International Journal of Engineering Science 66–67 (2013) 1–13

Contents lists available at SciVerse ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Vibration of nonlocal Kelvin–Voigt viscoelastic damped Timoshenko beams

Y. Lei [a], S. Adhikari [b,*], M.I. Friswell [b]

[a] College of Aerospace and Material Engineering, National University of Defence Technology, Changsha, Hunan 410073, PR China
[b] College of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, UK

ARTICLE INFO

Article history:
Received 10 December 2012
Received in revised form 9 February 2013
Accepted 11 February 2013
Available online 9 April 2013

Keywords:
Vibration
Viscoelastic
Nonlocal elasticity
Damping
Carbon nanotube
Timoshenko beam

ABSTRACT

This paper investigates the dynamic behavior of nonlocal viscoelastic damped nanobeams. The Kelvin–Voigt viscoelastic model, velocity-dependent external damping and Timoshenko beam theory are employed to establish the governing equations and boundary conditions for the bending vibration of nanotubes. Using transfer function methods (TFM), the natural frequencies and frequency response functions (FRF) are computed for beams with different boundary conditions. Unlike local structures, taking into account rotary inertia and shear deformation, the nonlocal beam has maximum frequencies, called the escape frequencies or asymptotic frequencies, which are obtained for undamped and damped nonlocal Timoshenko beams. Damped nonlocal beams are also shown to possess an asymptotic critical damping factor. Taking a carbon nanotube as a numerical example, the effects of the nonlocal parameter, viscoelastic material constants, the external damping ratio, and the beam length-to-diameter ratio on the natural frequencies and the FRF are investigated. The results demonstrate the efficiency of the proposed modeling and analysis methods for the free vibration and frequency response analysis of nonlocal viscoelastic damped Timoshenko beams.

1. Introduction

Static deformation, dynamic response, wave propagation and fluid–structure interactions of nano-structural elements such as nanowires, nanorods, nanobeams and nanoplates have attracted significant research attention over the past decade. Superior mechanical, electrical and thermal properties of nanostructures such as carbon nanotubes and grapheme and their potential engineering applications are the primary reasons behind this research interest. There are two major computational approaches available for the dynamic analysis of nanostructures, namely molecular dynamic (MD) simulation, and continuum mechanics based approaches. Among the mechanics based methods, the nonlocal continuum model (NCM) has received significant interest in recent years. Using the NCM length-scale effects may be included in a simple physically understandable way. Liew, Hu, and He (2008) presented the modeling of vibration and flexural wave propagation of single-walled carbon nanotubes (SWCNT) using the MD and continuum model. They highlighted that the nonlocal model can predict MD results better than the classical elasticity model when the nonlocal parameter was determined carefully for different situations. Murmu and Adhikari (2007) also compared the natural frequencies of SWCNT with tip mass and found that the nonlocal beam theory accurately predicts the frequencies from the MD simulations for certain nonlocal parameters. Wang, Zhang, Fei, and Murmu (2012) compared MD simulation results with nonlocal shell and beam model results for vibration analysis and found better agreement between the shell model and MD simulation. Lu, Lee, Lu, and Zhang (2007, 2006)

* Corresponding author. Tel.: +44 1792602088.
E-mail address: s.adhikari@swansea.ac.uk (S. Adhikari).

0020-7225/$ - see front matter
http://dx.doi.org/10.1016/j.ijengsci.2013.02.004

applied nonlocal Euler–Bernoulli and Timoshenko beam theories to investigate the free vibration and wave propagation of single- and multi-walled carbon nanotubes. Reddy (2007) presented the analytical solutions for bending, buckling and natural vibration of simply supported nonlocal beams using various beam theories, including Euler–Bernoulli, Timoshenko and Levinson. Thai (2012) proposed a nonlocal higher-order beam theory for the bending, buckling and vibration analysis of nanobeams. Torabi and Dastgerdi (2012) proposed an analytical method for the free vibration analysis of cracked nanobeams using Timoshenko beam theory. Liu and Yang (2012) researched elastic wave propagation in a single-layered graphene sheet on a two-parameter elastic foundation using nonlocal elasticity. Narendar and Gopalakrishnan (2012) used nonlocal continuum mechanics to describe ultrasonic flexural wave dispersion characteristics of monolayer graphene embedded in a polymer matrix. Wang and Li (2012) reviewed the recent progress in the dynamical analysis of nanostructures and focused on the vibration, wave propagation, and fluid–structure interaction of nanotubes.

A carbon nanotube (CNT) structure is effectively a beam with a hollow circular cross section, and the shear deflection and the rotatory inertia have significant impact on the dynamics. Timoshenko beam theory or higher-order theories should be adopted to analyze dynamic problems, especially for the high frequency response and wave propagation with higher wave numbers. Wang, Zhang, and He (2007) derived the governing equations and boundary conditions using Hamilton's principle for the vibration of nonlocal Timoshenko beams and analytically obtained the vibration frequencies of beams with various end conditions. Wang and Liew (2007) applied nonlocal continuum mechanics to analyze the static deformation of micro- and nano-structures using Euler–Bernoulli and Timoshenko beam theories and various closed-form solutions were presented. From the results it was concluded that the shear effect is evident for carbon nanotubes and the importance of applying high-order beam theory was justified. Shen, Li, Sheng, and Tang (2012a, 2012b) investigated the transverse vibration of nanotube-based micro-mass sensors using nonlocal Timoshenko beam theory. The transfer function method (TFM) was employed to obtain the closed-form solutions and it was noted that the non-local Timoshenko beam model is more appropriate than the nonlocal Euler–Bernoulli beam model for short SWCNT sensors. Ma, Gao, and Reddy (2008) developed a microstructure-dependent Timoshenko beam model with Poisson's effects based on a modified coupled stress theory. The static bending and free vibration problems of a simply-supported beam were solved. For beams with complex geometrical shapes (e.g. cross sectional variations) and boundary conditions, the analytical solution for the nonlocal Timoshenko beam becomes difficult and consequently numerical and approximate methods need to be adopted. Pradhan (2012) derived nonlocal Galerkin finite element equations for the analysis of CNTs with Timoshenko nonlocal beam theory and obtained the bending, buckling and free vibration results that were in good agreement with local and nonlocal results available in the literature. Roque, Ferreira, and Reddy (2011) studied the bending, buckling and free vibration of Timoshenko nanobeams with a meshless method. Ghannadpour, Mohammadi, and Fazilati (2013) the Ritz method to analyze the bending, buckling and vibration problems of nonlocal Euler–Bernoulli beams.

Since nanostructures have potential applications in many engineering fields with complex physical environments, some multi-field coupling effects, such as an embedded matrix, attached masses, moving loads, thermal-electrical–mechanical fields or magnetic fields were considered for buckling, vibration and wave propagation problems. Benzair et al. (2008) studied thermal effects on the vibration of single-walled carbon nanotubes using nonlocal Timoshenko beam theory. Wang, Li, and Kishimoto (2012) investigated the effects of axial load and an elastic matrix on flexural wave propagation in nanotubes with a nonlocal Timoshenko beam model. Murmu, McCarthy, and Adhikari (2012) applied an external longitudinal magnetic field and studied the vibration response of double-walled carbon nanotubes. Yan, Wang, and Zhang (2012) considered a double shell-potential flow model for the free vibration of fluid-filled single-walled carbon nanotubes. When nanostructures are in a humid or a magnetic environment, resting on a substrate (viscoelastic foundation), embedded in a polymer matrix or conveying a viscous fluid flow (Kiani, 2013), damping or viscoelastic effects will significantly affect the dynamics of nanostructures.

The damping characteristics of structures play a very important role in the dynamic analysis of structures. Damping mechanisms are often complex, and hence the damping is often approximated as proportional damping in engineering applications, with the damping parameters obtained from experiment. Although there are many papers that consider the dynamical behavior of CNTs and SLGSs using nonlocal elasticity theory, there are only a small number that consider nonlocal viscoelastic nanosystems (Arani, Shiravand, Rahi, & Kolahchi, 2012; Pouresmaeeli, Ghavanloo, & Fazelzadeh, 2013). The analysis of damped nanosystems has received very little attention (Adhikari, Mrumu, & McCarthy, 2013), although the existence of damping in nanomaterials has been recognised (Kim & Kim, 2011; Yadollahpour, Ziaei-Rad, & Karimzadeh, 2010). The combined effect of external damping and viscoelastic internal damping on the dynamical characteristics of nanosystems has not been investigated.

An accurate modeling of damping for the future generation of nano electro mechanical systems (NEMS) is vitally important for their design and analysis. As the scan rates of high-speed atomic force microscopes (AFM) increase, the need for more accurate estimation of damping characteristics becomes more apparent (Payton, Picco, Miles, Homer, & Champneys, 2012). Since AFM will remain as an important imaging tool for nanostructures, the understanding of damping will lead to superior image quality. The damping in nanosystems may arise due to the effect of external forces such as magnetic forces (Lee & Lin, 2010), interaction with the substrate, humidity and thermal effects (Chen, Ma, Liu, Zheng, & Xu, 2011). The characteristics of the damping could be elastic or, more generally, viscoelastic in nature. Another area where damping at the nanoscale is crucially important is nanosensors (Adhikari & Chowdhury, 2012). Vibration based nanoscale mass sensors exploit the frequency shift to detect mass. The quality of measurement of this frequency shift depends on the damping characteristics of the oscillators (Calleja, Kosaka, San Paulo, & Tamayo, 2012). In the case of biosensing (Calleja et al., 2012), cantilever sensors may need to operate within a fluidic environment. In that case, both internal as well as external damping is important. The study taken in this paper is motivated by these facts.

In this paper we consider a nonlocal viscoelastic constitutive model and external velocity-dependent damping model to analyze the dynamic characteristics of Timoshenko beams with different boundary conditions using the transfer function method (TFM). The complex eigenvalues and modes were computed and the influence of the nonlocal parameter, viscoelastic constants, external damping ratio and length-to-diameter ratio on the natural frequencies and frequency response functions (FRF) are investigated. In contrast to local structures, including rotary inertia in the analysis produces a maximum frequency, also known as the escape frequency or asymptotic frequency, in the response. We have obtained closed-from expressions for the asymptotic frequencies for undamped and damped Timoshenko beams. For damped nonlocal Timoshenko beams, the asymptotic critical damping factor is also obtained.

2. Nonlocal viscoelastic model

Based on Eringen's nonlocal elastic theory, the relationship between the nonlocal tress tensor and the local stress tensor for a linear and homogeneous elastic solid is

$$t_{kl} = \int_V k(x - x')\sigma_{kl} dV \tag{1}$$

where t_{kl} is the nonlocal elastic stress tensor, $k(|\bullet|)$ is the nonlocal kernel function and σ_{kl} is the stress tensor of local elasticity satisfying

$$\sigma_{kl} = \lambda_e \varepsilon_{rr}\delta_{kl} + 2\mu_e \varepsilon_{kl} \qquad \varepsilon_{kl} = (u_{k,l} + u_{l,k})/2 \tag{2}$$

This model can be in extended directly to a nonlocal viscoelastic model by

$$t^*_{kl} = \int_V k(|x - x'|)\sigma^*_{kl} dV \tag{3}$$

where t^*_{kl} and σ^*_{kl} are the nonlocal and local viscoelastic stress tensors. For a special kernel function, the differential form of the above equation for the nonlocal model may be obtained as

$$[1 - (e_0 a)^2 \nabla^2] t^*_{kl} = \sigma^*_{kl} \tag{4}$$

The extensional relaxation modulus for a general Maxwell model of linear and homogeneous viscoelastic solid is

$$E(t) = E + \sum_{m=1}^{N} E_m e^{-\frac{t}{\tau_m}} = E_0 - \sum_{m=1}^{N} E_m \left(1 - e^{-\frac{t}{\tau_m}}\right) \tag{5}$$

where $E_0 = E + \sum_{m=1}^{N} E_m$ is the initial extensional modulus and $\tau_m = \frac{\eta_m}{E_m}$ are the relaxation time constants.

Using the Boltzmann superposition principle, the integral constitutive relations for linear homogeneous viscoelastic Timoshenko beams may be expressed as

$$\sigma^*_{xx} = E(t)\varepsilon_{xx}(0) + \int_0^t E(t - \tau)\frac{\partial \varepsilon_{xx}(\tau)}{\partial \tau} d\tau \tag{6a}$$

$$\sigma^*_{xz} = G(t)\varepsilon_{xz}(0) + \int_0^t G(t - \tau)\frac{\partial \varepsilon_{xx}(\tau)}{\partial \tau} d\tau \tag{6b}$$

where $G(t)$ is the shear relaxation modulus of this viscoelastic solid. The nonlocal viscoelastic equations for Timoshenko beams may be written as

$$[1 - (e_0 a)^2 \nabla^2] t^*_{xx} = E_\infty \varepsilon_{xx}(t) + \int_0^t \sum_{m=1}^{N} E_m e^{\frac{\tau - t}{\tau_m}} \frac{\partial \varepsilon_{xx}(\tau)}{\partial \tau} d\tau \tag{7a}$$

$$[1 - (e_0 a)^2 \nabla^2] t^*_{xz} = G_\infty \varepsilon_{xx}(t) + \int_0^t \sum_{m=1}^{N} G_m e^{\frac{\tau - t}{\tau_m}} \frac{\partial \varepsilon_{xx}(\tau)}{\partial \tau} d\tau \tag{7b}$$

3. The nonlocal Kelvin–Voigt viscoelastic Timoshenko beam

For Timoshenko beam, the normal strain, ε_x, and shear strain, γ_{xz}, are given by

$$\varepsilon_{xx} = z\frac{\partial \theta}{\partial x}, \quad \gamma_{xz} = \theta + \frac{\partial w}{\partial x} \tag{8a, b}$$

The nonlocal viscoelastic constitutive relation for the beam with the Kelvin–Voigt viscoelastic model is

$$t_{xx} - (e_0a)^2 \frac{\partial^2 t_{xx}}{\partial x^2} = E\left(1 + \tau_d \frac{\partial}{\partial t}\right) z \frac{\partial \theta}{\partial x} \tag{9a}$$

$$t_{xz} - (e_0a)^2 \frac{\partial^2 t_{xz}}{\partial x^2} = G\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\theta + \frac{\partial w}{\partial x}\right) \tag{9b}$$

where E is the Young's modulus and G is the shear modulus, τ_d is the viscous damping coefficient, (e_0a) is the nonlocal characteristic parameter, w is the transverse deflection and θ is the rotation of the cross-section.

The equation of motion for the damped nonlocal viscoelastic Timoshenko beam, including external velocity-dependent damping, is

$$\rho A \frac{\partial^2 w}{\partial t^2} + C_1 \frac{\partial w}{\partial t} - \frac{\partial Q}{\partial x} = p \tag{10}$$

$$\rho I \frac{\partial^2 \theta}{\partial t^2} + C_2 \frac{\partial \theta}{\partial t} - \frac{\partial M}{\partial x} + Q = m \tag{11}$$

Here ρ is the mass density, I is the second moment of the cross section, A is the section area, C_1 is the displacement-velocity-dependent viscous damping coefficient, C_2 is the rotation-velocity dependent viscous damping coefficient, and p and m are external distributed loads.

The nonlocal bending moment, M, and the nonlocal shear force, Q, can be defined as

$$M = \int_A z t_{xx} dA \tag{12}$$

$$Q = \int_A t_{xz} dA \tag{13}$$

Integrating Eq. (9), using the definitions in the preceding equations, gives

$$M = (e_0a)^2 \frac{\partial^2 M}{\partial x^2} + EI\left(1 + \tau_d \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial x} \tag{14}$$

$$Q = (e_0a)^2 \frac{\partial^2 Q}{\partial x^2} + \kappa GA\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\theta + \frac{\partial w}{\partial x}\right) \tag{15}$$

Here κ is the shear correction factor that depends on the material and geometric parameters. From Eqs. (14) and (15), the nonlocal bending moment, M, and the nonlocal shear force, Q, are obtained, which are functions of the displacement w and rotation θ, and given by

$$M = EI\left(1 + \tau_d \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial x} + (e_0a)^2 \left[\rho A \frac{\partial^2 w}{\partial t^2} + C_1 \frac{\partial w}{\partial t} + \rho I \frac{\partial^3 \theta}{\partial x \partial t^2} + C_2 \frac{\partial^2 \theta}{\partial x \partial t} - p - \frac{\partial m}{\partial x}\right] \tag{16}$$

$$Q = \kappa GA\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\theta + \frac{\partial w}{\partial x}\right) + (e_0a)^2 \left[\rho A \frac{\partial^3 w}{\partial x \partial t^2} + C_1 \frac{\partial^2 w}{\partial x \partial t} - \frac{\partial p}{\partial x}\right] \tag{17}$$

The governing equation of nonlocal viscoelastic Timoshenko beam, are found by substituting Eqs. (16) and (17) into Eqs. (10) and (11), as

$$\rho A \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial^2 w}{\partial t^2} + C_1 \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial w}{\partial t} - \kappa GA\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) = \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] p \tag{18}$$

$$\rho I \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial^2 \theta}{\partial t^2} + C_2 \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial \theta}{\partial t} + \kappa GA\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\theta + \frac{\partial w}{\partial x}\right) - EI\left(1 + \tau_d \frac{\partial}{\partial t}\right) \frac{\partial^2 \theta}{\partial x^2} = \left[1 - (e_0a)^2 \frac{\partial^2}{\partial x^2}\right] m \tag{19}$$

When e_0a, C_1, C_2 and τ_d are all set to zero, these equations degenerate to the equations of motion for the classical elastic Timoshenko beam.

The boundary conditions at the ends at $x = 0$ and $x = L$ are given by:

Displacement:

$$w = w_B \quad \text{or} \quad Q = Q_B \tag{20a, b}$$

Rotation

$$\theta = \theta_B \quad \text{or} \quad M = M_B \tag{21a, b}$$

where w_B, θ_B, Q_B and M_B are known functions at the ends of beam which are selected based on the particular problem of interest.

4. Analysis of damped natural frequencies

The equations of motion derived in the last section can now be used to obtain the dynamic characteristics of the system. To obtain the natural frequencies assume that $w = We^{i\omega t}$ and $\theta = \Theta e^{i\omega t}$ and consider free vibration. Then,

$$-\rho A\omega^2\left[1-(e_0a)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]W + i\omega C_1\left[1-(e_0a)^2\frac{d^2}{\mathrm{d}x^2}\right]W - \kappa GA(1+i\omega\tau_d)\left(\frac{\mathrm{d}\Theta}{dx}+\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right) = 0 \tag{22}$$

$$-\rho I\omega^2\left[1-(e_0a)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\Theta + i\omega C_2\left[1-(e_0a)^2\frac{d^2}{dx^2}\right]\Theta + \kappa GA(1+i\omega\tau_d)\left(\Theta+\frac{\mathrm{d}W}{\mathrm{d}x}\right) - EI(1+i\omega\tau_d)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} = 0 \tag{23}$$

For free vibration, the boundary conditions described by Eqs. (20a,b) and (21a,b) can be expressed as

Displacement:

$$W = 0 \quad \text{or} \quad Q = \kappa GA(1+i\omega\tau_d)\left(\Theta+\frac{dW}{dx}\right) + (e_0a)^2\left[-\rho A\omega^2\frac{\mathrm{d}W}{\mathrm{d}x} + i\omega C_1\frac{\mathrm{d}W}{\mathrm{d}x}\right] = 0 \tag{24a, b}$$

Rotation:

$$\Theta = 0 \quad \text{or} \quad M = EI(1+i\omega\tau_d)\frac{\mathrm{d}\Theta}{\mathrm{d}x} + (e_0a)^2\left[-\rho A\omega^2 W + i\omega C_1 W - \rho I\omega^2\frac{\mathrm{d}\Theta}{\mathrm{d}x} + i\omega C_2\frac{\mathrm{d}\Theta}{dx}\right] = 0 \tag{25a, b}$$

The following non-dimensional terms are now introduced

$$\bar{x} = \frac{x}{L},\quad \overline{W} = \frac{W}{L},\quad \delta = \frac{I}{AL^2},\quad \beta = \frac{EI}{\kappa GAL^2},\quad \alpha = \frac{e_0a}{L},\quad c = L^2\sqrt{\frac{\rho A}{EI}}$$
$$\zeta_1 = \frac{C_1}{\rho A}c,\quad \zeta_2 = \frac{C_2}{\rho I}c,\quad \Omega = \omega c,\quad \tau = \frac{\tau_d}{c}$$

Then the equations of motion may be written as

$$-\Omega^2\beta\left[\overline{W}-\alpha^2\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\bar{x}^2}\right] + i\Omega\beta\zeta_1\left[\overline{W}-\alpha^2\frac{d^2\overline{W}}{d\bar{x}^2}\right] - (1+i\Omega\tau)\left(\frac{d\Theta}{d\bar{x}}+\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\bar{x}^2}\right) = 0 \tag{26}$$

$$-\Omega^2\delta\beta\left[\Theta-\alpha^2\frac{\mathrm{d}^2\Theta}{\mathrm{d}\bar{x}^2}\right] + i\Omega\delta\beta\zeta_2\left[\Theta-\alpha^2\frac{\mathrm{d}^2\Theta}{\mathrm{d}\bar{x}^2}\right] + (1+i\Omega\tau)\left(\Theta+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{x}}\right) - \beta(1+i\Omega\tau)\frac{\mathrm{d}^2\Theta}{\mathrm{d}\bar{x}^2} = 0 \tag{27}$$

Furthermore, the nonlocal bending moment M and shear force Q then become

$$\frac{Q}{\kappa GA} = (1+i\Omega\tau)\left(\Theta+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{x}}\right) + \alpha^2\beta\left[-\Omega^2\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{x}} + i\Omega\zeta_1\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{x}}\right] \tag{28}$$

$$\frac{ML}{EI} = (1+i\Omega\tau)\frac{\mathrm{d}\Theta}{\mathrm{d}\bar{x}} + \alpha^2\left[-\Omega^2\overline{W} + i\Omega\zeta_1\overline{W} - \delta\Omega^2\frac{\mathrm{d}\Theta}{\mathrm{d}\bar{x}} + i\Omega\zeta_2\delta\frac{\mathrm{d}\Theta}{\mathrm{d}\bar{x}}\right] \tag{29}$$

For undamped nonlocal elastic beam, $\zeta_1 = \zeta_2 = \tau_d = 0$, and consequently Eqs. (26) and (27) degenerate to the equations of motion for the nonlocal elastic Timoshenko beam as

$$(1-\Omega^2\beta\alpha^2)\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\bar{x}^2} + \Omega^2\beta\overline{W} + \frac{\mathrm{d}\Theta}{\mathrm{d}\bar{x}} = 0 \tag{30}$$

$$\beta(1-{}^2\delta\alpha^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}\bar{x}^2} - \frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{x}} - (1-\Omega^2\delta\beta)\Theta = 0 \tag{31}$$

Let $\overline{W} = \overline{W}_n e^{-ik\bar{x}}$ and $\Theta = i\Theta_n e^{-ik\bar{x}}$. Substituting these expressions into Eqs. (26) and (27) gives

$$[-\Omega^2\beta(1+\alpha^2k^2) + i\Omega\beta\zeta_1(1+\alpha^2k^2) + (1+i\Omega\tau)k^2]\overline{W}_n - (1+i\Omega\tau)k\Theta_n = 0 \tag{32}$$

$$[\Omega^2\delta\beta(1+\alpha^2k^2) - i\Omega\delta\beta\xi_2(1+\alpha^2k^2) - (1+i\Omega\tau) - \beta(1+i\Omega\tau)k^2]\Theta_n + (1+i\Omega\tau)k\overline{W}_n = 0 \tag{33}$$

Nontrivial solutions for $\overline{W}_n$ and Θ_n are obtained by setting the determinant of the matrix associated with these variables to zero, which gives the characteristic equation. After simplifying this can be written as

$$[-\Omega^2\beta(1+\alpha^2k^2)+i\Omega\beta\zeta_1(1+\alpha^2k^2)+(1+i\Omega\tau)k^2][\Omega^2\delta\beta(1+\alpha^2k^2)-i\Omega\delta\beta\xi_2(1+\alpha^2k^2)-(1+i\Omega\tau)-\beta(1+i\Omega\tau)k^2] + (1+i\Omega\tau)^2k^2=0 \quad (34)$$

This characteristic equation may be written as a quartic equation for ω as

$$a_1\Omega^4+ia_2\Omega^3+a_3\Omega^2+ia_4\Omega+a_5=0 \quad (35)$$

where

$$a_1=-\delta\beta^2\alpha^4k^4-2\delta\beta^2\alpha^2k^2-\delta\beta^2=-\delta\beta^2(1+\alpha^2k^2)^2$$
$$a_2=2\alpha^2\beta^2\zeta_1\delta k^2+\tau\beta\delta k^2+2\alpha^2\beta^2\zeta_2\delta k^2+\beta^2\delta\zeta_1+\beta^2\alpha^4\zeta_1\delta k^4+\beta^2\alpha^4\zeta_2\delta k^4+\tau\beta\alpha^2\delta k^4+\tau\beta^2\alpha^2k^4 + \tau\beta^2k^2+\tau\beta\alpha^2k^2+\beta^2\delta\zeta_2+\tau\beta$$
$$a_3=\tau^2\beta k^4+\zeta_1\tau\alpha^2\beta^2k^4+\zeta_1\tau\beta^2k^2+\delta\zeta_2\tau\beta\alpha^2k^4+\zeta_1\tau\beta\alpha^2k^2+\delta\zeta_2\tau\beta k^2+\zeta_1\tau\beta+\delta\zeta_1\zeta_2\alpha^4\beta^2k^4+\alpha^2\beta^2k^4 + 2\delta\zeta_1\zeta_2\alpha^2\beta^2k^2+\beta^2k^2+\delta\zeta_1\zeta_2\beta^2+\delta\beta\alpha^2k^4+\beta\alpha^2k^2+\delta\beta k^2+\beta$$
$$a_4=-\alpha^2\beta^2\zeta_1k^4-\beta^2\zeta_1k^2-\alpha^2\delta\beta\zeta_2k^4-\alpha^2\beta\zeta_1k^2-2\tau\beta k^4-\delta\beta\zeta_2k^2-\beta\zeta_1$$
$$a_5=-\beta k^4$$

This is a very general characteristic equation and some special cases of interest can be obtained. These include: the undamped beam where $\alpha=\zeta_1=\zeta_2=\tau=0$; the undamped nonlocal beam where $\zeta_1=\zeta_2=\tau=0$; the damped nonlocal elastic beam for $\tau=0$; and the nonlocal viscoelastic beam for $\zeta_1=\zeta_2=0$.

The roots of Eq. (35) are the eigenvalues of the nonlocal damped viscoelastic Timoshenko beam. Closed-form solutions of Eq. (35) are possible via Ferrari's formula. However, the resulting expressions are generally too complex to be physically meaningful and practically useful. Therefore, in the next section, we pursue a numerical approach to obtain the natural frequencies. The proposed Transfer Function method (TFM) is also general in that different boundary conditions may be considered. First, we consider some special cases of interest.

4.1. Undamped elastic beam

For an undamped local elastic beam, $\alpha=\zeta_1=\zeta_2=\tau=0$ and thus

$$-\delta\beta\Omega_L^4+[(\beta+\delta)k^2+1]\Omega_L^2-k^4=0 \quad (36)$$

This is the relationship between the natural frequencies and wavenumber of a classical Timoshenko beam.

For an undamped nonlocal elastic beam, $\zeta_1=\zeta_2=\tau=0$ and one obtains

$$-\delta\beta(1+\alpha^2k^2)^2\Omega_N^4+[(\beta+\delta)k^2+1](1+\alpha^2k^2)\Omega_N^2-k^4=0 \quad (37)$$

From Eqs. (36) and (37), for the same wavenumber k, we have a simple relationship of the form

$$\Omega_N^2=\frac{\Omega_L^2}{1+\alpha^2k^2} \quad (38)$$

For the natural frequencies of local and nonlocal beams, this equation is not valid in general. This is because the boundary conditions are different for local and nonlocal beams and the values of the wavenumber k are therefore also different. However, for simply supported and clamped beams this simple relationship is correct.

Solving Eq. (36) we obtain

$$\Omega_{L,1}^2(k)=\frac{(\beta+\delta)k^2+1+\sqrt{[(\beta+\delta)k^2+1]^2-4\delta\beta k^4}}{2\delta\beta} \quad (39a)$$

$$\Omega_{L,2}^2(k)=\frac{(\beta+\delta)k^2+1-\sqrt{[(\beta+\delta)k^2+1]^2-4\delta\beta k^4}}{2\delta\beta} \quad (39b)$$

Similarly, the solution of Eq. (37) gives

$$\Omega_{N,1}^2(\bar{k})=\frac{\Omega_{L,1}^2(\bar{k})}{1+\alpha^2\bar{k}^2} \quad (40a)$$

$$\Omega_{N,2}^2(\bar{k})=\frac{\Omega_{L,2}^2(\bar{k})}{1+\alpha^2\bar{k}^2} \quad (40b)$$

4.2. Undamped viscoelastic beam

For an undamped local viscoelastic beam we have $\alpha = \zeta_1 = \zeta_2 = 0$. Using these parameters we obtain

$$-\delta\beta\Omega_{dL}^4 + i\tau[(\beta+\delta)k^2+1]\Omega_{dL}^3 + [(\beta+\delta)k^2+\tau^2k^4+1]\Omega_{dL}^2 - 2i\tau k^4\Omega_{dL} - k^4 = 0 \tag{41}$$

This is the relationship between the natural frequencies and the wavenumbers of classical Timoshenko beams. Solving Eq. (41) we get

$$\Omega_{dL,1} = \Omega_{L,1}\left(i\frac{\Omega_{L,1}\tau}{2} + \sqrt{1-\frac{\Omega_{L,1}^2\tau^2}{4}}\right) \tag{42a}$$

$$\Omega_{dL,2} = \Omega_{L,2}\left(i\frac{\Omega_{L,2}\tau}{2} + \sqrt{1-\frac{\Omega_{L,2}^2\tau^2}{4}}\right) \tag{42b}$$

For an undamped nonlocal viscoelastic beam $\zeta_1 = \zeta_2 = 0$ and we have

$$-\delta\beta(1+\alpha^2k^2)^2\Omega_N^4 + i\tau[(\beta+\delta)k^2+1](1+\alpha^2k^2)\Omega_N^3 + \{[(\beta+\delta)k^2+1](1+\alpha^2k^2)+\tau^2k^4\}\Omega_N^2 - 2i\tau k^4 - k^4 = 0 \tag{43}$$

Solving Eq. (43), we get

$$\Omega_{dN,1} = \Omega_{L,1}\left(i\frac{\Omega_{N,1}\tau}{2} + \sqrt{1-\frac{\Omega_{N,1}^2\tau^2}{4}}\right) \tag{44a}$$

$$\Omega_{dN,2} = \Omega_{N,2}\left(i\frac{\Omega_{N,2}\tau}{2} \pm \sqrt{1-\frac{\Omega_{N,2}^2\tau^2}{4}}\right) \tag{44b}$$

4.3. Asymptotic analysis

In Eq. (35) when $k \to \infty$, an asymptotic equation for the natural frequency parameter Ω can be obtained as

$$\begin{aligned}&-(\beta\alpha^4\delta)\Omega^4 + i(\tau\beta\alpha^2 + \beta\delta\zeta_1\alpha^4 + \beta\delta\zeta_2\alpha^4 + \tau\alpha^2\delta)\Omega^3 + (\tau^2 + \tau\zeta_1\beta\alpha^2 + \delta\tau\zeta_2\alpha^2 + \delta\zeta_1\zeta_2\beta\alpha^4 + \beta\alpha^2 + \delta\alpha^2)\Omega^2\\ &\quad - i(\delta\zeta_2\alpha^2 + \zeta_1\beta\alpha^2 + 2\tau)\Omega - 1 = 0\end{aligned} \tag{45}$$

For an undamped nonlocal elastic beam $\zeta_1 = \zeta_2 = \tau = 0$ Eq. (45) becomes

$$-(\beta\alpha^4\delta)\Omega^4 + (\beta\alpha^2+\delta\alpha^2)\Omega^2 - 1 = 0 \tag{46}$$

Thus one obtains the following two asymptotic frequencies

$$\lim_{k\to\infty}\frac{\Omega_{k1}}{c} = \lim_{k\to\infty}\omega_{k1} = \frac{1}{c\alpha\sqrt{\delta}} = \frac{1}{e_0a}\sqrt{\frac{E}{\rho}} \tag{47a}$$

$$\lim_{k\to\infty}\frac{\Omega_{k2}}{c} = \lim_{k\to\infty}\omega_{k2} = \frac{1}{c\alpha\sqrt{\beta}} = \frac{1}{e_0a}\sqrt{\frac{\kappa G}{\rho}} \tag{47b}$$

It is known that the classical longitudinal and shear wave propagation velocities of the one-dimensional elastic solid are $c_1 = \sqrt{\frac{E}{\rho}}$ and $c_2 = \sqrt{\frac{G}{\rho}}$ respectively. The critical frequencies for the undamped nonlocal elastic Timoshenko beam are $\frac{1}{e_0a}c_1$ and $\frac{\sqrt{\kappa}}{e_0a}c_2$, respectively. These values are fundamental to the system and do not depend on the boundary conditions.

For an undamped nonlocal viscoelastic beam $\zeta_1 = \zeta_2 = 0$ and when $k \to \infty$ we obtain

$$-(\beta\alpha^4\delta)\Omega^4 + i(\tau\beta\alpha^2 + \tau\alpha^2\delta)\Omega^3 + (\tau^2+\beta\alpha^2+\delta\alpha^2)\Omega^2 - i(2\tau)\Omega - 1 = 0 \tag{48}$$

Applying factorization, this equation can be written as

$$(\beta\alpha^2\Omega^2 - i\tau\Omega - 1)(-\delta\alpha^2\Omega^2 + i\tau\Omega + 1) = 0 \tag{49}$$

and thus

$$\Omega_1 = i\frac{\tau}{2\beta\alpha^2} + \sqrt{-\frac{\tau^2}{4\beta^2\alpha^4} + \frac{1}{\beta\alpha^2}} \quad \text{and} \quad \Omega_2 = i\frac{\tau}{\delta\alpha^2} + \sqrt{-\frac{\tau^2}{4\delta^2\alpha^2} + \frac{1}{\delta\alpha^2}} \tag{50a, b}$$

The critical damping ratios are obtained by setting the oscillation frequency to zero to give

$$(\tau)_{\text{critical}} = 2\sqrt{\beta}\alpha \quad \text{and} \quad (\tau)_{\text{critical}} = 2\sqrt{\delta}\alpha \tag{51a, b}$$

For a damped nonlocal elastic beam, $\tau = 0$, and when $k \to \infty$ we obtain

$$-(\beta\alpha^4\delta)\Omega^4 + i(\beta\delta\zeta_1\alpha^4 + \beta\delta\zeta_2\alpha^4)\Omega^3 + (\delta\zeta_1\zeta_2\beta\alpha^4 + \beta\alpha^2 + \delta\alpha^2)\Omega^2 - i(\delta\zeta_2\alpha^2 + \zeta_1\beta\alpha^2)\Omega - 1 = 0 \quad (52)$$

Applying factorization, this equation can be written as

$$(\beta\alpha^2\Omega^2 - i\zeta_1\beta\alpha^2\Omega - 1)(-\delta\alpha^2\Omega^2 + i\delta\zeta_2\alpha^2\Omega + 1) = 0 \quad (53)$$

and thus

$$\Omega_1 = i\frac{\zeta_1}{2} + \sqrt{-\frac{\zeta_1^2}{4} + \frac{1}{\beta\alpha^2}} \quad \text{and} \quad \Omega_2 = i\frac{\zeta_2}{2} + \sqrt{-\frac{\zeta_2^2}{4} + \frac{1}{\delta\alpha^2}} \quad (54\text{a, b})$$

For this case, the critical damping ratios can be obtained as

$$(\zeta_1)_{\text{critical}} = \frac{2}{\sqrt{\beta}\alpha} \quad \text{and} \quad (\zeta_2)_{\text{critical}} = \frac{2}{\sqrt{\delta}\alpha} \quad (55\text{a, b})$$

Both the asymptotic frequencies and the critical damping ratios are functions of the material parameters only and independent of the boundary conditions.

5. Solution using the transfer function method (TFM)

Closed-from expressions derived in the previous section are only valid for certain special cases. For the general case, a numerical approach is required. In this section we propose the transfer function method (TFM) for nonlocal viscoelastic Timoshenko beams (Lei, Friswell, & Adhikar, 2006; Yang & Tan, 1992). To find the eigenvalues and frequency response functions (FRF) using the TFM, we define the state vector $\boldsymbol{\eta}(x, \Omega)$ and partition it as

$$\boldsymbol{\eta}(x, \Omega) = \left[\overline{W}, \frac{d\overline{W}}{d\bar{x}}, \Theta, \frac{d}{d\bar{x}}\right]^T \quad (56)$$

where the superscript T denotes matrix transpose. Eqs. (26) and (27) can be rewritten in compact state-space form as

$$\frac{d\boldsymbol{\eta}(\bar{x}, \Omega)}{d\bar{x}} = \boldsymbol{\Phi}(\Omega)\boldsymbol{\eta}(\bar{x}, \Omega) + \boldsymbol{g}(\bar{x}, \Omega) \quad (57)$$

with

$$\boldsymbol{\Phi}(\Omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\beta(\Omega^2 - i\Omega\zeta_1)}{\beta\alpha^2(\Omega^2 - i\Omega\zeta_1) - (1 + i\Omega\tau)} & 0 & 0 & \frac{1 + i\Omega\tau}{\beta\alpha^2(\Omega^2 + i\Omega\zeta_1) - (1 + i\Omega\tau)} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1 + i\Omega\tau}{\beta\delta\alpha^2(\Omega^2 - i\Omega\zeta_2) - (1 + i\Omega\tau)\beta} & \frac{\beta\delta(\Omega^2 - i\Omega\zeta_2) + 1 + i\Omega\tau}{\beta\delta\alpha^2(\Omega^2 - i\Omega\zeta_2) - (1 + i\Omega\tau)\beta} & 0 \end{bmatrix}$$

The vector $\boldsymbol{g}(\bar{x}, \Omega)$ is related to the distributed load, initial displacement and initial velocity of the beam and will vanish in the present study for the free vibration problem.

The boundary conditions can be expressed as

$$\boldsymbol{M}(\Omega)\boldsymbol{\eta}(0, \Omega) + \boldsymbol{N}(\Omega)\boldsymbol{\eta}(1, \Omega) = \boldsymbol{\gamma}(\Omega) \quad (58)$$

Here $\boldsymbol{M}(\Omega)$ and $\boldsymbol{N}(\Omega)$ are boundary condition set matrixes and $\boldsymbol{\gamma}(\Omega)$ is a vector determined by the force or displacement boundary conditions. For homogeneous boundary conditions $\boldsymbol{\gamma}(\Omega) = \mathbf{0}$.

For clamped–clamped beam the boundary condition set matrixes are given by

$$\boldsymbol{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (59\text{a, b})$$

For simply-supported beam one has

$$\boldsymbol{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha^2(i\Omega\zeta_1 - \Omega^2) & 0 & (1 + i\tau) + \alpha^2\delta(i\Omega\zeta_2 - \Omega^2) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (60\text{a})$$

$$\boldsymbol{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \alpha^2(i\Omega\zeta_1 - \Omega^2) & 0 & (1+i\Omega\tau) + \alpha^2\delta(i\Omega\zeta_2 - \Omega^2) & 0 \end{bmatrix} \tag{60b}$$

For a cantilever beam we have

$$\boldsymbol{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{61a}$$

$$\boldsymbol{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^2(i\Omega\zeta_1 - \Omega^2) & 0 & (1+i\Omega\tau) + \alpha^2\delta(i\Omega\zeta_2 - \Omega^2) & 0 \\ 0 & (1+i\Omega\tau) + \alpha^2\beta(i\Omega\zeta_1 - \Omega^2) & (1+i\Omega\tau) & 0 \end{bmatrix} \tag{61b}$$

For homogeneous boundary conditions $\gamma(\Omega) = \mathbf{0}$ and the solution of Eq. (57) can be expressed as

$$\boldsymbol{\eta}(\bar{x}, \Omega) = e^{\boldsymbol{\Phi}(\Omega)\bar{x}}\boldsymbol{\eta}_0(\Omega) \tag{62}$$

for a constant vector $\boldsymbol{\eta}_0$ that depends on the boundary conditions given by Eq. (58). For the free vibration of nonlocal beam, the natural frequencies Ω_k are the solution of the transcendental characteristic equation

$$\det[\boldsymbol{M}(\Omega) + \boldsymbol{N}(\Omega)e^{\Phi}] = 0 \tag{63}$$

Table 1
Comparison of the natural frequencies (in GHz) of a nanobeam with different boundary conditions and values of viscoelastic constant τ_d and nonlocal parameter α.

BCs	Undamped elastic beam			Kelvin–Voigt viscoelastic beam ($\zeta_1 = \zeta_2 = 0.0, \tau = 0.01$)		
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$
C-F	39.142	39.304	39.817	39.224+0.6850i	39.38+0.69246i	39.897+0.7156i
	230.14	215.74	183.09	228.92+23.391i	214.755+20.517i	182.49+14.804i
	589.23	487.01	349.81	568.92+153.37i	475.61+104.77i	345.62+54.051
	1040.8	736.93	465.04	924.27+478.53i	696.79+239.90i	455.12+95.535i
S–S	108.86	103.86	92.175	108.73+5.2349i	103.75+4.7647i	92.099+3.7532i
	406.70	344.37	253.25	400.09+73.069i	340.36+52.387i	251.66+28.331i
	834.37	607.19	391.03	775.63+307.53i	584.94+162.87i	385.15+67.544i
	1340.0	834.37	495.38	1080.0+793.16i	807.10+354.892i	483.38+108.41i
C–C	245.47	229.76	196.06	244.02+26.618i	228.60+23.356i	195.32+16.981i
	628.45	506.85	353.99	603.74+174.47i	493.95+113.47i	3.4961+55.354i
	1123.9	764.36	475.08	975.57+557.95i	719.47+258.09i	464.44+99.677i
	1682.3	967.85	623.69	1125.7+1250.2i	874.92+413.80i	599.55+171.83i

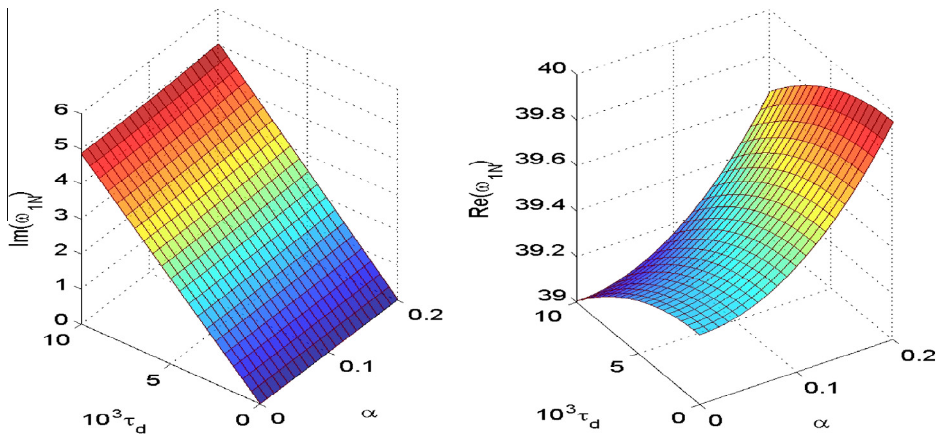


Fig. 1. The effect of the viscoelastic constant, τ_d, and the nonlocal parameter, α, on the first natural frequency.

The eigenfunction (mode shape) corresponding to Ω_k is defined as

$$\boldsymbol{\eta}(\bar{x}, \Omega_k) = e^{\Phi(\Omega_k)\bar{x}} \boldsymbol{\varphi}_k \tag{64}$$

where the non-zero vector $\boldsymbol{\varphi}_k$ satisfies

$$[\boldsymbol{M}(\Omega_k) + \boldsymbol{N}(\Omega_k)e^{\Phi}]\boldsymbol{\varphi}_k = 0 \tag{65}$$

The frequency response functions require the forcing to be specified, i.e. the function $\mathbf{g}(\bar{x}, \Omega)$. Then the solution of Eq. (57), that satisfies the boundary conditions given by Eq. (58), can be expressed as

$$\boldsymbol{\eta}(\bar{x}, \Omega) = \int_0^1 \boldsymbol{G}(\bar{x}, \xi, \Omega) g(\xi, \Omega) d\xi + \boldsymbol{H}(\bar{x}, \Omega)\boldsymbol{\gamma}(\Omega) \tag{66}$$

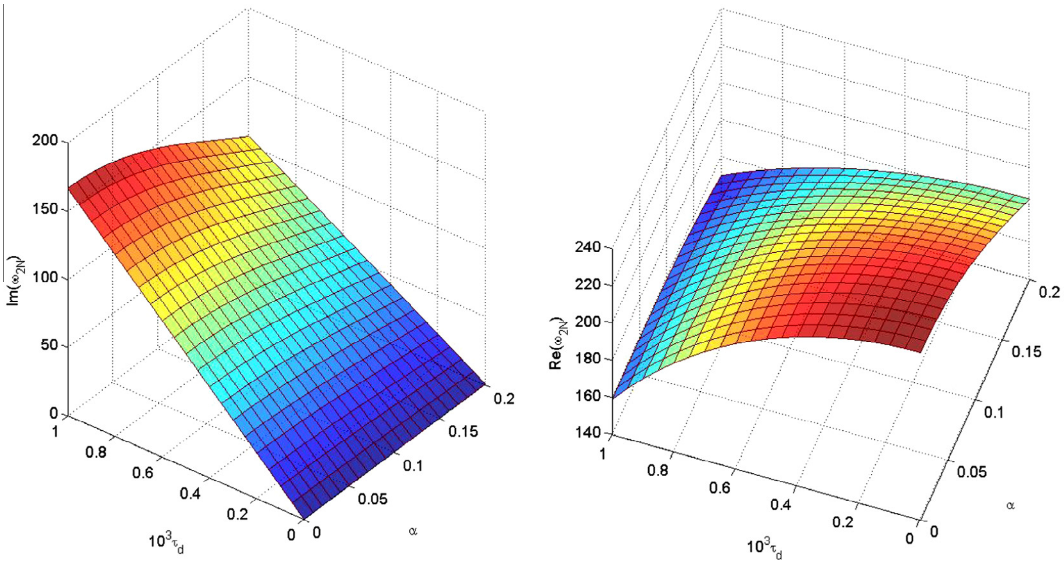


Fig. 2. The effect of the viscoelastic constant, τ_d, and the nonlocal parameter, α, on the second natural frequency.

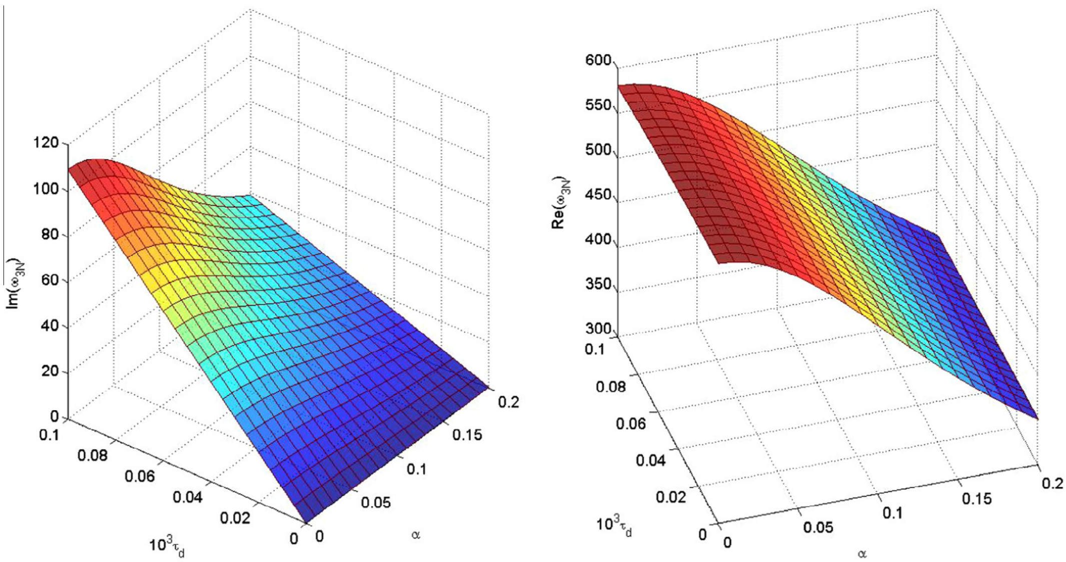


Fig. 3. The effect of the viscoelastic constant, τ_d, and the nonlocal parameter, α, on the third natural frequency.

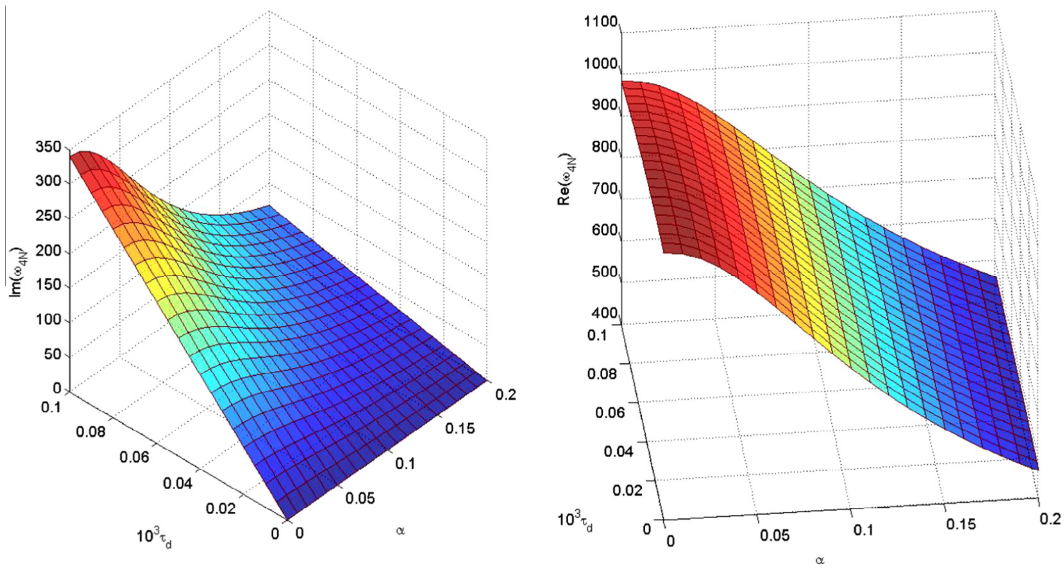
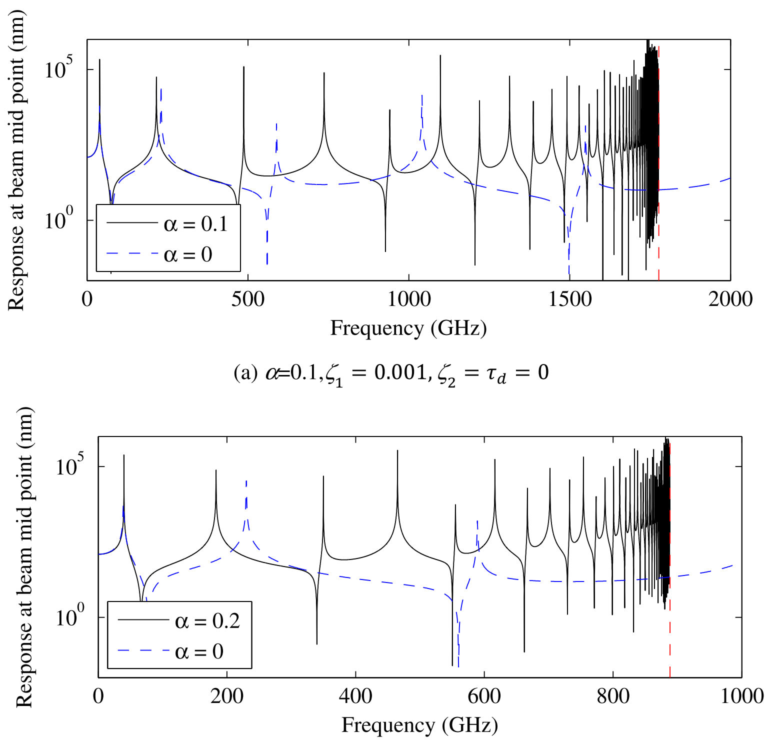


Fig. 4. The effect of the viscoelastic constant, τ_d, and the nonlocal parameter, α, on the fourth natural frequency.



(a) α=0.1, $\zeta_1 = 0.001, \zeta_2 = \tau_d = 0$

(b) α=0.2, $\zeta_1 = 0.001, \zeta_2 = \tau_d = 0$

Fig. 5. Frequency response functions for the local and nonlocal cantilever beam for two different values of the nonlocal parameter α. The cutoff frequency limit is shown by the dashed line.

where

$$\boldsymbol{G}(\bar{x}, \xi, \Omega) = \begin{cases} \boldsymbol{H}(\bar{x}, \Omega)\boldsymbol{M}(\Omega)e^{-\Phi\xi} & \bar{x} \geqslant \xi \\ -\boldsymbol{H}(\bar{x}, \Omega)\boldsymbol{N}(\Omega)e^{\Phi(1-\xi)} & \bar{x} < \xi \end{cases}$$

and

$$\boldsymbol{H}(\bar{x}, \Omega) = e^{-\boldsymbol{\Phi}\bar{x}}[\boldsymbol{M}(\Omega) + \boldsymbol{N}(\Omega)e^{\boldsymbol{\Phi}}]^{-1}$$

6. Numerical results and discussion

In this section, a single walled carbon nanotube (SWCNT) is used as an example, and the effects of the nonlocal parameter, the viscoelastic parameter and damping coefficients on the natural frequencies of a SWCNT cantilever beam are analyzed numerically. The basic parameters used in the calculations for the system are given as follows: Young's modulus E = 1 TPa, Poisson's ratio μ = 0.28, mass density ρ = 2.24 g/cm^3, diameter of the SWCNT d = 1.1 nm, effective tube thickness t = 0.342 nm, $\alpha = (e_0a/L) \in [0, 0.2]$, $\zeta_1 = \zeta_2[0, 0.02]$, $\tau \in [0, 0.01]$.

The natural frequencies of the undamped elastic and viscoelastic beams with three typical boundary conditions and different viscoelastic constant, τ_d, and nonlocal parameter, α, are given in Table 1. The natural frequencies of the undamped elastic local and nonlocal beams computed by TFM agree well with the results obtained in the literature (Shen et al., 2012a).

Figs. 1–4 show the variation of the imaginary and real parts of the first four natural frequencies of a cantilever nanobeam as functions of the viscoelastic constant, τ_d, and the nonlocal parameter, α. For the first natural frequency, the imaginary part increases linearly with τ_d and α has almost no effect. The real part increases slightly with α. For the higher natural frequencies, however, the real part decreases significantly with increasing values of α, which is consistent with observations for undamped Timoshenko beams (Shen et al., 2012b). As expected, the viscoelastic constant τ_d in general does not influence the real part significantly. The effect of τ_d on the imaginary parts is fairly linear for all of the four natural frequencies.

The frequency response functions (FRFs) of the damped beam are presented in Fig. 5. The input harmonic force is applied at the free end and the response in obtained at the middle point of the beam. The FRF of the local beam and the nonlocal beam with α = 0.1 and α = 0.2 are shown in Fig. 5(a) and (b) respectively. Observe the clustering of the vibration modes in the higher frequency range for the nonlocal beam. This strikingly different behavior is due to the existence of the asymptotic frequency discussed in Section 4. We have shown the upper cutoff frequency for the two cases, which are 1776.90 GHz and 888.45 GHz respectively.

7. Conclusions

The vibration characteristics of nonlocal viscoelastic damped nanobeams were investigated. The governing equation of motion and corresponding characteristic equation for the complex frequencies were derived. For certain boundary conditions and system parameters, closed-form analytical expressions of the complex frequencies were obtained by solving the underlying characteristic equation in an exact manner. Several physically intuitive special cases and asymptotic results were derived. The theory developed in the paper was applied to the dynamics of a single walled carbon nanotube. In the numerical examples, the effects of the nonlocal parameter, viscoelastic constants and the external damping parameter on the complex natural frequencies were discussed. The external damping parameter has predominantly linear effects on the natural frequencies. The real part of the natural frequencies generally decreases with increasing values of the nonlocal parameter. Some of the main theoretical contributions made in this paper include:

- Unlike local Timoshenko beams, nonlocal Timoshenko beams have two upper cut-off frequencies (asymptotic frequencies). It was shown that for undamped elastic Timoshenko beams, irrespective of the boundary conditions, the two upper cut-off frequencies are $(\omega_{k1})_{\max} = \frac{1}{e_0a}\sqrt{E/\rho}$ and $(\omega_{k2})_{\max} = \frac{1}{e_0a}\sqrt{\kappa G/\rho}$. Vibration of the system is not possible beyond these frequencies.
- For undamped nonlocal viscoelastic Timoshenko beams, asymptotically there exist two critical damping ratios $(\tau_1)_{\text{critical}} = 2e_0a\sqrt{\beta}/L$ and $(\tau_2)_{\text{critical}} = 2e_0a\sqrt{\delta}/L$ where $\beta = \frac{EI}{GAL^2}$ and $\delta = \frac{I}{AL^2}$.
- For damped nonlocal elastic Timoshenko beams, asymptotically there exist two critical damping ratios $(\zeta_1)_{\text{critical}} = \frac{2L}{e_0a\sqrt{\beta}}$ and $(\zeta_2)_{\text{critical}} = \frac{2L}{e_0a\sqrt{\delta}}$.

Frequency response functions obtained using the proposed transfer function method (TFM) verifies the existence of the upper cut-off frequencies. The transfer function method proposed here can be used as a general-purpose tool for the analysis of nonlocal damped Timoshenko beams with various boundary conditions.

Acknowledgment

YL gratefully acknowledges the supports of the National Natural Science Foundation (11272348) and the New Century Excellent Talents of Ministry of Education of China (NCET-08-148). SA acknowledges the support of the Royal Society of London for the Wolfson Research Merit Award.

References

Adhikari, S., & Chowdhury, R. (2012). Zeptogram sensing from gigahertz vibration: Graphene based nanosensor. *Physica E: Low-dimensional Systems and Nanostructures, 44*(7–8), 1528–1534.

Adhikari, S., Mrumu, T., & McCarthy M. A. (2013). Dynamic finite element analysis of axially vibrating nonlocal rods. *Finite Elements in Analysis and Design., 63*, 42–50.

Arani, A. G., Shiravand, A., Rahi, M., & Kolahchi, R. (2012). Nonlocal vibration of coupled DLGS systems embedded on Visco-Pasternak foundation. *Physica B, 407*, 4123–4131.

Benzair, A., Tounsi1, A., Besseghier, A., Heireche, H., Moulay, N., & Boumia, L. (2008). The thermal effect on vibration of single-walled carbon nanotubes using nonlocal Timoshenko beam theory. *Journal of Physics D: Applied Physics, 41*(22). 225404-1-10.

Calleja, M., Kosaka, P., San Paulo, A., & Tamayo, J. (2012). Challenges for nanomechanical sensors in biological detection. *Nanoscale, 4*, 4925–4938.

Chen, C., Ma, M., Liu, J., Zheng, Q., & Xu, Z. (2011). Viscous damping of nanobeam resonators: Humidity, thermal noise, and a paddling effect. *Journal of Applied Physics, 110*, 034320.

Ghannadpour, S. A. M., Mohammadi, B., & Fazilati J. (2013). Bending, buckling and vibration problems of nonlocal Euler beams using Ritz method. *Composite Structures, 96*, 584–589.

Kiani, K. (2013). Vibration behavior of simply supported inclined single-walled carbon nanotubes conveying viscous fluids flow using nonlocal Rayleigh beam model. *Applied Mathematical Modelling, 37*(4), 1836–1850.

Kim, Sun-Bae, & Kim, Ji-Hwan (2011). Quality factors for the nano-mechanical tubes with thermoelastic damping and initial stress. *Journal of Sound and Vibration, 330*(7), 1393–1402.

Lee, J., & Lin, C. (2010). The magnetic viscous damping effect on the natural frequency of a beam plate subject to an in-plane magnetic field. *Journal of Applied Mechanics –Transactions of the ASME, 77*(1), 011014.

Lei, Y., Friswell, M. I., & Adhikar, S. (2006). A Galerkin method for distributed systems with non-local damping. *International Journal of Solids and Structures, 43*(11–12), 3381–3400.

Liew, K. M., Hu, Yanggao, & He, X. Q. (2008). Flexural wave propagation in single-walled carbon nanotubes. *Journal of Computational and Theoretical Nanoscience, 5*(4), 581–586.

Liu, H., & Yang, J. L. (2012). Elastic wave propagation in a single-layered graphene sheet on two-parameter elastic foundation via nonlocal elasticity. *Physica E, 44*(7–8), 1236–1240.

Lu, pin, Lee, H. P., & Lu, C. (2006). Dynamic properties of flexural beam using a nonlocal elasticity model. *Journal of Applied Physics, 99*(7), 073510.

Lu, Pin, Lee, H. P., Lu, C., & Zhang, P. Q. (2007). Application of nonlocal beam models for carbon nanotubes. *International Journal of Solids and Structures, 44*(16), 5289–5300.

Ma, H. M., Gao, X. L., & Reddy, J. N. (2008). A microstructure-dependent Timoshenko beam model based on a modified couple stress theory. *Journal of the Mechanics and Physics of Solids, 56*(12), 3379–3391.

Murmu, T., & Adhikari, S. (2007). Nonlocal vibration of carbon nanotubes with attached buckyballs at tip. *Mechanics Research communications, 38*(1), 62–67.

Murmu, T., McCarthy, M. A., & Adhikari, S. (2012). Vibration response of double-walled carbon nanotubes subjected to an externally applied longitudinal magnetic field: A nonlocal elasticity approach. *Journal of Sound and Vibration, 331*(23), 5069–5086.

Narendar, S., & Gopalakrishnan, S. (2012). Nonlocal flexural wave propagation in an embedded graphene. *International Journal of Computers, 6*(1), 29–36.

Payton, D., Picco, L., Miles, M. J., Homer, M. E., & Champneys, A. R. (2012). Modelling oscillatory flexure modes of an atomic force microscope cantilever in contact mode whilst imaging at high speed. *Nanotechnology, 23*(26), 265702.

Pouresmaeeli, S., Ghavanloo, E., & Fazelzadeh S. A. (2013). Vibration analysis of viscoelastic orthotropic nanoplates resting on viscoelastic medium. *Composite Structures, 96*, 405–410.

Pradhan, S. C. (2012). Nonlocal finite element analysis and small scale effects of CNTs with Timoshenko beam theory. *Finite Elements in Analysis and Design, 50*(1), 8–20.

Reddy, J. N. (2007). Nonlocal continuum theories for buckling, bending and vibration of beams. *International Journal of Engineering Science, 45*(2–8), 288–307.

Roque, C. M. C., Ferreira, A. J. M., & Reddy, J. N. (2011). Analysis of Timoshenko nanobeams with a nonlocal formulation and meshless method. *International Journal of Engineering Science, 49*(9), 976–984.

Shen, Z. B., Li, X. F., Sheng, L. P., & Tang, G. J. (2012a). Transverse vibration of nanotube-based micro-mass sensor via Timoshenko beam. *Computational Materials Science, 53*(1), 340–346.

Shen, Z. B., Sheng, L. P., Li, X. F., & Tang, G. J. (2012b). Nonlocal Timoshenko beam theory for vibration of carbon nanotube-based biosensor. *Physica E: Low-Dimensional Systems and Nanostructures, 44*(7–8), 1169–1175.

Thai, Huu-Tai (2012). A nonlocal beam theory for bending, buckling and vibration of nanobeam. *International Journal of Engineering Science, 52*(3), 56–64.

Torabi, K., & Dastgerdi, J. Nafar (2012). An analytical method for free vibration analysis of Timoshenko beam theory applied to cracked nanobeams using nonlocal elasticity model. *Thin Solid Films, 520*(21), 6602–6695.

Wang, Z. Y., & Li, F. M. (2012). Dynamical properties of nanotubes with nonlocal continuum theory: a review. *Science China: Physics, Mechanics and Astronomy, 55*(7), 1210–1224.

Wang, Q., & Liew, K. M. (2007). Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures. *Physics Letter A, 363*(3), 236–242.

Wang, Y. Z., Li, F. M., & Kishimoto, K. (2012). *Transactions of the ASME Journal of Vibration and Acoustics, 134*(3). 031011-1-7.

Wang, C. Y., Zhang, J., Fei, Y. Q., & Murmu, T. (2012). Circumferential nonlocal effect on vibrating nanotubules. *International Journal of Mechanical Science, 58*(1), 86–90.

Wang, C. M., Zhang, Y. Y., & He, X. Q. (2007). Vibration of nonlocal Timoshenko beams. *Nanotechnology, 18*(10). 105401-1-9.

Yadollahpour, M., Ziaei-Rad, S., & Karimzadeh, F. (2010). Finite element modeling of damping capacity in nano-crystalline materials. *International Journal of Modeling, Simulation, and Scientific Computing, 1*(3), 421–433.

Yang, B., & Tan, C. A. (1992). Transfer functions of one-dimensional distributed parameter system. *Translation of ASME, Journal of Applied Mechanics, 59*(4), 1009–1014.

Yan, Y., Wang, W. Q., & Zhang, L. X. (2012). Free vibration of the fluid-filled single-walled carbon nanotube based on a double shell-potential flow model. *Applied Mathematical Modeling, 36*(12), 6146–6153.