



An iterative approach for nonproportionally damped systems

S. Adhikari*

College of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, UK

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ABSTRACT

Modal analysis of nonproportionally damped linear dynamic systems is considered. Dynamic response of such systems can be expressed by a modal series in terms of complex modes. Normally state-space based methods or approximate perturbation methods are necessary for the computation of complex modes. In this paper, an iterative method to calculate complex modes from classical normal modes for general linear systems is proposed. A simple numerical algorithm is developed to implement the iterative method. The new method is illustrated using a numerical example.

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1. Introduction

The equation of motion of an n -degree-of-freedom linear viscously damped system can be expressed by coupled differential equations as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t). \quad (1)$$

Here $\mathbf{u}(t) \in \mathbb{R}^n$ is the displacement vector, $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector, \mathbf{M} , \mathbf{K} , $\mathbf{C} \in \mathbb{R}^{n \times n}$ are respectively the mass matrix, stiffness and the viscous damping matrix. In general \mathbf{M} is a positive definite symmetric matrix, \mathbf{C} and \mathbf{K} are non-negative definite symmetric matrices. The natural frequencies ($\omega_j \in \mathbb{R}$) and the mode shapes ($\mathbf{x}_j \in \mathbb{R}^n$) of the corresponding undamped system can be obtained (Meirovitch, 1997) by solving the matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j, \quad \forall j = 1, 2, \dots, n. \quad (2)$$

The undamped eigenvectors satisfy an orthogonality relationship over the mass and stiffness matrices, that is

$$\mathbf{x}_k^T \mathbf{M}\mathbf{x}_j = \delta_{kj} \quad (3)$$

and

$$\mathbf{x}_k^T \mathbf{K}\mathbf{x}_j = \omega_j^2 \delta_{kj}, \quad \forall k, \quad j = 1, 2, \dots, n \quad (4)$$

where δ_{kj} is the Kronecker delta function. We construct the modal matrix

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \in \mathbb{R}^n. \quad (5)$$

The modal matrix can be used to diagonalize system (1) provided the damping matrix \mathbf{C} is simultaneously diagonalizable

with \mathbf{M} and \mathbf{K} . This condition, known as the proportional damping, originally introduced by Lord Rayleigh (Rayleigh, 1877) in 1877, is still in wide use today. The mathematical condition for proportional damping can be obtained from the commutative behaviour of the system matrices (Caughey and O'Kelly, 1965). This can be expressed as $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$ or equivalently $\mathbf{C} = \mathbf{M}\mathbf{f}(\mathbf{M}^{-1}\mathbf{K})$ as shown by Adhikari (2006). The concern of this paper is when this condition is not met, the most likely case for many practical applications. In particular, due to the recent developments in actively controlled structures and the increasing use of composite and smart materials, the need to consider general non-proportionally damped linear dynamic systems is more than ever before.

For nonproportionally damped systems, the modal damping matrix

$$\mathbf{C}' = \mathbf{X}^T \mathbf{C} \mathbf{X} \quad (6)$$

is not a diagonal matrix. Such problems can be solved using a spectral approach similar to the undamped or proportionally damped system by transforming Eq. (1) into a state-space form (Meirovitch, 1997). The state-space approach is not only computationally more expensive, it also lacks the physical insight provided by the classical normal mode based approach. Therefore, many authors have developed approximate methods in the original space. Rayleigh (1877) proposed a perturbation method which forms the basis of many contemporary approximation methods (Adhikari, 1999a; ElBeheiry, 2009; Adhikari, 1999b). It is now known that either the frequency separation between the normal modes (Hasselsman, 1976), often known as 'Hasselsman's criteria', or some form of diagonal dominance (Shahruz and Ma, 1988; Morzfeld et al., 2009; Adhikari, 2004; Morzfeld et al., 2008) in the modal damping matrix \mathbf{C}' is sufficient for neglecting modal coupling. In a recent work, Udawadia (2009) proved that for systems with non-repeated eigenvalues, the best

* Tel.: +44 01792 602088; fax: +44 01792 295676.

E-mail address: S.Adhikari@swansea.ac.uk

approximation of a diagonal modal damping matrix is simply to consider the diagonal of the \mathbf{C} matrix.

The eigenvalue problem corresponding to system (1) can be expressed as

$$[s_j^2 \mathbf{M} + s_j \mathbf{C} + \mathbf{K}] \mathbf{u}_j = 0, \quad \forall j = 1, 2, \dots, 2n \tag{7}$$

where $s_j \in \mathbb{C}$ are the eigenvalues and $\mathbf{u}_j \in \mathbb{C}^{2n}$ are the eigenvectors. Comprehensive details on this type of quadratic eigenvalue problem can be found in Tisseur and Meerbergen (2001). Since \mathbf{M} , \mathbf{K} and \mathbf{C} are all real matrices, the eigensolutions are either real or they appear in the complex conjugate pairs. In this paper we consider complex conjugate eigensolutions only as for stable systems such eigenvalues are of great practical importance. Using the eigensolutions, the frequency response function (FRF) can be obtained (see for example (Adhikari, 1999a; Tisseur and Meerbergen, 2001)) as

$$\mathbf{H}(i\omega) = \sum_{j=1}^n \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{i\omega - s_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{i\omega - s_j^*} \right] \text{ where} \tag{8}$$

$$\gamma_j = \frac{1}{\mathbf{u}_j^T [2s_j \mathbf{M} + \mathbf{C}] \mathbf{u}_j}.$$

Here $(\bullet)^*$ denotes complex conjugation and $(\bullet)^T$ denotes matrix transposition. This equation shows that if the complex eigensolutions s_j and \mathbf{u}_j can be obtained efficiently, the dynamic response can be obtained exactly using Eq. (8). In this paper an iterative approach is developed to obtain the complex eigensolutions of nonproportionally damped systems from the undamped eigensolutions.

2. Iterative approach for the eigensolutions

Ibrahimbegovic and Wilson (1989) have developed a procedure for analyzing non-proportionally damped systems using a subspace with a vector basis generated from the mass and stiffness matrices. Their approach avoids the use of complex eigensolutions. In the time domain, an iterative approach for solving the coupled equations was developed by Udwadia and Esfandiari (1990) based on updating the forcing term appropriately. In the method proposed here, we obtain the complex modes and complex frequencies in an iterative manner.

For distinct undamped eigenvalues $(\omega_l^2, \mathbf{x}_l, \forall l = 1, \dots, n)$, form a complete set of vectors. For this reason, \mathbf{u}_j can be expanded as a complex linear combination of \mathbf{x}_l . Thus, an expansion of the form

$$\mathbf{u}_j = \sum_{l=1}^n \alpha_l^{(j)} \mathbf{x}_l \tag{9}$$

may be considered. Without any loss of generality, we can assume that $\alpha_j^{(j)} = 1$ (normalization) which leaves us to determine $\alpha_l^{(j)}, \forall l \neq j$. Substituting the expansion of \mathbf{u}_j into the eigenvalue equation (7), one obtains the approximation error for the j -th mode as

$$\varepsilon_j = \sum_{l=1}^n s_j^2 \alpha_l^{(j)} \mathbf{M} \mathbf{x}_l + s_j \alpha_l^{(j)} \mathbf{C} \mathbf{x}_l + \alpha_l^{(j)} \mathbf{K} \mathbf{x}_l. \tag{10}$$

We use a Galerkin approach to minimize this error by viewing the expansion (9) as a projection in the basis functions $\mathbf{x}_l \in \mathbb{R}^n, \forall l = 1, 2, \dots, n$. Therefore, we make the error orthogonal to the basis functions, that is

$$\varepsilon_j \perp \mathbf{x}_k \text{ or } \mathbf{x}_k^T \varepsilon_j = 0, \quad \forall k = 1, 2, \dots, n. \tag{11}$$

Using the orthogonality property of the undamped eigenvectors described by (3) and (4) one obtains

$$s_j^2 \alpha_k^{(j)} + s_j \sum_{l=1}^n \alpha_l^{(j)} C'_{kl} + \omega_k^2 \alpha_k^{(j)} = 0, \quad \forall k = 1, \dots, n \tag{12}$$

where $C'_{kl} = \mathbf{x}_k^T \mathbf{C} \mathbf{x}_l$ are the elements of the modal damping matrix \mathbf{C}' defined in Eq. (6). The j -th equation of this set obtained by setting $k=j$ can be written as

$$(s_j^2 + s_j C'_{jj} + \omega_j^2) \alpha_j^{(j)} + s_j \sum_{l \neq j}^n \alpha_l^{(j)} C'_{jl} = 0. \tag{13}$$

Recalling that $\alpha_j^{(j)} = 1$ and \mathbf{C}' is a symmetric matrix, this equation can be rewritten as

$$s_j^2 + s_j \underbrace{\left(C'_{jj} + \sum_{l \neq j}^n \alpha_l^{(j)} C'_{lj} \right)}_{\gamma_j} + \omega_j^2 = 0 \tag{14}$$

where

$$\gamma_j = C'_{jj} + \mathbf{b}_j^T \mathbf{a}_j \tag{15}$$

$$\mathbf{b}_j = \{C'_{1j}, C'_{2j}, \dots, \{j\text{-th term deleted}\}, \dots, C'_{nj}\}^T \in \mathbb{R}^{(n-1)} \tag{16}$$

and

$$\mathbf{a}_j = \{\alpha_1^{(j)}, \alpha_2^{(j)}, \dots, \{j\text{-th term deleted}\}, \dots, \alpha_n^{(j)}\}^T \in \mathbb{C}^{(n-1)} \tag{17}$$

The vector \mathbf{a}_j is unknown and can be obtained by excluding the $j = k$ case in Eq. (12). Excluding this case one has

$$s_j^2 \alpha_k^{(j)} + s_j \left(C'_{kj} + \alpha_k^{(j)} C'_{kk} + \sum_{l \neq k \neq j}^n \alpha_l^{(j)} C'_{kl} \right) + \omega_k^2 \alpha_k^{(j)} = 0, \text{ or} \tag{18}$$

$$(s_j^2 + \omega_k^2 + C'_{kk}) \alpha_k^{(j)} + s_j \sum_{l \neq k \neq j}^n C'_{kl} \alpha_l^{(j)} = -s_j C'_{kj}, \quad \forall k = 1, \dots, n; \neq j.$$

These equations can be combined into a matrix form as

$$[\mathbf{P}_j - \mathbf{Q}_j] \mathbf{a}_j = \mathbf{b}_j. \tag{19}$$

In the above equation, the vectors \mathbf{a}_j and \mathbf{b}_j have been defined before. The matrices \mathbf{P}_j and \mathbf{Q}_j are defined as

$$\mathbf{P}_j = \text{diag} \left[\frac{s_j^2 + s_j C'_{11} + \omega_1^2}{-s_j}, \dots, \{j\text{-th term deleted}\}, \dots, \right. \tag{20}$$

$$\left. \times \frac{s_j^2 + s_j C'_{NN} + \omega_n^2}{-s_j} \right] \in \mathbb{C}^{(n-1) \times (n-1)},$$

and

$$\mathbf{Q}_j = \begin{bmatrix} 0 & C'_{12} & \dots & \{j\text{-th term deleted}\} & \dots & C'_{1n} \\ C'_{21} & 0 & \vdots & \vdots & \vdots & C'_{2n} \\ \vdots & \vdots & \vdots & \{j\text{-th term deleted}\} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C'_{n1} & C'_{n2} & \dots & \{j\text{-th term deleted}\} & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}. \tag{21}$$

From Eq. (19), \mathbf{a}_j should be obtained by solving the set of linear equations. Because \mathbf{P}_j is a diagonal matrix, one way to do this is

by using the Neumann expansion method (Adhikari, 1999a). Using the Neumann expansion we have

$$\mathbf{a}_j = [\mathbf{I}_{n-1} - \mathbf{P}_j^{-1} \mathbf{Q}_j]^{-1} \{\mathbf{P}_j^{-1} \mathbf{b}_j\} = [\mathbf{I}_{n-1} + \mathbf{R}_j + \mathbf{R}_j^2 + \mathbf{R}_j^3 + \dots] \mathbf{a}_{j0} \quad (22)$$

where \mathbf{I}_{n-1} is a $(n-1) \times (n-1)$ identity matrix,

$$\mathbf{R}_j = \mathbf{P}_j^{-1} \mathbf{Q}_j \in \mathbb{C}^{(n-1) \times (n-1)} \quad \text{and} \quad \mathbf{a}_{j0} = \mathbf{P}_j^{-1} \mathbf{b}_j \in \mathbb{C}^{(n-1)}. \quad (23)$$

Because \mathbf{P}_j is a diagonal matrix, its inversion can be carried out analytically and subsequently the closed-form expressions of the elements of \mathbf{a}_j can be obtained. Keeping one term in the series (22), the first-order expression of the elements of \mathbf{a}_j can be obtained as

$$\mathbf{a}_j \equiv \{\alpha_k^{(j)}\}_{\forall k \neq j} = \frac{s_j C'_{kj}}{\omega_k^2 + s_j^2 + s_j C'_{kk}}. \quad (24)$$

Similarly, the second-order expression of the elements of \mathbf{a}_j can be obtained as

$$\begin{aligned} \mathbf{a}_j \equiv \{\alpha_k^{(j)}\}_{\forall k \neq j} &= \frac{s_j C'_{kj}}{\omega_k^2 + s_j^2 + s_j C'_{kk}} \\ &+ \sum_{\substack{l=1 \\ l \neq j \neq k}}^n \frac{s_j^2 C'_{kl} C'_{lj}}{(\omega_k^2 + s_j^2 + s_j C'_{kk}) (\omega_l^2 + s_j^2 + s_j C'_{ll})}. \end{aligned} \quad (25)$$

The vector \mathbf{a}_j obtained using this way can be substituted back in the expression of the eigenvalues in (14), which in turn can be solved for s_j as

$$s_j = \frac{-\gamma_j \pm i \sqrt{4\omega_j^2 - \gamma_j^2}}{2}. \quad (26)$$

However, the vector \mathbf{a}_j is also a function of s_j . As a result γ_j becomes a function of s_j . This forms the basics of the iterative approach as from Eq. (26) one can write

$$s_j^{(r+1)} = \frac{-\gamma_j(s_j^{(r)})}{2} \pm i \sqrt{\frac{\omega_j^2 - \gamma_j^2(s_j^{(r)})}{4}}; \quad r = 0, 1, 2, \dots \quad (27)$$

For every iteration step, the vector \mathbf{a}_j gets updated based on new values of s_j using Eq. (24) or (25) depending on the order of terms retained in the series (22). The iteration can be started with the equivalent proportional damping assumption (Udwadia, 2009), namely

$$s_j^{(0)} = \frac{-C'_{jj}}{2} \pm i \sqrt{\frac{\omega_j^2 - C'_{jj}}{4}} \quad (28)$$

The iteration can be stopped when the successive values of s_j or \mathbf{a}_j do not change significantly. Once the final values of $\alpha_k^{(j)}$, $\forall k$ are obtained, the j -th complex mode \mathbf{u}_j can be obtained from the series (9).

The necessary and sufficient conditions for the convergence of the proposed method are difficult to obtain. Below we give a sufficient condition.

Proposition 1. A sufficient condition for the convergence of the proposed iterative method is that \mathbf{C}' is a diagonally dominant matrix.

Proof. During the iteration process, the value of s_j changes for different iteration steps. We aim to derive the condition for the convergence of series (22) for an arbitrary value of s_j . This will guarantee the convergence of the iterative method, no matter what the value of s_j . The complex matrix power series (22) converges if, and only if, for all the eigenvalues $\sigma_l^{(j)}$ of the matrix \mathbf{R}_j , the inequality

$|\sigma_l^{(j)}| < 1$ holds. Although this condition is both necessary and sufficient, checking convergence for all $j=1, \dots, n$ is not feasible for every iteration step. So we look for a sufficient condition which is relatively easy to check and which ensures convergence for all $j=1, \dots, n$.

For an arbitrary r -th iteration, let us denote the matrix \mathbf{R}_j defined in Eq. (23) as $\mathbf{R}_j^{(r)}$. Suppose the value of s_j for the r th iteration step is $s_j^{(r)}$. The kl -th element of the matrix $\mathbf{R}_j^{(r)}$ can be obtained as

$$\mathbf{R}_j^{(r)} = \frac{-s_j^{(r)} C'_{kl} (1 - \delta_{kl})}{\omega_k^2 + s_j^{(r)2} + s_j^{(r)} C'_{kk}}, \quad \forall k, l \neq j \quad (29)$$

Since a matrix norm is always greater than or equal to its maximum eigenvalue, it follows from the inequality $|\sigma_l^{(j)}| < 1$ that the convergence of the series is guaranteed if $\|\mathbf{R}_j^{(r)}\| < 1$. Writing the sum of absolute values of entries of $\mathbf{R}_j^{(r)}$ results in the following inequality as the required sufficient condition for the convergence

$$\sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\substack{l=1 \\ l \neq j}}^n \left| \frac{s_j^{(r)} C'_{kl}}{\omega_k^2 + s_j^{(r)2} + s_j^{(r)} C'_{kk}} \right| (1 - \delta_{lk}) < 1 \quad (30)$$

Dividing both the numerator and denominator by $s_j^{(r)}$, the above inequality can be written as

$$\sum_{k=1}^n \sum_{\substack{l=1 \\ l \neq i \neq k}}^n \frac{|C'_{kl}|}{|1/s_j^{(r)}(\omega_k^2 + s_j^{(r)2}) + C'_{kk}|} < 1 \quad (31)$$

Taking the maximum for all $k \neq j$, this condition can further be represented as

$$\max_{k \neq j} \frac{\sum_{l=1}^n |C'_{kl}|}{|1/s_j^{(r)}(\omega_k^2 + s_j^{(r)2}) + C'_{kk}|} < 1 \quad (32)$$

It is clear that (32) always holds if

$$\sum_{\substack{l=1 \\ l \neq j \neq k}}^n |C'_{kl}| < |C'_{kk}|, \quad \forall k \neq j \quad (33)$$

which in turn implies that, for all $j=1, \dots, n$, the inequality $\|\mathbf{R}_j^{(r)}\| < 1$ holds if \mathbf{C}' is a diagonally dominant matrix. It is important to note that the diagonal dominance of \mathbf{C}' is only a sufficient condition and the lack of it does not necessarily prevent convergence of the proposed iterative method. \square

3. Summary of the algorithm

In this section we propose a simple iterative algorithm to implement the idea developed in the previous section. We select a tolerance between the difference of the successive values of s_j , denoted by ε_m . A small value, say $\varepsilon_m = 0.001$ can be selected for numerical calculations. Considering that the undamped eigen-solutions (ω_j and \mathbf{x}_j) and the modal damping matrix \mathbf{C}' is known, the complex eigensolutions (s_j and \mathbf{u}_j) can be obtained using the following iterative algorithm:

Table 1

The complex eigenvalues of the system obtained using the proposed method is compared with the exact state-space method and the first-order perturbation method. The numbers in the parenthesis represent the percentage error.

Eigenvalue number	State-space (exact)	First-order perturbation	Proposed iterative method
1	-0.0103 + 0.6298i	-0.0125 + 0.6249i (0.8552)	-0.0114 + 0.6306i, (0.2018), 2 iterations
2	-0.0478 + 1.2407i	-0.1458 + 1.1547i (10.5023)	-0.0468 + 1.2379i, (0.2428), 6 iterations
3	-0.5252 + 1.2890i	-0.4250 + 1.5087i (17.3472)	-0.4314 + 1.3687i, (8.8436), 8 iterations

```

for  $j=1,2,\dots,n$  do
  Initialize  $\varepsilon = 100, r=0$ 
   $s_j^{(r)} = \frac{-C'_{jj}}{2} \pm i \sqrt{\frac{\omega_j^2 - C'^2}{4}}$ 
   $\mathbf{b}_j = \{C'_{1j}, C'_{2j}, \dots, [j\text{-th term deleted}], \dots, C'_{nj}\}^T$ 
  while  $\varepsilon > \varepsilon_m$  do
     $\mathbf{a}_j(s_j^{(r)}) \equiv \{\alpha_k^{(j)}\}_{k \neq j} = \frac{s_j^{(r)} C'_{kj}}{\omega_k^2 + s_j^{(r)2} + s_j^{(r)} C'_{kk}}$ 
     $\gamma_j = C'_{jj} + \mathbf{b}_j^T \mathbf{a}_j(s_j^{(r)})$ 
     $s_j^{(r+1)} = \frac{-\gamma_j(s_j^{(r)})}{2} \pm i \sqrt{\frac{\omega_j^2 - \gamma_j^2(s_j^{(r)})}{4}}$ 
     $\varepsilon = \frac{|s_j^{(r+1)} - s_j^{(r)}|}{|s_j^{(r)}|}$ 
     $r = r + 1$ 
  end while
   $\mathbf{u}_j = \sum_{k=1}^n \alpha_k^{(j)} \mathbf{x}_k$ 
end for
    
```

The algorithm is outlined for the first-order expression of $\alpha_k^{(j)}$ given by Eq. (24). However, the extension to the second or higher order expressions is straightforward. One simply needs to change the expression of $\mathbf{a}_j(s_j^{(r)})$ in this algorithm. If the higher-order terms are used, then less number of steps in the iteration are needed. Once the complex eigensolutions s_j and \mathbf{u}_j are obtained using this method for all j , the dynamic response such as the frequency response function can be obtained exactly using Eq. (8).

The computational complexity to solve algebraic eigenvalue problems scale cubically with the dimension (Wilkinson, 1988). Therefore, an estimation of the order of calculations needed to solve the eigenvalue problem of a matrix of size n is $O(n^3)$ for large n . This is the computational time for an undamped system of dimension n . Suppose the computational cost for the iteration of each eigen-solution pair (s_j and \mathbf{u}_j) is proportional to c_l . The value of c_l will be higher if more number of iterations are used. The total cost of the iteration would be in order nc_l . Adding these two, the order of calculations needed to approximate the eigensolutions with the

proposed method is $O(n^3 + C_l n)$. For a general nonproportionally damped system, the size of the state-space matrix is $2n$. Therefore, the computational time is in the order $O((2n)^3) = O(8n^3)$. It is important to note that the order of computation related to the iteration scale linearly with n as opposed to cubically in the case of direct state-space method. Therefore when n becomes large as expected for practical problems, the proposed method has a clear advantage as $O(8n^3) > O(n^3 + C_l n)$. The idealized computational efficiency can be obtained as

$$\frac{8n^3}{n^3 + C_l n} \rightarrow 8 \text{ (when } n \rightarrow \infty) \tag{34}$$

Next we illustrate this new method using a numerical example.

4. Numerical illustration

We consider a three-degree-of-freedom system with the mass, stiffness and damping matrices given by

$$\mathbf{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} \quad \text{and}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.75 & -1.75 \\ 0 & -1.75 & 1.75 \end{bmatrix}. \tag{35}$$

The undamped eigenvalues and eigenvectors are obtained as

$$\{\omega_1, \omega_2, \omega_3\} = \{0.6249, 1.1547, 1.5087\} \quad \text{and}$$

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & -1 \end{bmatrix}. \tag{36}$$

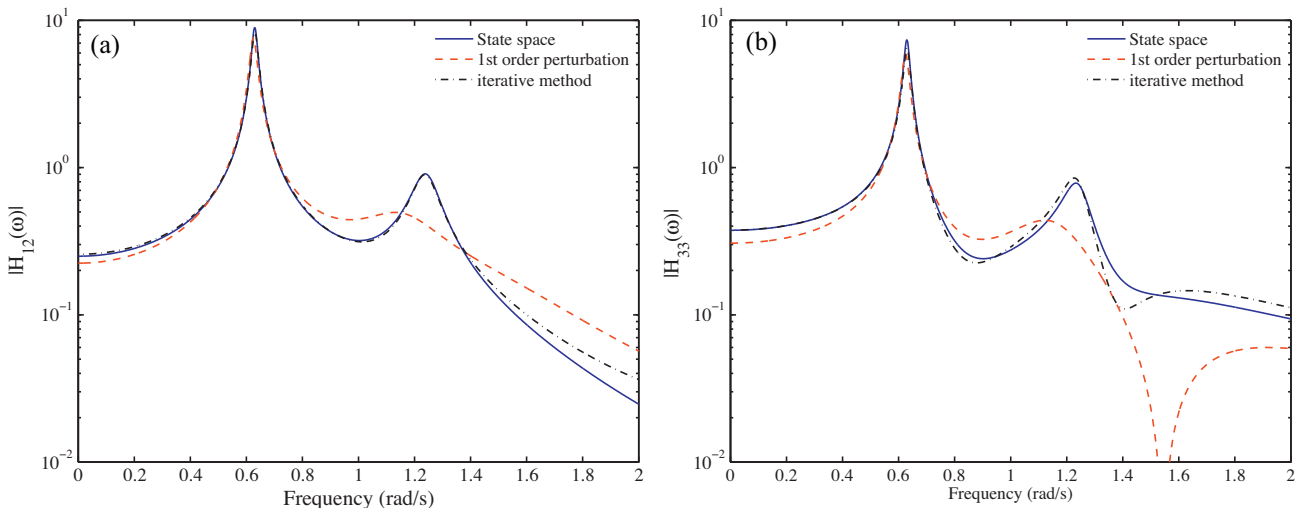


Fig. 1. A cross-FRF and a driving-point FRF of the system calculated using the exact state-space method, the first-order perturbation method and the proposed iterative method.

Note that the last two undamped eigenvalues are very close and therefore one would expect significant modal coupling. The eigenvalues obtained using the proposed iterative method is compared with the exact state-space method and the first-order perturbation method in Table 1. We have used the first-order expression of $\alpha_k^{(j)}$ given by Eq. (24) and considered the error tolerance to be $\varepsilon_m = 0.001$. For the three eigenvalues, respectively 2, 6 and 8 iterations are used. The first-order perturbation results are obtained from Eq. (28). The percentage errors are calculated with respect to the exact state-space results. Due to the high modal coupling, the first-order perturbation results corresponding to modes 2 and 3 have considerable errors (10.5023% and 17.3472%, respectively). Using the proposed iterative method, errors corresponding to these two modes reduce to 0.2428% and 8.8436% using 6 and 8 iterations, respectively. The computational time (obtained using Matlab's tic and toc command) for the state-space method is 0.8912×10^{-2} s. The computational times for the first-order perturbation and the proposed iterative method are respectively 0.2367×10^{-2} s and 0.2774×10^{-2} s. This implies that for this problem, the proposed approach is about 3.2 times more faster compared to the state-space method. As discussed before, theoretically the maximum computational efficiency can be upto 8 times. However, this can only be realized for very large systems. In Fig. 1 results two typical frequency response functions of the system calculated from Eq. (8) using the three methods are shown. The modal damping factors for the three modes are respectively 0.0164, 0.0386 and 0.4074. Due to the high damping in the third mode, the resonance peak is not visible. As expected, the first-order perturbation perform poorly around the second and third mode due to the significant modal coupling. From these results the relative accuracy of the proposed iterative method can be observed.

5. Conclusions

Due to the recent developments in actively controlled structures and the increasing use of composite and smart materials, there is a renewed interest to consider general nonproportionally damped linear dynamic systems. State-space based methods are normally used to address such problems. Such methods are computationally more expensive and often do not give the physical insight compared to the classical normal mode based method. In this paper a new iterative method has been proposed to obtain the complex eigensolutions of a general nonproportionally damped system from the undamped eigensolutions. It is assumed that all the eigenvalues are distinct and appear in complex conjugate pairs. The proposed method exploits a mathematical construction where complex eigenvalues and eigenvectors can be updated from their previous values in an iterative manner. A sufficient condition for the convergence of the proposed iterative method is derived. A simple algorithm is proposed to implement this method. The computational complexity of the proposed method is discussed

and compared with the state-space method. The applicability of the proposed method is investigated using an example with a high degree of modal coupling. Acceptable accuracy has been observed. Using the iterative method developed here, it is possible to obtain the eigenvalues, eigenvectors and consequently the dynamic response of nonproportionally damped systems by post-processing of the undamped eigenvalues and eigenvectors, which in turn can be obtained using a general purpose finite element software. Future work is necessary to extend this method to systems with repeated eigenvalues.

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