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Asymptotic frequencies of various damped nonlocal beams and plates

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ABSTRACT

A striking difference between the conventional local and nonlocal dynamical systems is that the later possess finite asymptotic frequencies. The asymptotic frequencies of four kinds of nonlocal viscoelastic damped structures are derived, including an Euler–Bernoulli beam with rotary inertia, a Timoshenko beam, a Kirchhoff plate with rotary inertia and a Mindlin plate. For these undamped and damped nonlocal beam and plate models, the analytical expressions for the asymptotic frequencies, also called the maximum or escape frequencies, are obtained. For the damped nonlocal beams or plates, the asymptotic critical damping factors are also obtained. These quantities are independent of the boundary conditions and hence simply supported boundary conditions are used. Taking a carbon nanotube as a numerical example and using the Euler–Bernoulli beam model, the natural frequencies of the carbon nanotubes with typical boundary conditions are computed and the asymptotic characteristics of natural frequencies are shown.

1. Introduction

Structures at the nano-scale dimension are subject to damping from external forces, such as magnetic forces (Lee and Lin, 2010), interaction with the substrate, humidity and thermal effects (Chen et al., 2011), and fluid damping (Bhiladvala and Wang, 2004). The Akhiezer damping phenomenon is found in nanostructures such as a nano-mechanical resonator with a nearly uniform strain field (Kunal and Aluru, 2011). The general characteristics of the damping may be elastic or viscoelastic and these characteristics play a vital role in the dynamic analysis of structures. At the nano-scale, the understanding of damping characteristics, for example in atomic force microscopes (AFM), is required to generate superior image quality at high scan rates. Further, damping at the nano-scale is important for vibration based nano-sensors (Adhikari and Chowdhury, 2012) for the accurate measurement of the frequency shift. Damping mechanisms are usually complex and often approximated as proportional damping in engineering applications, with the damping parameters found from experiment. Calleja et al. (2012) stated that accurate quantification of the damping is vital to understand the sensitivity of nano-scale mass sensors. Hence, the investigation of the dynamic response of damped beams and plates at the nano-scale is essential.

The development of nano-technology yields an increasing number of nano-scale devices, including nano-electro-mechanical systems (NEMS) and micro-electro-mechanical systems (MEMS). Basic structural elements such as beams and plates at the nano-scale are utilized as nano-structure components for NEMS and MEMS. Widely used nano-rods and nano-beams include carbon and boron nitride nanotubes, while popular nano-plates (i.e. two-dimensional structures) are graphene sheets and gold nano-plates. Two key computational methodologies available for the dynamic analysis of these nano-structures comprise molecular dynamic (MD) simulation and continuum mechanics based approaches. Both of these methods have drawbacks. Some realistic experiments are available, although controlling every parameter at the nano-scale is difficult. Continuum mechanics is gaining popularity, although the size and scale effects are important at the nano-scale, and cannot be ignored. Among the size-dependent mechanics methods, the nonlocal continuum model (Eringen, 1983) has received significant interest in recent years. By employing the nonlocal elasticity approach, length-scale effects may be included in a simple, but physically understandable, way in nano-structures. The nonlocal continuum theory has been compared with molecular dynamics simulations, and excellent agreement has been obtained (Ansari and Sahmani, 2012; Ansari et al., 2010; Murmu and Adhikari,

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2011). Nonlocal Euler–Bernoulli and Timoshenko beam models have been used for bending, vibration and buckling analysis of nano-structures (Ansari and Ramezannezhad, 2011; Benzair et al., 2008; Heireche et al., 2008; Li et al., 2011; Murmu and Adhikari, 2010; Murmu and Pradhan, 2009; Reddy, 2007; Reddy and Pang, 2008; Simsek, 2011; Thai, 2012; Yang et al., 2011, 2012). Lu et al. (2007) proposed nonlocal elastic plate theories. Nonlocal beam and plate models are widely used for carbon nano-tubes and graphene (Arash and Wang, 2012).

In this communication we consider viscoelastic damped nonlocal structures. The effect of viscoelastic damping on the vibration response has been investigated by several researchers. Lopez and Fernandez (2012) considered nonlocal viscoelastic damping patches in the bending vibrations of Euler–Bernoulli beams. The dynamics of Euler–Bernoulli beams and Kirchhoff plates with nonlocal damping, including time and hysteresis effects has been investigated (Friswell et al., 2007; Lei et al., 2006). Using the finite element method developed for nonlocal viscoelastic beam models, Friswell et al. (2007) studied the dynamics of beams with different boundary conditions. Although these investigations highlighted the role of viscoelasticity for macro-scale models, their applicability to the nano-scale remains an open question.

The appearance of asymptotic frequencies for nanostructures, using nonlocal mechanics, have been reported, including wave propagation in carbon nano-tubes via nonlocal continuum mechanics models (Wang, 2005). The derivation of asymptotic phase velocities and frequencies were presented. The asymptotic frequency is important as the scale coefficient or nonlocal parameter could be obtained from its known or measured value. Wang and Varadan (2007) performed similar work with nonlocal shell theory and estimated the scale coefficient based on the derived asymptotic frequency. Escape frequencies or asymptotic frequencies were also proposed for embedded graphene sheets (Narendar and Gopalakrishnan, 2012). The relationships between the asymptotic frequencies and the nonlocal scaling parameter were established.

The literature survey clearly shows that most studies investigating asymptotic frequencies on scale-dependent beams and plates considered undamped structures, and work on damped structures is very limited. For nonlocal elastic solids, asymptotic frequencies exist, in contrast to local elastic solids. In this communication, the asymptotic frequencies and asymptotic critical damping factors are derived and presented for four kinds of nonlocal viscoelastic damped structures. These structural elements include Euler–Bernoulli beams with rotary inertia, Timoshenko beams, Kirchhoff plates with rotary inertia, and Mindlin plates. Kirchhoff plate theory is a thin plate theory that ignores the effect of transverse shear deformation. Mindlin plate theory is a first-order shear-deformable plate theory, where the transverse shear deformation is significant in thick plates and shear-deformable plates. These structures may be applied to the nano-scale, including nonlocal effects. The asymptotic frequencies form an upper bound for the natural frequencies of these structures. The higher frequency mode shapes are not significantly affected by the boundary conditions of the structures; that is the asymptotic frequencies are equal for all boundary conditions for a given beam or plate theory. Hence simply supported boundary conditions are used to calculate the asymptotic frequencies (also called the maximum or escape frequencies) for the undamped and damped nonlocal beams and plates, as analytical solutions are most easily obtained for these boundary conditions. For the damped nonlocal beams or plates, the asymptotic critical damping factors are also obtained. A carbon nano-tube is taken as a numerical example, and using the Timoshenko beam model, the natural frequencies of the carbon nano-tubes with typical boundary conditions are computed, and the asymptotic characteristics of the natural frequencies are highlighted.

2. Equations of motion for damped nonlocal beams and plates

Lu et al. (2007) gave a brief review of nonlocal elasticity theory, and basic nonlocal beam and plate theories are reported in the literature (for example, Arash and Wang, 2012). According to nonlocal elasticity theory, the stress tensor at a point $\mathbf{r}$ is given by

$$\sigma = \int_V K(|\mathbf{r}' - \mathbf{r}|, \tau)\sigma_m(\mathbf{r}')d\mathbf{r}' \tag{1}$$

where K is the nonlocal modulus, $|\mathbf{r}' - \mathbf{r}|$ is the Euclidean distance and σ_m is the local macroscopic stress. The term τ is the material constant that depends on the internal and external lengths. Eq. (1) is in integral partial form and the resulting integro-partial differential equation is difficult to solve. For certain choices of the kernel function, this equation may be transformed into an easier differential form (Eringen, 1983; Reddy, 2007), for example

$$(1 - \tau^2 \iota^2 \nabla^2)\sigma = \sigma_m, \quad \tau = \frac{e_0 a}{\iota} \tag{2}$$

where e_0 is a material constant, a and ι denote the internal and external characteristic lengths, respectively.

Incorporating the nonlocal effects (Eringen, 1983), external damping, initial pre-stress and the Kelvin-Voigt viscoelastic model (or internal damping) into nano-beam and nano-plate models, the governing equations for the vibration analysis of different systems are obtained. The details of the derivation for beams is discussed in an earlier paper by the present authors (Lei et al., 2013a). Using Eq. (2) we list the equation of motion for four systems discussed in this communication.

2.1. Nonlocal viscoelastic Euler–Bernoulli beam with rotary inertia

Consider an Euler–Bernoulli beam with deflection w, at position x along the beam. Then we have the following equations:

$$\left(1 + \tau_d \frac{\partial}{\partial t}\right) EI \frac{\partial^4 w}{\partial x^4} + \rho A \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial^4 w}{\partial t^2} + C_1 \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial w}{\partial t} - \rho I \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + N_0 \left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right] \frac{\partial^2 w}{\partial x^2} = 0, \tag{3}$$

where E,I,ρ,A are Young's modulus, second moment of area, density and cross-sectional area of the beam. N_0 is the axial pre-stress. t denotes the time and τ_d is the viscoelastic time constant. C_1 is the displacement–velocity–dependent viscous damping coefficient. The term $e_0 a$ is the nonlocal parameter where a is the external characteristic length such as crack length, wavelength or sample length. The dynamics of a damped viscoelastic nonlocal Euler–Bernoulli beam was described by Lei et al. (2013b).

2.2. Nonlocal viscoelastic Timoshenko beam

The coupled equations of motion are given by

$$\rho A\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial^4 w}{\partial t^2}+C_1\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial w}{\partial t}-\kappa GA\left(1+\tau_d\frac{\partial}{\partial t}\right)\left(\frac{\partial\theta}{\partial x}+\frac{\partial^4 w}{\partial x^2}\right)+N_0\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial^2 w}{\partial x^2}=0 \quad (4)$$

$$\rho I\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial^4\theta}{\partial t^2}+C_2\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial\theta}{\partial x}+\kappa GA\left(1+\tau_d\frac{\partial}{\partial t}\right)\left(\theta+\frac{\partial w}{\partial x}\right)-EI\left(1+\tau_d\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2}=0, \quad (5)$$

where θ is the angle of rotation of the normal to the mid-surface of the beam, and w is the displacement of the mid-surface in the z direction. The terms κ and G are the Timoshenko shear coefficient and the shear modulus respectively. C_2 is the rotation-velocity dependent viscous damping coefficient.

2.3. Nonlocal viscoelastic Kirchhoff plate with rotary inertia

Using nonlocal plate theory (Pradhan and Phadikar, 2009) and incorporating viscoelasticity, the nonlocal viscoelastic plate equation is derived as

$$\left(1+\tau_d\frac{\partial}{\partial t}\right)D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-(e_0a)^2\rho h\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+(e_0a)^2\frac{\rho h^3}{12}\left(\frac{\partial^6 w}{\partial x^4\partial t^2}+2\frac{\partial^4 w}{\partial x^2\partial y^2\partial t^2}+\frac{\partial^6 w}{\partial y^4\partial t^2}\right)$$
$$+\rho h\frac{\partial^4 w}{\partial t^2}-\frac{\rho h^3}{12}\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+C_1\left[1-(e_0a)^2\left(\frac{\partial^6}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)\right]\frac{\partial w}{\partial t}-\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\left(N_0^{xx}\frac{\partial^2 w}{\partial x^2}+N_0^{yy}\frac{\partial^2 w}{\partial y^2}\right)=0 \quad (6)$$

where D is the bending rigidity, and h denotes the thickness of the plate. N_0^{xx} and N_0^{yy} are the in-plane forces in the x and y directions respectively.

2.4. Nonlocal viscoelastic Mindlin plate

The three coupled equations describing the motion of a Mindlin plate can be expressed as

$$\rho A\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]+\frac{\partial^2 w}{\partial t^2}+C_1\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\frac{\partial w}{\partial t}-\kappa^2 Gh\left(1+\tau_d\frac{\partial}{\partial t}\right)\left(\frac{\partial\theta_x}{\partial x}+\frac{\partial\theta_y}{\partial yx}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$
$$+\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\left(N_0^{xx}\frac{\partial^2 w}{\partial x^2}+N_0^{yy}\frac{\partial^2 w}{\partial y^2}\right)=0 \quad (7)$$

$$\rho\frac{h^3}{12}\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\frac{\partial^2\theta_x}{\partial t^2}+C_2\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial\theta_x}{\partial t}+\kappa^2 Gh\left(1+\tau_d\frac{\partial}{\partial t}\right)\left(\theta_x+\frac{\partial w}{\partial x}\right)$$
$$-D\left(1+\tau_d\frac{\partial}{\partial t}\right)\left[\frac{\partial^2\theta_x}{\partial x^2}+\frac{1}{2}(1-\nu)\frac{\partial^2\theta_x}{\partial y^2}+\frac{1}{2}(1+\nu)\frac{\partial^2\theta_x}{\partial x\partial y}\right]=0 \quad (8)$$

$$\rho\frac{h^3}{12}\left[1-(e_0a)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\frac{\partial^2\theta_y}{\partial t^2}+C_2\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\frac{\partial\theta_y}{\partial t}+\kappa^2 Gh\left(1+\tau_d\frac{\partial}{\partial t}\right)\left(\theta_y+\frac{\partial w}{\partial x}\right)$$
$$-D\left(1+\tau_d\frac{\partial}{\partial t}\right)\left[\frac{\partial^2\theta_y}{\partial y^2}+\frac{1}{2}(1-\nu)\frac{\partial^2\theta_y}{\partial x^2}+\frac{1}{2}(1+\nu)\frac{\partial^2\theta_y}{\partial x\partial y}\right]=0 \quad (9)$$

Here ν is the conventional Poisson's ratio. The derivation of nonlocal Mindlin plate theory is given, for example, in Ansari et al. (2010).

3. Natural frequency analysis for simply supported boundary conditions

The asymptotic frequencies are independent of boundary conditions (Adhikari et al., 2013), since near to the asymptotic frequency the wavelength of vibration becomes smaller compared to the length-scale of the system. Consequently the boundaries have less influence on the value of the asymptotic frequency. Hence we consider simply supported boundary conditions since these provide the simplest analytical solutions to obtain the natural frequencies.

3.1. Asymptotic frequency for the Euler–Bernoulli beam

Consider a solution of the form $w=W\sin(\beta x)e^{i\omega t}$, where i denotes the unit imaginary, and ω is the circular natural frequency. $W\sin(\beta x)$ represents the complex mode shape. Substituting this trial solution into Eq. (3), we have, for a nontrivial solution for W,

$$[\rho A+\rho A(e_0a)^2\beta^2+\rho I\beta^2+\rho I(e_0a)^2\beta^4]\omega^2-[EI\tau_d\beta^4+C_1(e_0a)^2\beta^2+C_1]i\omega-[EI\beta^4+N_0\beta^2+N_0(e_0a)^2\beta^4]=0 \quad (10)$$

The complex natural frequencies are obtained as

$$\omega_m = \frac{bi + \sqrt{4gc - b^2}}{2g} \tag{11}$$

where $\beta = m\pi/L$, $g = \rho A + \rho A(e_0 a)^2\beta^2 + \rho I\beta^2 + \rho I(e_0 a)^2\beta^4$, $b = EI\tau_d\beta^4 + C_1(e_0 a)^2\beta^2 + C_1$, $c = EI\beta^4 + N_0\beta^2 + N_0(e_0 a)^2\beta^4$.

When $\beta \to \infty$, the asymptotic frequency is obtained as the solution to the quadratic equation

$$\rho I(e_0 a)^2\omega^2 - EI\tau_d i\omega - EI + N_0(e_0 a)^2 = 0 \tag{12}$$

Thus the asymptotic frequency is

$$\omega_{asym} = \frac{\tau_d E i}{2(e_0 a)^2\rho} + \frac{1}{e_0 a}\sqrt{\frac{E}{\rho}}\sqrt{1 - \frac{\tau_d^2 E}{4(e_0 a)^2\rho}}, \quad \text{for} \quad N_0 = 0 \tag{13}$$

and

$$\omega_{asym} = \frac{1}{e_0 a}\sqrt{\frac{E}{\rho} - \frac{(e_0 a)^2 N_0}{\rho I}}, \quad \text{for} \quad \tau_d = 0. \tag{14}$$

3.2. Asymptotic frequency for the Timoshenko beam

Let $w = W\sin(\beta x)e^{i\omega t}$ and $\theta = i\Theta\cos(\beta x)e^{i\omega t}$. Substituting these expressions into Eqs. (4) and (5), we have

$$-\rho A[1 + (e_0 a)^2\beta^2]\omega^2 W + C_1[1 + (e_0 a)^2\beta^2]i\omega W + \kappa GA(1 + \tau_d i\omega)(\beta\Theta + \beta^2 W) - N_0[1 + (e_0 a)^2\beta^2]\beta^2 W = 0 \tag{15}$$

$$-\rho I[1 + (e_0 a)^2\beta^2]\omega^2\Theta + C_2[1 + (e_0 a)^2\beta^2]i\omega\Theta + \kappa GA(1 + \tau_d i\omega)(\Theta + \beta W) - EI(1 + \tau_d i\omega)\beta^2\Theta = 0 \tag{16}$$

For a nontrivial solution for W and Θ we have

$$\begin{aligned}&\{-\rho A[1 + (e_0 a)^2\beta^2]\omega^2 + C_1[1 + (e_0 a)^2\beta^2]i\omega + \kappa GA(1 + \tau_d i\omega)\beta^2 - N_0[1 + (e_0 a)^2\beta^2]\beta^2\} \times \{-\rho I[1 + (e_0 a)^2\beta^2]\omega^2 \\ &\quad + C_2[1 + (e_0 a)^2\beta^2]i\omega + \kappa GA(1 + \tau_d i\omega) + EI(1 + (\tau_d i\omega)\beta^2\} - [\kappa GA(1 + \tau_d i\omega)\beta^2] = 0\end{aligned} \tag{17}$$

When $\beta \to \infty$, the asymptotic frequency equation can be obtained from the following equations (Lei et al., 2013a).

(a) For $N_0 = 0$

In this case Eq. (17) becomes,

$$[-\rho A(e_0 a)^2\omega^2 + C_1(e_0 a)^2 i\omega + \kappa GA(1 + \tau_d i\omega)] \times [-\rho I(e_0 a)^2\omega^2 + C_2(e_0 a)^2 i\omega + EI(1 + \tau_d i\omega)] = 0$$

For the undamped nonlocal beam, when $C_1 = C_2 = \tau_d = 0$, there are two asymptotic frequencies given by

$$\omega_{\text{asym}} = \frac{1}{e_0 a}\sqrt{\frac{E}{\rho}} \quad \text{and} \quad \omega_{asym} = \frac{1}{e_0 a}\sqrt{\frac{\kappa G}{\rho}} \tag{18a,b}$$

For the nonlocal viscoelastic beam, when $C_1 = C_2 = 0$, there are two asymptotic frequencies given by

$$\Omega_{\text{asym}} = i\frac{\kappa G\tau_d}{2\rho(e_0 a)^2} + \frac{1}{e_0 a}\sqrt{\frac{\kappa G}{\rho}}\sqrt{1 - \frac{\kappa G\tau_d^2}{4\rho(e_0 a)^2}} \tag{19a}$$

$$\Omega_{\text{asym}} = i\frac{E\tau_d}{2\rho(e_0 a)^2} + \frac{1}{e_0 a}\sqrt{\frac{E}{\rho}}\sqrt{1 - \frac{E\tau_d^2}{4\rho(e_0 a)^2}} \tag{19b}$$

The critical damping ratios are obtained by setting the oscillation frequency to zero to give

$$(\tau_d)_{\text{critial}} = 2e_0 a\sqrt{\frac{\rho}{\kappa G}} \quad \text{and} \quad (\tau_d)_{\text{critial}} = 2e_0 a\sqrt{\frac{\rho}{E}} \tag{20a,b}$$

For a damped nonlocal elastic beam, $\tau_d = 0$, we obtain

$$\Omega_{\text{asym}} = i\frac{C_1}{2\rho A} + \sqrt{-\frac{C_1^2}{4(\rho A)^2} + \frac{\kappa G}{\rho(e_0 a)^2}} \quad \text{and} \quad \Omega_{\text{asym}} = i\frac{C_2}{2\rho I} + \sqrt{-\frac{C_2^2}{4(\rho I)^2} + \frac{E}{\rho(e_0 a)^2}} \tag{21a,b}$$

For this case, the critical damping ratios can be obtained as

$$(C_1)_{\text{critial}} = 2\frac{\rho A}{e_0 a} + \sqrt{\frac{\kappa G}{\rho}} \quad \text{and} \quad (C_2)_{\text{critial}} = 2\frac{\rho I}{e_0 a}\sqrt{\frac{E}{\rho}} \tag{22a,b}$$

(b) Case for $N_0 \neq 0$,

In this case Eq. (17) becomes

$$N_0(e_0a)^2[-\rho I(e_0a)^2\omega^2 + C_2(e_0a)^2 i\omega + EI(I + \tau_d i\omega)] = 0 \tag{23}$$

There is only one asymptotic frequency

$$\omega_{\text{asym}} = \frac{1}{e_0a}\sqrt{\frac{E}{\rho}}, \quad \text{for} \quad C_2 = \tau_d = 0 \tag{24}$$

$$\omega_{\text{asym}} = \frac{E\tau_d i}{2\rho(e_0a)^2} + \frac{1}{e_0a}\sqrt{\frac{E}{\rho}}\sqrt{1 - \frac{E\tau_d^2}{4\rho(e_0a)^2}}, \quad \text{for} \quad C_2 = 0 \tag{25}$$

$$\omega_{\text{asym}} = \frac{C_2 i}{2\rho I} + \sqrt{-\frac{C_2^2}{4(\rho I)^2} + \frac{E}{\rho(e_0a)^2}}, \quad \text{for} \quad \tau_d = 0 \tag{26}$$

These asymptotic frequencies are independent of the external damping parameter C_1.

3.3. Asymptotic frequency of the Kirchhoff plate

Let $w = W\sin(\beta x)\sin(\gamma y)e^{i\omega t}$. Substituting this expression into Eq. (6) to obtain the nontrivial solution for W, analyzing the asymptotic frequency for the Kirchhoff plate as $\beta \to \infty$ and $\gamma \to \infty$, and assuming $N_0^{xx} = N_0^{yy} = N_0$, gives

$$(1 + \tau_d i\omega)D - (e_0a)^2\omega^2\frac{\rho h^3}{12} + (e_0a)^2 N_0 = 0, \tag{27}$$

Thus

$$\omega_{\text{asym}} = \frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}} \quad \text{for} \quad N_0 = \tau_d = 0, \tag{28}$$

$$\omega_{\text{asym}} = \frac{\tau_d E i}{2(e_0a)^2 + \rho(1-\nu^2)} + \frac{1}{e_0a} + \sqrt{\frac{E}{\rho(1-\nu^2)}}\sqrt{1 - \frac{\tau_d^2 E}{4(e_0a)^2 + \rho(1-\nu^2)}}, \quad \text{for} \quad N_0 = 0, \tag{29}$$

$$\omega_{\text{asym}} = \frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)} + \frac{(e_0a)^2 N_0}{\rho h^3/12}}, \quad \text{for} \quad \tau_d = 0, \tag{30}$$

3.4. Asymptotic frequency for the Mindlin plate

Let $w = W\sin(\beta x)\sin(\gamma y)e^{i\omega t}$, $\theta_x = \Theta_{mn}\cos(\beta x)\sin(\gamma y)e^{i\omega t}$ and $\theta_y = \Xi_{mn}\sin(\beta x)\cos(\gamma y)e^{i\omega t}$. Substituting these expressions into Eqs. (7)–(9), considering nontrivial solutions for W, Θ_{mm} and $\Xi_{\underline{mm}}$, analyzing the asymptotic frequency of the Kirchhoff plate when $\beta \to \infty$ and $\gamma \to \infty$, and assuming $N_0^{xx} = N_0^{yy} = N_0$ gives the following expressions.

For $N_0 \neq 0$, $C_1 = C_2 = \tau_d = 0$

$$\left(\omega^2 - \frac{E}{(1-\nu^2)(e_0a)^2}\right)\left(\omega^2 + \frac{\kappa^2 G}{\rho(e_0a)^2}\right)\left(\omega^2 - \frac{G}{\rho(e_0a)^2}\right) = 0 \tag{31}$$

Hence,

$$\omega_{\text{asym}} = \frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}} \quad \text{and} \quad \omega_{\text{asym}} = \frac{1}{e_0a}\sqrt{\frac{G}{\rho}} \tag{32a,b}$$

For $N_0 = \tau_d = 0$

$$\left(\omega^2 + i\omega\frac{C_2}{h^3/12} - \frac{E}{(1-\nu^2)(e_0a)^2}\right)\left(\omega^2 + \frac{\kappa^2 G}{\rho(e_0a)^2}\right)\left(\omega^2 + i\omega\frac{C_2}{h^3/12} - \frac{G}{\rho(e_0a)^2}\right) = 0 \tag{33}$$

Hence,

$$\omega_{\text{asym}} = i\frac{6C_2}{\rho h^3} + \sqrt{-\left(\frac{6C_2}{\rho h^3}\right)^2 + \frac{E}{\rho(e_0a)^2(1-\nu^2)}} \quad \text{and} \quad \omega_{asym} = i\frac{6C_2}{\rho h^3} + \sqrt{-\left(\frac{6C_2}{\rho h^3}\right)^2 + \frac{G}{\rho(e_0a)^2}} \tag{34a,b}$$

For $N_0 = C_1 = C_2 = 0$

$$\left(\omega^2 - \frac{E(1+\tau_d i\omega)}{(1-\nu^2)(e_0a)^2}\right)\left(\omega^2 + \frac{\kappa^2 G}{\rho(e_0a)^2}\right)\left(\omega^2 - \frac{G(1+\tau_d i\omega)}{\rho(e_0a)^2}\right) = 0, \tag{35}$$

Hence,

$$\omega_{\text{asym}} = \frac{\tau_d E i}{2(e_0a)^2\rho(1-\nu^2)} + \frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}\sqrt{1 - \frac{\tau_d^2 E}{4(e_0a)^2\rho(1-\nu^2)}} \tag{36a}$$

Table 1
The asymptotic frequencies of the nonlocal beams and plates.

Structures	Asymptotic frequencies			
	Undamped		Damped	
	No pre-stress	Pre-stress	Internal damping	External damping
Euler beam	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho}-\frac{(e_0a)^2N_0}{\rho I}}$	$\frac{t_dEi}{2(e_0a)^2\rho}+\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}\sqrt{1-\frac{\tau_dE}{4(e_0a)^2\rho}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}$
Timoshenko beam	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}$ $\frac{1}{e_0a}\sqrt{\frac{\kappa G}{\rho}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}$	$\frac{\kappa G\tau_d i}{2\rho(e_0a)^2}+$ $\frac{1}{e_0a}\sqrt{\frac{\kappa G}{\rho}}\sqrt{1-\frac{\kappa Gr_d^2}{4\rho(e_0a)^2}}$ $\frac{E\tau_d i}{2\rho(e_0a)^2}+\frac{1}{e_0a}\sqrt{\frac{E}{\rho}}\sqrt{1-\frac{E\tau_d^2}{4\rho(e_0a)^2}}$	$i\frac{C_1}{2\rho A}+$ $\sqrt{-\frac{C_1^2}{4(\rho A)^2}+\frac{\kappa G}{\rho(e_0a)^2}}$ $i\frac{C_2}{2\rho A}+$ $\sqrt{-\frac{C_2^2}{4(\rho I)^2}+\frac{E}{\rho(e_0a)^2}}i$
Kirchhoff plate	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}+\frac{(e_0a)^2N_0}{\rho h^3/12}}$	$\frac{\tau_dEi}{2(e_0a)^2\rho(1-\nu^2)}$ $+\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}\sqrt{1-\frac{\tau_d^2E}{4(e_0a)^2\rho(1-\nu^2)}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}$
Mindlin plate	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}$ $\frac{1}{e_0a}\sqrt{\frac{G}{\rho}}$	$\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}$ $\frac{1}{e_0a}\sqrt{\frac{G}{\rho}}$	$\frac{\tau_dEi}{2(e_0a)^2\rho(1-\nu^2)}$ $+\frac{1}{e_0a}\sqrt{\frac{E}{\rho(1-\nu^2)}}\sqrt{1-\frac{\tau_d^2E}{4(e_0a)^2\rho(1-\nu^2)}}$ $\frac{\tau_dGi}{2(e_0a)^2\rho}+\frac{1}{e_0a}\sqrt{\frac{G}{\rho}}\sqrt{1-\frac{\tau_d^2G}{4(e_0a)^2\rho}}$	$i\frac{6C_2}{\rho h^3}+$ $\sqrt{-\left(\frac{6C_2}{\rho h^3}\right)^2+\frac{G}{\rho(e_0a)^2}}$ $i\frac{6C_2}{\rho h^3}+$ $\sqrt{-\left(\frac{6C_2}{\rho h^3}\right)^2+\frac{E}{\rho(e_0a)^2(1-\nu^2)}}$

Table 2
The critical damping ratios of the nonlocal beams and plates.

Structure	Critical damping ratio	
	Internal damping	External damping
Euler beam	$2e_0a\sqrt{\frac{\rho}{E}}$	–
Timoshenko beam	$2e_0a\sqrt{\frac{\rho}{\kappa G'}}$ $2e_0a\sqrt{\frac{\rho}{\kappa G}}$	$\frac{C_1}{\rho A}\leq\frac{1}{2e_0a}\sqrt{\frac{\kappa G}{\rho'}}$ $\frac{C_2}{\rho I}\leq\frac{1}{2e_0a}\sqrt{\frac{E}{\rho}}$
Kirchhoff plate	$2e_0a\sqrt{\frac{\rho(1-\nu^2)}{E}}$	–
Mindlin plate	$2e_0a\sqrt{\frac{\rho(1-\nu^2)}{E}}$ $2e_0a\sqrt{\frac{\rho}{G}}$	$\frac{C_2}{\rho\frac{h^3}{12}}\leq\frac{1}{e_0a}\sqrt{\frac{G}{\rho}}$ $\frac{C_2}{\rho\frac{h^3}{12}}\leq\frac{1}{e_0a}\sqrt{\frac{G}{\rho(1-\nu^2)}}$

$$\omega_{\text{asym}}=\frac{\tau_dGi}{2(e_0a)^2\rho}+\frac{1}{e_0a}\sqrt{\frac{G}{\rho}}\sqrt{1-\frac{\tau_d^2G}{4(e_0a)^2\rho}} \tag{36b}$$

3.5. Summary

Table 1 summarizes the important benchmark relations of the asymptotic frequencies of nonlocal beams and plates (Euler–Bernoulli beam, Timoshenko beam, Kirchhoff plate and Mindlin plate) for the damped and undamped cases. Non pre-stress and pre-stress conditions are considered with internal and external damping. Table 2 gives the critical damping ratio corresponding to the special cases discussed here.

4. Numerical example

In this section, a single walled carbon nanotube (SWCNT) is used as an example. The first 60 natural frequencies of the SWCNT, modeled using an Euler–Bernoulli beam model are analyzed numerically using the transfer functions method (Yang and Tan, 1992), for three typical boundary conditions, S–S (simply supported), C–C (clamped-clamped), C–F (clamped-free). The basic parameters used in the calculations for the system are: Young's modulus, E = 1 TPa, mass density, ρ = 2.24 g/cm^3, diameter of the SWCNT, d = 1.1 nm, effective tube thickness, t = 0.342 nm, $\alpha = (e_0a/L) = 0.2$, $\zeta_1 = 0$, $\tau_d = 10^{-5}$.

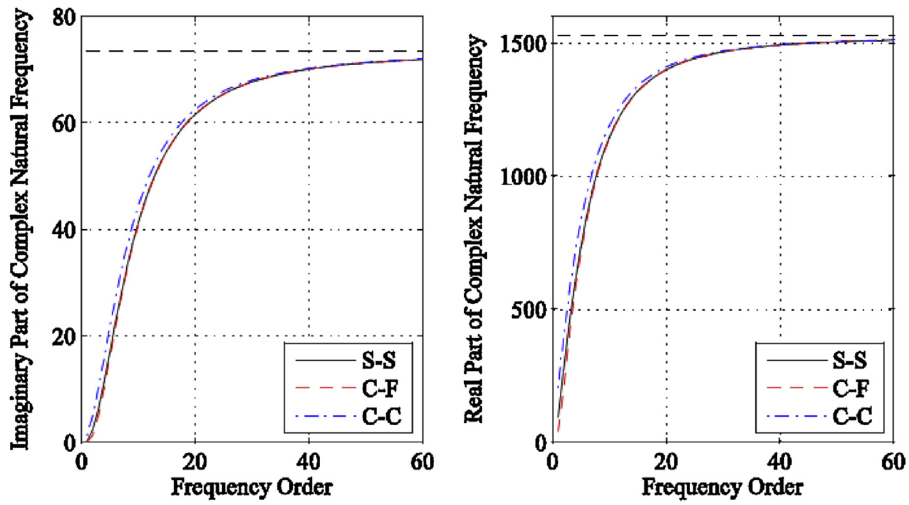

Fig. 1. The first 60 natural frequencies for an Euler–Bernoulli beam with S-S (simply supported), C–C (clamped–clamped), C–F (clamped–free) boundary conditions demonstrating the asymptotic convergence to the same limit for all boundary conditions.

For the SWCNT, the asymptotic frequency, which is independent of boundary conditions, is (1526.8 + 73.40i) Hz from Eq. (11) or Eq. (13). Fig. 1 shows the variation of the imaginary and real parts of the first 60 natural frequencies for the three boundary conditions for the nano-beam. The asymptotic characteristics of the natural frequencies of nonlocal beam (carbon nanotube) are clearly shown by the increasing order of the natural frequencies.

5. Conclusion

In this communication, the asymptotic frequencies of four types of nonlocal viscoelastic damped structures are derived, viz. Euler–Bernoulli beam with rotary inertia, a Timoshenko beam, a Kirchhoff plate with rotary inertia and a Mindlin plate. This upper 'cut-off' frequency is the maximum possible frequency of the system. This communication derives closed-form expressions of asymptotic frequencies for a class of damped nonlocal dynamical systems. The governing equations of motion of these four kinds of nonlocal viscoelastic damped structures are presented. Analytical expressions of the asymptotic frequencies and critical damping factors are derived, and Tables 1 and 2 summarize the contributions made. The natural frequencies of carbon nanotubes, modeled as a damped Euler–Bernoulli with typical boundary conditions, are computed using the transfer function method. The asymptotic characteristics of the natural frequencies are shown, and these asymptotic frequencies are shown to be independent of the boundary conditions. The derived closed-form expressions using the nonlocal theory can be employed for the analysis of micro and nano-scale beam and plate-like structures, in which the small-scale effects and damping become significant. These closed-form expressions of the asymptotic natural frequencies and critical damping factors can also serve as benchmarks for any future numerical studies.

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