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Vibration response of double-walled carbon nanotubes subjected to an externally applied longitudinal magnetic field: A nonlocal elasticity approach

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ABSTRACT

The magnetic properties of carbon nanotubes and their mechanical behaviour in a magnetic field have attracted considerable attention among the scientific and engineering communities. This paper reports an analytical approach to study the effect of a longitudinal magnetic field on the transverse vibration of a magnetically sensitive double-walled carbon nanotube (DWCNT). The study is based on nonlocal elasticity theory. Equivalent analytical nonlocal double-beam theory is utilised. Governing equations for nonlocal transverse vibration of the DWCNT under a longitudinal magnetic field are derived considering the Lorentz magnetic force obtained from Maxwell's relation. Numerical results from the model show that the longitudinal magnetic field increases the natural frequencies of the DWCNT. Both synchronous and asynchronous vibration phases of the tubes are studied in detail. Synchronous vibration phases of DWCNT are more affected by nonlocal effects than asynchronous vibration phases. The effects of a longitudinal magnetic field on higher natural frequencies are also presented. Vibration response of DWCNT with outer-wall stationary and single-walled carbon nanotube under the effect of longitudinal magnetic field are also discussed in the paper.

1. Introduction

The discovery of carbon nanotubes (CNTs) [1] has attracted enormous attention among the scientific community especially in the field of nanotechnology. This is due to the CNTs possessing novel mechanical, electronic, chemical and thermal conductivity properties. Because of these outstanding properties, CNTs have substantial promise as building blocks for nanoelectronics, nano-electromechanical systems (NEMS), micro-electromechanical systems (MEMS), nanodevices, nanocomposites, solar cells, space elevators etc. Based on the number of walls CNTs are designated as single-walled, double-walled or multi-walled. The study of the novel properties of single-walled and multi-walled CNTs is important for future materials development, and one of the important aspects is its behaviour in the presence of a magnetic field.

Magnetic field effects in nanotubes and nanoplates (graphene) could be important for exciting potential applications in nanotechnology such as in NEMS, MEMS, nanosensors, spintronics [2] and nanocomposites [3] etc. In recent years, research interest has grown on studying the magnetic properties of nanotubes, and behaviour of nanotubes subjected to an

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external magnetic field. Tsai et al. [4] has illustrated the magnetic properties of carbon nanotubes where the magnetism depends on the chiral angle and nanotube radius. The magnetic properties of nanotubes and their response in the presence of an external magnetic field are reported by Lopez et al. [5,6]. Chang and Lue [7] considered multiwall carbon nanotubes (MWCNTs) and investigated their magnetic properties by electron paramagnetic resonance and magnetization measurements. Reddy et al. [8] considered metal-filled multi-walled nanotubes and studied their magnetic properties by using vibrating sample magnetometry. Langer et al. [9] found that a magnetic field applied perpendicular to the tube axis increases the conductance and produces aperiodic fluctuations. Heremans et al. [10] carried out measurements of the magnetic moment and susceptibility of carbon nanotubes. They experimentally illustrated that magnetic moment and susceptibility behaviour is a function of magnetic field strength and temperature.

Studies on the influence of external magnetic fields on carbon nanotubes have also been reported. Jang and Sakka [11] illustrated the influence of the shape and size of carbon nanotubes (CNTs) on the alignment of multi-wall CNTs under a strong magnetic field. Bellucci et al. [12] investigated the influence of a perpendicular magnetic field on the transport properties of carbon nanotubes. Chen et al. [13] investigated the effects of an external magnetic field on the magnetic properties of nanotubes and nanowires. Kibalchenk et al. [14] examined the magnetic response of single-walled CNT under a magnetic field. Lee et al. [15] considered double-walled CNTs and illustrated discrete electronic states in a magnetic field. They confirmed that the electronic properties of double-walled CNTs strongly depend on the magnitude and direction of the magnetic field. Roche and Saito [16], Sebastiani and Kudin [17] and Zhang et al. [18] investigated the electronic properties of carbon nanotubes in the presence of a magnetic field. Wei and Wang [19] studied the different electromagnetic wave modes coupled in a longitudinal or transverse magnetic field. Mechanical studies, viz. vibration of single-walled and multi-walled nanotubes under a magnetic field have also been reported in the literature. Li et al. [20] investigated the effects of a magnetic field on the dynamic characteristics of multi-walled carbon nanotubes (MWNTs). Coupled dynamic equations of MWNTs subjected to a transverse magnetic field were developed. Wang et al. [21] investigated the effects of a longitudinal magnetic field on wave propagation in carbon nanotubes (CNTs) embedded in an elastic matrix. Dynamic equations were derived by considering the Lorentz magnetic forces. Wang et al. [22] presented an analytical method to investigate van der Waals interaction effects on vibration characteristics of multi-walled carbon nanotubes embedded in a matrix under a transverse magnetic field. Recently Narendar et al. [23] developed a nonlocal beam model to study wave propagation in single-walled CNT subjected to a longitudinal magnetic field.

Two obvious choices for the study of nanomaterials would be experiments and molecular dynamics simulations. However controlling every parameter in experiments at the nanoscale is difficult. In addition though molecular dynamics is popular for analysing CNTs and graphene, it takes into account every atom in the simulation and thus frequently requires a tremendous computational resource. These observations have led to the development of a number of theoretical and computational approaches on discrete and continuous models.

Experimental [24] and atomistic simulations [25] have evidenced a significant 'size-effect' in mechanical and physical properties for very 'small' structure. Size effects are related to the atoms and molecules that constitute the materials. Over the past decade, extensive literature has reported evidence of classical continuum models being able to predict the performance of 'large' nanostructures. For example, finite element based approaches have been proposed for dynamic analysis of double and multilayer carbon nanotubes [26,27]. However classical continuum mechanics is a scale-free theory and cannot account for effects arising from small size effects. The application of classical continuum models thus may be questionable in the analysis of 'smaller' nanostructures. Therefore, there have been research efforts to develop continuum models which incorporate small scale size effects. One widely used size-dependent theory is the nonlocal elasticity theory pioneered by Eringen [28].

In nonlocal elasticity theory [28], the small-scale effects are captured by assuming that the stress at a point is a function of the strains at all points in the domain. This is unlike classical elasticity theory, but is in accordance with atomic theory of lattice dynamics and experiment results from phonon dispersion. Nonlocal theory considers long-range inter-atomic interaction and yields results dependent on the size of a body. Some drawbacks of the classical continuum theory can be efficiently avoided and size-dependent phenomena can be reasonably explained by the nonlocal elasticity theory. The use of nonlocal elasticity in the mechanical analysis of nanostructures is extensively reported in the literature. A detail review of these various works can be found in [29].

Literature has shown that the carbon nanotubes can be modelled by both structural beam theories (viz. Euler Bernoulli beam theory, Timoshenko beam theory etc.) as well by shell theories. Wang and Zhang [30] proposed an elastic shell model for characterizing single walled carbon nanotubes. Cinefra et al. [31] illustrated some refined shell models for the vibration analysis of multi-walled carbon nanotubes. Liew and Wang [32] considered two developed elastic shell theories: Love's thin cylindrical shell theory and the Cooper Naghdi thick cylindrical shell theory for studying wave propagation in carbon nanotubes. Silvestre et al. [33] studied the buckling behaviour of single-walled carbon nanotubes (CNTs) via Donnell and Sanders shell models. Wang et al. [34] have shown cylindrical shell theories in predicting the critical buckling strains of axially loaded single-walled carbon nanotubes.

Similar to using shell theories for studying vibration and buckling of nanotubes, structural beam theories have also been used and proposed to predict the behaviour of nanotubes [35,36]. Like shell theories, structural beam theories when applied for CNT investigations are also found to predict the vibration behaviour of CNT reasonably well.

From the above discussions it is understood that the study of the mechanical behaviour of nanotubes subjected to an external magnetic field is important and requires attention. To the best of authors' knowledge the effects of magnetic fields

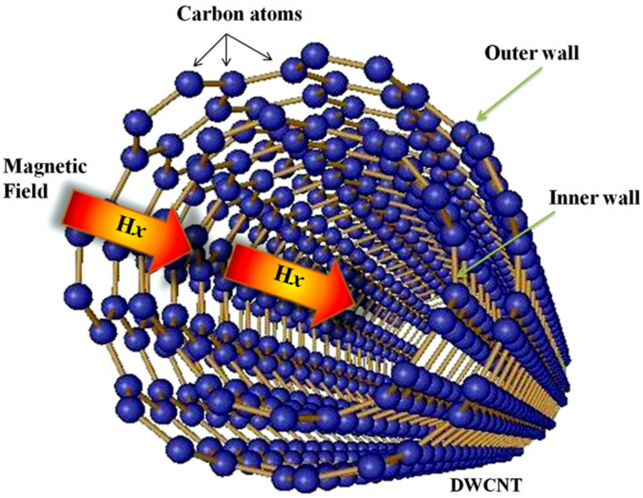

Fig. 1. Molecular diagram of armchair double-walled carbon nanotube subjected to axial magnetic field.

on the vibration characteristics of a double-walled carbon nanotubes (DWCNTs) using a nonlocal elasticity approach is unavailable in literature.

In the present paper, using structural beam theory we study the effects of a longitudinal magnetic field on the transverse vibration of a magnetically sensitive DWCNT (Fig. 1). Nonlocal elasticity is used to address the small scale effects. Nonlocal effects in the longitudinal direction are considered and the nonlocal effects in the circumferential direction of the CNTs are ignored. Equivalent nonlocal double-beam theory is utilised. Governing equations for nonlocal bending-vibration of DWCNTs under a longitudinal magnetic field are derived considering the Lorentz magnetic force obtained from Maxwell's relation. An analytical frequency relation is obtained to evaluate the natural frequencies of the DWCNTs as a function of a nonlocal parameter and magnetic field parameter. Both synchronous and asynchronous vibration phases of the tubes of DWCNTs are highlighted. Effects of the longitudinal magnetic field on higher natural frequencies are discussed. The validation of the present work is finally discussed.

2. Maxwell's relations

In this section we present Maxwell's relations. Denoting $\mathbf{J}$ as current density, $\mathbf{h}$ as distributing vector of the magnetic field, and $\mathbf{e}$ as strength vectors of the electric field, the Maxwell relations according to [37] is given as

$$\mathbf{J}=\nabla\times\mathbf{h} \tag{1}$$

$$\nabla\times\mathbf{e}=-\eta\frac{\partial\mathbf{h}}{\partial t} \tag{2}$$

$$\nabla\cdot\mathbf{h}=0 \tag{3}$$

$$\mathbf{e}=-\eta\left(\frac{\partial\mathbf{U}}{\partial t}\times\mathbf{H}\right) \tag{4}$$

$$\mathbf{h}=\nabla\times(\mathbf{U}\times\mathbf{H}) \tag{5}$$

Considering longitudinal magnetic field as a vector $\mathbf{H}=(H_x,0,0)$ acting on the DWCNT we have

$$\mathbf{h}=\nabla\times(\mathbf{U}\times\mathbf{H})=-H_x\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\mathbf{i}+H_x\frac{\partial v}{\partial x}\mathbf{j}+H_x\frac{\partial w}{\partial x}\mathbf{k} \tag{6}$$

where $\mathbf{U}$ is the displacement vector as $\mathbf{U}=(u,v,w)$. Here η is the magnetic field permeability, ∇ is the Hamiltonian operator and is expressed as $\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}$, and $(\mathbf{i},\mathbf{j},\mathbf{k})$ are the unit vectors. Thus we have

$$\mathbf{J}=\nabla\times\mathbf{h}=H_x\left(-\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial^2 w}{\partial x\partial y}\right)\mathbf{i}-H_x\left(\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)\mathbf{j}+H_x\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\mathbf{k}. \tag{7}$$

The Lorentz force induced by the longitudinal magnetic field is given as

$$\mathbf{f}=f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}=\eta(\mathbf{J}\times\mathbf{H})=\eta\left[0\mathbf{i}+H_x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)\mathbf{j}+H_x^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial y\partial z}\right)\mathbf{k}\right]. \tag{8}$$

Therefore the Lorentz forces along the x, y and z directions are

$$f_x = 0 \tag{9}$$

$$f_y = \eta H_x^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} \right) \tag{10}$$

$$f_z = \eta H_x^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z} \right). \tag{11}$$

For the present vibrational analysis in DWCNTs, we assume that $w = w(x,t)$ only, so that the Lorentz force in the z direction is written as [21], [23]

$$f_z = \eta H_x^2 \left(\frac{\partial^2 w}{\partial x^2} \right) \tag{12}$$

It should be noted that in the present study the effective Lorentz force is a function of magnetic permeability and H_x also.

3. Review of nonlocal elasticity

According to nonlocal elasticity theory [28], the basic equations for an isotropic linear homogenous nonlocal elastic body, neglecting the body force, are given as

$$\sigma_{ij,i} + \rho(f_j - \ddot{u}_j) = 0,$$

$$\sigma_{ij}(\mathbf{x}) = \int_{\mathbf{V}} \phi(|\mathbf{x} - \mathbf{x}'|, \alpha) \sigma_{ij}^c(\mathbf{x}') \mathrm{d}V(\mathbf{x}'),$$

$$\sigma_{ij}^c(\mathbf{x}') = C_{ijkl} \varepsilon_{kl}(\mathbf{x}'),$$

$$\varepsilon_{kl}(\mathbf{x}') = \frac{1}{2} \left(\frac{\partial u_i(\mathbf{x}')}{\partial x_j'} + \frac{\partial u_j(\mathbf{x}')}{\partial x_i'} \right) \tag{13}$$

The terms σ_{ij} σ_{ij}^c, ε_{kl}, C_{ijkl}, are the nonlocal stress, classical stress, classical strain and fourth order elasticity tensors, respectively. The volume integral is over the region $\mathbf{V}$ occupied by the body. The kernel function $\phi(|\mathbf{x} - \mathbf{x}'|, \alpha)$ is known as the nonlocal modulus. The nonlocal modulus acts as an attenuation function incorporating into the constitutive equations the nonlocal effects at the reference point $\mathbf{x}$ produced by local strain at the source $\mathbf{x}'$. The term $|\mathbf{x} - \mathbf{x}'|$ represents the distance in the Euclidean form and α is a material constant that depends on the internal (e.g., lattice parameter, granular size, distance between the C–C bonds) and external (e.g., crack length, wave length) characteristic lengths. One example of a nonlocal modulus, as given by Eringen [28], is

$$\phi(|\mathbf{x}|, \alpha) = (2\pi \ell^2 \alpha^2)^{-1} K_0(\sqrt{\mathbf{x} \cdot \mathbf{x}} / \ell \alpha) \tag{14}$$

The material constant α is defined as $\alpha = e_0 a / \ell$. Here e_0 is a constant for calibrating the model with experimental results and other validated models. The parameter e_0 is estimated such that the relations of the nonlocal elasticity model can provide a satisfactory approximation to the atomic dispersion curves of the plane waves obtained from atomistic lattice dynamics. According to Eringen [28], the value of e_0 is reported as 0.39. Details on the various values of nonlocal parameter e_0, as reported by various researchers, are discussed in [38]. The terms a and ℓ are the internal (e.g., lattice parameter, granular size, distance between C–C bonds) and external characteristics lengths (e.g., crack length, wave length) of the nanostructure, respectively.

4. Nonlocal Euler–Bernoulli beam equation

Eq. (13) is generally difficult to solve. Therefore by using Eq. (14), a differential form is popularly used as

$$(1 - \alpha^2 \ell^2 \nabla^2) \sigma_{ij}(\mathbf{x}) = \sigma_{ij}^c(\mathbf{x}) = C_{ijkl} \varepsilon_{kl}(\mathbf{x}) \tag{15}$$

Here ∇^2 is the Laplacian. For one-dimensional nanostructures such as nanotubes, Eq. (15) is simplified as

$$\left[1 - (e_0 a)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right] \sigma(x) = \sigma_{ij}^c(x) \tag{16}$$

Using the nonlocal relation (Eq. 16), the governing equation for a nonlocal Euler–Bernoulli beam equation under the influence of a longitudinal magnetic field can be written as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+kw(x,t)-(e_0a)^2k\frac{\partial^2 w(x,t)}{\partial x^2}-q(x,t)+(e_0a)^2\frac{\partial^2 q(x,t)}{\partial x^2}+m\frac{\partial^2 w(x,t)}{\partial t^2}-(e_0a)^2m\frac{\partial^4 w(x,t)}{\partial x^2\partial t^2}=0 \tag{17}$$

where w is the transverse deflection of the beam. The term q is considered as $q(x,t)=\int_A f_z dz=\eta AH_x^2\frac{\partial^2 w(x,t)}{\partial x^2}$ and is the magnetic force per unit length [23] acting on the beam. E, I and k are the Young's modulus of beam (CNT), area moment and spring modulus of the elastic medium, respectively. For the derivation of the beam equation based on nonlocal elasticity, ignoring surface effects, one can see Ref. [36].

5. Governing equation for vibration of DWCNT

Consider a DWCNT of length L subjected to longitudinal magnetic field, with strength H_x (Fig. 2). The DWCNT is modelled using nonlocal double Euler–Bernoulli beam theory [39,40] based on Eq. (17). The transverse deflection in the inner and outer wall of the DWCNT is denoted as w_1 and w_2, respectively. The equations of motion can be written as

Inner tube:

$$p=EI_1\frac{\partial^4 w_1}{\partial x^4}-\eta A_1H_x^2\frac{\partial^2 w_1}{\partial x^2}+\rho A_1\frac{\partial^2 w_1}{\partial t^2}-(e_0a)^2\left(\rho A_1\frac{\partial^4 w_1}{\partial x^2\partial t^2}-\eta A_1H_x^2\frac{\partial^4 w_1}{\partial x^4}-\frac{\partial^2 p}{\partial x^2}\right) \tag{18}$$

Outer tube:

$$-p=EI_2\frac{\partial^4 w_2}{\partial x^4}-\eta A_2H_x^2\frac{\partial^2 w_2}{\partial x^2}+\rho A_2\frac{\partial^2 w_2}{\partial t^2}-(e_0a)^2\left(\rho A_2\frac{\partial^4 w_2}{\partial x^2\partial t^2}-\eta A_2H_x^2\frac{\partial^4 w_2}{\partial x^4}+\frac{\partial^2 p}{\partial x^2}\right) \tag{19}$$

where p is given as

$$p=c(w_2-w_1) \tag{20}$$

Subscripts 1 and 2 denote the inner and outer tube, respectively. The term c is the van der Waals constant and is written as [41]

$$c=\frac{320(2R_1)\text{erg/cm}^2}{0.16g^2},\quad (g=0.142\text{ nm}) \tag{21}$$

where R_1 is the inner tube radius in cm.

The boundary condition of the DWCNT is assumed to be simply supported and considered as

$$w_1(0,t)=\frac{\partial^2 w_1(0,t)}{\partial x^2}=w_1(L,t)=\frac{\partial^2 w_1(L,t)}{\partial x^2}=0 \tag{22}$$

$$w_2(0,t)=\frac{\partial^2 w_2(0,t)}{\partial x^2}=w_2(L,t)=\frac{\partial^2 w_2(L,t)}{\partial x^2}=0 \tag{23}$$

The local and nonlocal boundary conditions are considered same for simply-supported case [42].

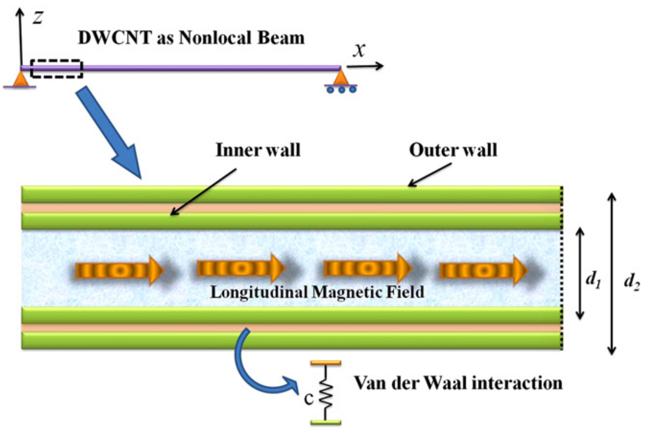

Fig. 2. Longitudinal magnetic field affected double-walled carbon nanotube (DWCNT) modelled as Euler–Bernoulli nonlocal beam.

6. Solution

6.1. Both walls of nanotube vibrating

The homogeneous governing partial differential Eqs. (18) and (19) of a DWCNT with boundary conditions Eqs. (22) and (23) can be solved by the Bernoulli––Fourier method assuming the solutions in the form

$$w_1(x,t)=\sum_{n=1}^{\infty}\sin(k_n x)U_{1n}(t) \tag{24}$$

$$w_2(x,t)=\sum_{n=1}^{\infty}\sin(k_n x)U_{2n}(t) \tag{25}$$

$$k_n=\frac{n\pi}{L}$$

where $U_{1n}(t)$ and $U_{2n}(t)$ are the unknown time functions, and $\sin(k_n x)$ is the known mode shape function for a simply supported single beam.

Substitution of Eqs. (24) and (25) into Eqs. (18) and (19) yields

$$\sum_{n=1}^{\infty}\left[\rho A_1(1+(e_0a)^2k_n^2)\frac{\partial^2 U_{1n}}{\partial t^2}+(EI_1k_n^4+\eta A_1H_x^2k_n^2+(e_0a)^2\eta A_1H_x^2k_n^4+c+c(e_0a)^2k_n^2)U_{1n}-c(1+(e_0a)^2k_n^2)U_{1n}\right]\sin(k_n x)=0 \tag{26}$$

$$\sum_{n=1}^{\infty}\left[\rho A_2(1+(e_0a)^2k_n^2)\frac{\partial^2 U_{2n}}{\partial t^2}+(EI_2k_n^4+\eta A_2H_x^2k_n^2+(e_0a)^2\eta A_2H_x^2k_n^4+c+c(e_0a)^2k_n^2)U_{2n}-c(1+(e_0a)^2k_n^2)U_{1n}\right]\sin(k_n x)=0 \tag{27}$$

We can write the above equations as

$$\frac{\partial^2 U_{1n}}{\partial t^2}+\left(\frac{R_1}{1+(e_0a)^2k_n^2}+S_1+T_1\right)U_{1n}-T_1U_{2n}=0, \tag{28}$$

$$\frac{\partial^2 U_{2n}}{\partial t^2}+\left(\frac{R_2}{1+(e_0a)^2k_n^2}+S_2+T_2\right)U_{2n}-T_2U_{1n}=0 \tag{29}$$

where we have considered

$$R_1=\frac{EI_1k_n^4}{\rho A_1},R_2=\frac{EI_2k_n^4}{\rho A_2}$$

$$S_1=S_2=S_{\mathrm{mf}}=\frac{\eta H_x^2k_n^2}{\rho}$$

$$T_1=\frac{c}{\rho A_1},T_2=\frac{c}{\rho A_2} \tag{30}$$

Assuming solutions of Eqs. (28) and (29) as

$$U_{1n}=W_{1n}\mathrm{e}^{\mathrm{i}\omega_n t}, \tag{31}$$

$$U_{2n}=W_{2n}\mathrm{e}^{\mathrm{i}\omega_n t} \tag{32}$$

where ω_n denotes the natural frequency of the DWCNT, and W_{1n} and W_{2n} represent the amplitude coefficients of the inner and outer tubes, respectively. Substituting them we get

$$\begin{bmatrix}\frac{R_1}{1+(e_0a)^2k_n^2}+S_1+T_1-\omega_n^2 & -T_1\\ -T_2 & \frac{R_2}{1+(e_0a)^2k_n^2}+S_2+T_2-\omega_n^2\end{bmatrix}\begin{Bmatrix}W_{1n}\\ W_{2n}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{33}$$

Nontrivial solutions for the constants W_{1n} and W_{2n} can be obtained only when the determinant of the coefficients in Eq. (33) vanishes. In this manner we have the characteristic equation as

$$\omega_n^4-\left(\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2\right)\omega_n^2+\frac{R_1R_2}{(1+(e_0a)^2k_n^2)^2}+\frac{R_1(T_2+S_{\mathrm{mf}})+R_2(T_1+S_{\mathrm{mf}})}{1+(e_0a)^2k_n^2}+(T_1+T_2)S_{\mathrm{mf}}+{S_{\mathrm{mf}}}^2=0 \tag{34}$$

The characteristic equation has the following two different real positive roots which are the frequencies of the DWCNT

$$\omega_{nI}^2=\frac{1}{2}\left[\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2-\sqrt{\begin{array}{c}\left(\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2\right)^2\\-4\left[\frac{R_1R_2}{\left(1+(e_0a)^2k_n^2\right)^2}+\frac{R_1(T_2+S_{\mathrm{mf}})+R_2(T_1+S_{\mathrm{mf}})}{1+(e_0a)^2k_n^2}+(T_1+T_2)S_{\mathrm{mf}}+{S_{\mathrm{mf}}}^2\right]\end{array}}\right] \tag{35}$$

$$\omega_{nII}^2=\frac{1}{2}\left[\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2+\sqrt{\begin{array}{c}\left(\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2\right)^2\\-4\left[\frac{R_1R_2}{\left(1+(e_0a)^2k_n^2\right)^2}+\frac{R_1(T_2+S)+R_2(T_1+S_{\mathrm{mf}})}{1+(e_0a)^2k_n^2}+(T_1+T_2)S_{\mathrm{mf}}+{S_{\mathrm{mf}}}^2\right]\end{array}}\right] \tag{36}$$

By simplifying the discriminant we have

$$\left(\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2\right)^2-4\left[\frac{R_1R_2}{\left(1+(e_0a)^2k_n^2\right)^2}+\frac{R_1(T_2+S_{\mathrm{mf}})+R_2(T_1+S_{\mathrm{mf}})}{1+(e_0a)^2k_n^2}+(T_1+T_2)S_{\mathrm{mf}}+{S_{\mathrm{mf}}}^2\right]$$
$$\equiv\left(\frac{R_1-R_2}{1+(e_0a)^2k_n^2}+T_1-T_2\right)^2+4T_1T_2>0 \tag{37}$$

and also we can see that

$$\frac{R_1R_2}{\left(1+(e_0a)^2k_n^2\right)^2}+\frac{R_1(T_2+S_{\mathrm{mf}})+R_2(T_1+S_{\mathrm{mf}})}{1+(e_0a)^2k_n^2}+(T_1+T_2)S_{\mathrm{mf}}+{S_{\mathrm{mf}}}^2>0 \tag{38}$$

So, we have the expression for the natural frequencies as

$$\omega_{nI}^2=\frac{1}{2}\left[\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2-\sqrt{\left(\frac{R_1-R_2}{1+(e_0a)^2k_n^2}+T_1-T_2\right)^2+4T_1T_2}\right] \tag{39}$$

$$\omega_{nII}^2=\frac{1}{2}\left[\frac{R_1+R_2}{1+(e_0a)^2k_n^2}+2S_{\mathrm{mf}}+T_1+T_2+\sqrt{\left(\frac{R_1-R_2}{1+(e_0a)^2k_n^2}+T_1-T_2\right)^2+4T_1T_2}\right] \tag{40}$$

Using Eqs. (39) and (40) the amplitude ratio of DWCNT under longitudinal magnetic field can be analytically obtained as

$$\gamma_n=\frac{W_{1n}}{W_{2n}}=\frac{T_1[1+(e_0a)^2k_n^2]}{R_1+[S_{\mathrm{mf}}+T_1-\omega_n^2][1+(e_0a)^2k_n^2]}=\frac{R_2+[S_{\mathrm{mf}}+T_2-\omega_n^2][1+(e_0a)^2k_n^2]}{T_2[1+(e_0a)^2k_n^2]} \tag{41}$$

So we have the amplitude ratios expressed as

$$\gamma_{nI}=\frac{1}{2T_2}\left[\frac{R_2-R_1}{1+(e_0a)^2k_n^2}+T_2-T_1+\sqrt{\left(\frac{R_1-R_2}{1+(e_0a)^2k_n^2}+T_1-T_2\right)^2+4T_1T_2}\right] \tag{42}$$

$$\gamma_{nII}=\frac{1}{2T_2}\left[\frac{R_2-R_1}{1+(e_0a)^2k_n^2}+T_2-T_1-\sqrt{\left(\frac{R_1-R_2}{1+(e_0a)^2k_n^2}+T_1-T_2\right)^2+4T_1T_2}\right] \tag{43}$$

Eqs. (31) and (32) can be written as

$$U_{1n}(t)=C_{1n}e^{i\omega_{nI}t}+C_{2n}e^{-i\omega_{nI}t}+C_{3n}e^{i\omega_{nII}t}+C_{4n}e^{-i\omega_{nII}t} \tag{44}$$

$$U_{2n}(t)=D_{1n}e^{i\omega_{nI}t}+D_{2n}e^{-i\omega_{nI}t}+D_{3n}e^{i\omega_{nII}t}+D_{4n}e^{-i\omega_{nII}t} \tag{45}$$

The above two equations by trigonometric functions yields

$$U_{1n}(t)=\sum_{k=I}^{II}A_{nk}\sin(\omega_{nk}t)+B_{nk}\cos(\omega_{nk}t) \tag{46}$$

$$U_{2n}(t)=\sum_{k=I}^{II}\gamma_{nk}(A_{nk}\sin(\omega_{nk}t)+B_{nk}\cos(\omega_{nk}t)) \tag{47}$$

where A_{nk} and B_{nk} (k=I, II) are constants. The transverse vibrations of the DWCNT under a longitudinal magnetic field can be described by

$$w_1(x,t)=\sum_{n=1}^{\infty}\sin(k_n x)\sum_{k=I}^{II}A_{nk}\sin(\omega_{nk}t)+B_{nk}\cos(\omega_{nk}t) \tag{48}$$

$$w_2(x,t)=\sum_{n=1}^{\infty}\sin(k_n x)\sum_{k=I}^{II}\gamma_{nk}(A_{nk}\sin(\omega_{nk}t)+B_{nk}\cos(\omega_{nk}t)) \tag{49}$$

Assuming the initial conditions as $w_1(x,0)=w_{10}(x)$; $w_2(x,0)=w_{20}(x)$; $\dot{w}_1(x,0)=\delta_{10}(x)$; $\dot{w}_2(x,0)=\delta_{20}(x)$, the constants can be determined as:

$$A_{nI}=\frac{1}{0.5L(\gamma_{nII}-\gamma_{nI})\omega_{nI}}\int_0^L(\gamma_{nII}\delta_{10}-\delta_{20})\sin(k_n x)\mathrm{d}x \tag{50}$$

$$A_{nII}=\frac{1}{0.5L(\gamma_{nI}-\gamma_{nII})\omega_{nII}}\int_0^L(\gamma_{nI}\delta_{10}-\delta_{20})\sin(k_n x)\mathrm{d}x \tag{51}$$

$$B_{nI}=\frac{1}{0.5L(\gamma_{nII}-\gamma_{nI})}\int_0^L(\gamma_{nII}w_{10}-w_{20})\sin(k_n x)\mathrm{d}x \tag{52}$$

$$B_{nII}=\frac{1}{0.5L(\gamma_{nI}-\gamma_{nII})}\int_0^L(\gamma_{nI}w_{10}-w_{20})\sin(k_n x)\mathrm{d}x \tag{53}$$

6.2. *Outer wall of nanotube fixed (w_2=0)*

Consider the mathematical case of DWCNT when the outer wall of the nanotube is fixed as shown in Fig. 3. The constrained DWCNT is subjected to longitudinal magnetic field.

The governing equation, using the nonlocal double beam theory of the constrained DWCNT, is given as

$$-cw_1=EI_1\frac{\partial^4 w_1}{\partial x^4}-\eta A_1H_x^2\frac{\partial^2 w_1}{\partial x^2}+\rho A_1\frac{\partial^2 w_1}{\partial t^2}-(e_0a)^2\left(\rho A_1\frac{\partial^4 w_1}{\partial x^2\partial t^2}-\eta A_1H_x^2\frac{\partial^4 w_1}{\partial x^4}+c\frac{\partial^2 w_1}{\partial x^2}\right) \tag{54}$$

Using the same procedure, the frequency of the inner nanotube for the given case is obtained as

$$\omega_{NI}=\sqrt{\left[\frac{R_1}{1+(e_0a)^2k_n^2}+S_{\mathrm{mf}}+T_1\right]} \tag{55}$$

6.3. *Single-walled carbon nanotube in longitudinal magnetic field*

Similarly we also consider the case of a single-walled carbon nanotube subjected to longitudinal field (Fig. 4). Here the effect of van der Waal's force is absent (c=0).

For a single-walled carbon nanotube the governing equation is given as

$$EI\frac{\partial^4 w}{\partial x^4}-\eta AH_x^2\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}-(e_0a)^2\left(\rho A\frac{\partial^4 w}{\partial x^2\partial t^2}-\eta AH_x^2\frac{\partial^4 w}{\partial x^4}\right)=0 \tag{56}$$

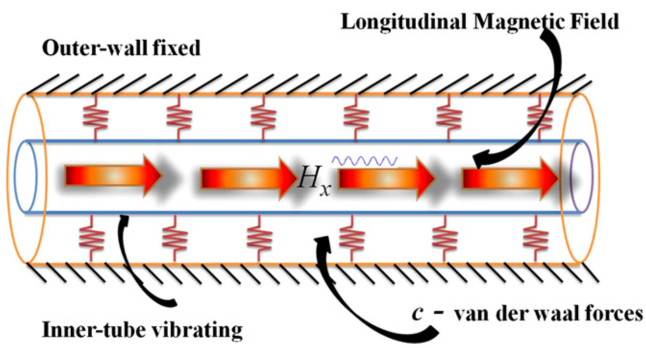

Fig. 3. Outer tube of the DWCNT is considered to be fixed.

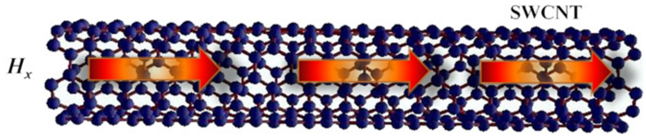

Fig. 4. Single-walled carbon nanotube subjected to longitudinal magnetic field.

The equation is similar to that obtained by Narendar et al. [23] for wave propagation in a CNT. The frequency of the single nanotube with the same diameter as the inner tube of the DWCNT is obtained as

$$\omega_{NI} = \sqrt{\left[\frac{R_1}{1+(e_0a)^2k_n^2}+S_{\mathrm{mf}}\right]} \tag{57}$$

Similarly the frequency of the single nanotube with diameter as outer tube (DWCNT) is obtained as

$$\omega_{NI} = \sqrt{\left[\frac{R_2}{1+(e_0a)^2k_n^2}+S_{\mathrm{mf}}\right]} \tag{58}$$

7. Molecular dynamics vs. nonlocal elasticity theory for CNTs

In various works reported in the literature, nonlocal elasticity theory applied to CNTs has been compared with molecular dynamics (MD) simulations, and good agreement is observed. However this theory strongly depends on the optimised value of the nonlocal parameter. Details on the various values of nonlocal parameter as reported by various researchers is discussed in [38]. Ansari and coworkers [43,44] used various nonlocal beams and shell theories for studying free vibration response of single-walled and double-walled carbon nanotubes (SWCNTs, DWCNTs) and compared with molecular dynamic simulation. The two approaches match at some nonlocal parameter values. According to Hu et al. [45], MD simulations indicate that the wave dispersion predicted by the nonlocal elastic cylindrical shell theory shows good agreement with that of the MD simulations in a wide frequency range up to the terahertz region. Khademolhosseini et al. [46] illustrated the superiority and accuracy of the nonlocal elasticity model over classical theories in predicting the size-dependent dynamic torsional response of SWCNTs by comparing their results with MD simulations. Recently Murmu and Adhikari [47] compared the frequency results of a cantilever SWCNT and found that the results from nonlocal beam theory match very well with the frequency from MD simulation for $e_0a=1.0$ nm. Further, Murmu and Adhikari [48] also reported that the frequency from nonlocal elasticity theory matches very well with MD simulations for CNT-based nanoscale biosensors for some optimised values of nonlocal parameter. Therefore nonlocal elasticity theory can be a reliable approach to predict the mechanical behaviour of nanotubes when an appropriate nonlocal parameter or scale coefficient is used.

8. Parameters used in the analysis

The inner and the outer tubes of DWCNT have diameters $d_1=0.7$ nm and $d_2=1.4$ nm [49]. The present model can be applied to the various nanotubes with different chirality, as the diameter of nanotubes is a function of tube chirality, which is expressed as

$$d_i = \frac{\mathrm{r}}{\pi}\sqrt{cn_i^2+cn_icm_i+cm_i^2},\ i=1,2 \tag{59}$$

where (cn,cm) is the chirality of the nanotube and $r=0.246$ nm.

The properties of carbon nanotubes are considered as Young's modulus, $E=1.0$ TPa; density, $\rho=2300$ kg/m^3, thickness, $h=0.35$ nm, Poisson's ratio as 0.3. It should be noted that the equivalent properties considered for the continuum approach (Young modulus, Poisson ratios, wall thickness and mass density) are idealization and not univocal for carbon nanotubes. This is because Young's modulus and Poisson's ratios, etc. are traditionally associated with the macroscopic-length scale [50]. Obtaining the accurate generalised coherent values of these properties of nanostructure material is thus a tedious task. Some applied physical value of CNT such as Young's modulus, shear modulus, density is reported in Mir et al. [51]. Using finite elements, Rouainia, and Djeghaba [52] carried out evaluation of longitudinal Young's modulus of single walled carbon nanotube (SWNT) reinforced concrete composite. Wang and Zhang [53] discussed the fundamental issues of the elastic properties and effective wall thickness of single-walled carbon nanotubes (SWCNTs). There is no agreement about the exact values of E, h, ρ of a carbon nanotube. To address the issues Vodenitcharova and Zhang [54] developed a continuum mechanics model to predict the effective thickness of a single-walled carbon nanotube and its corresponding Young's modulus. The ambiguity of different values of E, h of a carbon nanotube as reported by other researchers is also discussed in the paper [54].

The transverse areas and the bending rigidities of the nanotubes are determined as

$$A_i = \pi d_i h,\ EI = E\pi d_i^{\,3}h/8 \tag{60}$$

Using expression Eq. (21) the van der Waals constant is evaluated as $c = 0.0694$ TPa [41]. The magnetic permeability is assumed as $\xi = 4\pi \times 10^{-7}$. A Strong magnetic field is assumed for the study.

To examine the influence of the small length scale on vibrations of double-walled nanotubes in magnetic field, let us compare the local and nonlocal results. It follows that the ratios of the nonlocal results to the corresponding local results are, respectively,

$$(\mathrm{FR})_{\mathrm{I}} = \left(\frac{\mathrm{Freq}_{\mathrm{NL}}}{\mathrm{Freq}_{\mathrm{L}}}\right)_{\mathrm{I}}, \quad (\mathrm{FR})_{\mathrm{II}} = \left(\frac{\mathrm{Freq}_{\mathrm{NL}}}{\mathrm{Freq}_{\mathrm{L}}}\right)_{\mathrm{II}} \tag{61}$$

9. Numerical results and discussion

9.1. Small-scale effects

Fig. 5 shows the effect of small scale (scale coefficient, e_0a) on the frequency ratios of DWCNT, i.e., $(\mathrm{FR})_{\mathrm{I}}$ and $(\mathrm{FR})_{\mathrm{II}}$. The frequency ratios, $(\mathrm{FR})_{\mathrm{I}}$ and $(\mathrm{FR})_{\mathrm{II}}$ represent vibration phenomenon in the same phase (SP) and anti-phase (AP) modes, respectively. The SP and AP vibration behaviour depicts the vibration of inner and outer tubes of the DWCNT. SP frequency is the lower frequency while AP frequency is the higher frequency. These frequencies represent first natural frequencies. Curves have also been plotted in the figure considering with and without the effect of longitudinal magnetic field (LMF). A strong magnetic field has been used for the analysis, i.e., $H_x = 1 \times 10^8$ A/m. The scale coefficient is assumed in the range $0\ \mathrm{nm} \le e_0a \le 2\ \mathrm{nm}$ [55,56]. The shear deformation effect is neglected in the present study. The short DWCNT length of 10 nm is considered so as to highlight the small scale effects. At bigger lengths the small scale effects are lost since small scale effects are considered dependent in axial directions only.

From Fig. 5 it is observed that the numerical value of frequency ratios ($\mathrm{FR} = \mathrm{Freq}_{\mathrm{NL}}/\mathrm{Freq}_{\mathrm{L}}$) for all the curves decreases as the value of scale coefficient (nonlocal parameter), e_0a increases. This implies that the natural frequencies of a DWCNT incorporating the small scale effects are less than the natural frequencies of the same DWCNT without small scale effects (classical elasticity). This decrease in frequency of the simply-supported DWCNT can be attributed to the transverse force due to: (1) the curvature change in the nanotubes, and (2) the surface stress due to the nonlocal atom–atom interaction.

Comparison of $\mathrm{Freq}_{\mathrm{NL}}/\mathrm{Freq}_{\mathrm{L}}$ for same phase (SP) and anti-phase (AP) movement shows that SP vibration is more influenced by the small scale effects than the AP vibration. Frequencies of AP vibration are higher than the SP vibration. The small scale effects in the AP vibration are dampened by the active presence of van der Waal forces. However in SP vibration, van der Waal forces have little effect on frequencies as both the tubes of DWCNT vibrates as a single entity.

9.2. Presence of longitudinal magnetic field

Fig. 5 also shows the vibration of the DWCNT with and without the effect of a longitudinal magnetic field (LMF). From the figure it is observed that the frequencies of both the SP and AP vibration have higher frequencies in the presence of longitudinal magnetic field than without the magnetic field. The difference between frequencies of the SP vibrations with and without LMF is substantial. On the other hand, the difference between frequencies of the AP vibrations with and without LMF is negligible. This observation is mainly because of the effect of van der Waal forces. The difference between

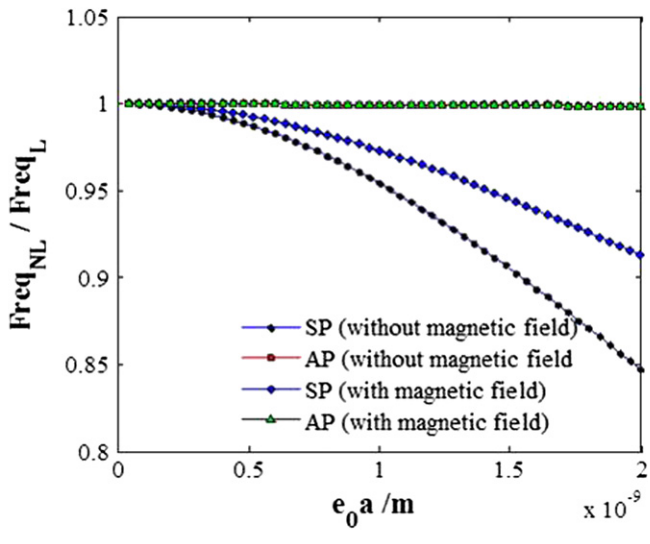

Fig. 5. Variation of first frequency ratios for DWCNT considering same- and anti-phase vibration with and without the longitudinal magnetic field.

frequencies of the AP vibrations with and without LMF is very small as can be seen from the zoomed in image in Fig. 6. As stated earlier the frequencies are relatively higher for AP vibration of DWCNT in the presence of LMF than without it. To show the difference we had considered length of short DWCNT as 5 nm. It should be noted the figure in Figs. 5 and 6 are not same. Here the shear effect has been neglected.

9.3. Higher natural frequencies

To see the vibration response of a DWCNT at higher natural frequencies, curves have been plotted for higher frequencies, i.e., $n=2,3$, and 4. The parameter used for the plots are the same as discussed in Sections 7 and 8.1 (i.e., DWCNT length is 10 nm). Figs. 7–9 show the vibration response of DWCNT for $n=2,3$, and 4, respectively. From the figure it is observed that the frequency ratios $Freq_{NL}/Freq_L$ decrease as the modes of natural frequencies increase. This implies that the frequencies of higher modes are more affected by small scale effects than the frequencies of lower modes.

The effect of a LMF is also shown in the figures. As the mode number increases the difference between $Freq_{NL}/Freq_L$ with and without the presence of LMF increases marginally. This is more prominent in the case of AP vibration phenomenon. The individual curves of AP vibration phenomenon with and without the presence of LMF are clearly distinguishable. Thus at higher modes, the effect of a LMF on AP vibration of DWCNT are significant and cannot be ignored. It is noted also that the

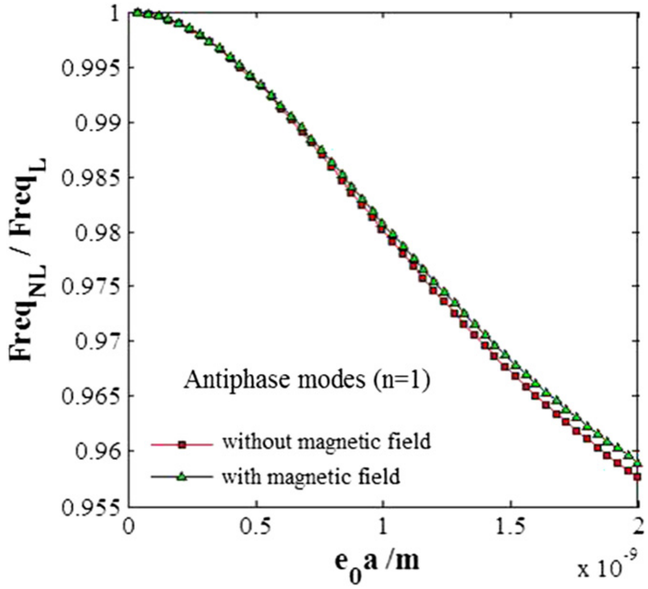

Fig. 6. Variation of first frequency ratios for DWCNT for anti-phases vibration with and without the longitudinal magnetic field.

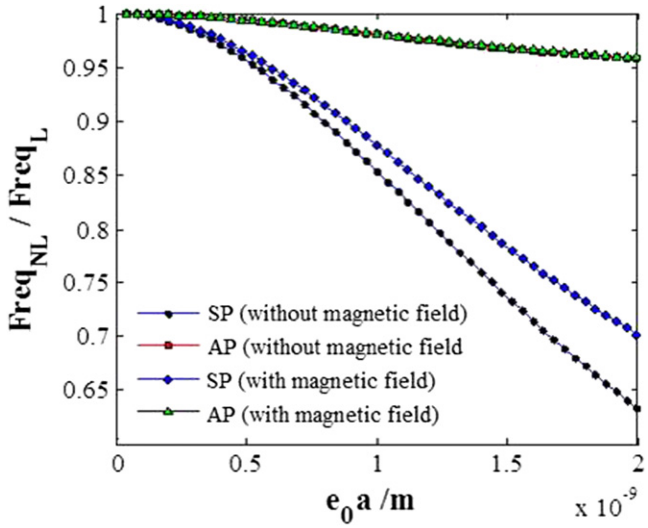

Fig. 7. Variation of frequency ratios ($n=2$) for DWCNT considering same- and anti phase vibration with and without the longitudinal magnetic field.

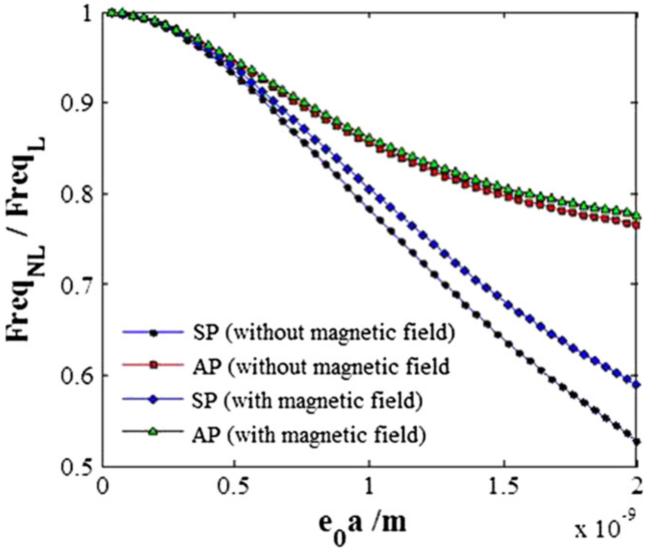

Fig. 8. Variation of frequency ratios ($n=3$) for DWCNT considering same- and anti-phase vibration with and without the longitudinal magnetic field.

frequencies for SP and AP vibration phenomenon are closer as mode numbers of frequencies increase. From Fig. 9 it is seen that the SP and AP vibration phenomenon with and without the presence of LMF overlap at higher scale coefficient e_0.

9.4. Effect of aspect ratio (L/d)

The effect of aspect ratio L/d on the vibration response of DWCNT with and without the presence of a LMF is examined. Fig. 10a–c show the vibration response of a DWCNT with and without the presence of a LMF for $L/d=20$, 30 and 50 The cross-sectional distortion behaviour of the DWCNT is ignored. The considered aspect ratios depict long slender DWCNTs and the tubular beam model is considered valid for the study and therefore the use of a shell model is not used in such a case.

SP and AP plots are also shown in the figures. From the figures it is observed that the effects of a LMF on both the SP and AP modes decrease as the aspect ratios L/d increases. Hence, for DWCNTs with higher L/d, the magnetic effects can be ignored at lower scale coefficient e_0a i.e., (at less scale effects).

9.5. Special cases of CNTs

In this section, some special cases of carbon nanotubes in the presence of a LMF are considered and compared with the general DWCNT model to see the relative effect. The special cases considered in the study include

(i) **DWCNT with outer tube stationary**
(ii) **SWCNT with diameter equal to inner tube diameter of the DWCNT**
(iii) **SWCNT with diameter equal to outer tube diameter of the DWCNT**

Here SWCNT denotes a single-walled carbon nanotube. The material properties of the SWCNT are same as that considered for the DWCNT.

Fig. 11 shows the effect of scale coefficient, e_0a on the frequency ratios of the DWCNT (for both SP and AP vibration modes) and on the special cases of carbon nanotubes subjected to a LMF. Fig. 11 includes sub figures Fig. 11a–f which depict the vibration responses of the carbon nanotubes at $n=1$, 2, 3, 4, 5 and 6. From the figure it is observed that the SP vibration is more influenced by small scale effects than the AP vibration. The frequencies of AP vibration are also higher than the SP vibration. The influence of small scale effects on the various cases is in the following order.

CASE 1 > CASE 2 > CASE 3 > CASE 4 > CASE 5.

The above Cases are defined in Table 1. The different cases of vibration of CNTs for different scale coefficients are more distinguishable at higher modes of vibration. Moreover for $n=6$ it is observed that the frequency for a DWCNT vibrating in same phase and SWCNT with diameter equal to inner tube diameter of DWCNT is identical at a certain value of scale coefficient.

9.6. Different strength of LMF

The effect of varying the strength of the LMF on the vibration response of a DWCNT is examined here. Fig. 12 shows the effects on both the SP and AP vibration modes. The strength of the LMF is considered varying from $H_x=1$ A/m to 10^{10} A/m,

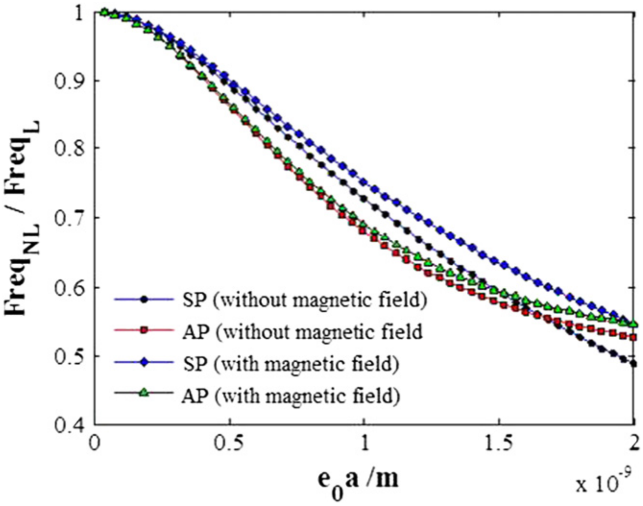

Fig. 9. Variation of frequency ratios ($n=4$) for DWCNT considering same- and anti-phase vibration with and without the longitudinal magnetic field.

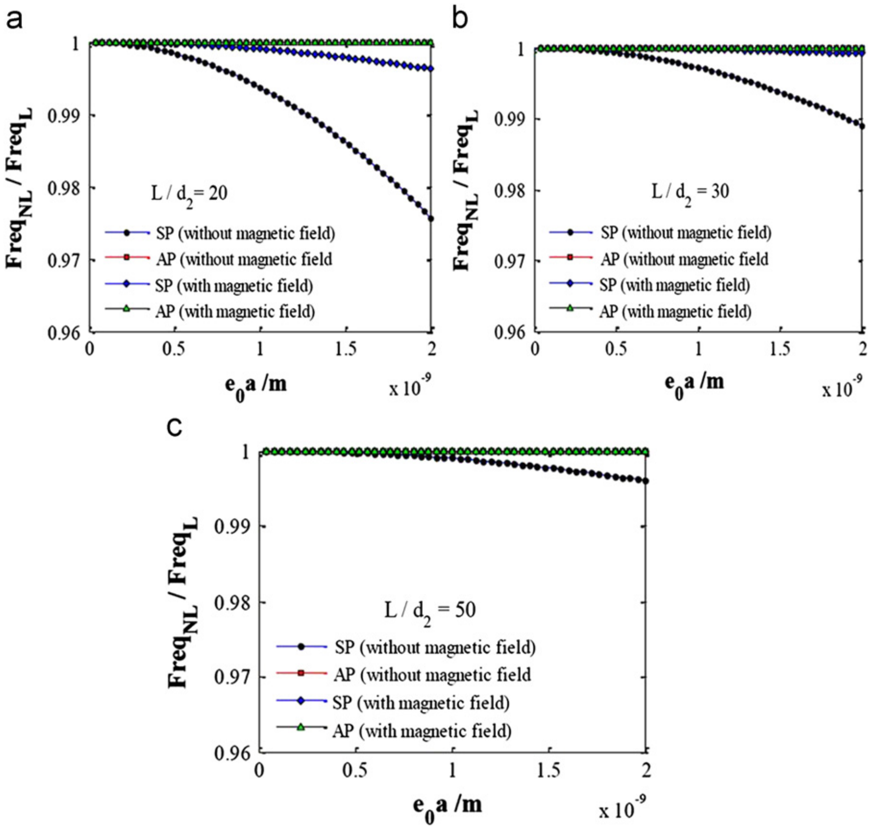

Fig. 10. Variation of frequency ratios for DWCNT considering same- and anti-phase vibration with and without the longitudinal magnetic field for different aspect ratios; (a) $L/d_2=20$, (b) $L/d_2=30$, (c) $L/d_2=50$.

and log-scale is used on the horizontal axis. From the figure it is observed that the $Freq_{NL}/Freq_L$ for the SP vibration mode increases as the magnitude of H_x increases. It is noted that the $Freq_{NL}/Freq_L$ increases with increase of $\log H_x$ prominently after $H_x = 10^7$A/m. It is also noted that the $Freq_{NL}/Freq_L$ of the AP mode is not affected by the increase of strength of LMF.

Figs. 13–15 show the effect of LMF strength on the vibration response of a DWCNT for higher natural frequencies, i.e., $=2$, $n=3$ and $n=4$, respectively. From the figures it is observed that the $Freq_{NL}/Freq_L$ for SP vibration mode increases as the

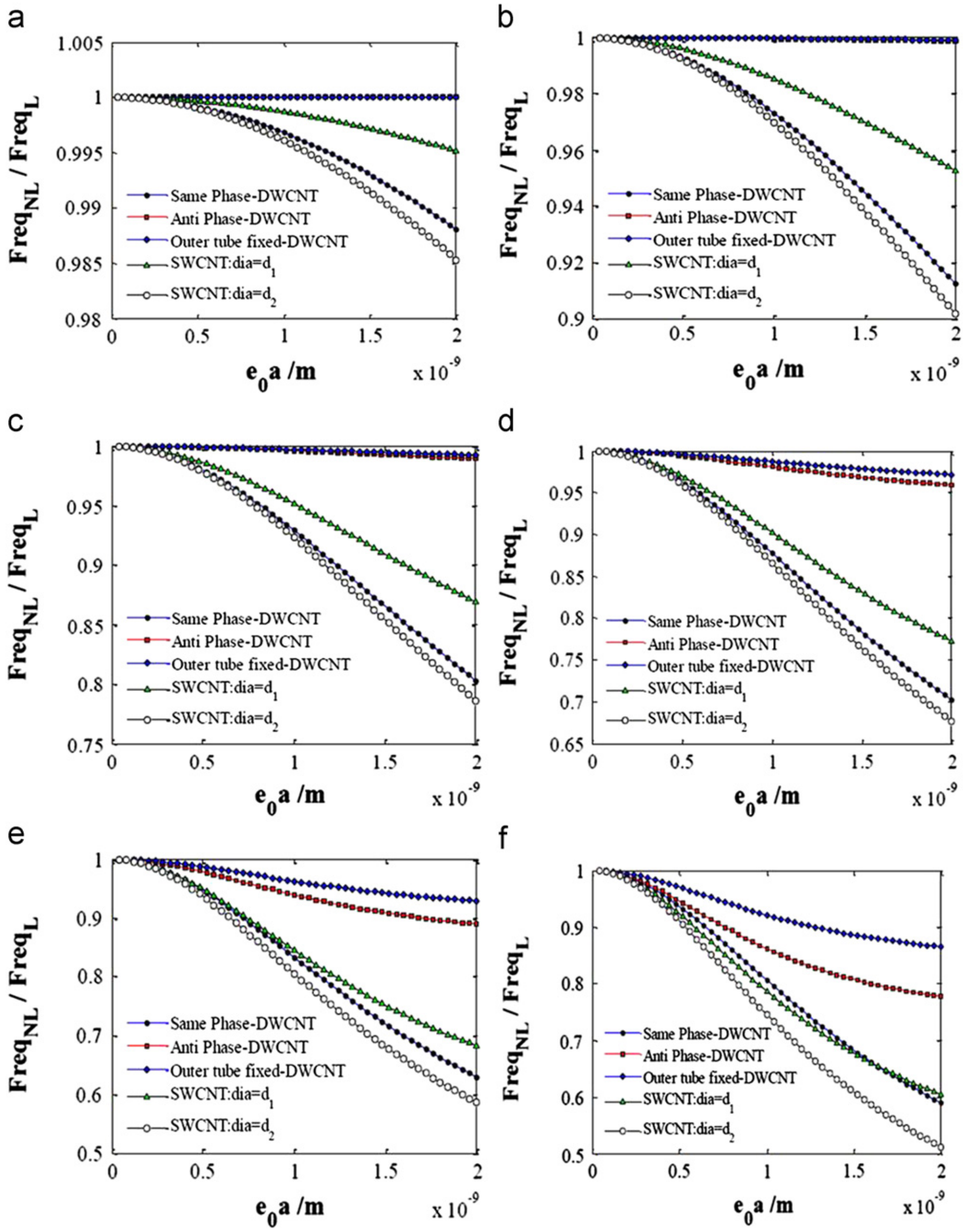

Fig. 11. Variation of frequency ratios for DWCNT considering same- and anti-phase vibration with and without the longitudinal magnetic field for different cases of CNTs; (a) $n=1$; (b) $n=2$; (c) $n=3$; (d) $n=4$; (e) $n=5$; (f) $n=6$.

Table 1
Different cases of Vibration of CNTs.

Cases	Conditions
CASE 1	SWCNT with diameter equal to outer tube diameter of DWCNT
CASE 2	DWCNT vibrating in same phase
CASE 3	SWCNT with diameter equal to inner tube diameter of DWCNT
CASE 4	DWCNT vibrating in anti-phase
CASE 5	DWCNT with outer tube stationary

magnitude of H_x increases for certain ranges of H_x. At higher frequencies, the nonlocal frequencies are below the local frequencies for all LMF strengths considered. It is also noted that at $n=2$, $n=3$ and $n=4$, the $Freq_{NL}/Freq_L$ increases with the increase of strength of LMF for AP vibration modes at higher span of strength of LMF.

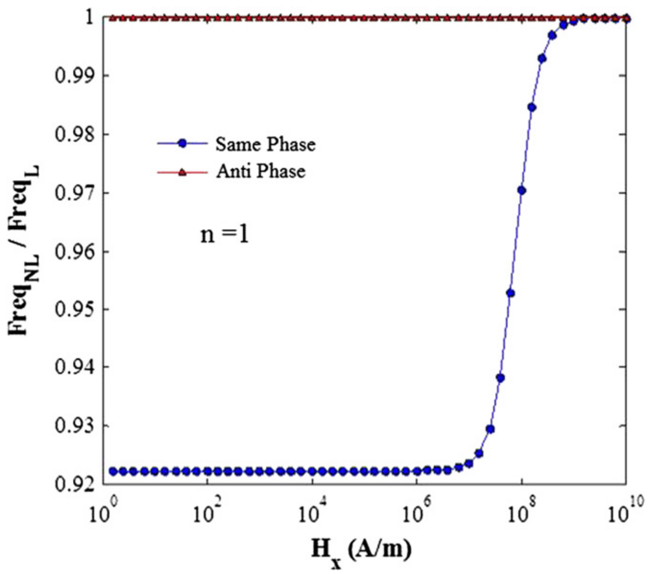

Fig. 12. Variation of frequency ratios field ($n=1$) against strength of longitudinal magnetic field for DWCNT considering same- and anti phase vibration.

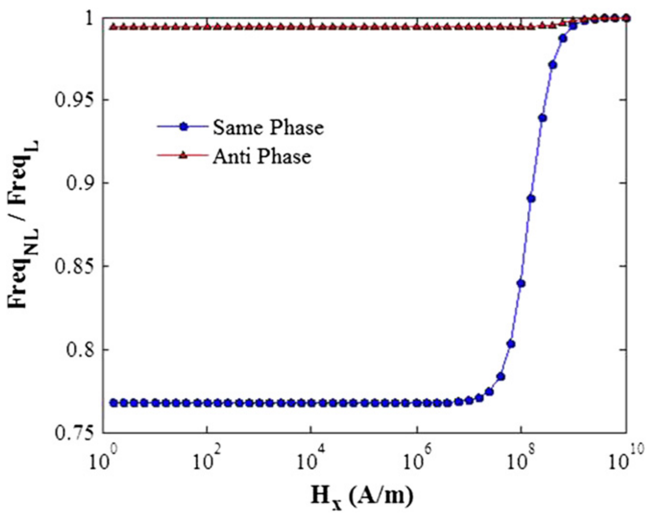

Fig. 13. Variation of frequency ratios field ($n=2$) against strength of longitudinal magnetic field for DWCNT considering same- and anti-phase vibration.

9.7. Discussion on validation of the work

Nonlocal elasticity theory is used to study the frequency analysis of DWCNTs under a magnetic field. Experimental or molecular dynamics results for the present study of vibration of a DWCNT subjected to longitudinal magnetic field are yet unavailable in the literature. Thus an exact comparison of the present results with existing experimental and MD results is difficult and represents scope for future study. However it is reported in the present study that the longitudinal magnetic field increases the natural frequency of the DWCNT. This may be attributed to the coupling effect of vibrating nanotubes and the longitudinal magnetic field. Further from energy standpoint, the DWCNT in a longitudinal magnetic field may possess lower magnetic energy. And it will hardly leave its equilibrium position if there is any slight disturbance, and no constrain of elastic deformation. The DWCNT will always tend to retreat to its position. Thus the rigidity of the DWCNT in a transverse field will increase. As a result the increasing of its natural frequency is reasonable. This is in-line with some studies in the literature with macroscopic beam and plate. For a macroscopic beam and plate under the effect of an in-plane magnetic field (non-transverse magnetic field) it is reported that the natural frequency increases as the strength of the in-plane magnetic field increases ([57], [58], [59] and [60]).

The present work can be extended to study the vibration of DWCNT under the effect of transverse and inclined magnetic field.

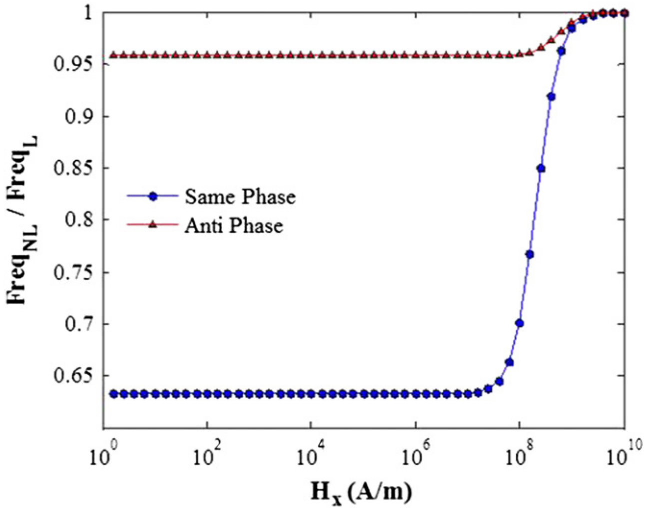

Fig. 14. Variation of frequency ratios field ($n=3$) against strength of longitudinal magnetic field for DWCNT considering same- and anti phase vibration.

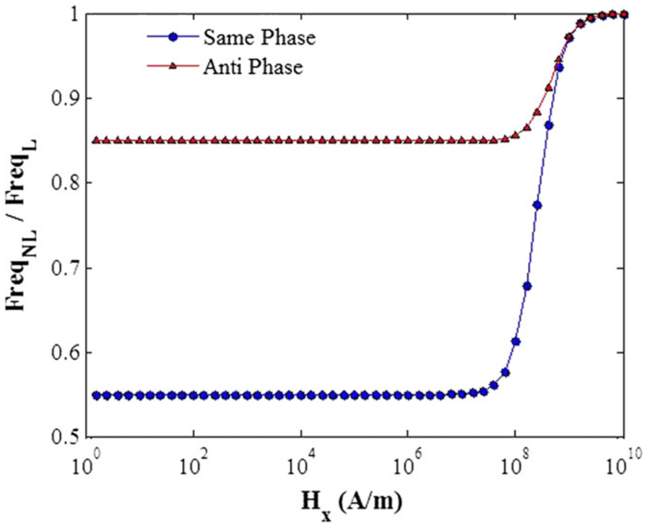

Fig. 15. Variation of frequency ratios field ($n=4$) against strength of longitudinal magnetic field for DWCNT considering same- and anti-phase vibration.

10. Conclusion

This paper presents an analytical model for studying the effects of a longitudinal magnetic field on the vibration of a magnetically sensitive double-walled carbon nanotube system (DWCNT). The dynamic equations of the DWCNT are derived utilising nonlocal elasticity theory. The two tubes of the nanotubes are defined as an equivalent nonlocal Euler–Bernoulli beam. Governing equations for nonlocal bending-vibration of the DWCNT under a longitudinal magnetic field are developed considering the Lorentz magnetic force obtained from Maxwell's relation. Analytical solutions are presented for the natural frequencies of the DWCNTS under a magnetic field. Results reveal that the nonlocal effects in the vibration response of the DWCNT are dampened by the presence of a longitudinal magnetic field. The presence of a longitudinal magnetic field increases the natural frequencies of the DWCNT. This is true for both same-phase and anti-phase modes of vibration. Further synchronous phase vibration of DWCNT is more influenced by the small scale effects than the anti-phase vibration.

The present study will hopefully be useful for further nonlocal elastic theoretical study and design of nanotubes in a magnetic field.

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References

[1] S. Iijima, Helical microtubules of graphitic carbon, *Nature* 354 (1991) 56–58.
[2] C. Schneider, B. Zhao, R. Kozhuharova, S. Groudeva-Zotova, T. Muhl, M. Ritschel, M. I, H. Vinzelberg, D. Elefant, A. Graff, A. Leonhardt, J. Fink, Towards molecular spintronics: magnetotransport and magnetism in carbon nanotube-based systems, *Diamond and Related Materials* 13 (2004) 215–220.
[3] A. Sharma, B. Tripathi, Y. Vijay, Dramatic improvement in properties of magnetically aligned CNT/polymer nanocomposites, *Journal of Membrane Science* 361 (2010) 89–95.
[4] C. Tsai, S. Chen, F. Shyu, C. Chang, M. Lin, Magnetization of carbon nanotubes, *Physica E-Low-Dimensional Systems & Nanostructures* 30 (2005) 86–92.
[5] F. Lopez-Urias, J. Rodriguez-Manzo, E. Munoz-Sandoval, M. Terrones, H. Terrones, Magnetic response in finite carbon graphene sheets and nanotubes, *Optical Materials* 29 (2006) 110–115.
[6] F. Lopez-Urias, J. Rodriguez-Manzo, M. Terrones, H. Terrones, Magnetic properties of carbon nanostructures, *International Journal of Nanotechnology* 4 (2007) 651–666.
[7] L. Chang, J. Lue, Magnetic properties of multi-walled carbon nanotubes, *Journal of Nanoscience and Nanotechnology* 9 (2009) 1956–1963.
[8] A. Reddy, M. Shaijumon, S. Ramaprabhu, Alloy hydride catalyst route for the synthesis of single-walled carbon nanotubes, multi-walled carbon nanotubes and magnetic metal-filled multi-walled carbon nanotubes, *Nanotechnology* 17 (2006) 5299–5305.
[9] L. Langer, V. Bayot, E. Grivei, J. Issi, J. Heremans, C. Olk, L. Stockman, C. VanHaesendonck, Y. Bruynseraede, Quantum transport in a multiwalled carbon nanotube, *Physical Review Letters* 76 (1996) 479–482.
[10] J. Heremans, C. Olk, D. Morelli, Magnetic-susceptibility of carbon structures, *Physical Review B* 49 (1994) 15122–15125.
[11] B. Jang, Y. Sakka, Influence of shape and size on the alignment of multi-wall carbon nanotubes under magnetic fields, *Materials Letters* 63 (2009) 2545–2547.
[12] S. Bellucci, J. Gonzalez, F. Guinea, P. Onorato, E. Perfetto, Magnetic field effects in carbon nanotubes, *Journal of Physics-Condensed Matter* 19 (2007).
[13] J. Chen, H. Liu, N. Ahmad, Y. Li, Z. Chen, W. Zhou, X. Han, Effect of external magnetic field on magnetic properties of Co–Pt nanotubes and nanowires, *Journal of Applied Physics* 109 (2011).
[14] M. Kibalchenko, M. Payne, J. Yates, Magnetic response of single-walled carbon nanotubes induced by an external magnetic field, *ACS Nano* 5 (2011) 537–545.
[15] C. Lee, Y. Hsue, R. Chen, T. Li, M. Lin, Electronic structures of finite double-walled carbon nanotubes in a magnetic field, *Journal of Physics-Condensed Matter* 20 (2008).
[16] S. Roche, R. Saito, Effects of magnetic field and disorder on the electronic properties of carbon nanotubes, *Physical Review B* 59 (1999) 5242–5246.
[17] D. Sebastiani, K. Kudin, Electronic response properties of carbon nanotubes in magnetic fields, *ACS Nano* 2 (2008) 661–668.
[18] Z. Zhang, W. Guo, Y. Guo, The effects of axial magnetic field on electronic properties of carbon nanotubes, *Acta Physica Sinica* 55 (2006) 6526–6531.
[19] L. Wei, Y. Wang, Electromagnetic wave propagation in single-wall carbon nanotubes, *Physics Letters A* 333 (2004) 303–309.
[20] S. Li, H. Xie, X. Wang, Dynamic characteristics of multi-walled carbon nanotubes under a transverse magnetic field, *Bulletin of Materials Science* 34 (2011) 45–52.
[21] H. Wang, K. Dong, F. Men, Y. Yan, X. Wang, Influences of longitudinal magnetic field on wave propagation in carbon nanotubes embedded in elastic matrix, *Applied Mathematical Modelling* 34 (2010) 878–889.
[22] X. Wang, J. Shen, Y. Liu, G. Shen, G. Lu, Rigorous van der Waals effect on vibration characteristics of multi-walled carbon nanotubes under a transverse magnetic field, *Applied Mathematical Modelling* 36 (2012) 648–656.
[23] S. Narendar, S. Gupta, S. Gopalakrishnan, Wave propagation in single-walled carbon nanotube under longitudinal magnetic field using nonlocal Euler–Bernoulli beam theory, *Applied Mathematical Modelling Corrected Proof* (2012).
[24] C. Kiang, M. Endo, P. Ajayan, G. Dresselhaus, M. Dresselhaus, Size effects in carbon nanotubes, *Physical Review Letters* 81 (1998) 1869–1872.
[25] C. Tang, W. Guo, C. Chen, Molecular dynamics simulation of tensile elongation of carbon nanotubes: temperature and size effects, *Physical Review B* 79 (2009).
[26] R. Chowdhury, C. Wang, S. Adhikari, Low frequency vibration of multiwall carbon nanotubes with heterogeneous boundaries, *Journal of Physics D: Applied Physics* 43 (2010).
[27] R. Chowdhury, C. Wang, S. Adhikari, F. Tong, Sliding oscillation of multiwall carbon nanotubes, *Physica E-Low-Dimensional Systems & Nanostructures* 42 (2010) 2295–2300.
[28] A. Eringen, On differential-equations of nonlocal elasticity and solutions of screw dislocation and surface-waves, *Journal of Applied Physics* 54 (1983) 4703–4710.
[29] B. Arash, Q. Wang, A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes, *Computational Materials Science* 51 (2012) 303–313.
[30] C. Wang, L. Zhang, An elastic shell model for characterizing single-walled carbon nanotubes, *Nanotechnology* 19 (2008).
[31] M. Cinefra, E. Carrera, S. Brischetto, Refined shell models for the vibration analysis of multiwalled carbon nanotubes, *Mechanics of Advanced Materials and Structures* 18 (2011) 476–483.
[32] K. Liew, Q. Wang, Analysis of wave propagation in carbon nanotubes via elastic shell theories, *International Journal of Engineering Science* 45 (2007) 227–241.
[33] N. Silvestre, C. Wang, Y. Zhang, Y. Xiang, Sanders shell model for buckling of single-walled carbon nanotubes with small aspect ratio, *Composite Structures* 93 (2011) 1683–1691.
[34] C. Wang, Z. Tay, A. Chowdhuary, W. Duan, Y. Zhang, N. Silvestre, Examination of cylindrical shell theories for buckling of carbon nanotubes, *International Journal of Structural Stability and Dynamics* 11 (2011) 1035–1058.
[35] J. Hsu, R. Chang, W. Chang, Resonance frequency of chiral single-walled carbon nanotubes using Timoshenko beam theory, *Physics Letters A* 372 (2008) 2757–2759.
[36] J. Reddy, S. Pang, Nonlocal continuum theories of beams for the analysis of carbon nanotubes, *Journal of Applied Physics* 103 (2008).
[37] J. Kraus, *Electromagnetics*, McGrawHill, Inc., USA, 1984.
[38] S. Narendar, S. Gopalakrishnan, Spectral finite element formulation for nanorods via nonlocal continuum mechanics, *Journal of Applied Mechanics and Transactions of the ASME* 78 (2011).
[39] A. Besseghier, A. Tounsi, M. Houari, A. Benzair, L. Boumia, H. Heireche, Thermal effect on wave propagation in double-walled carbon nanotubes embedded in a polymer matrix using nonlocal elasticity, *Physica E-Low-Dimensional Systems & Nanostructures* 43 (2011) 1379–1386.
[40] A. Tounsi, H. Heireche, H. Berrabah, A. Benzair, L. Boumia, Effect of small size on wave propagation in double-walled carbon nanotubes under temperature field, *Journal of Applied Physics* 104 (2008).
[41] L. Sudak, Column buckling of multiwalled carbon nanotubes using nonlocal continuum mechanics, *Journal of Applied Physics* 94 (2003) 7281–7287.
[42] M Xu, Free transverse vibrations of nano-to-micron scale beams. Proceedings of the Royal Society a-Mathematical Physical and Engineering Sciences 462 (2006) 2977–2995.
[43] R. Ansari, S. Sahmani, Small scale effect on vibrational response of single-walled carbon nanotubes with different boundary conditions based on nonlocal beam models, *Communications in Nonlinear Science and Numerical Simulation* 17 (2012) 1965–1979.
[44] R. Ansari, H. Rouhi, S. Sahmani, Calibration of the analytical nonlocal shell model for vibrations of double-walled carbon nanotubes with arbitrary boundary conditions using molecular dynamics, *International Journal of Mechanical Sciences* 53 (2011) 786–792.
[45] Y. Hu, K. Liew, Q. Wang, X. He, B. Yakobson, Nonlocal shell model for elastic wave propagation in single- and double-walled carbon nanotubes, *Journal of the Mechanics and Physics of Solids* 56 (2008) 3475–3485.

[46] F. Khademolhosseini, A.S. Phani, A. Nojeh, N. Rajapakse, Nonlocal continuum modeling and molecular dynamics simulation of torsional vibration of carbon nanotubes, *IEEE Transactions on Nanotechnology* 11 (2012) 34–43.
[47] T. Murmu, S. Adhikari, Nonlocal vibration of carbon nanotubes with attached buckyballs at tip, *Mechanics Research Communications* 38 (2011) 62–67.
[48] T. Murmu, S. Adhikari, Nonlocal frequency analysis of nanoscale biosensors, *Sensors and Actuators A: Physical* 173 (2012) 41–48.
[49] Y. Zhang, X. Liu, G. Liu, Thermal effect on transverse vibrations of double-walled carbon nanotubes, *Nanotechnology* 18 (2007).
[50] G. Odegard, T. Gates, L. Nicholson, K. Wise, Equivalent-continuum modeling of nano-structured materials, *Composites Science and Technology* 62 (2002) 1869–1880.
[51] M. Mir, A. Hosseini, G. Majzoobi, A numerical study of vibrational properties of single-walled carbon nanotubes, *Computational Materials Science* 43 (2008) 540–548.
[52] G. Rouainia, K. Djeghaba, Evaluation of young's modulus of single walled carbon nanotube (SWNT) reinforcedconcrete composite, *Journal of Engineering and Applied Sciences* 3 (2008) 504–515.
[53] C. Wang, L. Zhang, A critical assessment of the elastic properties and effective wall thickness of single-walled carbon nanotubes, *Nanotechnology* 19 (2008).
[54] T. Vodenitcharova, L. Zhang, Effective wall thickness of a single-walled carbon nanotube, *Physical Review B* 68 (2003).
[55] M. Aydogdu, Axial vibration of the nanorods with the nonlocal continuum rod model, *Physica E-Low-Dimensional Systems & Nanostructures* 41 (2009) 861–864.
[56] Q. Wang, C. Wang, The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes, *Nanotechnology* 18 (2007).
[57] W. Yang, H. Pan, D. Zheng, Q. Cai, Vibration and dynamic instability of ferromagnetic thin plates in magnetic fields, *Mechanics Research Communications* 26 (1999) 239–244.
[58] J. Lee, C. Lin, The magnetic viscous damping effect on the natural frequency of a beam plate subject to an in-plane magnetic field, *Journal of Applied Mechanics and Transactions of the ASME* 77 (2010).
[59] Y. Zhou, K. Miya, A theoretical prediction of natural frequency of a ferromagnetic beam plate with low susceptibility in an in-plane magnetic field, *Journal of Applied Mechanics and Transactions of the ASME* 65 (1998) 121–126.
[60] T. Takagi, J. Tani, Y. Matsubara, I. Mogi, Dynamic behavior of fusion structural components under strong magnetic-fields, *Fusion Engineering and Design* 27 (1995) 481–489.