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Hybrid perturbation-Polynomial Chaos approaches to the random algebraic eigenvalue problem

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ABSTRACT

The analysis of structures is affected by uncertainty in the structure's material properties, geometric parameters, boundary conditions and applied loads. These uncertainties can be modelled by random variables and random fields. Amongst the various problems affected by uncertainty, the random eigenvalue problem is specially important when analyzing the dynamic behavior or the buckling of a structure. The methods that stand out in dealing with the random eigenvalue problem are the perturbation method and methods based on Monte Carlo Simulation. In the past few years, methods based on Polynomial Chaos (PC) have been developed for this problem, where each eigenvalue and eigenvector are represented by a PC expansion. In this paper four variants of a method hybridizing perturbation and PC expansion approaches are proposed and compared. The methods use Rayleigh quotient, the power method, the inverse power method and the eigenvalue equation. PC expansions of eigenvalues and eigenvectors are obtained with the proposed methods. The new methods are applied to the problem of an Euler Bernoulli beam and a thin plate with stochastic properties.

1. Introduction

The algebraic eigenvalue problem arises in a variety of fields, for example, in buckling of columns and shells [1], vibration of elastic bodies [2] and electromagnetism. Accurate methods to calculate the eigensolutions of a deterministic matrix have been available for long (see, for example, [3]), but such is not the case when the matrix considered is random. When random matrices are considered, the joint pdf of eigenvalues is only available for some special random matrix distributions, as the Gaussian Orthogonal Ensemble [4] and Wishart matrices [5]. Randomness can be introduced in the system by random parameters (e.g. Young's modulus, mass density) and consequently propagated to the system matrices such as the mass and stiffness matrices. Using the stochastic finite element method [6], these matrices in turn can be represented by a linear combination of deterministic matrices, where the coefficients are random variables [7,8].

Several methods have been developed to solve the algebraic random eigenvalue problem. Methods dealing with large amounts of uncertainty are based on Monte Carlo Simulation (MCS). These strategies are based on ordering the samples depending on the distance between them and on calculating the eigenvalues of a sample using the ones of a close sample. This ordering can be based on algorithms from the traveling salesman problem and space reduction [9], component mode synthesis [10], or can be done in a tree-type data structure [11], and the relation between eigenvalues of close samples is obtained using different initialization strategies for the power method. The start-vector used is the result from the iteration process of the previous sample. The initialization strategies and size reduction methods reduce the computational time of MCS, but for smaller uncertainties, more efficient methods are available.

Methods that can be applied to small uncertainties are based on the perturbation method [12]. First applications date from the late sixties [13,14], and a series of modified methods have been developed. A comparison of several of these methods is given by Chen et al. [15]. Other perturbation-based methods use iterations or linear combination of deterministic and first order derivative of eigenvectors to deal with larger uncertainties or to allow reanalysis of structures [16–19]. Other methods available are based on crossing theory [20], Kronecker product [21], the dimensional decomposition method [22,23], asymptotic integral method [24,25], collocation methods [26], the use of interpolations, response surface methods and meta-models [27–29] and possibilistic approaches [30]. Williams [31] used an auxiliary function where the derivative of the eigenvector equals the eigenvalue multiplied by the eigenvector.

Several authors have applied Polynomial Chaos (PC) [6] based methods to the random algebraic eigenvalue problem. A PC expansion of eigenvalues and eigenvectors was obtained by Ghosh et al. [32] using MCS for the calculation of the coefficients

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of the expansion. Verhoosel et al. [33] developed an iterative procedure based on the inverse power method and Rayleigh quotient to obtain PC expansions of the eigensolutions. Ghanem and Ghosh [34] substituted eigenvalues and eigenvectors by their PC expansion in the eigenvalue problem. Coefficients were obtained from the nonlinear problem with the help of a norm equation for the eigenvectors. A modification of the previous method using enrichment functions was derived by Ghosh and Ghanem [35].

It can be observed that even if research has been carried out both on the perturbation and PC methods for the random eigenvalue problem, no method hybridizing both approaches is yet available. Efficient methods hybridizing PC and other methods have been proposed for the elliptic problem [36,37], where a reduction of the size of the linear system to be solved was achieved. The aim of the present paper is to gain efficiency on the PC algorithms for random eigenvalue problems through the use of results from the perturbation method. The outline of the paper is as follows. The basic theories of the perturbation method and PC are discussed respectively in Sections 2.1 and 2.2. PC expansion of eigenvalues is obtained in Section 3 using the Rayleigh quotient where eigenvectors are obtained from the perturbation method or from one of the methods developed to update eigenvectors. Four new methods, namely reduced spectral power method (RSPM), reduced spectral inverse power method (RSIPM), reduced spectral constrained coefficients method (RSCCM) and spectral constrained coefficients method (SCCM) are proposed to update the eigenvectors in Section 4. The four methods allow us to obtain an updated PC expansion of the eigenvectors and eigenvalues using Rayleigh quotient. A summary of the proposed methods is given in Section 5. A comparison of the methods is performed for the problem of a beam with stochastic properties in Section 6 and for a thin plate with stochastic properties in Section 7.

2. Stochastic Finite Element method for the random eigenvalue problem

The deterministic eigenvalue problem is given by the equation

$$\mathbf{A}\mathbf{u}^{(j)} = \lambda^{(j)}\mathbf{u}^{(j)}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{u}^{(j)}$ is the jth eigenvector, $\lambda^{(j)}$ is the corresponding eigenvalue and n is the degrees of freedom of the system. The system matrix is obtained from the generalized eigenvalue problem $\mathbf{K}\mathbf{y}^{(j)} = \lambda^{(j)}\mathbf{M}\mathbf{y}^{(j)}$ so that $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ and $\mathbf{u}^{(j)} = \mathbf{M}^{-1/2}\mathbf{y}^{(j)}$. In a dynamic problem, matrix $\mathbf{K}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix. The system matrix $\mathbf{A}$ is assumed to be symmetric. Randomness in the matrix $\mathbf{A}$ can be introduced by a parameter (e.g. Young's modulus) represented by a random field. The random field can be approximated with a finite set of random variables using a discretization procedure (see, e.g. [7,38]). For example, the random field is discretized using the Karhunen–Loève (KL) expansion [6] and truncated after M terms. Then, the system matrix $\mathbf{A}$ can be approximated by the following KL expansion

$$\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i \mathbf{A}_i. \tag{2}$$

Here $\mathbf{A}_0$ is the mean of the system matrix and $\mathbf{A}_i$ are the matrices obtained from using the eigenfunctions of the KL expansion in the Finite Element formulation of $\mathbf{A}$. It is observed that in a more general case, the random system matrix $\mathbf{A}$ can be approximated using a set of independent identically distributed random variables $\xi_1, \ldots, \xi_M$ such that

$$\mathbf{A} = \sum_{r=1}^{P} \Gamma_r \mathbf{A}_r, \tag{3}$$

where Γ_q are a set of P polynomials of increasing order in $\xi_1, \ldots, \xi_M$ orthogonal with respect to the pdf of the random variables $\xi_1, \ldots, \xi_M$. Generally they are chosen from the Wiener–Askey scheme of polynomials [39], but can be orthogonal with respect to an arbitrary probability density function [40]. In the next subsections, the perturbation method and the PC method are used to approximate the eigenvalues and eigenvectors of the stochastic system matrix.

2.1. Perturbation method for the random eigenvalue problem

Among the various methods developed to solve the random eigenvalue problem, the perturbation method is widely used due to its simplicity and computational efficiency. The different perturbation methods available to analyze the random eigenvalue problem are based on keeping different number of terms in the Taylor series expansions. The first order perturbation of the jth eigenvalue is given by

$$\lambda^{(j)} = \lambda_0^{(j)} + \sum_{i=1}^{M} \xi_i \frac{\partial \lambda^{(j)}}{\partial \xi_i} \quad \text{where} \quad \frac{\partial \lambda^{(j)}}{\partial \xi_i} = \mathbf{u}_{j0}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \mathbf{u}_{j0}. \tag{4}$$

For the case of (2), $\partial\mathbf{A}/\partial\xi_i = \mathbf{A}_i$. Perturbation methods can also be applied to eigenvectors, and the eigenvalues can then be obtained using the Rayleigh quotient. This approximation of eigenvalues is more accurate than the one obtained by directly applying the perturbation method via the Taylor series expansions [15]. If $\lambda_0^{(j)}$ and $\mathbf{u}_{j0}$ are the jth deterministic eigenvalue and the corresponding eigenvector, an expression for the first-order perturbation of the eigenvector can be given by Hasselman and Hart [14]

$$\mathbf{u}^{(j)} = \mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i}. \tag{5}$$

The deterministic eigenvectors satisfy the following properties

$$\mathbf{u}_{j0}^T \mathbf{u}_{j0} = 1 \quad \text{and} \quad \mathbf{u}_{j0}^T \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = 0. \tag{6}$$

Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of deterministic eigenvectors [41,13], so that

$$\mathbf{u}_{ji} = \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = \sum_{m=1, m\neq j}^{N} \alpha_{jim} \mathbf{u}_{m0} \quad \text{where} \quad \alpha_{jim} = \frac{1}{\lambda_0^{(j)} - \lambda_0^{(m)}} \mathbf{u}_{m0}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \mathbf{u}_{j0}. \tag{7}$$

For the case of (2), $\partial\mathbf{A}/\partial\xi_i = \mathbf{A}_i$. This equation is used when all deterministic eigenvectors are calculated. If only a limited number of eigenvectors were calculated, other methods described by Nelson [42] could be applied. The case of complex or repeated eigenvalues is not dealt with here. The perturbation method for such cases is derived, for example in [43–46]. For the case of repeated eigenvalues, the space corresponding to a given eigenvalue is the space spanned by its two eigenvectors, so that the proposed methods would be valid for the eigenvalues that are not repeated but not for the repeated one. The case of veering of modes is dealt with, for example in [47,48], and the proposed method does not allow to deal with this problem.

2.2. Polynomial Chaos approach for the random eigenvalue problem

Uncertainty is represented by a finite set of random variables $\{\xi_1, \ldots, \xi_M\}$ defined on the probability space $(\Xi, \mathcal{B}_\Xi, P_\xi)$. Any random quantity of interest of the system considered is then defined on this probability space, in particular, eigenvalues and eigenvectors. The eigensolutions are assumed to have finite second-order moments, and can be represented in the space of square integrable functions $\mathcal{L}^2(\Xi, dP_\xi)$, and a basis of functions Γ_k in $\mathcal{L}^2(\Xi, dP_\xi)$ can be defined. The representations of $\lambda^{(j)}$ and $\mathbf{u}^{(j)}$ on the basis functions Γ_k truncated after P terms can be given by

$$\lambda^{(j)}(\xi_1,\dots\xi_M)=\sum_{k=1}^{P}\lambda_{jk}\Gamma_k(\xi_1,\dots\xi_M), \tag{8}$$

$$\mathbf{u}^{(j)}(\xi_1,\dots\xi_M)=\sum_{k=1}^{P}\mathbf{u}_k^{(j)}\Gamma_k(\xi_1,\dots\xi_M), \tag{9}$$

where λ_{jk} and $\mathbf{u}_k^{(j)}$ are unknowns and the basis functions considered here are PC [6]. The basis functions Γ_k are obtained from the tensor product of several univariate orthogonal polynomials up to a fixed total-order specification o. The total number of polynomials used as basis functions for order o can be obtained as $P=(M+o)!/(M!o!)$ [7]. For example, if the random variables $\xi_1,\dots,\xi_M$ are Gaussian, the polynomials generally used are Hermite polynomials. The univariate Hermite polynomials of order m can be calculated from the equation $H_m(\xi_i)=(-1)^m\exp(\xi_i^2/2)\partial^m\exp(-\xi_i^2/2)/\partial^m\xi_i$, where each polynomial depends on a Gaussian random variable ξ_i.

PC has already been used in the context of the algebraic random eigenvalue problem by Ghanem and Ghosh [34]. In [34], eigenvalues and eigenvectors were substituted by their expansions, i.e., $\lambda^{(j)}$ and $\mathbf{u}^{(j)}$ are replaced in Eq. (1) by their expansions from Eqs. (8) and (9). The resulting equation is projected on the basis in $\mathcal{L}^2(\Xi,\mathcal{B}_\Xi,P_\xi)$ through the Galerkin method, that is, the equation is multiplied by each basis function Γ_p and subsequently the mean of the equation is taken. The resulting system of equations of size $P\times n$ relating the unknowns λ_{jl} and $\mathbf{u}_l^{(j)}$ can be given by

$$\left(\sum_{r=1}^{P}\mathbf{e}_{0r}\otimes\mathbf{A}_r\right)\begin{bmatrix}\mathbf{u}_1^{(j)}\\ \vdots\\ \mathbf{u}_P^{(j)}\end{bmatrix}=\sum_{l=1}^{P}\lambda_{jl}\mathbf{e}_{0l}\otimes\mathrm{I}\begin{bmatrix}\mathbf{u}_1^{(j)}\\ \vdots\\ \mathbf{u}_P^{(j)}\end{bmatrix}. \tag{10}$$

For the case of (2), the matrix on the left hand side reduces to $\left(\mathbf{c}_0\otimes\mathbf{A}_0+\sum_{i=1}^{M}\mathbf{c}_{1i}\otimes\mathbf{A}_i\right)$. Here the elements in the kth row and pth column of the $P\times P$ matrices $\mathbf{c}_0,\mathbf{c}_{1i}$ and $\mathbf{e}_{0l}$ are respectively $c_{0kp}=\mathrm{E}[\Gamma_k\Gamma_p]$, $c_{1ikp}=\mathrm{E}[\xi_i\Gamma_k\Gamma_p]$, with $i=1,\dots M$ and $e_{0lkp}=\mathrm{E}[\Gamma_k\Gamma_p\Gamma_l]$ with $l=1,\dots P$, and $\otimes$ denotes the Kronecker product. If the norm of the eigenvectors is also prescribed, the nonlinear system of equations can be solved with iterative techniques. A good initial approximation is often needed to obtain a fast convergence.

An iterative procedure to obtain PC expansions of eigenvalues and eigenvectors has been proposed by Verhoosel et al. [33] based on the inverse power method. The algorithm of the iteration can be concisely described by the following steps:

1. $\lambda_{(q+1)}^{(j)}=\left(\mathbf{u}_{(q)}^{(j)}\right)^T(\mathbf{A}_0+\sum_{i=1}^{M}\xi_i\mathbf{A}_i)\mathbf{u}_{(q)}^{(j)}$.
2. $\mathbf{u}_{(q+1)}^{(j)}=\left(\lambda_{(q+1)}^{(j)}-\lambda_0^{(j)}\right)\left[\mathbf{A}_0+\sum_{i=1}^{M}\xi_i\mathbf{A}_i-\lambda_0^{(j)}\mathbb{I}\right]^{-1}\mathbf{u}_{(q)}^{(j)}$.
3. $\mathbf{u}_{(q+1)}^{(j)}\to\dfrac{\mathbf{u}_{(q+1)}^{(j)}}{\left\|\mathbf{u}_{(q+1)}^{(j)}\right\|_{L_S^2}}$ where $\left\|\mathbf{u}_{(q+1)}^{(j)}\right\|_{L_S^2}=\sqrt{\sum_{k=1}^{P}\mathrm{E}\left[\Gamma_k^2\right]\left(\mathbf{u}_{k(q+1)}^{(j)}\right)^T\mathbf{u}_{k(q+1)}^{(j)}}$.
4. Define errors $\epsilon_{1_{q+1}}=\|[\mathbf{A}_0+\sum_{i=1}^{M}\xi_i\mathbf{A}_i-\lambda_{(q+1)}^{(j)}\mathbb{I}]\|_{L_S^2}$ and $\epsilon_{2_{(q+1)}}=\dfrac{|V_{\lambda_{(q+1)}^{(j)}}-V_{\lambda_{(q)}^{(j)}}|}{|V_{\lambda_{(q)}^{(j)}}|}$ with $V_{\lambda_{(q)}^{(j)}}=\dfrac{\sqrt{\sum_{k=1}^{P}\lambda_{jk(q)}^2\mathrm{E}\left[\Gamma_k^2\right]}}{\lambda_{j1(q)}}$, the coefficient of variation of the eigenvalue.

For all the steps of the iterative procedure where it is needed, the coefficients of the PC expansions are obtained using the Galerkin method. The subscripts (q) and $(q+1)$ denote the number of the iteration. That is, $\lambda_{(q)}^{(j)}$ is the PC expansion of the jth eigenvalue at iteration q, while $\lambda_{jk(q)}$ denotes the kth coefficient of the expansion, so that $\lambda_{(q)}^{(j)}=\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k$. Similarly, $\mathbf{u}_{(q)}^{(j)}$ is the PC expansion of the jth eigenvector at iteration q, so that $\mathbf{u}_{(q)}^{(j)}=\sum_{k=1}^{P}\mathbf{u}_{k(q)}^{(j)}\Gamma_k$.

The two methods discussed here can be used to obtain the eigenvalue and eigenvector coefficients λ_{jk} and $\mathbf{u}_k^{(j)}$ of the PC expansions given by Eqs. (8) and (9). From these PC expansions, the pdf of eigenvalues and eigenvectors can be obtained by a Monte Carlo Simulation. This is done by sampling the set of independent Gaussian random variables $\xi_1,\dots,\xi_M$. The corresponding values of the basis functions Γ_k are calculated and subsequently introduced in the PC expansions of eigenvalues and eigenvectors given by Eqs. (8) and (9). A numerical approximation to the pdf of eigenvalues from the samples of eigenvalues can then be obtained. Moments of eigenvalues and eigenvectors can also be derived from Eqs. (8) and (9). The first and second moments of eigenvalues are given by

$$\mathrm{E}\left[\lambda^{(j)}\right]=\sum_{k=1}^{P}\lambda_{jk}\mathrm{E}[\Gamma_k]=\lambda_{j1}, \tag{11}$$

$$\mathrm{E}\left[\left(\lambda^{(j)}\right)^2\right]=\sum_{k,l=1}^{P}\lambda_{jk}\lambda_{jl}\mathrm{E}[\Gamma_k\Gamma_l]=\sum_{k=1}^{P}\lambda_{jk}^2\mathrm{E}\left[\Gamma_k^2\right]. \tag{12}$$

Similarly, the first and second moments of the eigenvectors can be calculated

$$\mathrm{E}\left[\mathbf{u}^{(j)}\right]=\sum_{k=1}^{P}\mathbf{u}_k^{(j)}\mathrm{E}[\Gamma_k]=\mathbf{u}_1^{(j)}, \tag{13}$$

$$\mathrm{E}\left[\mathbf{u}^{(j)}\left(\mathbf{u}^{(j)}\right)^T\right]=\sum_{k,l=1}^{P}\mathbf{u}_k^{(j)}\left(\mathbf{u}_l^{(j)}\right)^T\mathrm{E}[\Gamma_k\Gamma_l]=\sum_{k=1}^{P}\mathbf{u}_k^{(j)}\left(\mathbf{u}_k^{(j)}\right)^T\mathrm{E}\left[\Gamma_k^2.\right] \tag{14}$$

3. Rayleigh quotient method for the Polynomial Chaos expansion of eigenvalues

The Rayleigh quotient can be used to obtain an approximation to the eigenvalue $\lambda^{(j)}$ if an approximation to eigenvector $\mathbf{u}^{(j)}$ is available

$$\lambda^{(j)}=\frac{\left(\mathbf{u}^{(j)}\right)^T\mathbf{A}\mathbf{u}^{(j)}}{\left(\mathbf{u}^{(j)}\right)^T\mathbf{u}^{(j)}}. \tag{15}$$

This method to obtain eigenvalues is similar to step 1 of the algorithm by Verhoosel et al. [33], but here, the eigenvectors do not need to be normalized as the Rayleigh quotient already removes the effect of the norm of eigenvectors on eigenvalues through the denominator. The eigenvalues of the random algebraic eigenvalue problem can be expanded with a PC expansion such that the jth eigenvalue is given by Eq. (8). Substituting this expansion in the expression of the Rayleigh quotient in Eq. (15) leads to

$$\left(\sum_{k=1}^{P}\lambda_{jk}\Gamma_k\right)\left(\mathbf{u}^{(j)}\right)^T\mathbf{u}^{(j)}=\left(\mathbf{u}^{(j)}\right)^T\left(\sum_{r=1}^{P}\Gamma_r\mathbf{A}_r\right)\mathbf{u}^{(j)} \tag{16}$$

In the following subsections, the eigenvectors $\mathbf{u}^{(j)}$ will be substituted by different approximations, allowing to obtain the coefficients λ_{jk} from a linear system of equations. It is noted that, once a description of eigenvectors is available, two approaches to calculate the PC expansion of eigenvalues are available. Here, the expansion is obtained using Galerkin method, but non-intrusive methods could also be applied [49].

3.1. Perturbation of the eigenvectors

Using the first-order perturbation of the jth eigenvector, the corresponding eigenvalue can be approximated using the Rayleigh quotient by substituting $\mathbf{u}^{(j)}$ from Eqs. (5) and (7) into Eq. (16)

$$\left(\sum_{k=0}^{P}\lambda_{jk}\Gamma_k\right)\left(\mathbf{u}_{j0}^T\mathbf{u}_{j0}+2\sum_{i=1}^{M}\xi_i\mathbf{u}_{j0}^T\mathbf{u}_{ji}+\sum_{i,g=1}^{M}\xi_i\xi_g\mathbf{u}_{ji}^T\mathbf{u}_{gi}\right)$$
$$=\sum_{r=1}^{P}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{j0}\Gamma_r+2\sum_{r=1}^{P}\sum_{g=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{jg}\xi_g\Gamma_r+\sum_{r=1}^{P}\sum_{g,h=1}^{M}\mathbf{u}_{jg}^T\mathbf{A}_r\mathbf{u}_{jh}\xi_g\xi_h\Gamma_r. \tag{17}$$

The Galerkin method is applied to Eq. (17) (i.e. the equation is multiplied by the pth PC basis function Γ_p and mean of the equation is taken) and the resulting equation is simplified using properties given by Eq. (6). Then, coefficients λ_{jk} of the PC expansion of $\lambda^{(j)}$ can be obtained from the equations

$$\sum_{k=1}^{P}\lambda_{jk}\left(c_{0kp}+\sum_{i,g=1}^{M}\mathbf{u}_{ji}^T\mathbf{u}_{jg}c_{2igkp}\right)=\sum_{r=1}^{P}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{j0}c_{0rp}+2\sum_{r=1}^{P}\sum_{g=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{jg}c_{1grp}+\sum_{r=1}^{P}\sum_{g,h=1}^{M}\mathbf{u}_{jg}^T\mathbf{A}_r\mathbf{u}_{jh}c_{2ghrp}. \tag{18}$$

This equation reduces, for the case of $\mathbf{A}$ given by Eq. (2), to

$$\sum_{k=1}^{P}\lambda_{jk}\left(c_{0kp}+\sum_{i,g=1}^{M}\mathbf{u}_{ji}^T\mathbf{u}_{jg}c_{2igkp}\right)=\lambda_0^{(j)}d_{0p}+\sum_{i=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}d_{1ip}+\sum_{i,g=1}^{M}\left(2\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{jg}+\mathbf{u}_{jg}^T\mathbf{A}_0\mathbf{u}_{ji}\right)d_{2igp}+\sum_{i,g,h=1}^{M}\mathbf{u}_{jg}^T\mathbf{A}_i\mathbf{u}_{jh}d_{3ighp}, \tag{19}$$

where $p=1,\ldots,P$. Matrix $\mathbf{c}_0$ has already been defined as a diagonal matrix with element in the kth row and pth column given by $c_{0kp}=\mathrm{E}[\Gamma_k\Gamma_p]$. Similarly, the $P\times P$ matrix $\mathbf{c}_{2ig}$ has elements $c_{2igkp}=\mathrm{E}[\Gamma_k\Gamma_p\xi_i\xi_g]$ and column vectors of length P $\mathbf{d}_0,\mathbf{d}_{1i},\mathbf{d}_{2ig}$ and $\mathbf{d}_{3igh}$ have their pth term given by $d_{0p}=\mathrm{E}[\Gamma_p],d_{1ip}=\mathrm{E}[\Gamma_p\xi_i],d_{2igp}=\mathrm{E}[\Gamma_p\xi_i\xi_g]$ and $d_{3ighp}=\mathrm{E}[\Gamma_p\xi_i\xi_g\xi_h]$ respectively. This notation for matrices $\mathbf{c}$ and $\mathbf{d}$ is such that for each matrix the mean of the product of, respectively, one and two polynomials Γ_k is calculated. The numerical subindex (e.g. 1, 2) indicate the number of random variables multiplying those polynomials inside the mean operator. Eq. (19) can be represented by the linear system of equations

$$\left(\mathbf{c}_0+\sum_{i,g=1}^{M}\mathbf{u}_{ji}^T\mathbf{u}_{jg}\mathbf{c}_{2ig}\right)\begin{bmatrix}\lambda_{j1}\\ \vdots\\ \lambda_{jP}\end{bmatrix}=\lambda_0^{(j)}\mathbf{d}_0+\sum_{i=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}\mathbf{d}_{1i}+\sum_{i,g=1}^{M}\left(2\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{jg}+\mathbf{u}_{jg}^T\mathbf{A}_0\mathbf{u}_{ji}\right)\mathbf{d}_{2ig}+\sum_{i,g,h=1}^{M}\mathbf{u}_{jg}^T\mathbf{A}_i\mathbf{u}_{jh}\mathbf{d}_{3igh} \tag{20}$$

and the coefficients λ_{jk} of the eigenvalue expansion can be obtained by solving the $P\times P$ linear system given by Eq. (20).

3.2. Reduced Polynomial Chaos eigenvectors

The PC expansion of an eigenvector is given by Eq. (9). A possible approach to reduce the size of the system is to assume the vectors $\mathbf{u}_k^{(j)}$ can be expressed as a linear combination of the vectors $\mathbf{u}_{j0}$ and $\mathbf{u}_{ji}$, such that

$$\mathbf{u}_k^{(j)}=a_{0k}^{(j)}\mathbf{u}_{j0}+\sum_{i=1}^{M}a_{ik}^{(j)}\mathbf{u}_{ji}. \tag{21}$$

The reduction is performed when $M+1<n$, otherwise, the system obtained is equivalent to the full system where the eigenvector is given in Eq. (9), for $M+1=n$, or the vectors $\mathbf{u}_{j0},\mathbf{u}_{ji}$ are linearly dependent, for $M+1>n$. In the cases where $M+1\geqslant n$, the system is not reduced, but a projection of the eigenvectors on the deterministic eigenvectors can be performed. This procedure will be developed in Section 3.3. The idea of expanding the random eigenvectors into a basis formed by the deterministic eigenvector and its first derivative has already been used by Nair and Keane [19], and an accurate approximation to eigenvectors was obtained. Substituting Eq. (21) in Eq. (9), the eigenvector can be expanded as

$$\mathbf{u}^{(j)}=\left(\sum_{k=1}^{P}a_{0k}^{(j)}\Gamma_k\right)\mathbf{u}_{j0}+\sum_{i=1}^{M}\left(\sum_{k=1}^{P}a_{ik}^{(j)}\Gamma_k\right)\mathbf{u}_{ji}=\mathbf{U}_j\begin{bmatrix}\sum_{k=1}^{P}a_{0k}^{(j)}\Gamma_k\\ \vdots\\ \sum_{k=1}^{P}a_{Mk}^{(j)}\Gamma_k\end{bmatrix}, \tag{22}$$

where the rectangular matrix $\mathbf{U}_j$ is formed with the columns of the deterministic eigenvector and its derivatives with respect to each random variable

$$\mathbf{U}_j=[\,\mathbf{u}_{j0}\quad\mathbf{u}_{j1}\quad\ldots\quad\mathbf{u}_{jM}\,]\in\mathbb{R}^{n\times(M+1)}. \tag{23}$$

This expansion will be used in the next section, where coefficients $a_{0k}^{(j)},a_{ik}^{(j)}$ will be obtained. After obtaining the set of coefficients $a_{0k}^{(j)},a_{ik}^{(j)}$, the PC expansion of the jth eigenvalue is obtained using the Rayleigh quotient. That is, the new eigenvectors from Eq. (22) are substituted in Eq. (16)

$$\begin{aligned}\left(\sum_{k=0}^{P}\lambda_{jk}\Gamma_k\right)&\left(\sum_{l,m=1}^{P}a_{0m}^{(j)}a_{0l}^{(j)}\Gamma_l\Gamma_m\mathbf{u}_{j0}^T\mathbf{u}_{j0}+2\sum_{i=1}^{M}\sum_{l,m=1}^{P}a_{0m}^{(j)}a_{gl}^{(j)}\Gamma_l\Gamma_m\mathbf{u}_{j0}^T\mathbf{u}_{ji}\right.\\&\left.+\sum_{i,g=1}^{M}\sum_{l,m=1}^{P}a_{im}^{(j)}a_{gl}^{(j)}\Gamma_l\Gamma_m\mathbf{u}_{ji}^T\mathbf{u}_{gi}\right)=\sum_{r,l,m=1}^{P}a_{0m}^{(j)}a_{0l}^{(j)}\Gamma_r\Gamma_l\Gamma_m\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{j0}\\&+2\sum_{i=1}^{M}\sum_{r,l,m=1}^{P}a_{0m}^{(j)}a_{il}^{(j)}\Gamma_r\Gamma_l\Gamma_m\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{ji}\\&+\sum_{i,g=1}^{M}\sum_{r,l,m=1}^{P}a_{im}^{(j)}a_{gl}^{(j)}\Gamma_r\Gamma_l\Gamma_m\mathbf{u}_{jg}^T\mathbf{A}_r\mathbf{u}_{ji}.\end{aligned} \tag{24}$$

As in the previous subsection, the equation is simplified using properties given by Eq. (6). Then, the equation is multiplied by Γ_p and mean is taken. The coefficients of the eigenvalue expansion can be retrieved from the set of equations

$$\begin{aligned}\sum_{k=1}^{P}\lambda_{jk}&\sum_{l,m=1}^{P}g_{lmpk}\left(a_{0l}^{(j)}a_{0m}^{(j)}+\sum_{i,g=1}^{M}a_{im}^{(j)}a_{gl}^{(j)}\mathbf{u}_{ji}^T\mathbf{u}_{jg}\right)\\&=\sum_{r,l,m=1}^{P}g_{rlmp}\left(a_{0l}^{(j)}a_{0m}^{(j)}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{j0}+2\sum_{i=1}^{M}a_{0m}^{(j)}a_{il}^{(j)}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{ji}\right.\\&\left.+\sum_{g,i=1}^{M}a_{gm}^{(j)}a_{il}^{(j)}\mathbf{u}_{jg}^T\mathbf{A}_r\mathbf{u}_{ji}\right).\end{aligned} \tag{25}$$

If the system matrix is given by Eq. (2), the previous equation reduces to

$$\begin{aligned}\sum_{k=1}^{P}\lambda_{jk}&\sum_{l,m=1}^{P}g_{lmpk}\left(a_{0l}^{(j)}a_{0m}^{(j)}+\sum_{i,g=1}^{M}a_{im}^{(j)}a_{gl}^{(j)}\mathbf{u}_{ji}^T\mathbf{u}_{jg}\right)\\&=\sum_{l,m=1}^{P}a_{0l}^{(j)}a_{0m}^{(j)}\left(\lambda_{0j}e_{0lmp}+\sum_{i=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}e_{1ilmp}\right)\\&+\sum_{l,m=1}^{P}a_{0l}^{(j)}\sum_{i,g=1}^{M}a_{im}^{(j)}2\mathbf{u}_{j0}^T\mathbf{A}_g\mathbf{u}_{ji}e_{1glmp}\\&+\sum_{l,m=1}^{P}\sum_{i,g}^{M}a_{im}^{(j)}a_{gl}^{(j)}\left(\mathbf{u}_{ji}^T\mathbf{A}_0\mathbf{u}_{jg}e_{0lmp}+\sum_{h=1}^{M}\mathbf{u}_{ji}^T\mathbf{A}_h\mathbf{u}_{jg}e_{1hlmp}\right).\end{aligned} \tag{26}$$

Here $p=1,\ldots,P,e_{0lmp}=\mathrm{E}[\Gamma_l\Gamma_m\Gamma_p],e_{1glmp}=\mathrm{E}[\Gamma_m\Gamma_l\Gamma_p\xi_g],\ g_{klmp}=\mathrm{E}[\Gamma_k\Gamma_l\Gamma_m\Gamma_p]$. As formerly, the set of coefficients λ_{jk} can be found solving a linear equation similar to the one obtained in Eq. (20). The set of coefficients $a_{0k}^{(j)},a_{ik}^{(j)}$ have yet not been defined. They are calculated in the next section, following three approaches, namely, the reduced spectral power method, the reduced spectral inverse method and the reduced spectral constrained coefficients method.

3.3. Polynomial Chaos eigenvectors projected on deterministic eigenvectors

The PC expansion of an eigenvector is given by Eq. (9). The vectors $\mathbf{u}_k^{(j)}$ can be expressed as a linear combination of the deterministic eigenvectors $\mathbf{u}_{j0}$, such that

$$\mathbf{u}_k^{(j)} = \sum_{i=1}^{n}\left(\sum_{k=1}^{P} a_k^{(ji)}\Gamma_k\right)\mathbf{u}_{i0} = \mathbf{U}_0\begin{bmatrix}\sum_{k=1}^{P} a_k^{(j1)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P} a_k^{(jn)}\Gamma_k\end{bmatrix} = \mathbf{U}_0(\mathbf{I}_n\otimes[\Gamma_1\ldots\Gamma_P])\begin{bmatrix}a_1^{(j1)}\\ \vdots \\ a_P^{(j1)}\\ \vdots \\ a_P^{(jn)}\end{bmatrix}, \tag{27}$$

where $\mathbf{U}_0$ is the matrix of deterministic eigenvectors. This calculation of the Rayleigh quotient uses the coefficients $a_{0k}^{(j)}, a_{ik}^{(j)}$ obtained from the spectral constrained coefficients method obtained in Section 4.4. After obtaining the set of coefficients $a_{0k}^{(j)}, a_{ik}^{(j)}$, the PC expansion of the jth eigenvalue is obtained using the Rayleigh quotient. That is, the new eigenvectors from Eq. (27) are substituted in Eq. (16)

$$\left(\sum_{k=1}^{P}\lambda_{jk}\Gamma_k\right)\left(\sum_{k=1}^{n}\sum_{l,m=1}^{P} a_m^{(jk)}a_l^{(jk)}\Gamma_l\Gamma_m\right) = \sum_{r,l,m=1}^{P}\sum_{g,k=1}^{n}\mathbf{u}_{g0}^T\mathbf{A}_r\mathbf{u}_{k0}a_m^{(jk)}a_l^{(jg)}\Gamma_l\Gamma_m\Gamma_r. \tag{28}$$

Eq. (28) is multiplied by Γ_p and mean is taken, leading to

$$\left(\sum_{k=1}^{n}\sum_{l,m=1}^{P} a_m^{(jk)}a_l^{(jk)}\mathbf{g}_{lm}\right)\begin{bmatrix}\lambda_{j1}\\ \vdots \\ \lambda_{jP}\end{bmatrix} = \sum_{r,l,m=1}^{P}\sum_{g,k=1}^{n}\mathbf{v}_{g0}^T\mathbf{A}_r\mathbf{v}_{k0}a_m^{(jk)}a_l^{(jg)}\mathbf{g}_{lmr}, \tag{29}$$

which, if the system matrix is given by a KL expansion, reduces to

$$\left(\sum_{k=1}^{n}\sum_{l,m=1}^{P} a_m^{(jk)}a_l^{(jk)}\mathbf{g}_{lm}\right)\begin{bmatrix}\lambda_{j1}\\ \vdots \\ \lambda_{jP}\end{bmatrix} = \sum_{k=1}^{n}\lambda_0^{(k)}\sum_{l,m=1}^{P} a_m^{(jk)}a_l^{(jk)}\mathbf{e}_{0lm} + \sum_{g,k=1}^{n}\sum_{i}^{M}\mathbf{v}_{g0}^T\mathbf{A}_i\mathbf{v}_{k0}^T\sum_{l,m=1}^{P} a_m^{(jk)}a_l^{(jg)}\mathbf{e}_{1ilm}, \tag{30}$$

where $\mathbf{g}_{lm}$ is a matrix indexed by (k,p) and $\mathbf{e}_{0lm}, \mathbf{e}_{1ilm}$ and $\mathbf{g}_{lmr}$ are vectors indexed by p. The set of coefficients $a_k^{(ji)}$ have yet not been defined. They are calculated in the next section, following the spectral constrained coefficient method.

4. Updating of the eigenvectors

When considering a deterministic eigenvalue problem, several iterative methods are available to approximate each eigenvector with a desired accuracy [3]. The power method is based on the fact that a symmetric matrix $\mathbf{B}$ with eigenvalues χ_i and eigenvectors $\mathbf{x}_i$ multiplied s times by itself can be expressed in terms of the sth power of its eigenvalues through $\mathbf{B}^s = \sum_{i=1}^{n}\chi_i^s\mathbf{x}_i\mathbf{x}_i^T$. If the eigenvalues have distinct values, the expression is dominated by $\sum_{i=1}^{r}\chi_i^s\mathbf{x}_i\mathbf{x}_i^T$ where χ_i are the r larger or dominant eigenvalues. Therefore, the power method is an iterative method that allows the obtention of the eigenvector corresponding to the largest eigenvalue through $\mathbf{B}^s\mathbf{v}$, where $\mathbf{v}$ is the start-vector of the iteration algorithm, and s is the step of the iteration. A shift of origin (i.e. $(\mathbf{B}-p\mathbb{I})$, where $p\in\mathbb{R}$) allow convergence of $(\mathbf{B}-p\mathbb{I})^s\mathbf{v}$ to different eigenvectors. When the inverse of $\mathbf{B}$ is considered, it is observed that $\mathbf{B}^{-s}$ is dominated by $\sum_{i=1}^{r}(1/\chi_i^s)\mathbf{x}_i\mathbf{x}_i^T$ where χ_i are the r smallest eigenvalues. If the power method is applied to the matrix $(\mathbf{B}-p\mathbb{I})^{-1}$, the inverse power method is obtained, and the product $(\mathbf{B}-p\mathbb{I})^{-s}\mathbf{v}$ converges to the eigenvector corresponding to the eigenvalue closest to p [3]. In a deterministic system, these two methods allow to update a given approximation to an eigenvector. In the following subsections, the power and inverse power methods are extended to the stochastic case for updating the PC expansions of the eigenvectors. Two other methods are also developed to update eigenvectors, based on the eigenvalue equation.

4.1. Reduced spectral power method

The power method has been used in the context of spectral stochastic finite element method (SSFEM) by Lee and Singh [21] to obtain the mean and covariance of the eigenvectors. Only one iteration was performed, so that the random eigenvectors were obtained by multiplying the deterministic eigenvectors by the system stochastic matrix from Eq. (2). Here, the equation used to derive the reduced spectral power method (RSPM) is based on the deterministic power method equation

$$\lambda_{(q)}^{(j)}\mathbf{u}_{(q+1)}^{(j)} = \mathbf{A}\mathbf{u}_{(q)}^{(j)}, \tag{31}$$

where the subscripts (q) and $(q+1)$ indicate the number of the iteration. Substituting the PC expansion of eigenvalues from Eq. (8) into Eq. (31) leads to

$$\left(\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k\right)\mathbf{u}_{(q+1)}^{(j)} = \mathbf{A}\mathbf{u}_{(q)}^{(j)}, \tag{32}$$

where the initial approximation to the eigenvector is given by

$$\mathbf{u}_{(0)}^{(j)} = \mathbf{u}_{j0} + \sum_{i=1}^{M}\xi_i\mathbf{u}_{ji}. \tag{33}$$

Substituting Eq. (9) into Eq. (32) and applying the Galerkin method to the resulting equation would lead to a $P\times n$ linear system from where the unknown vectors $\mathbf{u}_k^{(j)}$ could be retrieved. A reduction in the system is performed by projecting the qth iteration of the jth eigenvector $\mathbf{u}_{(q)}^{(j)}$ into the subspace defined by the deterministic eigenvector and its derivatives, as in Eq. (21). In the iterative case, eigenvector and constants $\mathbf{u}^{(j)}, a_{0k}^{(j)}$ and $a_{ik}^{(j)}$ are obtained from an iteration and, therefore, are expressed respectively as $\mathbf{u}_{(q)}^{(j)}, a_{0k,(q)}^{(j)}$ and $a_{ik,(q)}^{(j)}$. The system is reduced from a $P\times n$ system to a $(M+1)\times P$ system, and the unknowns of the reduced system are the set of coefficients $a_{0k}^{(j)}, a_{ik}^{(j)}$ for $k=1,\ldots,P$ and $i=1,\ldots M$. Eq. (32) projected in the subspace $\mathrm{span}\{\mathbf{u}_{j0},\ldots,\mathbf{u}_{ji}\}$ can be rewritten as

$$\left(\sum_{l=1}^{P}\lambda_{jl(q)}\Gamma_l\right)\mathbf{U}_j^T\mathbf{U}_j\begin{bmatrix}\sum_{k=1}^{P} a_{0k,(q+1)}^{(j)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P} a_{Mk,(q+1)}^{(j)}\Gamma_k\end{bmatrix} = \mathbf{U}_j^T\mathbf{A}\mathbf{U}_j\begin{bmatrix}\sum_{k=1}^{P} a_{0k,(q)}^{(j)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P} a_{Mk,(q)}^{(j)}\Gamma_k\end{bmatrix}, \tag{34}$$

The iterative algorithm can be derived easily by multiplying Eq. (34) by Γ_p and taking the mean

$$\sum_{l=1}^{P}\lambda_{jl(q)}\mathbf{U}_j^T\mathbf{U}_j\begin{bmatrix}\sum_{k=1}^{P} a_{0k,(q+1)}^{(j)}\mathrm{E}[\Gamma_k\Gamma_l\Gamma_p] \\ \vdots \\ \sum_{k=1}^{P} a_{Mk,(q+1)}^{(j)}\mathrm{E}[\Gamma_k\Gamma_l\Gamma_p]\end{bmatrix} = \mathrm{E}\left[\Gamma_p\mathbf{U}_j^T\mathbf{A}\mathbf{U}_j\begin{bmatrix}\sum_{k=1}^{P} a_{0k,(q)}^{(j)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P} a_{Mk,(q)}^{(j)}\Gamma_k\end{bmatrix}\right]. \tag{35}$$

An expression for $\mathbf{u}_{(1)}^{(j)}$ is calculated, which corresponds to one iteration. An accurate solution is expected from one iteration as the first-

order perturbation of eigenvectors are close to the exact solution. Furthermore, accuracy might be compromised if a large number of iterations are performed. This is so because the deterministic power method converges to the eigenvector corresponding to the largest eigenvalue when a considerable number of iterations are performed. The first iteration is obtained from Eq. (35) where $a_{01,(0)} = a_{i2,(0)} = 1$ and all the other coefficients are 0 and $\lambda_{jl(0)}$ are obtained from Eq. (20). The vector of unknown coefficients $\mathbf{a} = [a^{(j)}_{01(1)} \cdots a^{(j)}_{0M(1)}\ a^{(j)}_{11(1)} \cdots a^{(j)}_{ip(1)} \cdots a^{(j)}_{MP(1)}]^T$ is the solution to the linear system

$$\left(\sum_{l=1}^{P}(\lambda_{jl(0)}\mathbf{U}_j^T\mathbf{U}_j \otimes \mathbf{e}_{0l})\right)\mathbf{a} = \mathbf{f}. \tag{36}$$

The $P \times P$ matrices $\mathbf{e}_{0l}$ are such that their kth row and pth element is e_{0lkp} and

$$\mathbf{f} = \begin{bmatrix} \sum_{r=1}^{P}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{j0}\mathbf{c}_{0r} + \sum_{i}^{M}\sum_{r=1}^{P}\mathbf{u}_{j0}^T\mathbf{A}_r\mathbf{u}_{jk}\mathbf{c}_{1ir} \\ \sum_{r=1}^{P}\mathbf{u}_{j1}^T\mathbf{A}_r\mathbf{u}_{j0}\mathbf{c}_{0r} + \sum_{i}^{M}\sum_{r=1}^{P}\mathbf{u}_{j1}^T\mathbf{A}_r\mathbf{u}_{jk}\mathbf{c}_{1ir} \\ \vdots \\ \sum_{r=1}^{P}\mathbf{u}_{jM}^T\mathbf{A}_r\mathbf{u}_{j0}\mathbf{c}_{0r} + \sum_{i}^{M}\sum_{r=1}^{P}\mathbf{u}_{jM}^T\mathbf{A}_r\mathbf{u}_{jk}\mathbf{c}_{1ir} \end{bmatrix} \tag{37}$$

can be rewritten, for the case of a system matrix given by a KL expansion, as

$$\mathbf{f} = \begin{bmatrix} \lambda_0^{(j)}\mathbf{d}_0 + \sum_{i=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}\mathbf{d}_{1i} + \sum_{i,k=1}^{M}\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{jk}\mathbf{d}_{2ik} \\ \sum_{i=1}^{M}\left(\mathbf{u}_{j1}^T\mathbf{A}_0\mathbf{u}_{ji} + \mathbf{u}_{j1}^T\mathbf{A}_i\mathbf{u}_{0j}\right)\mathbf{d}_{1i} + \sum_{i,k=1}^{M}\mathbf{u}_{j1}^T\mathbf{A}_i\mathbf{u}_{jk}\mathbf{d}_{2ik} \\ \vdots \\ \sum_{i=1}^{M}\left(\mathbf{u}_{jM}^T\mathbf{A}_0\mathbf{u}_{ji} + \mathbf{u}_{jM}^T\mathbf{A}_i\mathbf{u}_{0j}\right)\mathbf{d}_{1i} + \sum_{i,k=1}^{M}\mathbf{u}_{jM}^T\mathbf{A}_i\mathbf{u}_{jk}\mathbf{d}_{2ik} \end{bmatrix}. \tag{38}$$

The deterministic power method can update the eigenvectors if the initial approximation is close to the solution, but as the number of iterations increases, the method converges to the eigenvector corresponding to the largest eigenvalue. The reduced spectral power method is likely to suffer from the same drawback. The reduced spectral inverse power method is derived in the next subsection to try to overcome this difficulty.

4.2. Reduced spectral inverse power method

The reduced spectral inverse power method (RSIPM) developed here is based on the deterministic inverse power method equation

$$\left(\mathbf{A} - \lambda_0^{(j)}\mathbf{I}\right)\mathbf{u}^{(j)}_{(q+1)} = \left(\lambda^{(j)}_{(q)} - \lambda_0^{(j)}\right)\mathbf{u}^{(j)}_{(q)}, \tag{39}$$

where the subscripts (q) and $(q+1)$ indicate the number of the iteration. Substitution of the PC expansion of eigenvalues from Eq. (8) into Eq. (39) leads to

$$\left(\mathbf{A} - \lambda_0^{(j)}\mathbf{I}\right)\mathbf{u}^{(j)}_{(q+1)} = \left(\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k - \lambda_0^{(j)}\right)\mathbf{u}^{(j)}_{(q)}, \tag{40}$$

where the initial approximation to the eigenvector is given by Eq. (33). As in the previous subsection, an approximation to $\mathbf{u}^{(j)}_{(q+1)}$ could be obtained by applying the Galerkin method to Eq. (40). This approximation requires the solution of a $P \times n$ linear system. A reduction of the size of the system is achieved by projecting the equation in the subspace $\mathrm{span}\{\mathbf{u}_{j0}, \ldots, \mathbf{u}_{ji}\}$ where the size of the new system is $(M+1) \times P$. The approximation to the jth eigenvector $\mathbf{u}^{(j)}_{(q+1)}$ can be expressed as the eigenvector from Eq. (22) where subscripts $(q+1)$ are added to $\mathbf{u}^{(j)}, a^{(j)}_{0k}$ and $a^{(j)}_{ik}$. The projection of Eq. (40) leads to

$$\left(\mathbf{U}_j^T\mathbf{A}\mathbf{U}_j - \lambda_0^{(j)}\mathbf{U}_j^T\mathbf{U}_j\right)\begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q+1)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q+1)}\Gamma_k \end{bmatrix} = \left(\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k - \lambda_0^{(j)}\right)\mathbf{U}_j^T\mathbf{U}_j \times \begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q)}\Gamma_k \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q)}\Gamma_k \end{bmatrix}. \tag{41}$$

The unknowns of the iteration can be obtained by multiplying Eq. (41) by Γ_p and taking the mean

$$\begin{aligned} \sum_{r=1}^{P}\mathbf{U}_j^T\mathbf{A}_r\mathbf{U}_j\begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q+1)}\mathrm{E}[\Gamma_r\Gamma_k\Gamma_p] \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q+1)}\mathrm{E}[\xi_i\Gamma_k\Gamma_p] \end{bmatrix} - \lambda_0^{(j)}\mathbf{U}_j^T\mathbf{U}_j\begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q+1)}\mathrm{E}[\Gamma_k\Gamma_p] \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q+1)}\mathrm{E}[\Gamma_k\Gamma_p] \end{bmatrix} \\ = \sum_{l=1}^{P}\lambda_{jk(q)}\mathbf{U}_j^T\mathbf{U}_j\begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q)}\mathrm{E}[\Gamma_k\Gamma_l\Gamma_p] \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q)}\mathrm{E}[\Gamma_k\Gamma_l\Gamma_p] \end{bmatrix} - \lambda_0^{(j)}\mathbf{U}_j^T\mathbf{U}_j\begin{bmatrix} \sum_{k=1}^{P}a^{(j)}_{0k,(q)}\mathrm{E}[\Gamma_k\Gamma_p] \\ \vdots \\ \sum_{k=1}^{P}a^{(j)}_{Mk,(q)}\mathrm{E}[\Gamma_k\Gamma_p] \end{bmatrix}. \end{aligned} \tag{42}$$

For the first iteration, the approximation to the jth eigenvalue $\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k$ can be obtained from Eq. (20), that is, after applying the Rayleigh quotient using the first-order perturbation of eigenvectors. The coefficients used for the first iteration are $a_{01,(0)} = a_{i2,(0)} = 1$ and all the other coefficients of vector $\mathbf{a}$ are 0. The unknown coefficients $a_{0k,(1)}$ and $a_{ik,(1)}$ can be found by solving a linear system of equations where

$$\mathbf{S} = \left(\sum_{r=1}^{P}(\mathbf{U}_j^T\mathbf{A}_r\mathbf{U}_j) \otimes \mathbf{e}_{0q} - \lambda_0^{(j)}(\mathbf{U}_j^T\mathbf{U}_j) \otimes \mathbf{c}_0\right) \tag{43}$$

reduces, if the system matrix is given by a KL expansion, to

$$\mathbf{S} = \left(\sum_{i=1}^{M}(\mathbf{U}_j^T\mathbf{A}_i\mathbf{U}_j) \otimes \mathbf{c}_{1i} + (\mathbf{U}_j^T(\mathbf{A}_0 - \lambda_0^{(j)}\mathbf{I})\mathbf{U}_j) \otimes \mathbf{c}_0\right) \tag{44}$$

such that

$$\mathbf{S}\begin{bmatrix} a_{01(1)} \\ \vdots \\ a_{0M(1)} \\ a_{11(1)} \\ \vdots \\ a_{ip(1)} \\ \vdots \\ a_{MP(1)} \end{bmatrix} = \begin{bmatrix} \lambda_{j1(0)}\mathrm{E}\left[\Gamma_1^2\right] - \lambda_0^{(j)}\mathrm{E}[\Gamma_1] \\ \vdots \\ \lambda_{jP(0)}\mathrm{E}\left[\Gamma_P^2\right] - \lambda_0^{(j)}\mathrm{E}[\Gamma_P] \\ \sum_{g=1}^{M}(\mathbf{u}_{j1})^T\mathbf{u}_{jg}\left(\sum_{k=1}^{P}\lambda_{jk(0)}\mathbf{c}_{1gk1} - \lambda_0^{(j)}\mathbf{d}_{1g1}\right) \\ \vdots \\ \sum_{g=1}^{M}(\mathbf{u}_{ji})^T\mathbf{u}_{jg}\left(\sum_{k=1}^{P}\lambda_{jk(0)}\mathbf{c}_{1gkp} - \lambda_0^{(j)}\mathbf{d}_{1gp}\right) \\ \vdots \\ \sum_{g=1}^{M}(\mathbf{u}_{jM})^T\mathbf{u}_{jg}\left(\sum_{k=1}^{P}\lambda_{jk(0)}\mathbf{c}_{1gkP} - \lambda_0^{(j)}\mathbf{d}_{1gP}\right) \end{bmatrix}. \tag{45}$$

In the next section, a third method to update the eigenvectors is proposed.

4.3. Reduced spectral constrained coefficients method

The reduced spectral constrained coefficients method (RSCCM) developed here is based on the eigenvalue equation

$$\left(\mathbf{A}-\lambda_{(q)}^{(j)}\mathbf{I}\right)\mathbf{u}_{(q+1)}^{(j)}=\mathbf{0}_n, \tag{46}$$

where the subscripts (q) and $(q+1)$ indicate the number of the iteration and $\mathbf{0}_n=[0\dots 0]^T\in\mathbb{N}^n$. The quantity $\lambda_{(q)}^{(j)}$ is the PC expansion of the jth eigenvalue from the qth iteration. Substitution of the PC expansion of eigenvalues from Eq. (8) into Eq. (46) leads to

$$\left(\mathbf{A}-\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k\mathbf{I}\right)\mathbf{u}_{(q+1)}^{(j)}=\mathbf{0}_n, \tag{47}$$

where the initial approximation to the eigenvalues is given by Eq. (20). As in the previous subsections, an approximation to $\mathbf{u}_{(q+1)}^{(j)}$ could be obtained by applying the Galerkin method to Eq. (47) and assuming a term of $[\mathbf{u}_{1(q+1)}^{(j)}\dots\mathbf{u}_{P(q+1)}^{(j)}]$ is known and equal to one. This assumption allows the removal of the singularity of the linear matrix. The 'constrained' term, that is the term set to 1, is the element whose value is expected to be the largest. Heuristically, it is assumed that this term is the term of $\mathbf{u}_{1(q+1)}^{(j)}$ in the position of the largest term of the deterministic eigenvector. If another term of the PC expansion of the eigenvector were to be larger, our assumption of the position of the largest term of the vector would not be correct. Therefor, the method would be repeated by constraining this term, that is, the term in the position of the obtained largest term, to one. This approximation requires the solution of a $P\times(n-1)$ linear system.

A reduction to the size of the system is achieved by projecting the equation in the subspace $\text{span}\{\mathbf{u}_{j0},\dots,\mathbf{u}_{ji}\}$ where the size of the new system is $(M+1)\times n$. This projection is equivalent to express the jth eigenvector $\mathbf{u}_{(q+1)}^{(j)}$ as the eigenvector from Eq. (22) where subscripts $(q+1)$ are added to $\mathbf{u}^{(j)}, a_{0k}^{(j)}$ and $a_{ik}^{(j)}$. The projection of Eq. (47) leads to

$$\left(\mathbf{U}_j^T\mathbf{A}\mathbf{U}_j-\sum_{k=1}^{P}\lambda_{jk(q)}\Gamma_k\mathbf{U}_j^T\mathbf{U}_j\right)\begin{bmatrix}\sum_{k=1}^{P}a_{0k,(q+1)}^{(j)}\Gamma_k\\ \vdots\\ \sum_{k=1}^{P}a_{Mk,(q+1)}^{(j)}\Gamma_k\end{bmatrix}=\mathbf{0}_{M+1}. \tag{48}$$

The unknowns of the iteration can be obtained by multiplying Eq. (48) by Γ_p and taking the mean

$$\left(\sum_{i=1}^{P}\left(\mathbf{U}_j^T\mathbf{A}_i\mathbf{U}_j-\lambda_{ji(q)}\mathbf{U}_j^T\mathbf{U}_j\right)\right)\begin{bmatrix}\sum_{k=1}^{P}a_{0k,(q+1)}^{(j)}\mathrm{E}[\Gamma_i\Gamma_k\Gamma_p]\\ \vdots\\ \sum_{k=1}^{P}a_{Mk,(q+1)}^{(j)}\mathrm{E}[\Gamma_i\Gamma_k\Gamma_p]\end{bmatrix}=\mathbf{0}_{M+1}. \tag{49}$$

Reordering the rows and columns obtained from the previous system considering $k=1,\dots,P$, the coefficients $[a_{01},\dots,a_{0M},a_{11},\dots,a_{ip},\dots,a_{MP}]$ are the solution of the system

$$\left(\sum_{i=1}^{P}(\mathbf{U}_j^T\mathbf{A}_i\mathbf{U}_j-\lambda_{ji(q)}(\mathbf{U}_j^T\mathbf{U}_j))\otimes\mathbf{e}_{0i}\right)\mathbf{a}=\mathbf{0}_{P(M+1)}. \tag{50}$$

If the system matrix is given by a KL expansion,this system of equations reduces to

$$\left((\mathbf{U}_j^T\mathbf{A}_0\mathbf{U}_j)\otimes\mathbf{c}_0+\sum_{i=1}^{M}(\mathbf{U}_j^T\mathbf{A}_i\mathbf{U}_j)\otimes\mathbf{c}_{1i}-\sum_{i=1}^{P}\lambda_{ji(q)}(\mathbf{U}_j^T\mathbf{U}_j)\otimes\mathbf{e}_{0i}\right)\mathbf{a}=\mathbf{0}_{P(M+1)}. \tag{51}$$

As in the case of the full system, a term of the vector $\mathbf{a}=[a_{01(q+1)}^{(j)}\dots a_{MP(q+1)}^{(j)}]$, namely $a_{01(q+1)}^{(j)}$, is assumed to be equal to one and an $(M+1)\times(n-1)$ linear system is obtained. For the first iteration, the approximation to the jth eigenvalue $\sum_{k=1}^{P}\lambda_{jk}\Gamma_k$ can be obtained from Eq. (20), that is, after applying the Rayleigh quotient using the first-order perturbation of eigenvectors.

4.4. Spectral constrained coefficients method

The spectral constrained coefficients method (SCCM) developed here is based on the eigenvalue equation given by Eq. (46), where the eigenvector $\mathbf{u}_{(q+1)}^{(j)}$ is projected in the set of deterministic eigenvectors $\mathbf{U}_0$, as in Eq. (27), such that

$$\mathbf{U}_0^T\left(\mathbf{A}-\lambda_{(q)}^{(j)}\mathbf{I}\right)\mathbf{U}_0\begin{bmatrix}\sum_{k=1}^{P}a_{k(q+1)}^{(j1)}\Gamma_k\\ \vdots\\ \sum_{k=1}^{P}a_{k(q+1)}^{(jn)}\Gamma_k\end{bmatrix}=\mathbf{0}_n. \tag{52}$$

It is assumed that the vector of coefficient multiplying the jth eigenvector is $[a_{1(q+1)}^{(jj)}\dots a_{P(q+1)}^{(jj)}]=[1\ 0\dots\ 0]$. The rectangular matrix of eigenvectors where the jth eigenvector is not included is denoted by $\mathbf{U}_{0j}$ and the resulting system is given by

$$\left(\sum_{k=1}^{P}\left(\mathbf{U}_{0j}^T\mathbf{A}_k\mathbf{U}_{0j}-\lambda_{jk}\mathbf{I}\right)\Gamma_k\right)\begin{bmatrix}\sum_{k=1}^{P}a_{k(q+1)}^{(j1)}\Gamma_k\\ \vdots\\ \sum_{k=1}^{P}a_{k(q+1)}^{(j(j-1))}\Gamma_k\\ \sum_{k=1}^{P}a_{k(q+1)}^{(j(j+1))}\Gamma_k\\ \vdots\\ \sum_{k=1}^{P}a_{k(q+1)}^{(jn)}\Gamma_k\end{bmatrix}=-\begin{bmatrix}\sum_{i=1}^{P}\mathbf{u}_{10}^T\mathbf{A}_i\mathbf{u}_{j0}\xi_i\\ \vdots\\ \sum_{i=1}^{P}\mathbf{u}_{(j-1)0}^T\mathbf{A}_i\mathbf{u}_{j0}\xi_i\\ \sum_{i=1}^{P}\mathbf{u}_{(j+1)0}^T\mathbf{A}_i\mathbf{u}_{j0}\xi_i\\ \vdots\\ \sum_{i=1}^{P}\mathbf{u}_{N0}^T\mathbf{A}_i\mathbf{u}_{j0}\xi_i\end{bmatrix}. \tag{53}$$

As formerly, the system of equations is multiplied by Γ_p and mean is taken

$$\left(\sum_{k=1}^{P}(\mathbf{U}_{0j}^T\mathbf{A}_k\mathbf{U}_{0j}-\lambda_{jk(q)}\mathbf{I})\otimes\mathbf{e}_{0k}\right)\begin{bmatrix}a_{1(q+1)}^{(j(1))}\\ \vdots\\ a_{P(q+1)}^{(j(j-1))}\\ a_{1(q+1)}^{(j(j+1))}\\ \vdots\\ a_{P(q+1)}^{(jn)}\end{bmatrix}=-\sum_{i=1}^{M}\left(\mathbf{U}_0^T\mathbf{A}_i\mathbf{u}_{j0}\right)\otimes\mathbf{d}_{1i}. \tag{54}$$

If the system matrix can be expressed through a KL expansion, the previous equation reduces to

$$\left(\Lambda_{0j}\otimes\mathbf{c}_0+\sum_{i=1}^{M}\mathbf{U}_{0j}^T\mathbf{A}_i\mathbf{U}_{0j}\otimes\mathbf{c}_{1i}-\sum_{k=1}^{P}\lambda_{jk(q)}\mathbf{I}\otimes\mathbf{e}_{0k}\right)\begin{bmatrix}a_{1(q+1)}^{(j(1))}\\ \vdots\\ a_{P(q+1)}^{(j(j-1))}\\ a_{1(q+1)}^{(j(j+1))}\\ \vdots\\ a_{P(q+1)}^{(jn)}\end{bmatrix}$$

$$=-\sum_{i=1}^{M}\left(\mathbf{U}_0^T\mathbf{A}_i\mathbf{u}_{j0}\right)\otimes\mathbf{d}_{1i}, \tag{55}$$

where Λ_{0j} is a diagonal matrix whose diagonal elements are the set of deterministic eigenvalues $\lambda_0^{(j)}$. The initial approximation $\lambda_{(0)}^{(j)}$ is jth diagonal term from $\mathbf{U}_0^T\mathbf{A}\mathbf{U}_0$. In the next section, the proposed methods are summarized.

5. Summary of the proposed methods

Three hybrid perturbation-PC approximations and an approximation based on a projection on the deterministic eigenvectors basis are proposed for the solution of the algebraic random eigenvalue problem arising in structural dynamics. The random eigenvalues are firstly approximated with a PC expansion using the first-order perturbation of eigenvectors in the Rayleigh quotient. These results can be further improved if the eigenvectors are updated using the reduced spectral power method, the reduced spectral inverse power method, the reduced spectral constrained coefficients method or the spectral constrained coefficients method (RSPM, RSIPM, RSCCM and SCCM), respectively. These methods allow us to obtain PC expansions of the eigenvectors. These expansions are then used in the Rayleigh quotient to obtain an improved PC expansion of each eigenvalue. Alternatively, direct MCS can be performed on the Rayleigh quotient to obtain the moments of the eigenvalues. The proposed methods can be implemented by the following steps:

1. Calculate $\mathbf{A}_0$ and the system KL expansion matrices $\mathbf{A}_i\ \forall i=1,\ldots,M$.
2. Obtain the deterministic eigenvalues and eigenvectors $\lambda_0^{(j)},\mathbf{u}_{0j}$.
3. Use Eq. (7) to obtain the first-order perturbation of eigenvectors $\mathbf{u}^{(j)}=\mathbf{u}_{0j}+\sum_{i=1}^{M}\xi_i\partial\mathbf{u}^{(j)}/\partial\xi_i$.
4. Calculate the PC expansion of eigenvalues using the Rayleigh quotient from Eq. (20), obtained using the first-order perturbation of eigenvectors and the truncated Karhunen–Loève expansion of the stiffness matrix.
5. Calculate a new approximation to eigenvectors using one of the four proposed methods
   - The coefficients of the PC expansions involved in each eigenvector approximation are obtained with the reduced spectral power method (RSPM) from Eq. (36).
   - The coefficients of the PC expansions involved in each eigenvector approximation are obtained with the reduced spectral inverse power method (RSIPM) from Eq. (45).
   - The coefficients of the PC expansions involved in each eigenvector approximation are obtained with the reduced spectral constrained coefficients method (RSCCM) from Eq. (50) or Eq. (51) if the system matrix is given by a KL expansion.
   - The coefficients of the PC expansions involved in each eigenvector approximation are obtained with the spectral constrained coefficients method (SCCM) from Eq. (54) or Eq. (55) if the system matrix is given by a KL expansion.
6. Calculate the PC expansion of eigenvalues using Rayleigh quotient as in step 4 if another step of the iteration is going to be performed.
7. Calculate the first and second moments of the eigenvalues using Eq. (11) and (12) or through direct MCS of the Rayleigh quotient using the PC expansion of the eigenvectors.

6. Numerical example: Euler–Bernoulli beam with stochastic properties

A clamped-free beam with equation of motion

$$-\frac{\partial^2}{\partial x^2}\left(EI(x,\theta)\frac{\partial^2 v}{\partial x^2}\right)-\rho A\frac{\partial^2 v}{\partial t^2}+p_y=0$$

$$\text{Throughout}\quad -L/2\leqslant x\leqslant L/2, \tag{56}$$

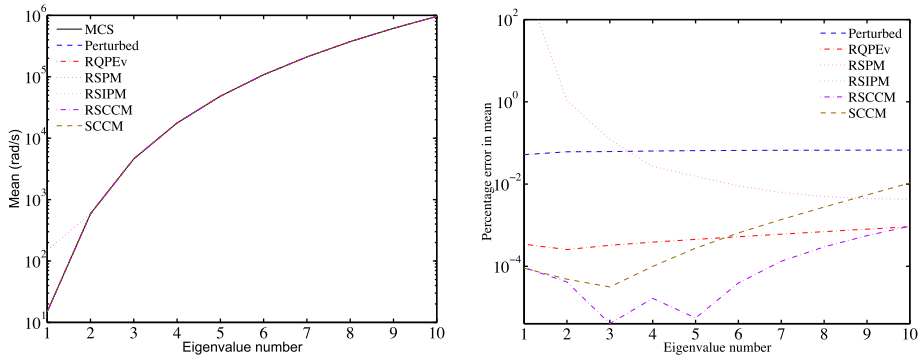
with v the vertical displacement of the centroidal axis, p_y the vertical load applied to the beam and $EI(x,\theta)$ is represented by a random field. The system is shown in Fig. 1, where the values of the parameters appearing in the equation are given. The system is discretized with the finite element method, using 100 elements. Details of the method can be found, for example, in the book by Dawe [50]. Uncertainty is introduced in the system by a Gaussian random field representing the bending rigidity, that is, $w(x,\theta)=EI$ with mean $\mathrm{E}[w(x,\theta)]=5.7520$ kg.m^2. The discretization of $w(x,\theta)$ is done by the KL decomposition of the exponential autocorrelation function $C(x_1,x_2)=e^{-|x_1-x_2|/L}$, that is $w(x,\theta)=\sum_{i=1}^{\infty}\lambda_i f_i\xi_i$ where the eigenvalues and eigenfunctions of the correlation function $\lambda_i,f_i(x_1)$ are obtained after solving the equation $\int_{-L/2}^{L/2}C(x_1,x_2)f(x_2)dx_2=\lambda f(x_1)$ and ξ_i is a set of Gaussian independent random variable. The KL expansion is truncated after $M=5$ terms, so that the corresponding KL expansion of the stiffness matrix is $\mathbf{A}=\mathbf{A}_0+\sum_{i=1}^{5}\xi_i\mathbf{A}_i$. More details on the KL expansion of the autocorrelation function used can be found in [6]. The effect of two different standard deviations of the random field is studied. These standard deviations are respectively 7% and 15% of the mean of the random field. Standard deviations of 7% and 15% are considered with the view that they represent the cases of low and high randomness without producing a swapping of the modes. The maximum order of the Hermite polynomial used is 4, so that $P=126$ polynomials are used as basis functions. Only the results corresponding to the first 10 eigenvalues are considered as the chances of modal overlap increases for higher modes.

Results obtained with the proposed methods are compared against the results from first-order perturbation method from Eq. (4) and MCS with 5000 samples. Convergence of the MCS results is ensured by choosing this number of samples. The moments are obtained by performing MCS on the Rayleigh quotients using PC expansion of eigenvectors and 5000 samples. Figs. 2 and 3 show the mean and standard deviation of the eigenvalues obtained with the first order perturbation and proposed methods for a standard deviation of 7% of the mean of the random field. In these figures the variation of percentage error of the corresponding quantities with respect to MCS results are also shown. The percentage errors are explicitly reported in Tables 1 and 2 to show the relative accuracies of the different methods.

Now we consider the case of higher uncertainty. Figs. 4 and 5 show the mean and standard deviation of the eigenvalues obtained with the first order perturbation and the proposed methods for a standard deviation of 15% of the mean of the random field.

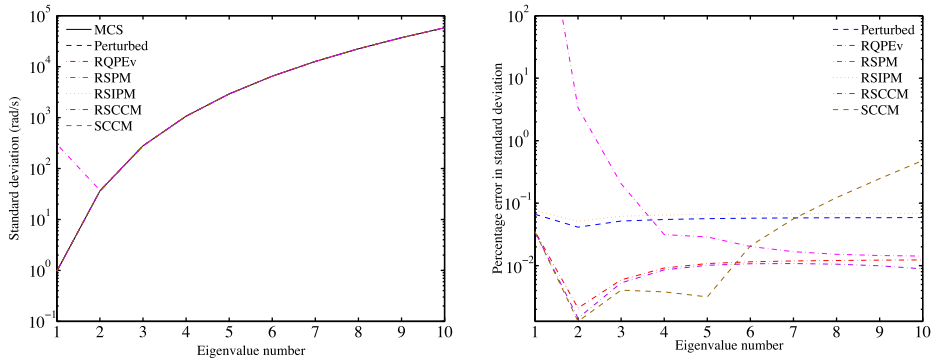
The variation of percentage error with different modes with respect to MCS results are also shown. The percentage errors are again explicitly given in Tables 3 and 4 to show the relative accuracies of the different methods. As expected, in general the errors are larger

Fig. 1. Euler–Bernoulli beam with spatially varying random bending rigidity $w(x,\theta)$. The length of the beam is $L=1.65$ m, the section area is $A=8.2123e-5$ m^2, the density is $\rho=7800$ kg/m^3 and the mean of the bending rigidity random field is $\mathrm{E}[w(x,\theta)]=5.7520$ kg m^2.



(a) Mean of the eigenvalues. (b) Percentage error in the mean.

Fig. 2. Mean and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, first-order perturbation method (Perturbed), Rayleigh quotient using first-order perturbation of eigenvectors (RQPEv) the proposed reduced spectral power method (RSPM), reduced spectral inverse power method (RSIPM), reduced spectral constrained coefficient method (RSCCM) and spectral constrained coefficient method (SCCM). The standard deviation of the discretized random field is 7% of the mean value.



(a) Standard deviation of the eigenvalues. (b) Percentage error in the standard deviation.

Fig. 3. Standard deviation and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. The standard deviation of the random field is 7% of the mean value.

Table 1
Percentage errors in mean obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the discretized random field is 7% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	5.191e−2	3.441e−4	8.992e+2	5.190e−2	9.358e−5	9.012e−5
$\lambda^{(2)}$	6.079e−2	2.562e−4	1.077e+0	6.079e−2	4.161e−5	4.877e−5
$\lambda^{(3)}$	6.158e−2	3.244e−4	1.206e−1	6.158e−2	4.050e−6	3.156e−5
$\lambda^{(4)}$	6.372e−2	3.872e−4	2.686e−2	6.372e−2	1.660e−5	9.959e−5
$\lambda^{(5)}$	6.504e−2	4.519e−4	1.553e−2	6.504e−2	5.550e−6	2.764e−4
$\lambda^{(6)}$	6.584e−2	5.236e−4	9.030e−3	6.584e−2	3.941e−5	6.475e−4
$\lambda^{(7)}$	6.633e−2	6.053e−4	6.235e−3	6.633e−2	1.333e−4	1.373e−3
$\lambda^{(8)}$	6.665e−2	6.982e−4	4.975e−3	6.665e−2	2.969e−4	2.762e−3
$\lambda^{(9)}$	6.687e−2	8.026e−4	4.436e−3	6.687e−2	5.569e−4	5.427e−3
$\lambda^{(10)}$	6.702e−2	9.188e−4	4.274e−3	6.702e−2	9.474e−4	1.067e−2

than the ones from the previous case where the amount of randomness in the system was comparatively lower.

It can be observed that the error of the RSIPM and the first-order perturbation method are very close. This is due to the fact that, if the random variables are set to zero, the matrix of the RSIPM becomes singular. For small standard deviation, it is heuristically expected that the projection of the updated eigenvector on the deterministic eigenvector is much larger than the projection on the other vectors. An eigenvector updated with RSIPM is consequently likely to be close to the related deterministic eigenvector. The first-order perturbation method can be considered as the result of applying the Rayleigh quotient using the deterministic eigenvectors. Consequently, it is expected that both methods lead to similar results.

For the RSPM it can be observed that the error of the first eigenvalue is the highest of all. It has already been commented that the power method converges to the largest eigenvalues, and this fact is likely to be the reason behind the error in the approximation to the first eigenvalues. Consequently, the RSPM is expected to lead to good results for the larger eigenvalues, which are not studied here.

We note that the SRCCM and the SCCM results are more accurate than the ones obtained with the Rayleigh quotient using

Table 2
Percentage errors in standard deviation obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the discretized random field is 7% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	6.637e−2	3.550e−2	3.101e+4	7.637e−2	3.492e−2	3.491e−2
$\lambda^{(2)}$	4.129e−2	2.095e−3	3.372e+0	5.128e−2	1.442e−3	1.285e−3
$\lambda^{(3)}$	5.174e−2	5.921e−3	2.050e−1	6.174e−2	5.311e−3	4.053e−3
$\lambda^{(4)}$	5.472e−2	9.169e−3	3.143e−2	6.471e−2	8.549e−3	3.806e−3
$\lambda^{(5)}$	5.650e−2	1.083e−2	2.884e−2	6.650e−2	1.015e−2	3.165e−3
$\lambda^{(6)}$	5.747e−2	1.154e−2	2.026e−2	6.747e−2	1.072e−2	2.074e−2
$\lambda^{(7)}$	5.804e−2	1.190e−2	1.674e−2	6.804e−2	1.081e−2	5.613e−2
$\lambda^{(8)}$	5.840e−2	1.211e−2	1.523e−2	6.840e−2	1.057e−2	1.230e−1
$\lambda^{(9)}$	5.864e−2	1.225e−2	1.454e−2	6.864e−2	9.965e−3	2.472e−1
$\lambda^{(10)}$	5.881e−2	1.236e−2	1.423e−2	6.880e−2	8.899e−3	4.813e−1

Table 3
Percentage errors in mean obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the discretized random field is 15% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	2.436e−1	6.603e−3	1.694e+4	2.436e−1	1.037e−3	9.479e−4
$\lambda^{(2)}$	2.846e−1	6.281e−3	2.324e+1	2.846e−1	3.263e−4	4.351e−4
$\lambda^{(3)}$	2.884e−1	7.245e−3	2.638e+0	2.884e−1	1.856e−4	5.955e−5
$\lambda^{(4)}$	2.986e−1	8.433e−3	5.809e−1	2.986e−1	3.142e−4	1.837e−4
$\lambda^{(5)}$	3.048e−1	9.715e−3	3.356e−1	3.048e−1	1.530e−5	2.338e−4
$\lambda^{(6)}$	3.086e−1	1.116e−2	1.940e−1	3.086e−1	1.080e−3	2.371e−4
$\lambda^{(7)}$	3.109e−1	1.280e−2	1.330e−1	3.109e−1	3.284e−3	2.144e−4
$\lambda^{(8)}$	3.124e−1	1.465e−2	1.053e−1	3.124e−1	7.208e−3	1.705e−4
$\lambda^{(9)}$	3.134e−1	1.669e−2	9.297e−2	3.134e−1	1.360e−2	1.039e−4
$\lambda^{(10)}$	3.141e−1	1.891e−2	8.865e−2	3.141e−1	2.189e−2	9.950e−6

Table 4
Percentage errors in standard deviation obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the discretized random field is 15% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	2.782e−1	8.210e−2	2.258e+5	2.882e−1	6.755e−2	6.727e−2
$\lambda^{(2)}$	2.591e−1	1.134e−2	2.222e+2	2.690e−1	6.361e−3	6.800e−3
$\lambda^{(3)}$	2.875e−1	2.097e−2	1.026e+0	2.975e−1	3.262e−3	2.751e−3
$\lambda^{(4)}$	3.012e−1	2.915e−2	5.953e−1	3.112e−1	1.024e−2	9.579e−3
$\lambda^{(5)}$	3.090e−1	3.392e−2	4.719e−1	3.190e−1	1.262e−2	1.285e−2
$\lambda^{(6)}$	3.134e−1	3.676e−2	2.779e−1	3.234e−1	1.077e−2	1.385e−2
$\lambda^{(7)}$	3.161e−1	3.898e−2	1.862e−1	3.261e−1	4.393e−3	1.377e−2
$\lambda^{(8)}$	3.178e−1	4.105e−2	1.428e−1	3.278e−1	8.731e−3	1.292e−2
$\lambda^{(9)}$	3.190e−1	4.312e−2	1.222e−1	3.290e−1	3.322e−2	1.131e−2
$\lambda^{(10)}$	3.198e−1	4.527e−2	1.130e−1	3.298e−1	7.636e−2	8.809e−3

first-order perturbation eigenvectors. This observation is particularly true for the mean results. For standard deviation, the results for the three methods are very similar for the case of lower standard deviation of $w(x,\theta)$, although SCCM performs better for few eigenvalues. For the case of higher standard deviation of $w(x,\theta)$, SCCM performs better than the other two methods. Based on relative accuracy of the different methods, it is finally observed that the most promising method aimed at updating eigenvectors from first-order perturbation method is SCCM.

7. Numerical example: thin plate with stochastic properties

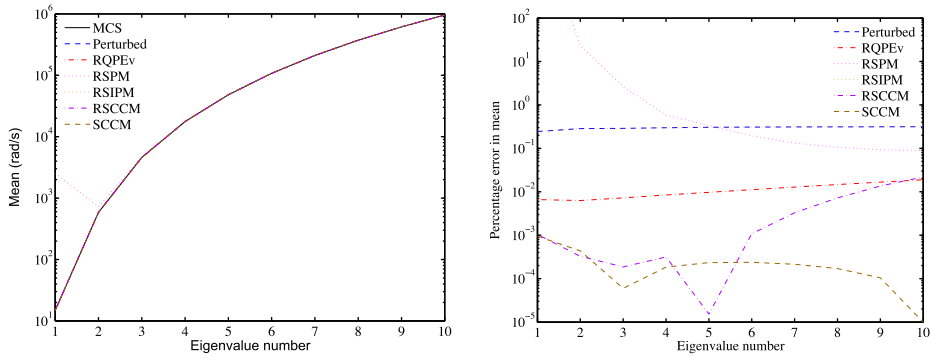
A clamped-free thin plate is considered, where its equation of motion is given by Kirchhoff–Love plate theory

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-2\rho h\ddot{w}+p=0 \quad \text{where} \quad \begin{cases} Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \\ Q_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial y} \end{cases}. \tag{57}$$

In the above equation M_x, M_y, M_{xy} are the bending moments per unit length, and are given by

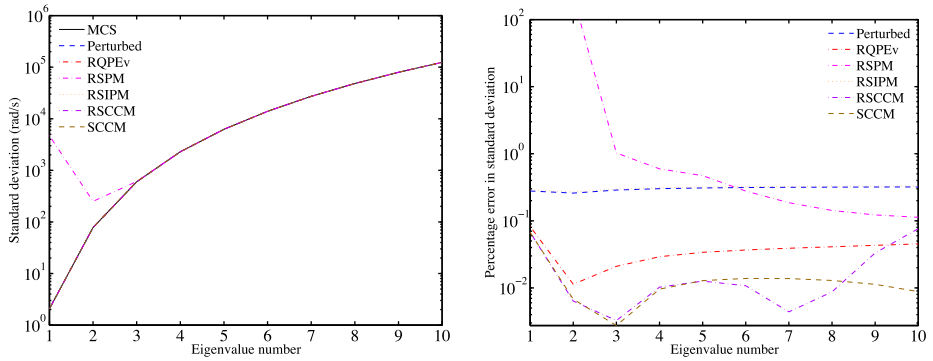
$$\mathcal{M}=\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}=-D(x,y,\theta)\begin{bmatrix} \left[\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right] \\ \left[\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right] \\ (1-\nu)\frac{\partial^2 w}{\partial x\partial y} \end{bmatrix}, \tag{58}$$

where $D(x,y,\theta)$ is the random bending rigidity, ν is Poisson's ratio, p is the force applied to the plate and w is the vertical displacement. The system is shown in Fig. 6, where values of the parameters appearing in the equation are provided. The system is discretized with the finite element method, using a rectangular mesh with $n_x=10$ elements on the length of the plate and $n_y=6$ elements on its width. Details of the method can be found, for example, in the book by Dawe [50]. Uncertainty is introduced in the system by a set of independent Gaussian random variables ξ_i, each one



(a) Mean of the eigenvalues. (b) Percentage error in the mean.

Fig. 4. Mean and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. The standard deviation of the random field is 15% of the mean value.



(a) Standard deviation of the eigenvalues. (b) Percentage error in the standard deviation.

Fig. 5. Standard deviation and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. All the standard deviations of Rayleigh quotients have been obtained with MCS using 5000 samples. The standard deviation of the discretized random field is 15% of the mean value.

affecting a different substructure of the plate. The plate is divided into four substructures and the bending rigidity of each of them is modeled as uncertain using the relationship $D_i = D_0(1+\epsilon\xi_i)$, with $D_0 = Eh^3/12(1-\nu^2)$ and $i = 1,\ldots,4$. The plate, along with the division of four substructures, can be seen in Fig. 6. The first substructure is composed by the first five elements in the x direction and first three in the y direction. The global stiffness matrix of the system can then be represented by $\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{4}\xi_i\mathbf{A}_i$, where $\mathbf{A}_0$ is the deterministic system matrix and $\mathbf{A}_i$ are obtained by considering only the elements of a given substructure. It should be noted that the standard deviation ϵ is included within the coefficient matrices $\mathbf{A}_i$. The effect of two different standard deviations of the random variables is studied. These standard deviations are respectively $\epsilon = 0.07$ and $\epsilon = 0.15$. As formerly, standard deviations of 7% and 15% are considered with the view that they represent the cases of low and high randomness without producing a swapping of the modes. The maximum order of the Hermite polynomial used is 4, so that $P = 70$ polynomials are used as basis functions. Only the results corresponding to the first 10 eigenvalues are considered as the chances of modal overlap increases for higher modes.

Results obtained with the proposed methods are compared against the results from first-order perturbation method from Eq. (4) and MCS with 5000 samples. Convergence of the MCS results is ensured by choosing this number of samples. The moments are obtained by performing MCS on the Rayleigh quotients using PC expansion of eigenvectors and 5000 samples. Figs. 7 and 8 show the mean and standard deviation of the eigenvalues obtained with the first order perturbation and proposed methods for a standard deviation of 7% of the mean of the random variables. In these figures the variation of percentage error of the corresponding quantities with respect to MCS results are also shown. The percentage errors are explicitly reported in Tables 5 and 6 to show the relative accuracies of the different methods.

For the case of a standard deviation of 15% of the mean of the random variables D_i, Figs. 9 and 10 show the mean and standard

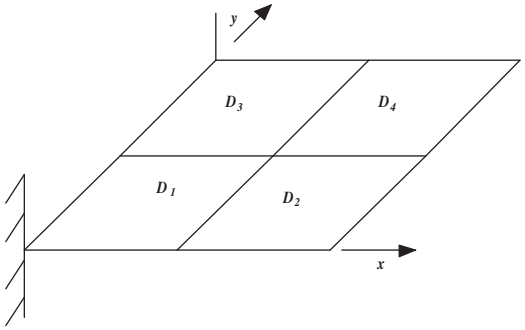
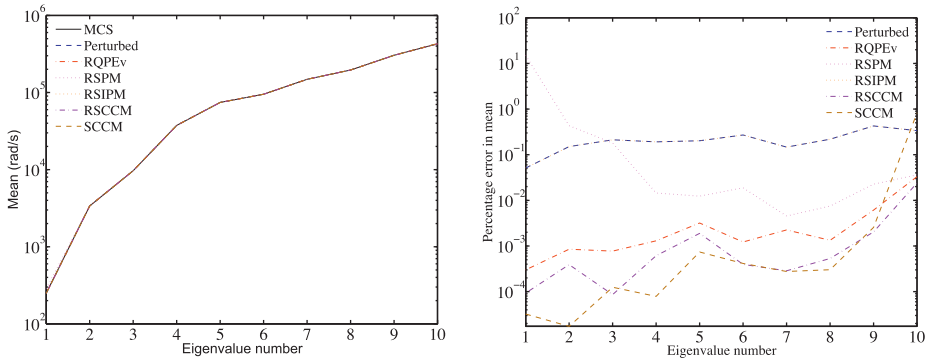
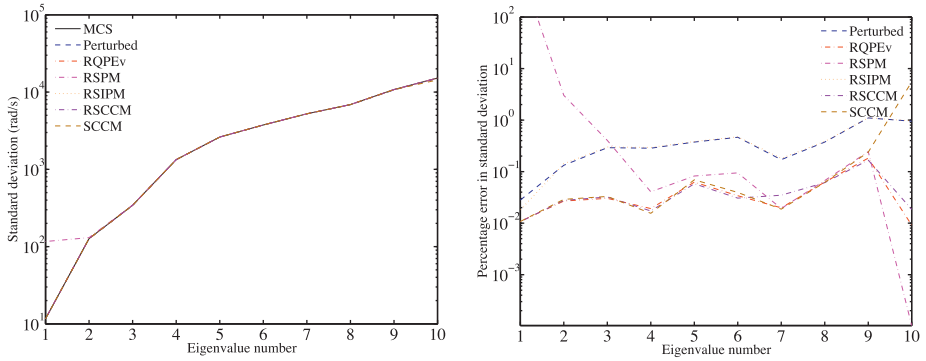


Fig. 6. Kirchhoff–Love plate with $D(x,y,\theta)$ given by $D_i = D_0(1+\epsilon\xi_i)$ for each substructure. The length $L_x = 1.0$ m, width $L_y = 0.6$ m, Poisson's ratio $\nu = 0.3$, modulus of elasticity $E = 200$ GPa, thickness $h = 3$ mm and density $\rho = 7860$ kg/m^3 when the mean system is considered.



(a) Mean of the eigenvalues. (b) Percentage error in the mean.

Fig. 7. Mean and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, first-order perturbation method (Perturbed), Rayleigh quotient using first-order perturbation of eigenvectors (RQPEv) the proposed reduced spectral power method (RSPM), reduced spectral inverse power method (RSIPM), reduced spectral constrained coefficient method (RSCCM) and spectral constrained coefficient method (SCCM). The standard deviation of the discretized random variables is 7% of the mean value.



(a) Standard deviation of the eigenvalues. (b) Percentage error in the standard deviation.

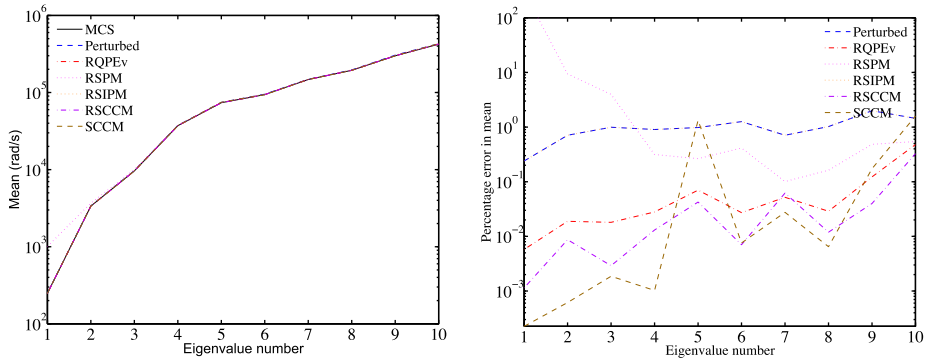
Fig. 8. Standard deviation and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. The standard deviation of the random variables is 7% of the mean value.

Table 5
Percentage errors in mean obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the random variables is 7% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	5.107e−2	3.001e−4	1.509e+1	5.107e−2	9.311e−5	3.232e−5
$\lambda^{(2)}$	1.506e−1	8.505e−4	4.325e−1	1.506e−1	3.916e−4	1.737e−5
$\lambda^{(3)}$	2.120e−1	7.736e−4	1.807e−1	2.120e−1	8.601e−5	1.248e−4
$\lambda^{(4)}$	1.917e−1	1.298e−3	1.449e−2	1.917e−1	6.129e−4	7.883e−5
$\lambda^{(5)}$	2.025e−1	3.185e−3	1.238e−2	2.025e−1	1.888e−3	7.403e−4
$\lambda^{(6)}$	2.704e−1	1.223e−3	1.885e−2	2.704e−1	3.949e−4	4.149e−4
$\lambda^{(7)}$	1.467e−1	2.251e−3	4.536e−3	1.467e−1	2.854e−4	2.762e−4
$\lambda^{(8)}$	2.185e−1	1.333e−3	7.507e−3	2.185e−1	5.322e−4	3.022e−4
$\lambda^{(9)}$	4.268e−1	6.113e−3	2.266e−2	4.268e−1	2.037e−3	2.555e−3
$\lambda^{(10)}$	3.368e−1	3.303e−2	3.620e−2	3.368e−1	2.412e−2	8.304e−1

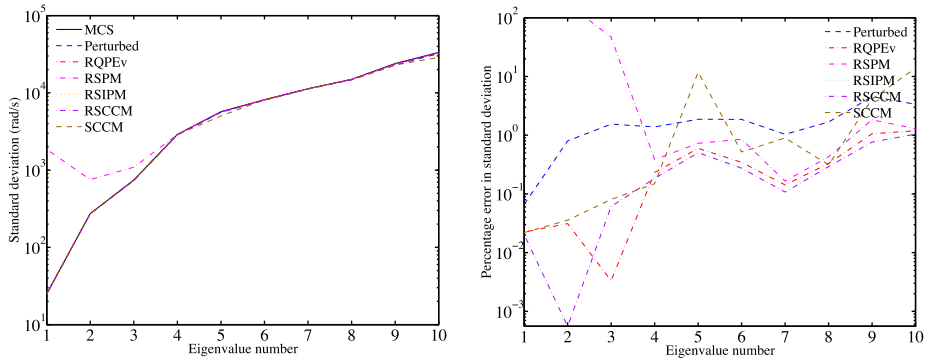
Table 6
Percentage errors in standard deviation obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the random variables is 7% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	2.812e−2	1.082e−2	8.969e+2	1.811e−2	1.071e−2	1.077e−2
$\lambda^{(2)}$	1.326e−1	2.689e−2	3.017e+0	1.426e−1	2.762e−2	2.892e−2
$\lambda^{(3)}$	2.908e−1	3.028e−2	4.031e−1	3.007e−1	3.221e−2	3.302e−2
$\lambda^{(4)}$	2.843e−1	1.918e−2	4.057e−2	2.943e−1	1.723e−2	1.553e−2
$\lambda^{(5)}$	3.745e−1	6.247e−2	8.215e−2	3.844e−1	5.818e−2	6.927e−2
$\lambda^{(6)}$	4.636e−1	3.406e−2	9.456e−2	4.735e−1	3.059e−2	3.949e−2
$\lambda^{(7)}$	1.707e−1	1.984e−2	1.925e−2	1.807e−1	3.490e−2	1.870e−2
$\lambda^{(8)}$	3.748e−1	6.264e−2	6.717e−2	3.848e−1	6.113e−2	6.195e−2
$\lambda^{(9)}$	1.101e+0	1.829e−1	2.444e−1	1.111e+0	1.667e−1	2.387e−1
$\lambda^{(10)}$	9.523e−1	9.147e−3	1.032e−4	9.622e−1	1.849e−2	5.505e+0



(a) Mean of the eigenvalues. (b) Percentage error in the mean.

Fig. 9. Mean and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. The standard deviation of the random variables is 15% of the mean value.



(a) Standard deviation of the eigenvalues. (b) Percentage error in the standard deviation.

Fig. 10. Standard deviation and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, Perturbed, RQPEv, RSPM, RSIPM, RSCCM and SCCM. All the standard deviations of Rayleigh quotients have been obtained with MCS using 5000 samples. The standard deviation of the random variables is 15% of the mean value.

Table 7
Percentage errors in mean obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the random variables is 15% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	2.397e−1	5.774e−3	3.052e+2	2.397e−1	1.143e−3	2.260e−4
$\lambda^{(2)}$	7.094e−1	1.887e−2	9.392e+0	7.094e−1	8.680e−3	6.156e−4
$\lambda^{(3)}$	9.930e−1	1.808e−2	3.961e+0	9.930e−1	2.910e−3	1.862e−3
$\lambda^{(4)}$	9.039e−1	2.799e−2	3.142e−1	9.039e−1	1.306e−2	1.026e−3
$\lambda^{(5)}$	9.804e−1	7.018e−2	2.652e−1	9.804e−1	4.247e−2	1.332e+0
$\lambda^{(6)}$	1.259e+0	2.718e−2	4.131e−1	1.259e+0	7.032e−3	7.573e−3
$\lambda^{(7)}$	7.082e−1	5.194e−2	1.008e−1	7.082e−1	6.192e−2	2.750e−2
$\lambda^{(8)}$	1.021e+0	2.886e−2	1.624e−1	1.021e+0	1.194e−2	6.461e−3
$\lambda^{(9)}$	1.993e+0	1.229e−1	4.831e−1	1.993e+0	3.995e−2	1.732e−1
$\lambda^{(10)}$	1.445e+0	4.774e−1	5.438e−1	1.445e+0	3.263e−1	1.592e+0

Table 8
Percentage errors in standard deviation obtained using the proposed methods for the first ten eigenvalues. The standard deviation of the discretized random field is 15% of the mean value.

Eigenvalue number	Perturbed	RQPEv	RSPM	RSIPM	SRCCM	SCCM
$\lambda^{(1)}$	6.671e−2	2.246e−2	7.305e+3	5.671e−2	2.012e−2	2.201e−2
$\lambda^{(2)}$	8.021e−1	3.122e−2	1.753e+2	8.120e−1	5.569e−4	3.573e−2
$\lambda^{(3)}$	1.544e+0	3.329e−3	4.659e+1	1.554e+0	6.104e−2	8.079e−2
$\lambda^{(4)}$	1.384e+0	2.318e−1	3.848e−1	1.394e+0	1.841e−1	1.491e−1
$\lambda^{(5)}$	1.869e+0	5.965e−1	7.282e−1	1.879e+0	5.037e−1	1.182e+1
$\lambda^{(6)}$	1.858e+0	3.449e−1	8.519e−1	1.868e+0	2.738e−1	5.179e−1
$\lambda^{(7)}$	1.037e+0	1.423e−1	1.655e−1	1.047e+0	1.079e−1	8.992e−1
$\lambda^{(8)}$	1.677e+0	3.303e−1	4.297e−1	1.687e+0	2.914e−1	3.134e−1
$\lambda^{(9)}$	4.542e+0	1.061e+0	1.833e+0	4.551e+0	7.637e−1	4.138e+0
$\lambda^{(10)}$	3.358e+0	1.189e+0	1.288e+0	3.368e+0	1.042e+0	1.382e+1

deviation of the eigenvalues obtained with the first order perturbation and the proposed methods.

The variation of percentage error with different modes with respect to MCS results are also shown. The percentage errors are again explicitly given in Tables 7 and 8 to show the relative accuracies of the different methods. As formerly, in general the errors are larger than the ones from the previous case where the amount of randomness in the system was comparatively lower.

The results observed for the plate problem are quantitatively similar to the ones of the beam problem. That is, results from the RSIPM and the first-order perturbation method are very close, the RSPM leads to the highest error for the smallest eigenvalues, and the SRCCM and the SCCM results are more accurate than the ones obtained with the Rayleigh quotient using first-order perturbation eigenvectors for the mean results and very similar for standard deviation.

8. Conclusions

Four methods, namely the reduced spectral power method (RSPM), the reduced spectral inverse power method (RSIPM), the reduced spectral constrained coefficients method (RSCCM) and the spectral constrained coefficients method (SCCM), have been proposed and compared to improve the accuracy of the approximate solution of the random eigenvalue problem for symmetric matrices. The first two methods are based on approaches used in the context of the deterministic eigenvalue problem to obtain eigenvectors, namely, the power method and the inverse power method. The next two methods are based on the equation of the eigenvalue problem. The deterministic methods to update the eigenvectors are adapted to the stochastic case through a projection in a truncated set of basis functions of a Hilbert space. These basis functions are the multivariate Hermite polynomials used in the PC method. The four proposed methods lead to a Polynomial Chaos (PC) expansion of eigenvectors. Furthermore, a size reduction of the equations is achieved by assuming that, in the PC expansion of eigenvectors, the coefficient vectors belong to the subspace spanned by the deterministic vector and its derivatives with respect to the random variables. Eigenvalues are obtained from the updated eigenvectors using the Rayleigh quotient, where the eigenvectors are given by the PC expansion obtained from one of the four proposed methods. A PC expansion of each eigenvalue is proposed by projecting the Rayleigh quotient into the PC basis functions. Numerical results suggest that the Rayleigh quotient using first-order perturbed eigenvectors outperforms RSPM and RSIPM when considering the smallest eigenvectors. The only methods leading to an improvement of the results would then be SCCM and RSCCM. Numerical results on stochastic beam and plate problems indicate that the reduced spectral power method leads to a better approximation of the larger eigenvalues.

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