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Dynamic characteristics of damped viscoelastic nonlocal Euler–Bernoulli beams

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ABSTRACT

The dynamic characteristics of damped viscoelastic nonlocal beams are studied in this paper. The Kelvin–Voigt and three-parameter standard viscoelastic models, velocity-dependent external damping and nonlocal Euler–Bernoulli beam theory are employed to establish the governing equations of motion for the bending vibration of nanobeams. A transfer function method (TFM) is developed to obtain closed-form and uniform solution for the vibration analysis of Euler–Bernoulli beams with different boundary conditions. New analytical expressions for critical viscoelastic parameters, damping parameters and limiting frequencies are obtained. Considering a carbon nanotube as a numerical example, the effects of the nonlocal and viscoelastic constants on the natural frequencies and damping factors are discussed. The results demonstrate the efficiency of the proposed modeling and analysis methods for free vibration analysis of viscoelastic damped nonlocal Euler–Bernoulli beams.

1. Introduction

Knowledge of the dynamic response of damped beam structures is necessary in many areas of mechanical, civil, aerospace and nano engineering. The beam is a simple model of a one dimensional continuous system. This simple structure is important from theoretical and engineering points of view, and thus dynamic problems involving one-dimensional continuous beams have drawn huge attention.

Recently, beams have been used as nanostructure components for nanoelectromechanical (NEMS) and microelectromechanical systems (MEMS). In nano-electromechanical systems (NEMS) and nanostructures, similar to macro-scale beams, nanobeams such as carbon nanotubes, boron nanotubes, nanorods can be subjected to damping. The damping may arise due to the effect of external forces such as magnetic forces (Lee and Lin, 2010), interaction with the substrate, humidity and thermal effects (Chen et al., 2011). The characteristics of the damping could be elastic or, more generally, viscoelastic in nature. The study of small-scale damped beams is necessary for the dynamic response analysis of nano-electromechanical systems (NEMS). In particular, accurate quantification of damping is vitally important to understand the sensitivity of nano-scale mass sensors (Calleja et al., 2012).

At the nano-scale, experimental (Bauer et al., 2011; Kiang et al., 1998; Xiao and Hou, 2006; Zienert et al., 2010) and atomistic simulations (Chowdhury et al., 2010; Tang et al., 2009, 2008) have illustrated significant size-effects in mechanical and physical properties. Size effects are related to the atoms and molecules that constitute the materials. The applications of classical continuum models are questionable in the analysis of 'smaller' structures. Therefore, there have been investigations (Akgoz and Civalek, 2011, 2012; Beni et al., 2011; Kong et al., 2009; Simsek, 2010a) to broaden continuum models to incorporate small-scale size effects. One widely applied size-dependent theory is the nonlocal elasticity theory pioneered by Eringen (1983).

In nonlocal elasticity theory, the small-scale effects are captured by assuming that the stress at a point is a function of the strains at all points in the domain (Eringen, 1983). This is unlike classical elasticity theory, but is in accordance with the atomic theory of lattice dynamics and experiment results from phonon dispersion. Nonlocal theories consider long-range inter-atomic interaction and yields results dependent on the size of a body. Drawbacks of the classical continuum theory can be efficiently avoided and size-dependent phenomena can be reasonably explained by nonlocal elasticity theory. The nonlocal theory has been compared to molecular dynamics simulations, and excellent agreement has been found (Ansari and Sahmani, 2012; Ansari et al., 2010; Murmu and Adhikari, 2011). Using this theory, the present classical elasticity beam theories were augmented to incorporate the size-effects

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(nonlocal effects) at the small scale. Nonlocal Euler–Bernoulli beam and Timoshenko beam models (Civalek et al., 2009; Ansari and Ramezannezhad, 2011; Benzair et al., 2008; Heireche et al., 2008; Li et al., 2011; Murmu and Adhikari, 2010; Murmu and Pradhan, 2009; Reddy, 2007; Reddy and Pang, 2008; Simsek, 2011; Thai, 2012; Yang et al., 2011, 2012) were used for bending, vibration and buckling analysis. Considering nonlocal elasticity formulation and meshless method analysis of Timoshenko nanobeams were also reported (Roque et al., 2011). Some interesting studies on vibration of carbon nanotubes by using nonlocal elasticity and beam theories are reported by Simsek (2010a,b) and Zidour et al. (2012). The study includes vibration of nanotubes under action of a moving harmonic load and temperature effects, respectively. Other applications of nonlocal elasticity in the mechanical analysis of nanostructures have been extensively reported in the literature (Aghababaei and Reddy, 2009; Aksencer and Aydogdu, 2011; Ansari et al., 2011a, 2011b, 2010; Aydogdu, 2009; Aydogdu and Filiz, 2011; Hsu et al., 2011; Murmu and Adhikari, 2011; Wang, 2005). The mechanism of nonlocal effects in graphene was described by Wang et al. (2011). Arash and Wang (2012) recently reviewed the application of nonlocal elasticity beam and plate models to carbon nanotubes and graphene.

Several investigations have considered the dynamic analysis of macro-scale viscoelastic damped beams (without size-effects). Kapur et al. (1977) reported on the dynamic response of viscoelastically damped beams where the viscoelastic material is represented by those of a four element viscoelastic model. Teng and Hu (2001) studied the effects of frequency on the damping characteristics for viscoelastic laminated beams and proposed design parameters by employing the Ross–Kerwin–Ungar (RKU) model. Palmeri and Adhikari (2011) studied the transverse vibrations of double-beam systems, made of two outer elastic beams continuously joined by an inner viscoelastic layer using a Galerkin-type state-space approach. The effect of viscoelastic damping on the vibration response is also reported in the literature. Lopez and Fernandez (2012) used nonlocal viscoelastic damping patches to reduce the bending vibrations of Euler–Bernoulli beams; the damping force at a given point depended on the time history and the velocities within a spatial domain. A dynamic analysis of Euler–Bernoulli beams and Kirchhoff plates along with nonlocal damping comprising time and hysteresis effect was reported by Lei et al. (2006). The damping force in a non-local model is the weighted average of the velocity field over the spatial domain, determined by a kernel function based on distance measures. They showed that the damping model has a significant impact on the modal damping ratios of the structure. Friswell et al. (2007) further analyzed the dynamics of beams with different boundary conditions using the finite element method developed for non-local viscoelastic beam models. Although these works help to explain the role of viscoelasticity for macro-scale models, their applicability to the nanoscale remains an open question.

From the above literature survey it is noticed that the study of damped viscoelastic Euler–Bernoulli beams has not been performed using size-dependent theories. All real-life systems, including nanoscale systems, have some damping. In the context of nano electro mechanical systems (NEMS), an accurate and general modeling of damping is important for their design and analysis. In the context of high-speed atomic force microscopes (AFM), the need for more accurate estimation of damping characteristics becomes more apparent (Payton et al., 2012) with the increase of scan rates. The understanding of damping will lead to superior image quality, as AFMs will remain as one of the fundamental imaging tool for nanostructures. The effect of external forces such as magnetic forces (Lee and Lin, 2010), interaction with the substrate, humidity and thermal effects (Chen et al., 2011) also give rise to damping effects in a variety of nanostructures. The consideration of damping at the nanoscale is crucially important for nanosensors (Murmu and Adhikari, 2012). Vibration based nanoscale mass-sensors utilize the shift in the measured frequency to detect the mass of tiny particles. The quality of measurement of this frequency shift depends on the damping characteristics of the oscillators (Calleja et al., 2012). In the case of biosensing, cantilever sensors may need to operate within a fluidic environment. In that case, both internal as well as external damping is important. The study taken in this paper is motivated by these observations.

Viscoelasticity can be viewed as non-locality in the time domain (Adhikari and Wagner, 2004). This is a general damping model and conventional viscous damping can be identified as a special case. In the absence of detailed measurements, this general approach is more useful for nanoscale structures. This paper aims to understand the role of nonlocality in space and time simultaneously in the context of nanobeams. We study the free vibration of damped viscoelastic Euler–Bernoulli beams for different boundary conditions. First, the equation of motion is developed from stress considerations for the general nonlocal multi-parameter viscoelastic model. The characteristic equation governing the natural frequencies of this general model is developed. Two special cases of this general model, namely the nonlocal Kelvin–Voigt viscoelastic model and the nonlocal three-parameter standard viscoelastic model are discussed in detail. Several physically insightful asymptotic results for the nonlocal Kelvin–Voigt viscoelastic model are derived. A Transfer Function Method (TFM) is proposed for the free vibration analysis for general boundary conditions. The theoretical results derived in this paper are illustrated using an example of a single walled carbon nanotube (SWCNT).

2. Nonlocal viscoelastic model

Based on Eringen's nonlocal elastic theory (Eringen, 1983), the relationship between the nonlocal stress tensor and the local stress tensor for a linear and homogeneous elastic solid can be expressed as

$$t_{kl} = \int_V k(|x - x'|, \phi)\sigma_{kl}\mathrm{d}V \quad (1)$$

where t_{kl} is nonlocal elastic stress tensor and $k(|\bullet|)$ is the nonlocal kernel function. The term ϕ is $\tau = e_0 l_i / l_e$, where l_i and l_e are the internal and external characteristic length respectively. e_0 is a constant appropriate to each material. The term σ_{kl} is the stress tensor for local elasticity given by

$$\sigma_{kl} = \lambda_e \varepsilon_{rr}\delta_{kl} + 2\mu_e\varepsilon_{kl}, \quad \varepsilon_{kl} = (u_{k,l} + u_{l,k})/2 \quad (2)$$

where λ and μ being the Lame's constant.

This approach can be extended directly to a nonlocal viscoelastic model as

$$t^*_{kl} = \int_V k(|x - x'|, \phi)\sigma^*_{kl}\mathrm{d}V \quad (3)$$

where $t^*{}_{kl}$ and $\sigma^*{}_{kl}$ are nonlocal and local viscoelastic stress tensor respectively. The integral constitutive relationship for a linear and homogeneous viscoelastic solid is given by

$$\sigma^*_{kl} = G_{klij}(t)\varepsilon_{ij}\left(0^+\right) + \int_0^t G_{klij}(t-\tau)\ \frac{\partial \varepsilon_{ij}(\tau)}{\partial \tau}\mathrm{d}\tau \quad (4)$$

where G_{klij} is a tensor stress relaxation function.

The constitutive equation for nonlocal viscoelastic solids can be obtained by combining nonlocal elasticity and viscoelasticity models. Therefore, for nonlocal viscoelastic solids, the nonlocal stress can be expressed as

$$t_{kl}^{*} = \int_V k(|x-x'|,\phi)\left[G_{klij}(t)\varepsilon_{ij}\left(0^{+}\right)+\int_0^t G_{klij}(t-\tau)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}d\tau\right]dV \tag{5}$$

For the special kernel function introduced by Eringen (1983), the differential form of above equation for nonlocal model can be obtained as

$$\left[1-(e_0a)^2\nabla^2\right]t_{kl}^{*} = G_{klij}(t)\varepsilon_{ij}\left(0^{+}\right)+\int_0^t G_{klij}(t-\tau)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}d\tau \tag{6}$$

This particular form of nonlocal model has been used extensively in literature (Arash and Wang, 2012). Many authors have proposed methods to identify the nonlocal parameter e_0a from molecular dynamics (MD) simulations. Duan et al. (2007) calculated the values of the nonlocal parameter in the context of vibration of carbon nanotubes modeled as Timoshenko beams. Their results demonstrates that a nonzero value fits the MD data better than when $e_0a = 0$ (that is the local case). More recently Liang and Han (2012) derived an analytical expression for the nonlocal parameter for carbon nanotubes modeled as thin cylindrical shells. They also concluded that nonzero values of e_0a should be used. Another way to view the nonlocal elasticity is that it is a generalization of the classical local elasticity, which is recovered as the special case when the nonlocal parameter goes to zero. By considering a nonlocal viscoelastic model in Eq. (6) we include generality in the time domain in addition to the spatial domain.

A schematic representation of a general Maxwell model for linear viscoelasticity is shown in Fig. 1. The extensional relaxation modulus for this model is given by

$$E(t) = E_\infty + \sum_{m=1}^{N} E_m e^{-\frac{t}{\tau_m}} = E_0 - \sum_{m=1}^{N} E_m\left(1-e^{-\frac{t}{\tau_m}}\right) \tag{7}$$

where $E_0 = E_\infty + \sum_{m=1}^{N} E_m$is initial extensional modulus and $\tau_m = \eta_m/E_m$ are the relaxation times.

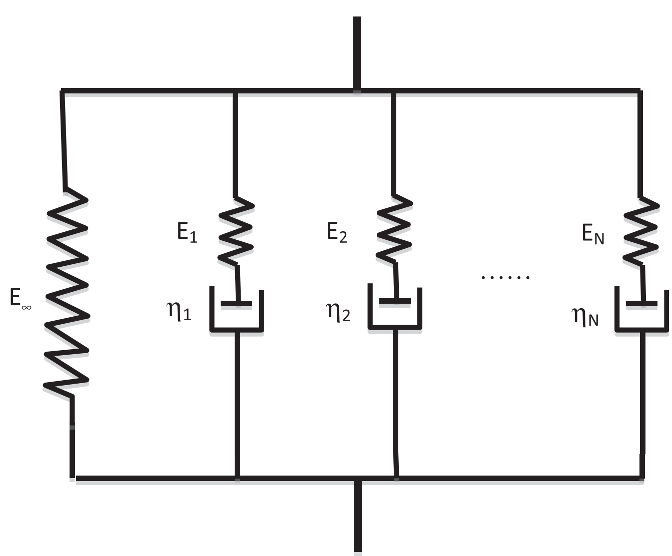

Fig. 1. The general Maxwell model for linear viscoelasticity.

By the Boltzmann superposition principle, the integral constitutive equation for linear homogeneous viscoelastic Euler–Bernoulli beams can be equivalently expressed as

$$\sigma_{xx}^{*} = E(t)\varepsilon_{xx}(0) + \int_0^t E(t-\tau)\frac{\partial\varepsilon_{xx}(\tau)}{\partial\tau}d\tau \tag{8}$$

Following the extension given in Eq. (6), for the nonlocal viscoelastic for Euler–Bernoulli one has

$$\left[1-(e_0a)^2\nabla^2\right]t_{xx}^{*} = E(t)\varepsilon_{xx}(0) + \int_0^t E(t-\tau)\frac{\partial\varepsilon_{xx}(\tau)}{\partial\tau}d\tau \tag{9}$$

or

$$\left[1-(e_0a)^2\nabla^2\right]t_{xx}^{*} = E_\infty\,\varepsilon_{xx}(t) + \int_0^t \sum_{m=1}^{N} E_m e^{\frac{\tau-t}{\tau_m}}\frac{\partial\varepsilon_{xx}(\tau)}{\partial\tau}d\tau \tag{10}$$

In the next section we derive the equation of motion based on this relationship.

3. Governing equations

For an Euler–Bernoulli beam, the axial strain is given by

$$\varepsilon_{xx} = -y\frac{\partial^2 w}{\partial x^2} \tag{11}$$

When the external velocity-dependent non-viscous damping force is considered, the equation of motion for a damped nonlocal viscoelastic Euler–Bernoulli beam can be expressed by the following integro-partial-differential equation

$$\begin{aligned} &E_\infty I\frac{\partial^4 w}{\partial x^4} + \int_0^t \sum_{m=1}^{N} E_m I e^{\frac{\tau-t}{\tau_m}}\frac{\partial^5 w(x,\tau)}{\partial x^4\partial\tau}d\tau \\ &+ C_2\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\int_0^t \mu e^{-\mu(t-\tau)}\frac{\partial w}{\partial\tau}d\tau \\ &+ \rho A\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2 w}{\partial t^2} = \left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]p \end{aligned} \tag{12}$$

where w is the transverse deflection, I is the second moment of the cross sectional area A, C_2 is the velocity-dependent viscous damping coefficient. For an undamped nonlocal elastic ($C_2 = 0, \tau_m \to 0$) Euler–Bernoulli beam, the equation of motion (12) can be simplified to the special case (Reddy, 2007) as

$$E_\infty I\frac{\partial^4 w}{\partial x^4} + \rho A\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2 w}{\partial t^2} = \left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]p \tag{13}$$

Equation (12) considered here is therefore a generalization of the conventional undamped nonlocal Euler–Bernoulli beam model.

In addition to the equation of motion, the boundary conditions must also be defined. The ends of the beams may be pinned or fixed, in which case the boundary conditions may be applied directly to the displacement. However if the ends of the beam are free, either in displacement or rotation, then the boundary conditions must be based on zero bending moment or shear force. Hence, the bending moment, M, and the shear force, Q, are required and may be calculated as

$$M = -E_\infty I\frac{\partial^2 w}{\partial x^2} - \int_0^t \sum_{m=1}^{N} E_m I e^{\frac{\tau-t}{\tau_m}} \frac{\partial^3 w(x,\tau)}{\partial x^3 \partial \tau} d\tau + (e_0 a)^2\left[\rho A\frac{\partial^2 w}{\partial t^2} + C_2\int_0^t \mu e^{-\mu(t-\tau)} \frac{\partial w}{\partial \tau} d\tau - p\right] \quad (14)$$

and

$$Q = -E_\infty I\frac{\partial^3 w}{\partial x^3} - \int_0^t \sum_{m=1}^{N} E_m I e^{\frac{\tau-t}{\tau_m}} \frac{\partial^4 w(x,\tau)}{\partial x^4 \partial \tau} d\tau + (e_0 a)^2\left[\rho A\frac{\partial^3 w}{\partial x \partial t^2} + C_2\int_0^t \mu e^{-\mu(t-\tau)} \frac{\partial^2 w}{\partial x \partial \tau} d\tau - \frac{\partial p}{\partial x}\right] \quad (15)$$

We now wish to determine the natural frequencies for the beam model. Hence we assume solutions of the form $w = W(x)\exp(i\omega t)$, and consider free vibration. In this case ω is a complex natural frequency and $W(x)$ is the corresponding mode shape. Note that since the system is damped, complex eigenvalues could be used instead of complex natural frequencies. Then, Eq. (12) becomes

$$\left(E_\infty I + \sum_{m=1}^{N} E_m I\frac{i\omega\tau_m}{1+i\omega\tau_m}\right)\frac{d^4 W}{dx^4} + \left[\rho A\omega^2 - C_2\frac{i\omega\mu}{i\omega+\mu}\right](e_0 a)^2\frac{d^2 W}{dx^2} + \left[C_2\frac{i\omega\mu}{i\omega+\mu} - \rho A\omega^2\right]W = 0 \quad (16)$$

and the nonlocal bending moment or shear force become

$$M = -\left(E_\infty I + \sum_{m=1}^{N} E_m I\frac{i\omega\tau_m}{1+i\omega\tau_m}\right)\frac{d^2 W}{dx^2} + (e_0 a)^2\left(-\rho A\omega^2 + C_2\frac{i\omega\mu}{i\omega+\mu}\right)W \quad (17)$$

$$Q = -\left(E_\infty I + \sum_{m=1}^{N} E_m I\frac{i\omega\tau_m}{1+i\omega\tau_m}\right)\frac{d^3 W}{dx^3} + (e_0 a)^2\left(-\rho A\omega^2 + C_2\frac{i\omega\mu}{i\omega+\mu}\right)\frac{dW}{dx} \quad (18)$$

Let $\zeta = C_2/\rho A$ and $c = 1/L^2\sqrt{EI/\rho A}$. We also introduce the following non-dimensional terms

$$\overline{x} = \frac{x}{L}, \quad \overline{W} = \frac{W}{L}, \quad \alpha = \frac{e_0 a}{L} \quad (19)$$

The equation of motion can be rewritten in terms of these quantities as

$$c^2\left(1 + i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{d^4\overline{W}}{d\overline{x}^4} + \left[\omega^2 - i\omega\zeta\frac{\mu}{i\omega+\mu}\right]\alpha^2\frac{d^2\overline{W}}{d\overline{x}^2} + \left[i\omega\zeta\frac{\mu}{i\omega+\mu} - \omega^2\right]\overline{W} = 0 \quad (20)$$

This is a fourth-order ordinary differential equation for the function $\overline{W}$.The coefficients of this equation are complex valued and functions of the frequency parameter ω. The bending moment M and the shear force Q can be also written in terms of the non-dimensional parameters as

$$M = \rho A L^3\left[-c^2\left(1 + i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{d^2\overline{W}}{d\overline{x}^2} + \alpha^2\left[i\omega\zeta\frac{\mu}{i\omega+\mu} - \omega^2\right]\overline{W}\right] \quad (21)$$

and

$$Q = \rho A L^2\left[-c^2\left(1 + i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{d^3\overline{W}}{d\overline{x}^3} + \alpha^2\left[i\omega\zeta\frac{\mu}{i\omega+\mu} - \omega^2\right]\frac{d\overline{W}}{dx}\right] \quad (22)$$

The mode shape $\overline{W}(\overline{x})$ is obtained as the solution to Eq. (20) that satisfies the required boundary conditions. Since Eq. (20) is a fourth order ordinary differential equation in $\overline{x}$ with constant coefficients the solution is a sum of four terms of the form $\overline{W} = \overline{W}_n e^{-i\beta\overline{x}}$. Hence the equation governing the eigenvalues, Eq. (20), becomes

$$c^2\left(1 + i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\beta^4 + \left[i\omega\zeta\frac{\mu}{i\omega+\mu} - \omega^2\right]\alpha^2\beta^2 + \left[i\omega\zeta\frac{\mu}{i\omega+\mu} - \omega^2\right] = 0 \quad (23)$$

For some boundary conditions we can determine the required solutions for β; for example pinned–pinned boundary conditions give $\beta = 1/2k\pi$ for the k th mode. General boundary conditions are solved using the Transfer Function Method described later. Assuming that β is known, then Eq. (23) may be written as polynomial of order $(N+2)$ for the eigenvalues ω. Hence there are $N+2$ roots, which are either real or occur in complex conjugate pairs depending on the material parameters, external damping coefficient and boundary conditions. The dynamic characteristic of the motion will be different for different types of roots.

4. Nonlocal Kelvin–Voigt viscoelasticity model

Suppose that the external damping is viscous, that is $\mu \to \infty$. The Kelvin–Voigt viscoelastic model assumes that $N = 1$ and $E_1 = \infty$. Hence, substituting into Eq. (23), and after some care applying the condition $E_1 = \infty$, we have

$$-\left(1 + \alpha^2\beta^2\right)\omega^2 + \left(\zeta + \alpha^2\beta^2\zeta + \tau_d c^2\beta^4\right)i\omega + c^2\beta^4 = 0 \quad (24)$$

where we have defined $\tau_d = \eta_1/E_\infty$. This is a quadratic equation in ω and can be solved to obtain the natural frequencies to give

$$\omega_n = \frac{\left(\zeta + \alpha^2\beta^2\zeta + \tau_d c^2\beta^4\right)i \mp \sqrt{-\left(\zeta + \alpha^2\beta^2\zeta + \tau_d c^2\beta^4\right)^2 + 4c^2\beta^4\left(1 + \alpha^2\beta^2\right)}}{2\left(1 + \alpha^2\beta^2\right)} \quad (25)$$

or

$$\omega_n = \left[\frac{\zeta}{2} + \frac{c^2\beta^4}{2\left(1+\alpha^2\beta^2\right)}\tau_d\right] i \mp \sqrt{-\left[\frac{\zeta}{2} + \frac{c^2\beta^4}{2\left(1+\alpha^2\beta^2\right)}\tau_d\right]^2 + \frac{c^2\beta^4}{1+\alpha^2\beta^2}} \quad (26)$$

Equation (26) is a general expression of the complex natural frequencies. The following special cases are of interest from a physical point of view:

(i) For an undamped, local elastic beam, $\alpha = \tau_d = \zeta = 0$. Then the natural frequencies are given by

$$\omega_n = c\beta^2 \quad (27)$$

This is the well-known classical result for Euler–Bernoulli beams.

(ii) For a damped, local elastic beam, $\alpha = \tau_d = 0$. Then the complex natural frequencies are

$$\omega_n = i\frac{\zeta}{2} + c\beta^2\sqrt{1 - \left(\frac{\zeta}{2c\beta^2}\right)^2} \quad (28)$$

This is the result for a classical viscously damped Euler–Bernoulli beam.

(iii) For a local viscoelastic beam, $\alpha = \zeta = 0$. Then the complex natural frequencies are

$$\omega_n = \left[i\frac{c\beta^2\tau_d}{2} + \sqrt{1 - \left(\frac{c\beta^2\tau_d}{2}\right)^2}\right] c\beta^2 \quad (29)$$

The imaginary part of the natural frequencies is given by $c^2\beta^4\tau_d/2$. Note that $c\beta^2$is the natural frequency of the local elastic beam and usually is GHz for carbon nanotubes. Therefore, its square $c^2\beta^4$ would be very large and even with a very small value of τ_d, the imaginary part would still be very large. This means that the damping ratio will be very high. For an oscillatory solution the real part of the complex natural frequency must be non-zero. The critical value of the parameter τ_d to obtain non-oscillatory solutions is $(\tau_d)_{\text{crit}} = 2/c\beta^2$, which depends on the natural frequency considered (i.e. the value of β) and is a small value.

(iv) For an undamped, nonlocal beam, $\tau_d = \zeta = 0$. Then the natural frequencies are given by

$$\omega_n = c\beta^2\sqrt{\frac{1}{1+\alpha^2\beta^2}} \quad (30)$$

This corresponds to the expression for the natural frequencies available in literature for undamped nonlocal Euler–Bernoulli beams (Lu et al., 2006).

(v) For a damped nonlocal elastic beam, $\tau_d = 0$. Then the complex natural frequencies are

$$\omega_n = \left[\frac{i\zeta}{2} + \sqrt{-\frac{\zeta^2}{4} + \frac{c^2\beta^4}{1+\alpha^2\beta^2}}\right] \quad (31)$$

Similarly to Eq. (28) for the local beam the imaginary part is just half of the external damping parameter ζ, and is independent of the other parameters. Compared to Eq. (30), the real part is only slightly different due to the damping parameter ζ. Asymptotically the real part tends to $c\beta/\alpha$ for high order frequencies (that is, when β is large).

(vi) For a nonlocal viscoelastic beam, $\zeta = 0$. Then the complex natural frequencies are

$$\omega_n = \frac{c^2\beta^4}{1+\alpha^2\beta^2}\left[\frac{\tau_d}{2}i + \sqrt{-\frac{\tau_d^2}{4} + \frac{1+\alpha^2\beta^2}{c^2\beta^4}}\right] \quad (32)$$

Similarly to Eq. (29), for damped nonlocal viscoelastic beams, the imaginary part of the natural frequencies, $(c^2\beta^4/1+\alpha^2\beta^2)(\tau_d/2)$, is high, decreases with increasing nonlocal parameter and tends to $(c^2\beta^2/\alpha^2)(\tau_d/2)$ for high order frequencies (when β is very large). The real parts of the complex natural frequencies also decrease with increasing nonlocal parameter, but the rate of increase is smaller than the imaginary part. There is also a critical value of the parameter τ_d which gives a non-oscillatory response, given by $(\tau_d)_{\text{crit}} = 2/c\beta^2\sqrt{1+\alpha^2\beta^2}$.

(vii) For a damped viscoelastic beam, $\alpha = 0$. Then the complex natural frequencies are

$$\omega_n = \left[\frac{\zeta}{2} + \frac{c^2\beta^4}{2}\tau_d\right] i + \sqrt{-\left[\frac{\zeta}{2} + \frac{c^2\beta^4}{2}\tau_d\right]^2 + c^2\beta^4} \quad (33)$$

Comparing with τ_d, the effect of the external damping parameter ζ on each natural frequency is the same. With the increasing order of the frequency (that is higher values of β), the effect of ζ will decrease relatively. Table 1 summarizes the limiting results of the frequencies and critical damping parameters corresponding to the special cases discussed here.

5. Nonlocal three-parameter standard viscoelasticity model

Suppose that the external damping is viscous, that is $\mu \to \infty$. The three-parameter standard viscoelasticity model is given by a single term, i.e. $N = 1$. Defining $\tau_1 = \eta_1/E_1$ and $\nu = E_1/E_\infty$, then Eq. (23) becomes

$$c^2\left(1 + \frac{i\omega\nu\tau_1}{1+i\omega\tau_1}\right)\beta^4 + \left[i\omega\zeta - \omega^2\right]\alpha^2\beta^2 + \left[i\omega\zeta - \omega^2\right] = 0 \quad (34)$$

which may be rewritten as

$$i\tau_1\omega^3 + (1+\zeta\tau_1)\omega^2 - \left[\zeta + \frac{c^2\beta^4\tau_1(1+\nu)}{1+\alpha^2\beta^2}\right] i\omega - \frac{c^2\beta^4}{1+\alpha^2\beta^2} = 0 \quad (35)$$

Table 1
Limiting results for the natural frequencies and damping parameters.

Parameter values	Limiting results
$\alpha = 0$ and $= 0$	$(\tau_d)_{\text{crit}} = 2/c\beta^2$ $\omega_{\text{nonlocal}} \to c^2\beta^4\tau_d i$
$\alpha = 0$	$(\tau_d)_{\text{crit}} = 2/c\beta^2 - \zeta/c^2\beta^4$
$\zeta = 0$	$(\tau_d)_{\text{crit}} = \frac{2}{c\beta^2}\sqrt{1+\alpha^2\beta^2} \to \frac{2\alpha}{c\beta}$ $\omega_{\text{nonlocal}} \to (c^2\beta^4\tau_d/1+\alpha^2\beta^2)i$
$\tau_d = 0$	$\omega_{\text{nonlocal}} \to c\beta/\alpha$ and $\omega_{\text{local}}/\omega_{\text{nonlocal}} \to \beta\alpha$

Equation (35) is a cubic equation for the complex natural frequency ω. Adhikari (2005) discussed the characteristics of this kind of eigenvalue equation arising in the context of single-degree-of-freedom non-viscously damped oscillators. Closed-form solutions of Eq. (35) are possible via Cardano's formula. However, the resulting expressions are generally too complex to be physically meaningful. Therefore, in the next section we pursue a numerical approach to obtain the complex natural frequencies. The proposed Transfer Function method (TFM) is also general in terms of the different boundary conditions.

6. Transfer Function method (TFM) for nonlocal beams

The Transfer Function method (TFM) (Yang and Tan, 1992) is suitable to analyze the dynamics of one-dimensional structures. To find the eigenvalues using the transfer function method, we define the state vector $\eta(x,\omega)$ as

$$\boldsymbol{\eta}(x,\omega) = \left[\overline{W}, \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{x}}, \frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\overline{x}^2}, \frac{\mathrm{d}^3\overline{W}}{\mathrm{d}\overline{x}^3}\right]^T \tag{37}$$

Equation (20) can be rewritten in a matrix form as

$$\frac{d\boldsymbol{\eta}(\overline{x},\omega)}{d\overline{x}} = \boldsymbol{\Phi}(\omega)\boldsymbol{\eta}(\overline{x},\omega) \tag{38}$$

where

$$\boldsymbol{\Phi}(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\omega^2 - i\omega\zeta\dfrac{\mu}{i\omega+\mu}}{c^2\left(1+i\omega\sum_{m=1}^{N}\dfrac{E_m}{\mathrm{E}_\infty}\dfrac{\tau_m}{1+i\omega\tau_m}\right)} & 0 & \dfrac{\alpha^2\left(i\omega\zeta\dfrac{\mu}{i\omega+\mu} - \omega^2\right)}{c^2\left(1+i\omega\sum_{m=1}^{N}\dfrac{E_m}{\mathrm{E}_\infty}\dfrac{\tau_m}{1+i\omega\tau_m}\right)} & 0 \end{bmatrix} \tag{39}$$

$$\boldsymbol{N}(\omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^2\left[i\omega\zeta\dfrac{\mu}{i\omega+\mu} - \omega^2\right] & 0 & -c^2\left(1+i\omega\sum\limits_{m=1}^{N}\dfrac{E_m}{E_\infty}\dfrac{\tau_m}{1+i\omega\tau_m}\right) & 0 \\ 0 & \alpha^2\left[i\omega\zeta\dfrac{\mu}{i\omega+\mu} - \omega^2\right] & 0 & -c^2\left(1+i\omega\sum\limits_{m=1}^{N}\dfrac{E_m}{E_\infty}\dfrac{\tau_m}{1+i\omega\tau_m}\right) \end{bmatrix} \tag{43}$$

The homogeneous boundary conditions can be expressed as

$$\boldsymbol{M}(\omega)\boldsymbol{\eta}(0,\omega) + \boldsymbol{N}(\omega)\boldsymbol{\eta}(1,\omega) = \boldsymbol{0} \tag{40}$$

Here $\boldsymbol{M}(\omega)$and $\boldsymbol{N}(\omega)$ are boundary condition set matrices at the ends of the beam given by $\overline{x} = 0$ and $\overline{x} = 1$ respectively. Several examples will be given to highlight the approach; other examples are then easily derived. For clamped–clamped boundary conditions

$$\boldsymbol{M}(\omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{N}(\omega) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{41}$$

where the first two rows in (40) gives zero displacement and rotation at $\overline{x} = 0$, and the third and fourth rows gives zero displacement and rotation at $\overline{x} = 1$. For beams with simple support at $\overline{x} = 0$ the second row of $\boldsymbol{M}(\omega)$ ensures that the bending moment is zero, so that

$$\boldsymbol{M}(\omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha^2\left[i\omega\zeta\dfrac{\mu}{i\omega+\mu} - \omega^2\right] & 0 & -c^2\left(1+i\omega\sum\limits_{m=1}^{N}\dfrac{E_m}{E_\infty}\dfrac{\tau_m}{1+i\omega\tau_m}\right) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

For a cantilever beam, $\boldsymbol{M}(\omega)$ is given by Eq. (41), and $\boldsymbol{N}(\omega)$ is obtained by ensuring that both the bending moment and shear force are zero at $\overline{x} = 1$. Thus

The solution of Eq. (38) can be expressed as

$$\boldsymbol{\eta}(\overline{x},\omega) = e^{\boldsymbol{\Phi}(\omega)\overline{x}}\boldsymbol{\eta}_0(\omega) \tag{44}$$

for a constant vector $\boldsymbol{\eta}_0$ that depends on the boundary conditions given by Eq. (40). For the free vibration of nonlocal beam, the natural frequencies ω_k are the solution of the transcendental characteristic equation

$$\det\left[\boldsymbol{M}(\omega) + \boldsymbol{N}\left(\omega\right)e^{\boldsymbol{\Phi}}\right] = 0 \tag{45}$$

The eigenfunction (mode shape) corresponding to ω_k is defined as

$$\boldsymbol{\eta}(\overline{x}, \omega) = e^{\boldsymbol{\Phi}(\omega_k)}\boldsymbol{\varphi}_k \tag{46}$$

where the non-zero vector $\boldsymbol{\varphi}_k$ satisfies

$$\left[\boldsymbol{M}(\omega_k) + \boldsymbol{N}(\omega_k)e^{\boldsymbol{\Phi}}\right]\boldsymbol{\varphi}_k = 0 \tag{47}$$

7. Numerical results and discussion

In this section, a single walled carbon nanotube (SWCNT) is taken as an example. The effects of the nonlocal parameter, the viscoelastic parameter and the damping coefficients on the natural frequencies are analyzed. Different boundary conditions on the SWCNT are considered. The basic parameters of the system used for the numerical calculations are as follows: length $L = 11$ nm, Young's modulus E (or E_∞ in the three parameter viscoelastic model) = 1 TPa, mass density $\rho = 2.24$ g/cm^3, diameter of the SWCNT $d = 1.1$ nm, effective tube thickness $t = 0.342$ nm and nonlocal parameter $\alpha=(e0a/L)\in[0,0.2]$. For the Kelvin–Voigt model, $\tau_d\in[0,\ 10^{-4}]$ns. For the three-parameter standard viscoelastic solid $N = 1$, $E_1/E_\infty = 1$ and $\tau_1\in[0,\ 10^{-3}]$ns. Without any loss of generality we set the viscous damping coefficient to $\zeta = 0$, as the effect on the complex natural frequencies is similar to the local beam model and therefore well understood.

7.1. Kelvin–Voigt viscoelastic beam

The natural frequencies of undamped elastic and viscoelastic beam with three typical boundary conditions and different viscoelastic constant τ_d and nonlocal parameter α are given in Table 2. It can be observed that the natural frequencies of the undamped elastic local and nonlocal beam computed by TFM, agree with the results in the literature (Lu et al., 2006).

Figs. 2–5 present the imaginary and the real parts of the first four natural frequencies of the nanobeam with simply supported boundary conditions as functions of parameters τ_d and α. These two parameters quantify nonlocality in time and space respectively. Fig. 2 shows that the effects of both of τ_d and α on the imaginary part (relating to damping ratio) of the first frequency are small. The imaginary part increases linearly with τ_d to about 3.5%, and remains almost constant with respect to α. On the other hand, the real part

Table 2
Comparison of the natural frequencies (GHz) of the nano-beam with different boundary conditions and viscoelastic constant τ_d and nonlocal parameter α

BCs	Undamped elastic beam			Kelvin–Voigt viscoelastic beam ($\zeta = 0.0$, $\tau_d = 10^{-4}$)		
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$
C–F	39.797	39.969	40.514	39.794 + 0.49756*i*	39.966 + 0.50189*i*	40.511 + 0.51566*i*
	249.40	234.07	198.94	248.63 + 19.541*i*	233.43 + 17.212*i*	198.55 + 12.433*i*
	698.33	577.97	416.67	681.32 + 153.20*i*	568.37 + 104.95*i*	413.09 + 54.544*i*
	1368.4	969.89	613.41	1235.5 + 588.31*i*	923.77 + 295.53*i*	601.91 + 118.21*i*
S–S	111.71	106.58	94.589	111.64 + 3.9205*i*	106.52 + 3.5683*i*	94.547 + 2.8108*i*
	446.84	378.36	278.24	442.42 + 62.728*i*	375.67 + 44.973*i*	277.17 + 24.321*i*
	1005.4	731.65	471.18	953.93 + 317.56*i*	712.06 + 168.18*i*	465.99 + 69.746*i*
	1787.4	1113.0	660.79	1479.0 + 1003.6*i*	1042.7 + 389.14*i*	646.39 + 137.17*i*
C–C	278.55	251.81	202.36	277.67 + 22.160*i*	251.15 + 18.109*i*	202.02 + 11.695*i*
	767.82	603.87	412.38	749.13 + 168.37*i*	594.82 + 104.15*i*	409.51 + 48.568*i*
	1505.4	1012.9	623.13	1359.2 + 647.25*i*	969.55 + 292.99*i*	613.19 + 110.90*i*
	2223.6	1437.0	829.71	1584.5 + 1623.9*i*	1310.4 + 589.75*i*	806.08 + 196.61*i*

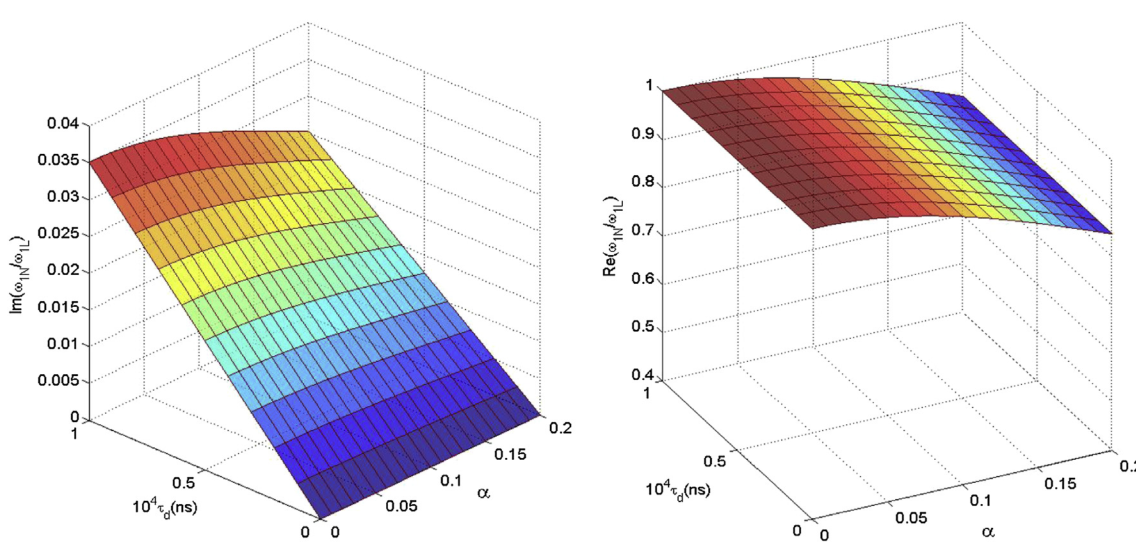

Fig. 2. The imaginary and real parts of the first eigenvalue of the Kelvin–Voigt viscoelastic beam.

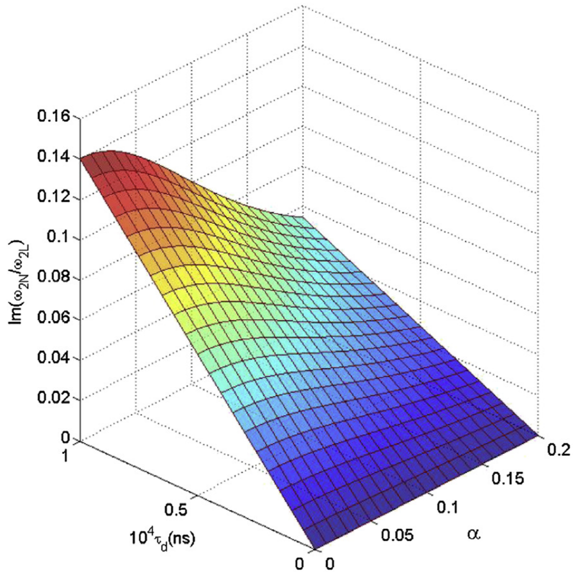

Fig. 3. The imaginary and real parts of the second eigenvalue of the Kelvin–Voigt viscoelastic beam.

remains almost constant with respect to τ_d and decreases about 15% when α changes from 0 to 0.2. For higher natural frequencies, the effect of τ_d is fairly linear, but the effect of α is comparatively nonlinear. With the increasing values of α, both the imaginary and the real parts show a decreasing trend.

7.2. Three-parameter standard viscoelastic beam with simply supported boundary conditions

The three-parameter standard viscoelastic beam discussed earlier in section 5 is considered here. For this model the characteristic equation is given by a cubic polynomial. We use the proposed transfer function method to obtain the natural frequencies. Figs. 6–9 show the imaginary and the real parts of the first four natural frequencies of the nanobeam with simply supported boundary conditions as functions of the relaxation time $\tau_1 \in [0, 10^{-3}]$ns and the nonlocal parameter $\alpha \in [0,0.2]$.

In contrast to the effects of τ_d on the natural frequencies discussed in the previous subsection, the effect of τ_1 is nonlinear. In Fig. 6, the imaginary part increases with τ_1 increasing from 0 to 10^{-3} ns, while the real part remains relatively constant. In Figs. 7–9, there is a noticeable maximum of the imaginary part with respect to the parametric variability of the relaxation time τ_1. The maximum value occurs near $\omega_k\tau_1 = 1$. The variation of the imaginary part with respect to the nonlocal parameter α is comparatively less pronounced for lower frequencies. For higher frequencies, however, a significant decrease in the value is observed for higher values of α. The real part of the complex natural frequencies is relatively insensitive with respect to the relaxation time τ_1. This is expected as the relaxation time does not influence the oscillation

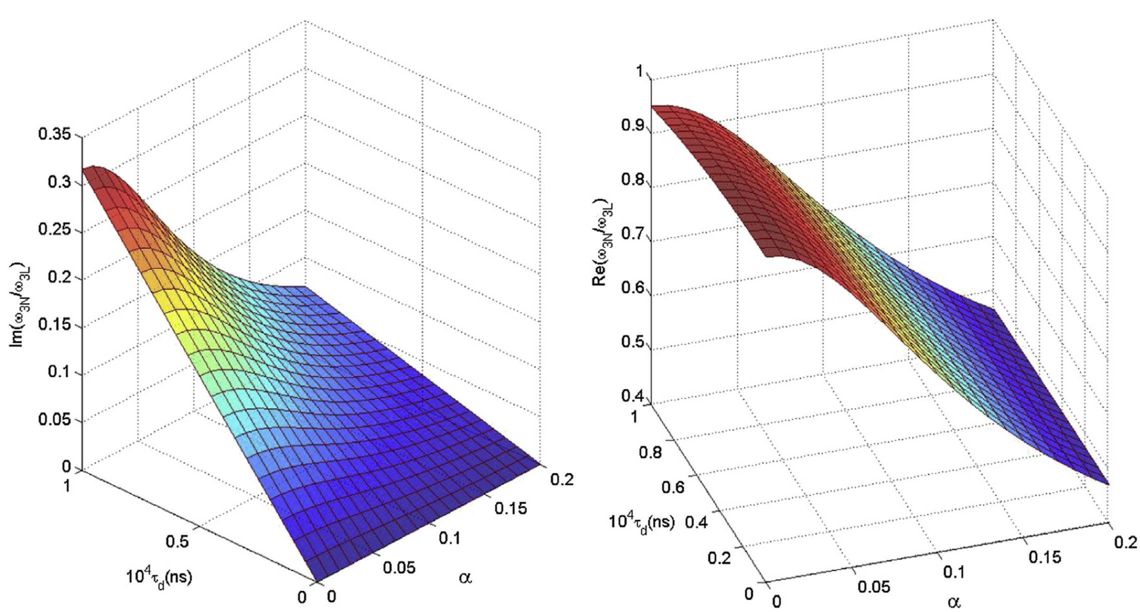

Fig. 4. The imaginary and real parts of the third eigenvalue of the Kelvin–Voigt viscoelastic beam.

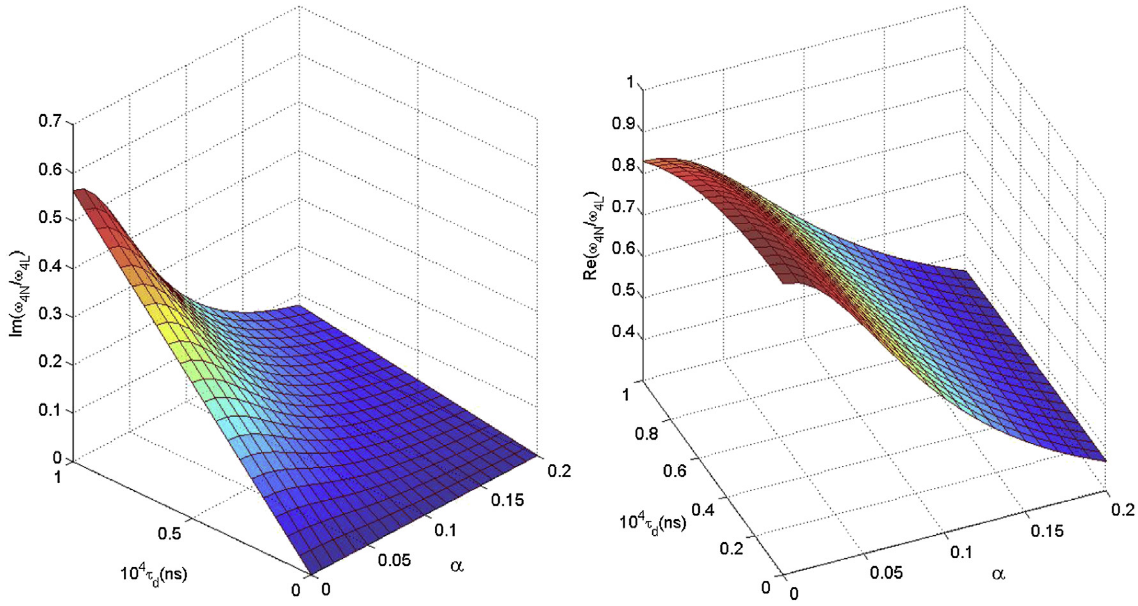

Fig. 5. The imaginary and real parts of the fourth eigenvalue of the Kelvin–Voigt viscoelastic beam.

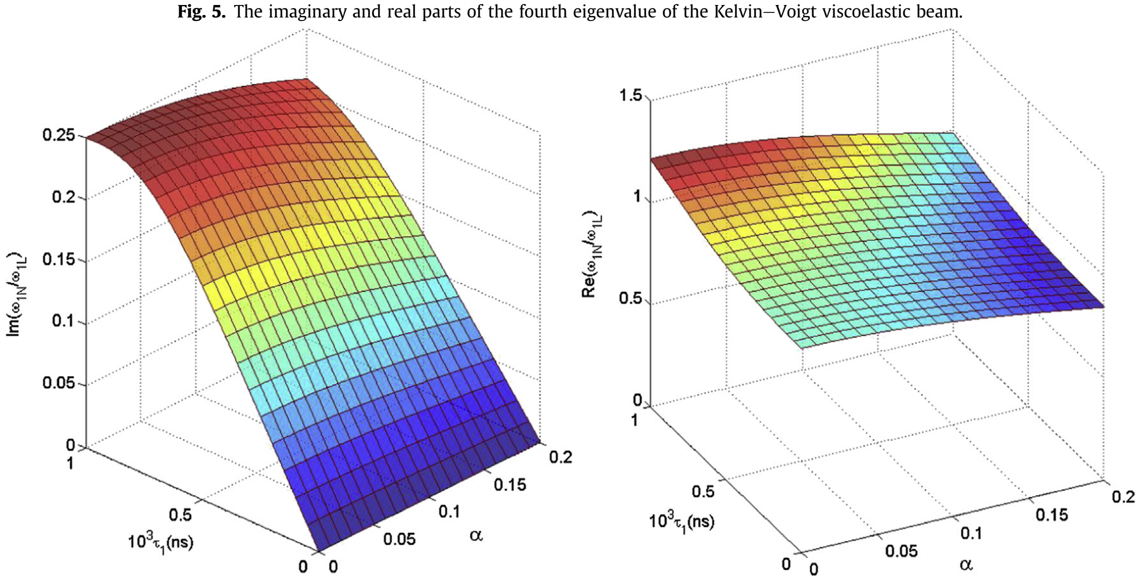

Fig. 6. The imaginary and real parts of the first eigenvalue of the three-parameter standard viscoelastic beam.

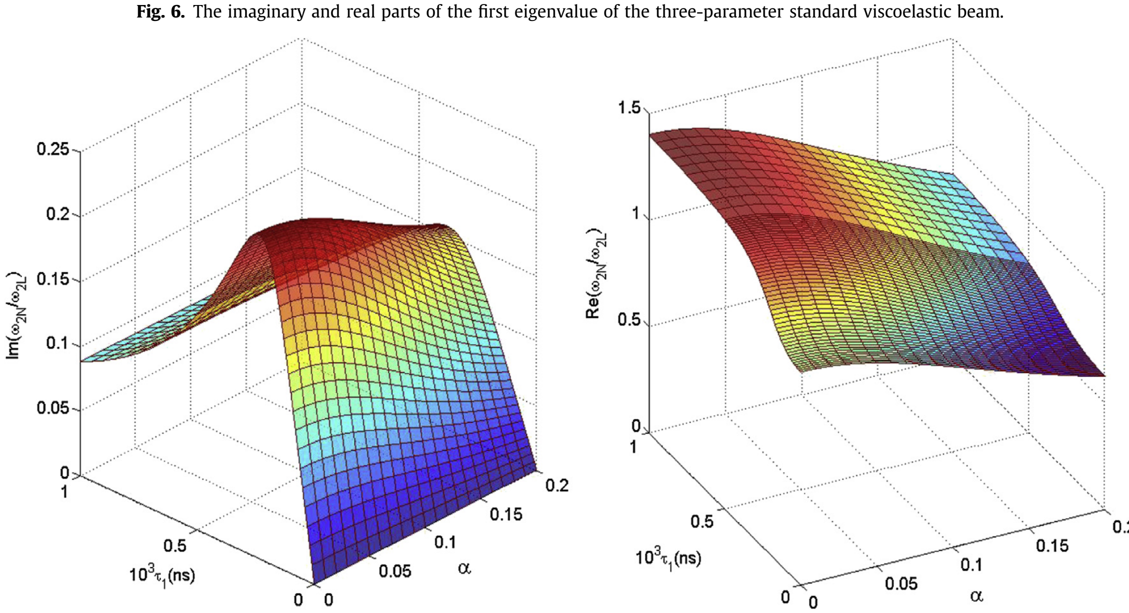

Fig. 7. The imaginary and real parts of the second eigenvalue of the three-parameter standard viscoelastic beam.

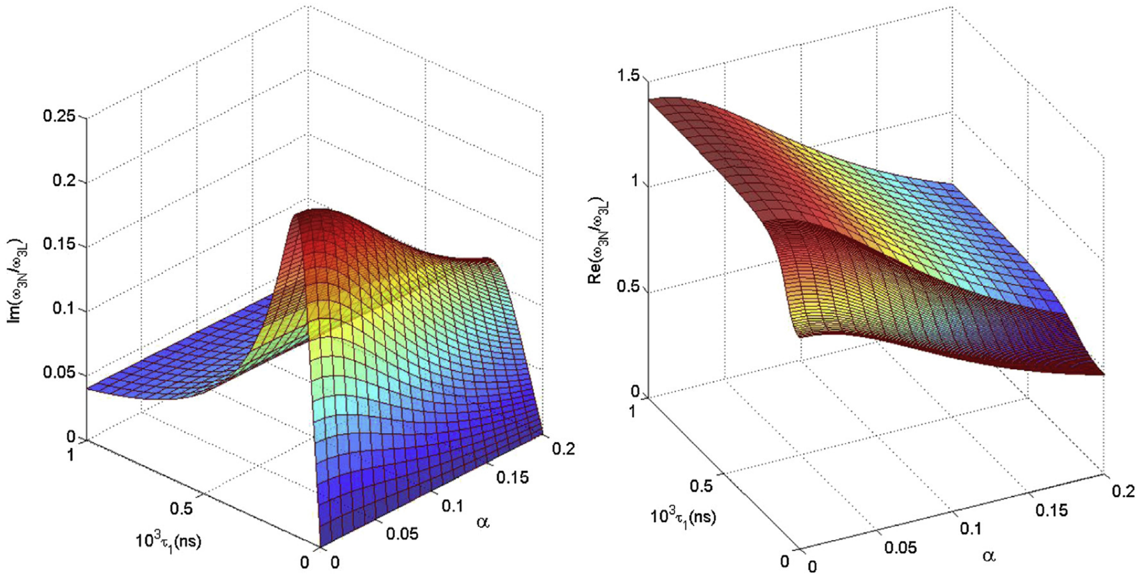

Fig. 8. The imaginary and real parts of the third eigenvalue of the three-parameter standard viscoelastic beam.

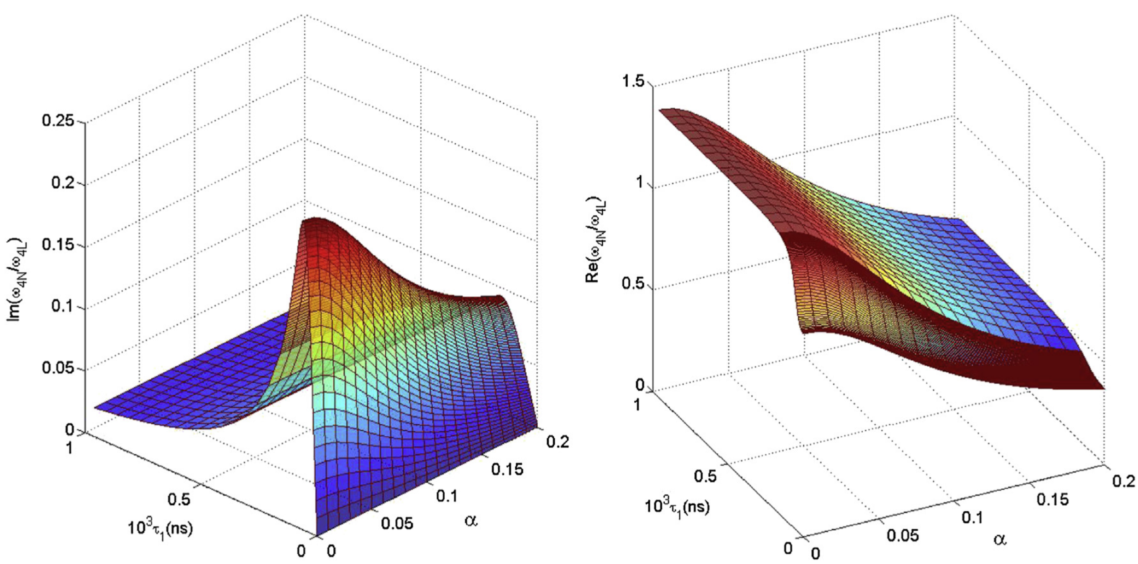

Fig. 9. The imaginary and real parts of the fourth eigenvalue of the three-parameter standard viscoelastic beam.

frequency significantly. For increasing values of α, there is a clear decreasing trend in the real part of the frequencies. The decrease is sharper for the higher frequencies. This implies that for the higher values of the nonlocal parameter, a three-parameter standard viscoelastic Euler–Bernoulli beam effectively becomes softer.

8. Conclusions

In this paper, the free vibration of a damped nonlocal viscoelastic Euler–Bernoulli beam was investigated. The governing equation of motion and corresponding characteristic equation for the complex natural frequencies were derived. For certain boundary conditions and system parameters, closed-form analytical expressions of the complex frequencies were obtained by solving the underlying characteristic equation in an exact manner. Several physically intuitive special cases and asymptotic results were derived. It was shown that the classical local undamped and damped Euler–Bernoulli beams as well as the undamped nonlocal Euler–Bernoulli beam arise as special cases of the general expressions proposed in the paper.

For general boundary conditions, a closed-form and uniform solution was proposed using the Transfer Function Method (TFM). The theory developed in the paper was applied to the dynamics of a single walled carbon nanotube. In the numerical examples, the effects of the nonlocal parameter, viscoelastic constants and the external damping parameter on the complex natural frequencies were discussed. Some conclusions arising from the numerical results can be summarized as follows:

- The external damping parameters have simple effects on the natural frequencies and the dependence with nonlocal parameter is not strong.
- For the Kelvin–Voigt model, the imaginary part of the complex natural frequency increases almost linearly with the

viscoelastic parameter. There is only a small dependence of the viscoelastic parameter on the real of the complex natural frequencies.

- For the three parameter standard viscoelastic model, the viscoelastic parameters have a complicated effect on the natural frequencies, especially for high natural frequencies.
- The nonlocal parameters decrease the sensitivity of the viscoelastic parameter on the damped natural frequencies.

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