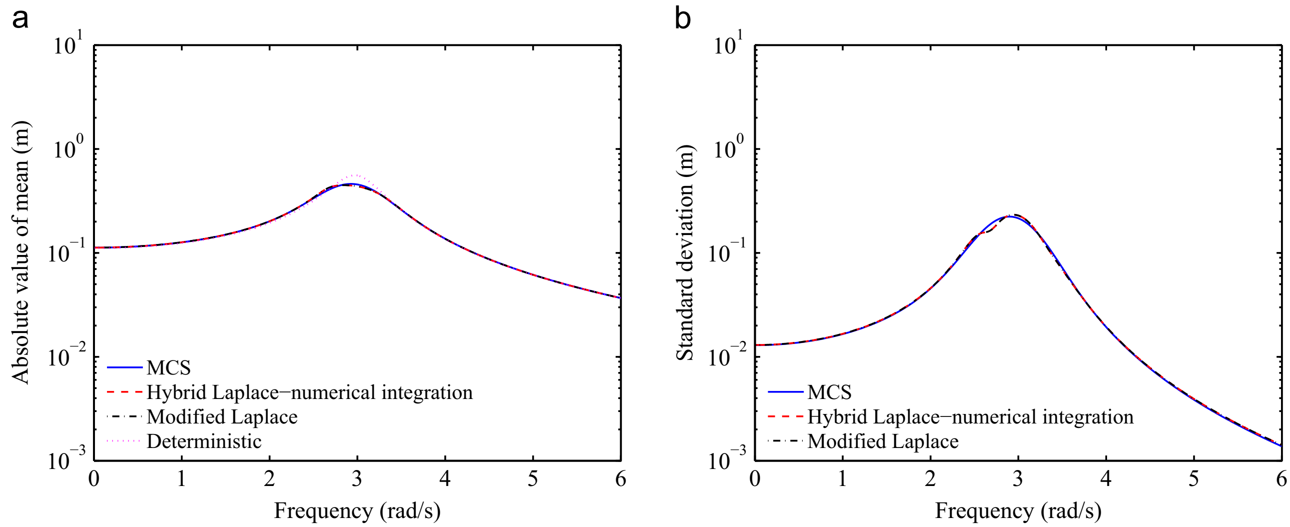

Fig. 7. Mean and standard deviation of the absolute value of the transfer function for normal distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

5.3. Gamma distribution

The probability density function $f_x(x)$ and $f_x'(x)/f_x(x)$ of the gamma distribution, defined in the interval $[0, \infty]$, are given by

$$f_x(x) = \frac{x^{\alpha_g - 1}}{\Gamma(\alpha_g)\beta_g^{\alpha_g}} e^{-x/\beta_g}, \quad \frac{f_x{}'(x)}{f_x(x)} = \frac{\alpha_g - 1}{x} - \frac{1}{\beta_g}. \tag{71}$$

Relationships between mean μ_x, variance $\sigma_x{}^2$ and parameters α_g and β_g are given by

$$\mu_x = \alpha_g \beta_g, \quad \sigma_x^2 = \alpha_g \beta_g^2 \tag{72}$$

$$\alpha_g = \frac{\mu_x^2}{\sigma_x^2}, \quad \beta_g = \frac{\sigma_x^2}{\mu_x}. \tag{73}$$

This pdf can be obtained with the maximum entropy principle, where the constraints g_i imposed on the entropy equation are that the random variable belongs to a given interval and the means of the random variable x and of $\ln x$ are known. As formerly, θ_a is the real solution of the third order polynomial

$$b_{1_a} x^3 + b_{2_a} x^2 + b_{3_a} x + b_{4_a} = 0 \tag{74}$$

with coefficients

$$b_{1_a} = -\frac{1}{\beta_g}, \quad b_{2_a} = -\frac{\bar{\zeta}_\omega}{\beta_g} + \alpha_g + a - 3,$$
$$b_{4_a} = \omega^4(\alpha_g + a - 1), \quad b_{3_a} = \bar{\zeta}_\omega\left(\alpha_g + a - 2\right) - \frac{\omega^4}{\beta_g} \tag{75}$$

and $\bar{\zeta}_\omega = 2\omega^2(2\zeta_n^2 - 1)$, at which $y(x)$ is maximum. A third-order polynomial has three solutions. Among these solutions, the ones that can be considered as plausible solutions to our problem are real and satisfy $x > 0$, $y'(x, a) = 0$ and $y''(x, a) < 0$. As we have already stated, one of the solutions is close to μ, and a second plausible solution is close to $x_{h2} = \omega^2(1 - 2\zeta_n^2)$. The third real solution of the polynomial would then be the relative minimum point situated between the two relative maximums, and verifies $y''(x, a) > 0$. Otherwise, two of the solutions are complex and there is only one real solution. We can then find an approximation to the solutions using Newton iteration method to (74). As indicated in Section 4.2.2, Laplace's method will be a good option to approximate the integrals depending on the shape of $y(x)$. The second derivative of $y(x, a)$ is given at (51) where $\bar{h}(x, \omega)$ is given by (52)

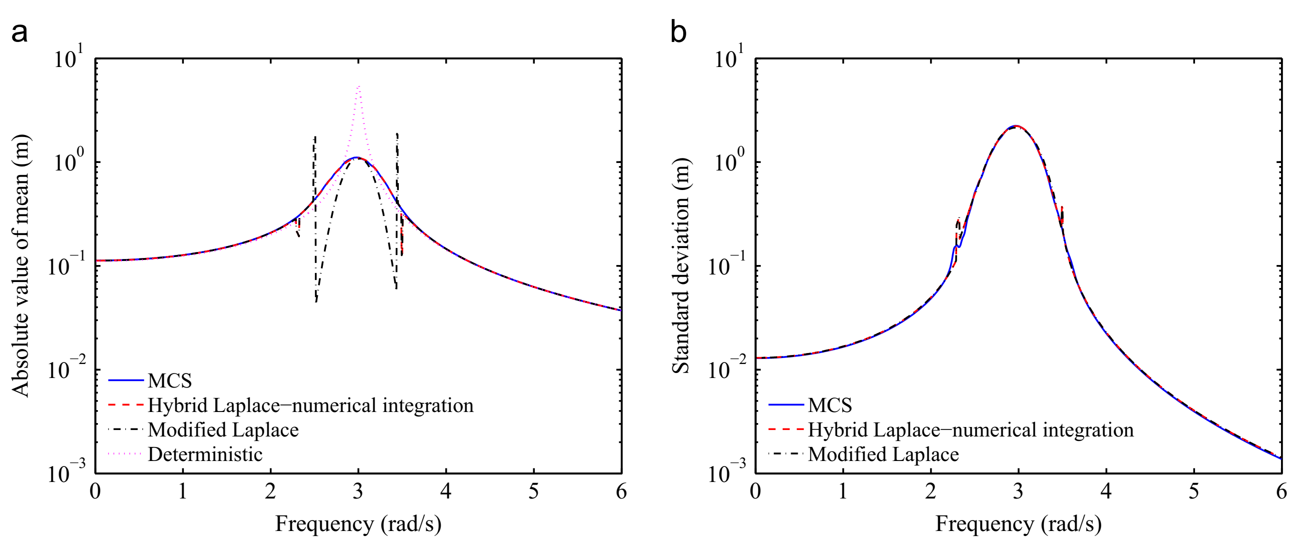

Fig. 8. Mean and standard deviation of the absolute value of the transfer function for normal distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

and

$$\bar{f}(x) = \frac{1-\alpha_g}{x^2}. \tag{76}$$

From (45), an analytical approximation to the integrals appearing in the expressions of the two first moments of the absolute value of the FRF is given by

$$I(\omega, a) \backsim \frac{\sqrt{2\pi}\,\theta_a^{\alpha_g-1+a} e^{-\theta_a/\beta_g}}{\Gamma(\alpha_g)\beta_g^{\alpha_g}((\theta_a-\omega^2)^2+4\zeta_n^2\omega^2\theta_a)}\left[\frac{\alpha_g-1-a}{\theta_a^2}+\bar{h}(\theta_a)\right]^{-1/2} \tag{77}$$

where a is given in Eqs. (47)–(49). Parameters α_g and β_g can be derived from (73) if the mean μ_x and the standard deviation σ_x of ${\omega_n}^2$ are known. Plots of approximations for an SDOF with $\mu_x = 9$ and $\sigma_x = 1$ are displayed, in Fig. 9 for $\zeta_n = 0.1$ and in Fig. 10 for $\zeta_n = 0.01$. Again note the difference between the results for small and high damping.

5.4. Lognormal distribution

A lognormal distribution is a probability distribution obtained by taking the exponential of a normal distribution of mean μ and standard deviation σ, $N(\mu, \sigma)$. The probability density function $f_x(x)$, and $f'_x(x)/f_x(x)$ of lognormal distribution, defined in the interval $(0, \infty)$, is given by

$$f_x(x) = \frac{e^{-(\ln x-\mu)^2/2\sigma^2}}{x\sqrt{2\pi\sigma^2}}, \quad \frac{f'_x(x)}{f_x(x)} = -\frac{\sigma^2 \ln x - \mu}{x\sigma^2} \tag{78}$$

with mean μ_x and variance σ_x of lognormal distribution related to μ and σ by

$$\mu_x = e^{\mu+\sigma^2/2}, \quad \sigma_x^2 = (e^{\sigma^2}-1)e^{2\mu+\sigma^2} \tag{79}$$

$$\mu = \ln \mu_x - \frac{1}{2}\ln\left(1+\frac{\sigma_x}{\mu_x^2}\right), \quad \sigma = \ln\left(\frac{\sigma_x}{\mu_x^2}+1\right). \tag{80}$$

This pdf can be obtained with the maximum entropy principle, where the constraint g_i imposed on the entropy equation is that the mean of $\ln x$ and of $(\ln x)^2$ are known. From (50) parameter θ_a can be identified as the solution to the equation

$$y'(\theta_a, a) = -\frac{\sigma^2+\ln\theta_a-\mu}{\theta_a\sigma^2}+\frac{a}{\theta_a}-\frac{(2(\theta_a-\omega^2)+4\zeta_n^2)}{(\theta_a-\omega^2)^2+4\zeta_n^2\theta_a} = 0. \tag{81}$$

As indicated in Section 4.2.2, we will assume here that one solution, θ_{a_μ}, is close to the mean of the distribution and that another relative maximum of $y(x)$, $\theta_{a_{\omega^2}}$, can arise together with a relative minimum situated between those two maximums. We can then find an approximation to these solutions using Newton method. From (45) and depending on the shape of $y(x)$, an analytical approximation to the integrals appearing in the expressions of $\mathrm{E}[h]$ and of $\mathrm{E}[|h|^2]$ allows to find an approximation to those moments. This analytical approximation to integrals is given by

$$I(\omega, a) \backsim \frac{\theta_a^{a-1} e^{-(\ln\theta_a-\mu)^2/2\sigma^2}}{\sqrt{\sigma^2}((\theta_a-\omega^2)^2+4\zeta_n^2\omega^2\theta_a)} \left[\frac{\sigma^2(a-1)-\ln\theta_a+\mu}{\sigma^2\theta_a^2}+\bar{h}(\theta_a)\right]^{-1/2} \tag{82}$$

with a given in Eqs. (47)–(49). Parameters μ and σ can be found if μ_x and σ_x are known. Plots of the approximations for an SDOF with $\mu_x = 9$ and $\sigma_x = 1$ are displayed, in Fig. 11 for $\zeta_n = 0.1$ and in Fig. 12 for $\zeta_n = 0.01$. Like the previous pdfs, the difference in results between the low and high damping can also be seen here.

6. Multiple-degrees-of-freedom (MDOF) systems

6.1. Response calculation

The aim of this section is to extend the results of SDOF systems to MDOF systems with certain assumptions. Applying finite element method to structural dynamic systems leads, generally, to an MDOF problem where a displacement vector $\mathbf{u}$ is the unknown. The equation of motion of a linear dynamical system in the frequency domain can be expressed as

$$\left(-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}\right)\mathbf{u} = \mathbf{f}. \tag{83}$$

Here ω is the frequency, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the mass, damping and stiffness matrices of the system, $\mathbf{f}$ is the forcing vector in the frequency domain and $\mathbf{u}$ is the response vector.

The frequency response vector of the MDOF system can be given by (see, for example, [16])

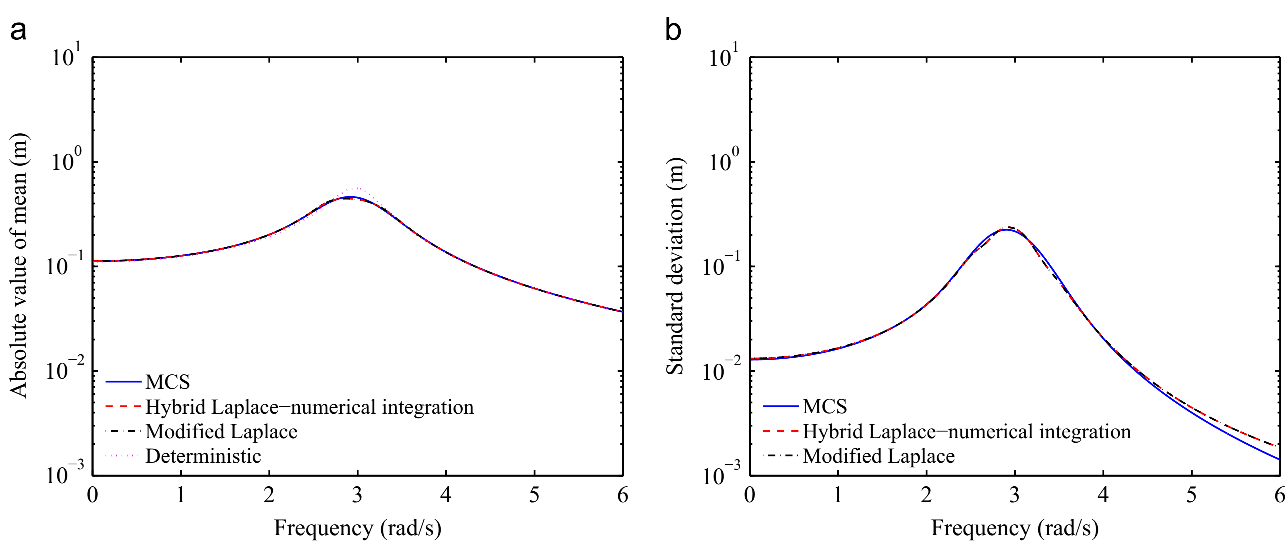

Fig. 9. Mean and standard deviation of the absolute value of the transfer function for gamma distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

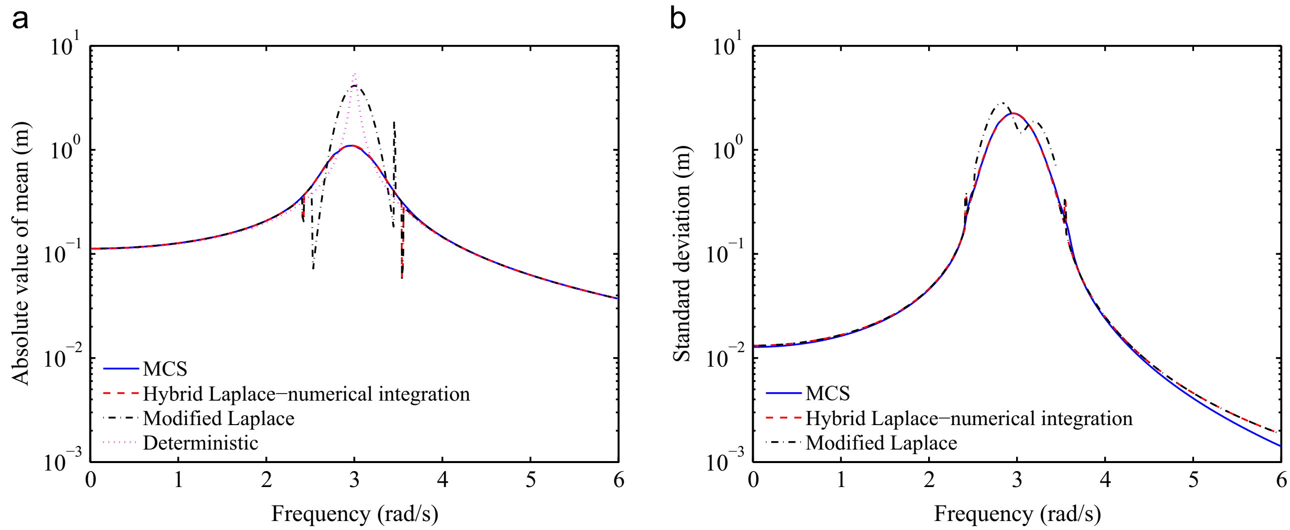

Fig. 10. Mean and standard deviation of the absolute value of the transfer function for gamma distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

$$\mathbf{u} = \boldsymbol{\Phi}[-\omega^2\mathbf{I} + \mathrm{i}\omega 2\zeta\boldsymbol{\Omega} + \boldsymbol{\Omega}^2]^{-1}\boldsymbol{\Phi}^T\bar{\mathbf{f}} = \boldsymbol{\Phi}\mathbf{H}'\boldsymbol{\Phi}^T\mathbf{f} \tag{84}$$

where $\boldsymbol{\Phi}$ is the matrix of eigenvectors (modal matrix) and $\boldsymbol{\Omega}$ is a diagonal matrix of natural frequencies of the system. The frequency response can be expressed as

$$\begin{aligned}\mathbf{u} &= \boldsymbol{\Phi}\left[\sum_{j=1}^{N}\frac{e_j e_j^T}{-\omega^2 + 2\mathrm{i}\omega\zeta_j\omega_j + \omega_j^2}\right]\boldsymbol{\Phi}^T\mathbf{f}\\ &= \boldsymbol{\Phi}\left[\sum_{j=1}^{N}\frac{e_j e_j^T}{-\omega^2 + 2\mathrm{i}\omega\zeta_j\omega_j + \omega_j^2}\right]\mathbf{F}\end{aligned} \tag{85}$$

where e_j is the jth unit vector, or jth column of an identity matrix, and matrix $\mathbf{H}'$ is therefore diagonal. We denote by $\mathbf{u}^*$ and $\mathbf{H}'^*$ the complex conjugate of $\mathbf{u}$ and $\mathbf{H}'$ respectively. The jth diagonal element of matrix $\mathbf{H}'$ is denoted by h'_j, and h'^*_j is its complex conjugate. Vector $\boldsymbol{\Phi}_i$ is the ith row of matrix $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}_{i_j}$ is its jth element. The jth element of vector $\mathbf{F} = \boldsymbol{\Phi}^T\mathbf{f}$ is denoted by F_j. Uncertainty is introduced by the diagonal terms of $\mathbf{H}'$, and therefore, all other vectors and matrices are deterministic. From (85), an expression of u_i, the ith term of vector $\mathbf{u}$, and of $|\mathrm{u}_i|^2$ can be derived

$$\mathrm{u}_i = \sum_{j=1}^{N}\boldsymbol{\Phi}_{i_j}h'_j F_j \tag{86}$$

$$|\mathrm{u}_i|^2 = \mathrm{u}_i^T\mathrm{u}_i^* = \sum_{j=1}^{N}\sum_{k=1}^{N}F_j h'_j\boldsymbol{\Phi}_{i_j}\boldsymbol{\Phi}_{i_k}h'^*_k F_k. \tag{87}$$

Denoting $C_j = F_j\boldsymbol{\Phi}_{i_j}$, the expression of $|\mathrm{u}_i|^2$ can be simplified

$$\begin{aligned}|\mathrm{u}_i|^2 &= \sum_{j=1}^{N}C_j^2 h'_j h'^*_j + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}C_j C_k\left(h'_j h'^*_k + h'^*_j h'_k\right) = \sum_{j=1}^{N}C_j^2|h'_j|^2 \\ &\quad + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}C_j C_k 2\Re(h'_j h'^*_k).\end{aligned} \tag{88}$$

Mean of $|\mathrm{u}_i|^2$ is equal to the second moments of $|\mathrm{u}_i|$ and u_i.

6.2. Mean of real and imaginary parts of the response

Expressions of real and imaginary parts of u_i can be derived from (86)

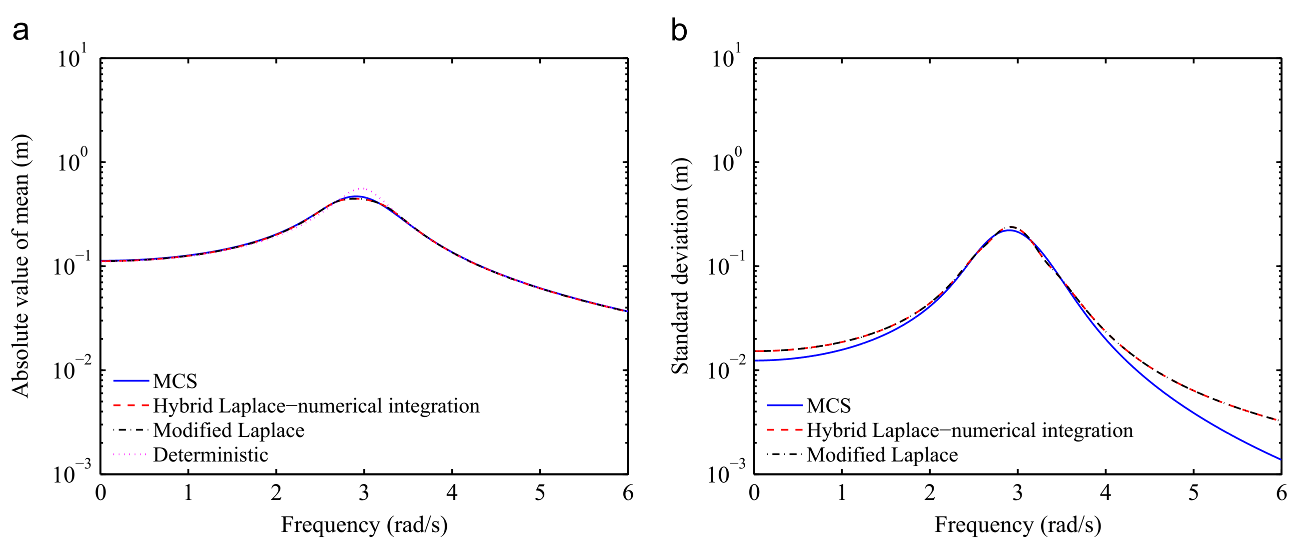

Fig. 11. Mean and standard deviation of the absolute value of the transfer function for lognormal distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

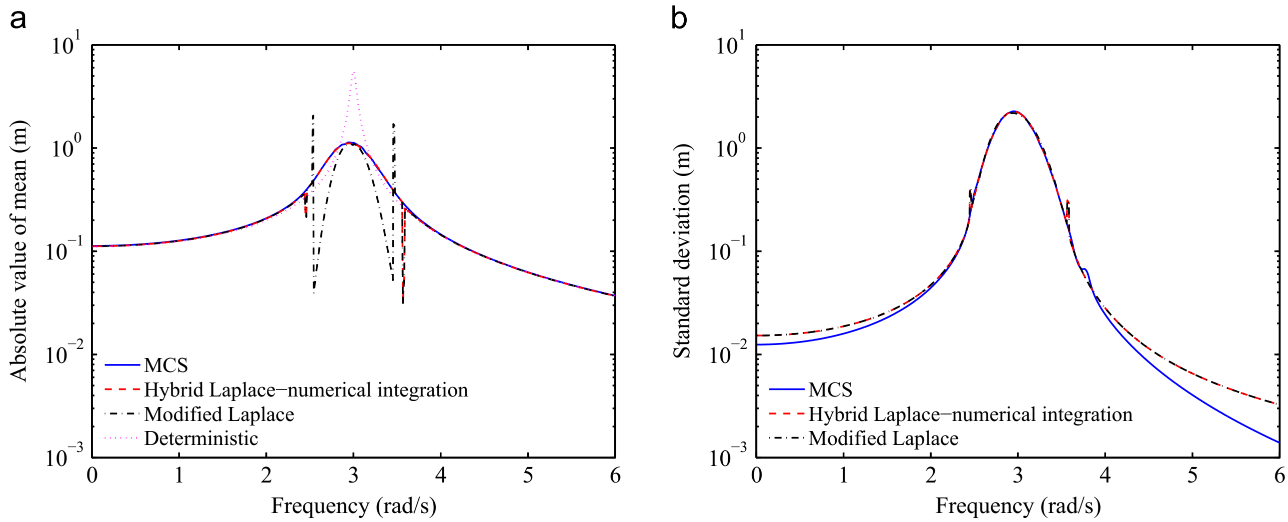

Fig. 12. Mean and standard deviation of the absolute value of the transfer function for lognormal distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

$$\Re(\mathrm{u}_i) = \sum_{j=1}^{N} C_j \Re\left(h_j'\right), \quad \Re\left(h_j'\right) = \frac{\omega_j^2 - \omega^2}{(\omega_j^2 - \omega^2)^2 + 4\zeta_j^2\omega^2\omega_j^2} \tag{89}$$

$$\Im(\mathrm{u}_i) = \sum_{j=1}^{N} C_j \Im\left(h_j'\right), \quad \Im\left(h_j'\right) = \frac{-2\zeta_j\omega\omega_j}{(\omega_j^2 - \omega^2)^2 + 4\zeta_j^2\omega^2\omega_j^2}. \tag{90}$$

Mean is a linear operator, therefore, the mean of each product $C_j\Re(h_j')$, $C_j\Im(h_j')$, $C_j^2|h_j'|^2$ and $C_jC_k2\Re(h_j'h_k'^*)$ is needed. A well-known theorem of random matrix theory [17] states that the joint density functions of the latent roots $l_1, \ldots, l_n$ of a $n \times n$ positive definite matrix $\mathbf{A}$ with density function $f(\mathbf{A})$ is given by

$$\frac{\pi^{n^2/2}}{\Gamma_n(m/2)} \prod_{i<j}^{m} (l_i - l_j) \int_{O(n)} f(\mathbf{\Omega L \Omega}^T)(d\mathbf{\Omega}) \tag{91}$$

where $\mathbf{\Omega}$ is an orthogonal matrix of the orthogonal group $O(n)$ verifying $\mathbf{A} = \mathbf{\Omega L \Omega}^T$, with the diagonal matrix $\mathbf{L} = diag(l_1, \ldots, l_n)$. The eigenvalues and eigenvectors of $\mathbf{A}$ are independent if $f(\mathbf{A}) = f(\mathbf{\Psi A \Psi}^T)$ with $\mathbf{\Psi}$ being an orthogonal matrix, and, for some distributions as Wishart matrix distribution, they are asymptotically independent. Furthermore, applying the maximum entropy principle to obtain joint distribution of eigenvalues and eigenvectors where no data is available on the joint pdf leads to independent eigenvalues and eigenvectors. It is assumed that eigenvectors and eigenvalues are independent. The independence of eigenvalues and eigenvectors of a positive definite matrix implies that

$$\begin{aligned} \mathrm{E}\left[C_j\Re\left(h_j'\right)\right] &= \mathrm{E}\left[C_j\right]\mathrm{E}\left[\Re\left(h_j'\right)\right] \\ \mathrm{E}\left[C_j\Im\left(h_j'\right)\right] &= \mathrm{E}\left[C_j\right]\mathrm{E}\left[\Im\left(h_j'\right)\right] \end{aligned} \tag{92}$$

where $\mathrm{E}[C_j] = \sum_{k=1}^{n} \mathrm{E}[\Phi_{ij}\Phi_{kj}]\mathbf{f}_k$ can be obtained from the second moment of the jth eigenvalue, with $\mathbf{f}_k$ being the kth element of the forcing term. Only means $\mathrm{E}[\Re(h_j')]$ and $\mathrm{E}[\Im(h_j')]$ remain to be calculated to obtain $\mathrm{E}[\Re(\mathrm{u}_i)]$ and $\mathrm{E}[\Im(\mathrm{u}_i)]$. These means are given by

$$\mathrm{E}\left[\Re\left(h_j'\right)\right] = I_{2_j} - \omega^2 I_{1_j} \tag{93}$$

$$\mathrm{E}\left[\Im\left(h_j'\right)\right] = -2\zeta_j\omega I_{3_j}. \tag{94}$$

If uncorrelated random variables are assumed, integrals I_{1_j}, I_{2_j} and I_{3_j} are given by

$$I_{1_j} = \int_{\mathcal{D}} \frac{f_x}{(x_j - \omega^2)^2 + 4\zeta_j^2\omega^2 x_j} dx_j \tag{95}$$

$$I_{2_j} = \int_{\mathcal{D}} \frac{x_j f_x}{(x_j - \omega^2)^2 + 4\zeta_j^2\omega^2\omega_j^2} dx_j \tag{96}$$

$$I_{3_j} = \int_{\mathcal{D}} \frac{\sqrt{x_j} f_x}{(x_j - \omega^2)^2 + 4\zeta_j^2\omega^2 x_j} dx_j. \tag{97}$$

An analytical approximation to these integrals could be given, depending on the shape of the integrand, by Eqs. (70), (77) and (82) for normal, gamma and lognormal distributions respectively. Parameter $a=0$ for I_{1_j}, $a=1$ for I_{2_j} and $a = 1/2$ for I_{3_j}.

6.3. Variance of the response

Expressions of the second moment of response can be derived from (88), remembering that mean is a linear operator and that eigenvalues and eigenvectors are assumed to be independent

$$\begin{aligned} m_2 = \mathrm{E}\left[|\mathrm{u}_i|^2\right] &= \sum_{j=1}^{N} \mathrm{E}\left[C_j^2\right]\mathrm{E}\left[|h_j'|^2\right] \\ &+ \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} \mathrm{E}\left[C_jC_k\right]2\mathrm{E}\left[\Re(h_j'h_k'^*)\right] \end{aligned} \tag{98}$$

$$\mathrm{E}\left[|\mathrm{u}_i|^2\right] = \sum_{j=1}^{N} \mathrm{E}\left[C_j^2\right]I_{1_j} + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} \mathrm{E}\left[C_jC_k\right]2\mathrm{E}\left[\Re(h_j'h_k'^*)\right] \tag{99}$$

$$\mathrm{E}\left[\Re(h_j'h_k'^*)\right] = \left(I_{2_j} - \omega^2 I_{1_j}\right)\left(I_{2_k} - \omega^2 I_{1_k}\right) + 4\zeta_j\zeta_k\omega^2 I_{3_j}I_{3_k}. \tag{100}$$

Means $\mathrm{E}[C_j^2]$ can be obtained from the fourth moment of the jth eigenvector and $\mathrm{E}[C_jC_k]$ from the fourth moment of the eigenvector matrix Φ. The second moment is given by

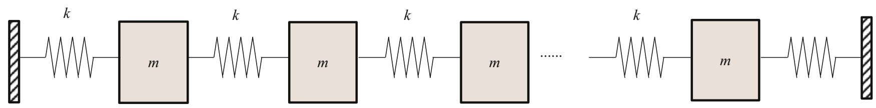

Fig. 13. Linear array of N spring–mass oscillators, $N=20$, $m=1$ kg and $k=350$ N/m. A proportional damping model with damping factor 0.1 and 0.01 is assumed.

$$m_2 = \mu_{u_i}^2 + \sigma_{u_i}^2 \tag{101}$$

where mean of response and squared value of mean are given by

$$\mu_{u_i} = \mathrm{E}\left[\Re(u_i)\right] + i\mathrm{E}\left[\Im(u_i)\right] \tag{102}$$

$$\mu_{u_i}^2 = \mathrm{E}\left[\Re(u_i)\right]^2 + \mathrm{E}\left[\Im(u_i)\right]^2. \tag{103}$$

And finally the variance can be obtained as

$$\sigma_{u_i}^2 = \mathrm{E}\left[|u_i|^2\right] - \mu_{u_i}^2. \tag{104}$$

In the next section, numerical results are shown for an MDOF system with random eigenvalues with the different distributions already discussed.

6.4. Numerical example

A proportionally damped system consisting of a linear array of spring–mass oscillators is considered to illustrate the proposed approach. Fig. 13 shows the model system. N masses, each of the nominal mass m, are connected by springs of nominal stiffness k. The system considered uses the mean of eigenvalues and eigenvectors obtained from the deterministic mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$, and forcing vector $\mathbf{f}$ as

$$\mathbf{M} = m\mathbf{I}, \quad \mathbf{K} = k\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{105}$$

with $\mathbf{I}$ being the identity matrix. The number of degrees of freedom of the system is 20, therefore, matrices $\mathbf{M}$ and $\mathbf{K}$ become 20×20 matrices. Mass and stiffness constants are given by $m=1$ kg and $k=350$ N/m. The eigenvectors used are the ones obtained from the deterministic matrices $\mathbf{M}$ and $\mathbf{K}$ matrices, that is, they are considered deterministic, or with central moments equal to zero. It is observed that the eigenvalue problem $k\mathbf{K} = \lambda m\mathbf{I}$ here reduces to $\mathbf{K} = (\lambda m/k)\mathbf{I}$, such that the eigenvectors are deterministic and the pdf of the eigenvalues depends on that of m/k. Mean eigenvalues are obtained from the deterministic matrices $\mathbf{M}$ and $\mathbf{K}$, and the standard deviation σ of each eigenvalue is given by a percentage of the mean of the considered eigenvalue. Each percentage is obtained through a sample of the uniform distribution $U(10, 15)$. Damping factors are assumed to be all equal to $\zeta_j = 0.1$ or to $\zeta_j = 0.01$. The number of samples for MCS is 10,000. Results are obtained for normal distribution with damping factors $\zeta_j = 0.1$ in Fig. 14 and $\zeta_j = 0.01$ in Fig. 15. Figs. 16 and 17 show Gamma distribution results for $\zeta_j = 0.1$ and $\zeta_j = 0.01$ respectively. Lognormal distribution results for $\zeta_j = 0.1$ and $\zeta_j = 0.01$ are respectively displayed in Figs. 18 and 19. In all the figures, a comparison between mean and standard deviation obtained through approximations and MCS is facilitated. Uniform distribution results are not shown as an analytical expression to integrals is available and results match exactly MCS results.

7. Results and discussion

7.1. Discussion of the proposed analytical methods

In this paper, the mean and the variance of frequency response function of single and multiple-degrees-of-freedom systems are calculated from the pdf of independent eigenvalues. This method needs the calculation of three integrals per frequency and degree of freedom, namely the ones appearing in Eqs. (47)–(49). Unfortunately, exact analytical integration is only available when the random variable, i.e. the squared natural frequency, has uniform distribution. Therefore, the main problem of the method is the calculation of the integrals. Numerical calculation of the integrals or evaluation through MCS can become computationally expensive for systems with large degrees of freedom. This problem can be overcome if integrals can be approximated using one of the two methods proposed in this paper.

The first method, referred as hybrid Laplace-numerical

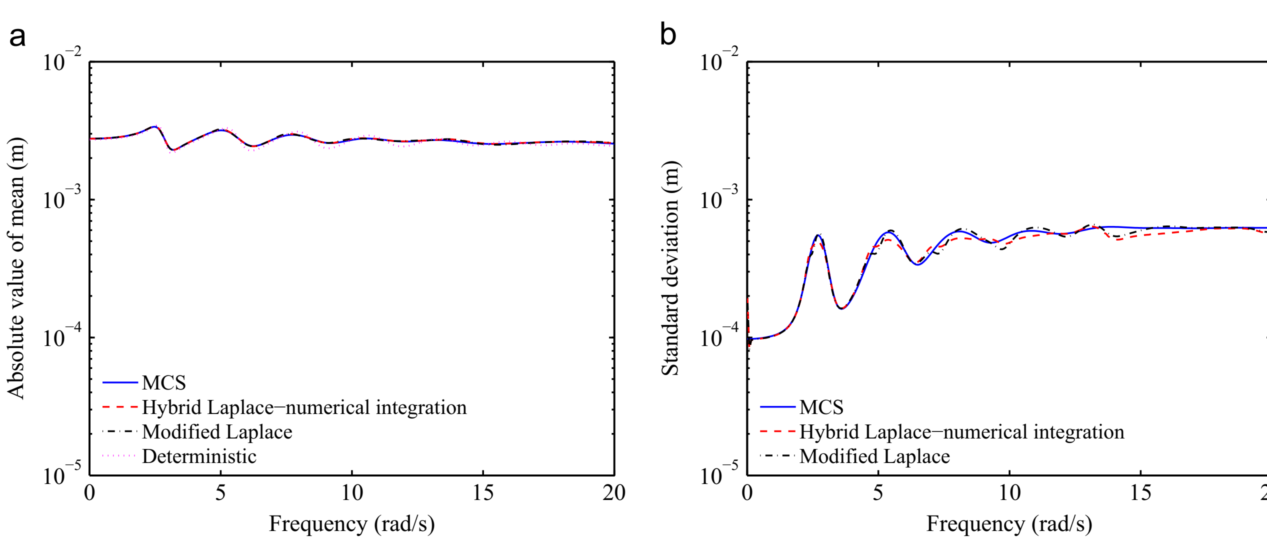

Fig. 14. Mean and standard deviation of the absolute value of the transfer function for normal distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

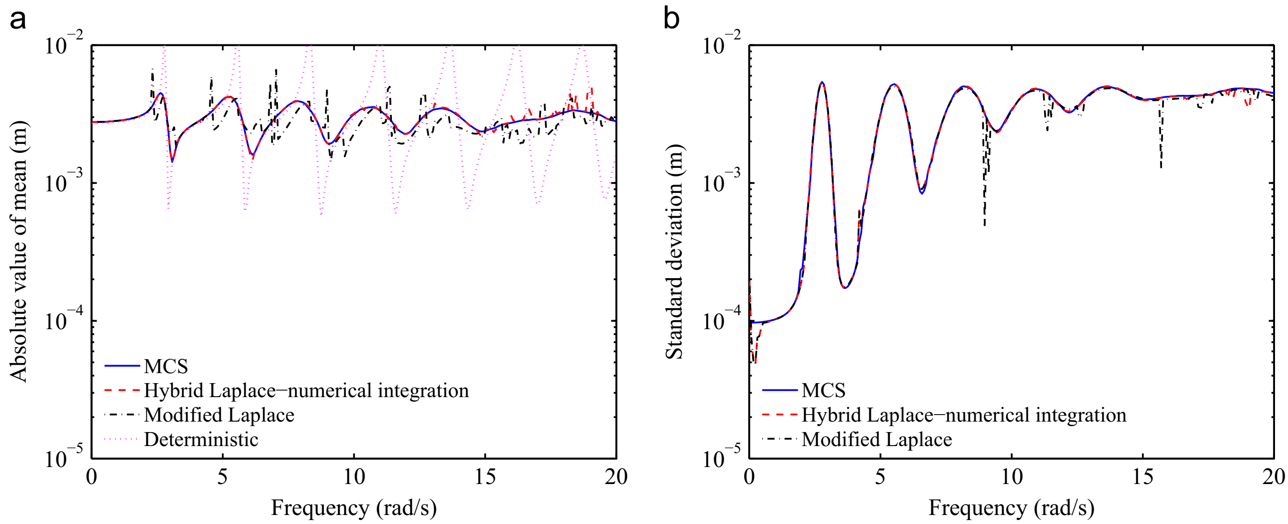

Fig. 15. Mean and standard deviation of the absolute value of the transfer function for normal distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

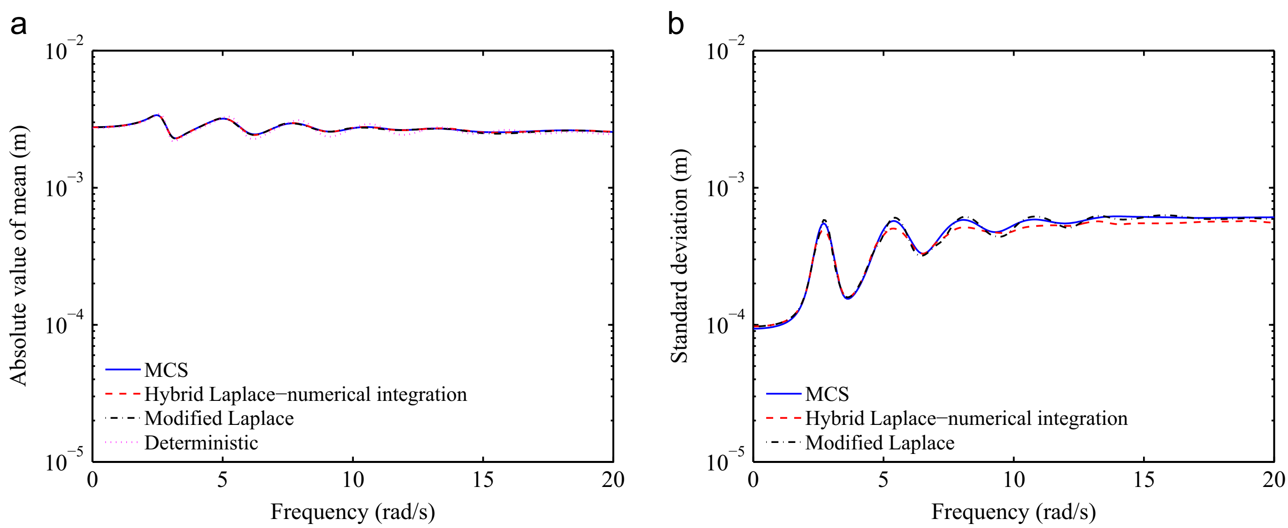

Fig. 16. Mean and standard deviation of the absolute value of the transfer function for gamma distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

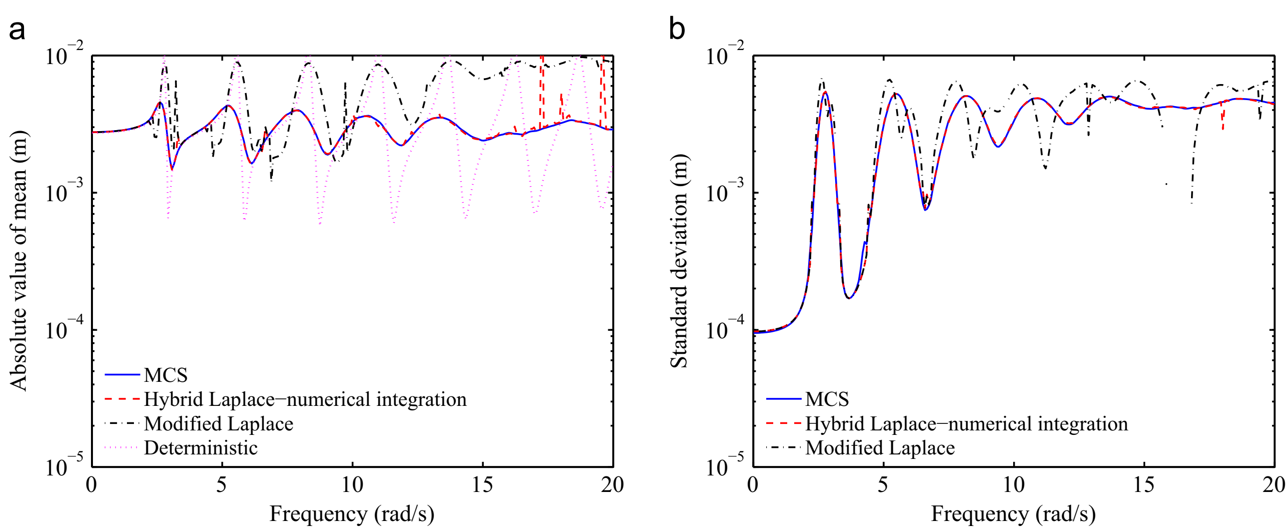

Fig. 17. Mean and standard deviation of the absolute value of the transfer function for gamma distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

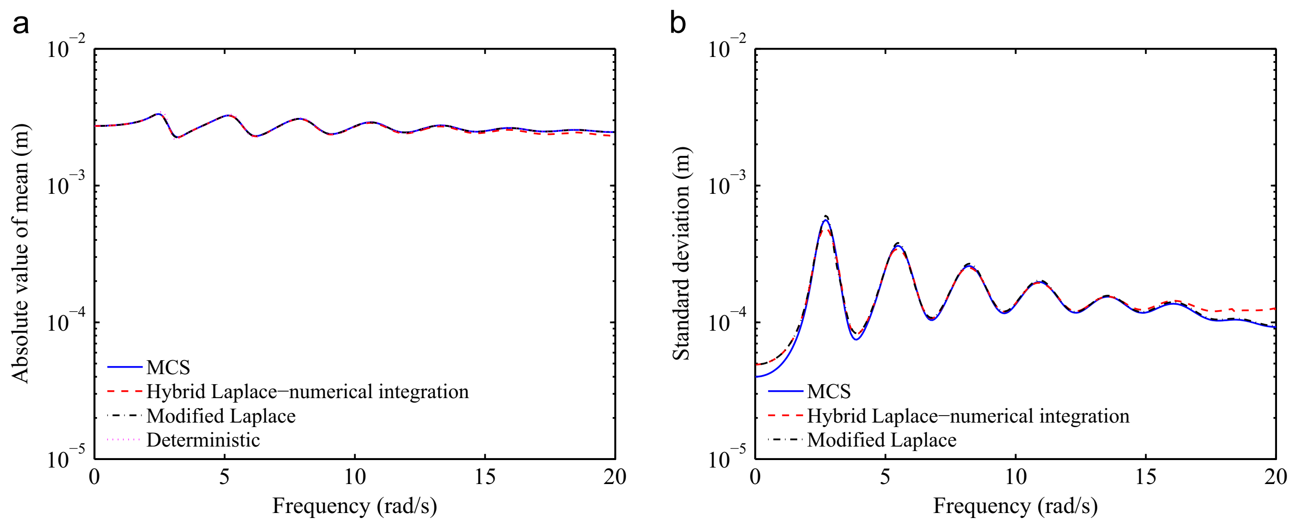

Fig. 18. Mean and standard deviation of the absolute value of the transfer function for lognormal distribution with $\zeta_n = 0.1$. (a) Absolute value of mean. (b) Standard deviation.

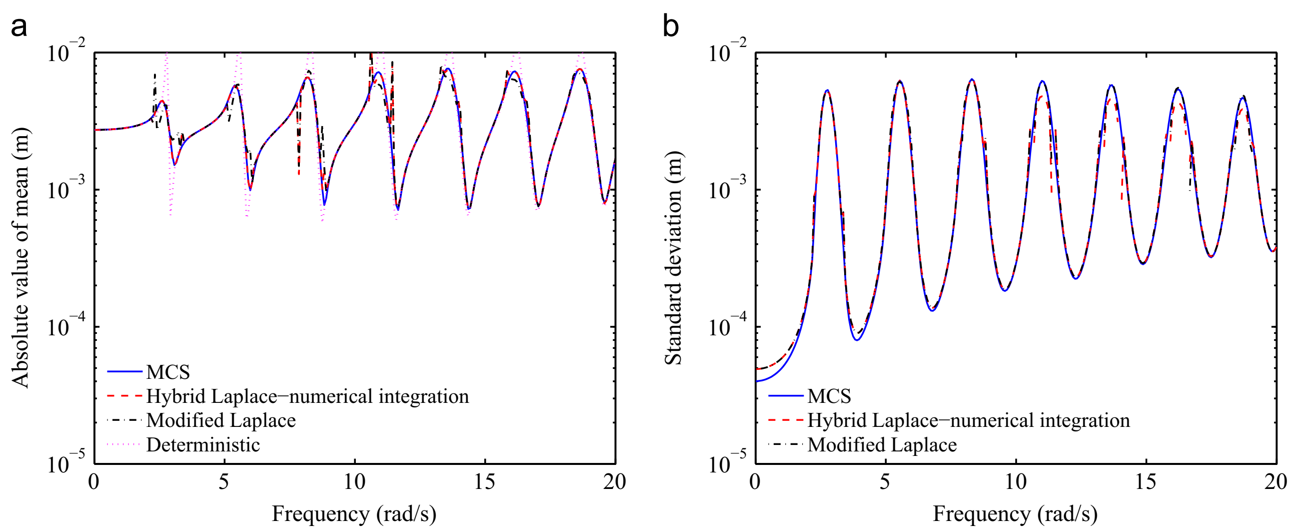

Fig. 19. Mean and standard deviation of the absolute value of the transfer function for lognormal distribution with $\zeta_n = 0.01$. (a) Absolute value of mean. (b) Standard deviation.

integration, is a hybrid method between Laplace's method and numerical integration. Laplace's method is used to approximate the integrals at those frequencies where the method is expected to give a good approximation, and numerical integration is used for the remaining frequencies. The second method, i.e. hybrid modified Laplace, also approximates the integrals with Laplace's method when it is supposed to give a good approximation. The remaining integrals are approximated, when possible, with a modified Laplace's method and with numerical integration otherwise. For the frequencies where the modified Laplace's method can be applied, it is observed that Laplace's method provides an approximation smaller than the exact value, while the modified Laplace's method provides mostly a larger approximation. The modified Laplace's approximation gives good approximation at resonance points both for lognormal and normal distribution, but results are too large when dealing with the gamma distribution.

It is observed, in all figures where the mean of the system is calculated, that the difference between mean and deterministic system is larger at frequencies in the neighborhood of resonance frequencies than at other frequencies. This is due to the fact that each deterministic FRF corresponding to a sample of the natural frequency ω_n has a sharp peak at frequency $\omega_n\sqrt{1 - 2\zeta_n^2}$. The effect of taking the mean is equivalent to add up those FRFs with peaks at different frequencies and dividing the result by the number of samples. As a result, the mean appears more damped than the deterministic system in the neighborhood of the mean natural frequency and is closer to the deterministic system at other frequencies. It is also observed that for gamma distribution, the modified Laplace's method leads to results significatively larger than the ones from MCS. This is due to the fact that, while for other distributions the methods provide a good approximation for the resonance frequency, for gamma distribution, the method leads to a result close to the deterministic response. The accuracy of the modified Laplace's method is therefore dependent on the pdf of the random variable.

Overall, the level of damping has a significant impact on Laplace's method for both SDOF and MDOF systems. When the systems are reasonably damped (about 10% damping), all the proposed methods work well and the results agree with each other. But the situation changes dramatically when damping becomes small (about 1% or smaller). In this case only numerical integration is able to produced results which agree with the direct Monte Carlo simulation results, at frequencies in the vicinity of resonance.

Therefore, one of the key conclusions from this work is that care should be taken for dynamic analysis of stochastic systems with very light damping.

7.2. Summary of numerical results

It can be observed that damping has an important effect on the standard deviation of the response. The higher is the damping, the smaller is the standard deviation compared to the mean. This is observed for both SDOF and MDOF systems, but is more evident for MDOF systems. This effect is independent of the distribution of the random variable. Comparing results of mean and standard deviation for SDOF systems, it is observed that mean and standard deviation of all the pdfs give similar results. On the other side, this is only observed for frequencies near the first natural frequency for MDOF systems. For higher frequencies, mean of the FRF for normal distribution appears more damped than the ones obtained with other distributions, and standard deviation is generally larger than the one for other distributions. Values of mean and standard deviation of FRF for MDOF for lognormal, gamma and uniform distribution are almost coincident for every frequency.

8. Conclusions

This paper considers qualitative and quantitative analysis of response statistics of damped stochastic linear dynamical systems. The mean response of the system is effectively more damped than the underlying baseline system. Assuming uniform distribution of the squared natural frequency, a closed-from expression of the equivalent damping of the mean response is obtained as

$$\zeta_e \approx 3^{1/4}\sqrt{\epsilon\zeta_n/\pi} \tag{106}$$

where ϵ is the standard deviation of the squared natural frequency and ζ_n is the damping ratio. This simple expression qualitatively explains many key general observations on the dynamic response.

For quantitative analysis of dynamic response statistics, considering the distribution of the system eigenvalues, two novel semi-analytical methods, namely hybrid Laplace-numerical integration and hybrid modified Laplace methods, have been proposed. The proposed methods have been extended to general multiple degree of freedom systems assuming uncorrelated eigensolutions. Due to the semi-analytical nature of the results, the proposed methods can offer computational advantages over direct Monte Carlo simulations for structures with very large number of degrees of freedom.

Mean of the real and imaginary parts of the response vector and second moment of its absolute value are calculated making use of the proposed methods. Four probability density functions, namely uniform, normal, log-normal and gamma, are considered for the natural frequencies. Exact closed-form expressions for the response moments have been obtained for the uniform distribution. It is observed that the accuracy of the proposed method depends on the pdf of the random variable, on the damping factor and on the frequency at which the integral is evaluated. In general lightly damped systems show less accuracy compared to systems with more damping. The assumption of uncorrelated eigenvalues is likely to be valid for low frequencies only, where less overlap between the eigenvalues is expected. Future works will consider dynamic response in the medium and higher frequencies where joint pdf of eigenvalues and eigenvectors need to be considered.

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