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Derivatives of Complex Eigenvectors Using Nelson's Method

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Nomenclature

c_i, d_i	=	scalars used to calculate the eigenvector derivatives
$\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K}$	=	mass, damping, and stiffness matrices
s_i	=	system eigenvalues
$\boldsymbol{u}_i, \boldsymbol{v}_i$	=	left and right system eigenvectors
$\boldsymbol{x}_i, \boldsymbol{y}_i$	=	vectors used to calculate the eigenvector derivatives
θ	=	system parameter
$\{\ \}_e$	=	eth element of a vector

Introduction

THE calculation of derivatives of natural frequencies and mode shapes with respect to model parameters is vital for design optimization, model updating, fault detection, and many other applications.[1,2] The methods to calculate these derivatives are well established for undamped structures. Fox and Kapoor[3] calculated the derivative of the eigenvectors by expressing these derivatives as a linear combination of the undamped eigenvectors. Nelson[4] introduced the approach, extended in this Note, where only the eigenvector of interest was required. Adhikari[1] extended the method of Fox and Kapoor to systems with nonproportional damping using the mass, damping, and stiffness matrices directly. Lee et al.[5] calculated the eigenvector derivatives of self-adjoint systems using a similar approach to Nelson. This Note extends Nelson's method to nonproportionally damped systems with complex modes. This method has the great advantage that only the eigenvector of interest is required. The method proposed by Adhikari[1] obtained the eigenvector derivative as a linear combination of all of the eigenvectors. For large-scale structures, with many degrees of freedom, obtaining all of the eigenvectors is a computationally expensive task. Both self-adjoint and non-self-adjoint systems are considered.

The eigenvalues and corresponding right and left eigenvectors of the standard equations of motion in structural dynamics in second-order form are given by the solutions of

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\boldsymbol{u}_i = 0 \tag{1}$$

$$\boldsymbol{v}_i^T\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right) = 0 \tag{2}$$

Often the structural matrices are symmetric, but here we allow the possibility that the matrices are asymmetric. For the self-adjoint case (symmetric matrices) the left and right eigenvectors are equal, $\boldsymbol{u}_i = \boldsymbol{v}_i$. Also, the eigenvalues and eigenvectors must occur in complex conjugate pairs because the structural matrices are real. Furthermore, in this Note we assume that the eigenvalues are distinct.

The eigenvectors are not unique in the sense that any scalar (complex) multiple of an eigenvector is also an eigenvector. There are numerous ways of introducting a normalization to ensure uniqueness. For undamped systems mass normalization is the most common. A useful normalization for damped systems is

$$\boldsymbol{v}_i^T[s_i\boldsymbol{M} + (1/s_i)\boldsymbol{K}]\boldsymbol{u}_i = \boldsymbol{v}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{u}_i = 1 \tag{3}$$

which ensures that the eigenvectors have the equivalent scaling to the measured eigenvectors.[6,7] One disadvantage of this scaling is that "real" modes, produced by proportionally damped models, are multiplied by a complex scalar.

Eigenvalue Derivatives

Adhikari[1] obtained the derivatives of the eigenvalues for the self-adjoint case. Here we extend the approach to the general case. Differentiating Eq. (1) with respect to the parameter θ gives

$$\left(s_i^2\frac{\partial\boldsymbol{M}}{\partial\theta} + s_i\frac{\partial\boldsymbol{D}}{\partial\theta} + \frac{\partial\boldsymbol{K}}{\partial\theta}\right)\boldsymbol{u}_i + (2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{u}_i\frac{\partial s_i}{\partial\theta} + \left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\frac{\partial\boldsymbol{u}_i}{\partial\theta} = 0 \tag{4}$$

Equation (4) is now premultiplied by $\boldsymbol{v}_i^T$. The third term is then zero from Eq. (2), and the scaling makes the coefficient of the eigenvalue derivative unity. Thus

$$\frac{\partial s_i}{\partial\theta} = -\boldsymbol{v}_i^T\left(s_i^2\frac{\partial\boldsymbol{M}}{\partial\theta} + s_i\frac{\partial\boldsymbol{D}}{\partial\theta} + \frac{\partial\boldsymbol{K}}{\partial\theta}\right)\boldsymbol{u}_i \tag{5}$$

Eigenvector Derivatives: Self-Adjoint Case

In the self-adjoint case the left and right eigenvectors are equal, and the eigenvector derivative satisfies, from Eq. (4),

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\frac{\partial\boldsymbol{u}_i}{\partial\theta} = \boldsymbol{h}_i \tag{6}$$

where the vector $\boldsymbol{h}_i$ consists of the first two terms in Eq. (4) and all of these quantities are now known. Equation (6) cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues this matrix has a null space of dimension one. Following Nelson's approach, the eigenvector derivative is written as

$$\frac{\partial\boldsymbol{u}_i}{\partial\theta} = \boldsymbol{x}_i + c_i\boldsymbol{u}_i \tag{7}$$

where $\boldsymbol{x}_i$ and c_i have to be determined. These quantities are not unique because any multiple of the eigenvector can be added to $\boldsymbol{x}_i$. A convenient choice is to identify the element of maximum magnitude in $\boldsymbol{u}_i$ and make the corresponding element in $\boldsymbol{x}_i$ equal to zero. Although other elements of $\boldsymbol{x}_i$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Substituting Eq. (7) into Eq. (6) gives

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\boldsymbol{x}_i = \boldsymbol{F}_i\boldsymbol{x}_i = \boldsymbol{h}_i \tag{8}$$

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Table 1 Eigenvalues and eigenvectors and their derivatives for the example

Quantity	Pair 1	Derivatives	Pair 2	Derivatives
Eigenvalues	-1.728 ± 3.175j	0.0441 ± 0.0441j	-0.272 ± 3.903j	-0.0441 ± 0.0758j
Right eigenvectors	0.318 ∓ 0.200j	0.0113 ∓ 0.00279j	0.0702 ∓ 0.204j	-0.0114 ∓ 0.00502j
	-0.123 ± 0.298j	0.0172 ± 0.00275j	0.257 ∓ 0.148j	0.00717 ∓ 0.000506j
Left eigenvectors	0.318 ∓ 0.200j	0.0113 ∓ 0.00279j	0.117 ∓ 0.340j	-0.0190 ∓ 0.00836j
	-0.0741 ± 0.179j	0.0103 ± 0.00165j	0.257 ∓ 0.148j	0.00717 ∓ 0.000506j

This can be solved, including the constraint on the zero element of $\boldsymbol{x}_i$, by solving the equivalent problem

$$\begin{bmatrix} \boldsymbol{F}_{i11} & 0 & \boldsymbol{F}_{i13} \\ 0 & 1 & 0 \\ \boldsymbol{F}_{i31} & 0 & \boldsymbol{F}_{i33} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_{i1} \\ x_{i2} \quad (=0) \\ \boldsymbol{x}_{i3} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{h}_{i1} \\ 0 \\ \boldsymbol{h}_{i3} \end{Bmatrix} \tag{9}$$

where $\boldsymbol{F}_i$ is defined in Eq. (8) and has the row and column corresponding to the zeroed element of $\boldsymbol{x}_i$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices and hence is computationally efficient.

It only remains to compute the scalar constant c_i to obtain the eigenvector derivative. For this the normalization equation must be used. Differentiating Eq. (3) with $\boldsymbol{v}_i = \boldsymbol{u}_i$, substituting Eq. (7), and rearranging produces

$$c_i = -\boldsymbol{u}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{x}_i - \frac{1}{2}\boldsymbol{u}_i^T\left(2\boldsymbol{M}\frac{\partial s_i}{\partial\theta} + 2s_i\frac{\partial \boldsymbol{M}}{\partial\theta} + \frac{\partial \boldsymbol{D}}{\partial\theta}\right)\boldsymbol{u}_i \tag{10}$$

Eigenvector Derivatives: Non-Self-Adjoint Case

Two problems arise in the non-self-adjoint case: the left and right eigenvector derivatives must be calculated simultaneously, and extra constraints must be introduced for the relative scaling of the left and right eigenvectors. The fact that the scaling given by Eq. (3) is insufficient to give unique eigenvectors can be demonstrated by multiplying the left eigenvector by any scalar and dividing the right eigenvector by the same scalar. The derivatives of the right eigenvectors are written as in Eq. (7), and the vector $\boldsymbol{x}_i$ is calculated as before. The derivatives of the left eigenvectors are written as

$$\frac{\partial \boldsymbol{v}_i}{\partial\theta} = \boldsymbol{y}_i + d_i\boldsymbol{v}_i \tag{11}$$

The vector $\boldsymbol{y}_i$ is obtained in a similar manner to $\boldsymbol{x}_i$. Equation (2) is differentiated, and Eq. (11) is used to obtain

$$\boldsymbol{y}_i^T\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right) = \boldsymbol{y}_i^T\boldsymbol{F}_i = \boldsymbol{g}_i \tag{12}$$

where

$$\boldsymbol{g}_i = -\boldsymbol{v}_i^T\left(s_i^2\frac{\partial \boldsymbol{M}}{\partial\theta} + s_i\frac{\partial \boldsymbol{D}}{\partial\theta} + \frac{\partial \boldsymbol{K}}{\partial\theta}\right) - \boldsymbol{v}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\frac{\partial s_i}{\partial\theta} \tag{13}$$

As for the right eigenvectors, the vector and scalar in Eq. (11) are not unique, but the same procedure of setting one of the elements of $\boldsymbol{y}_i$ to zero can be used.

It remains to compute the scalars c_i and d_i, using the eigenvector normalization. Differentiating Eq. (3) with respect to the parameter θ, and substituting the expressions for the eigenvector derivatives Eqs. (7) and (11) produces

$$c_i + d_i = -\boldsymbol{v}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{x}_i - \boldsymbol{y}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{u}_i - \boldsymbol{v}_i^T\left(2\boldsymbol{M}\frac{\partial s_i}{\partial\theta} + 2s_i\frac{\partial \boldsymbol{M}}{\partial\theta} + \frac{\partial \boldsymbol{D}}{\partial\theta}\right)\boldsymbol{u}_i \tag{14}$$

It remains to impose a constraint on the relative magnitudes of the eigenvectors. The best approach is to set one element in both eigenvectors to be equal. This element is arbitrary, but should be chosen so that this element has a large magnitude in both the left and the right eigenvectors. One possibility is to multiply the magnitudes of the corresponding elements of both eigenvectors and choose the largest product. Suppose that vector element number e is chosen. Then

$$\{\boldsymbol{u}_i\}_e = \{\boldsymbol{v}_i\}_e, \qquad \left\{\frac{\partial \boldsymbol{u}_i}{\partial\theta}\right\}_e = \left\{\frac{\partial \boldsymbol{v}_i}{\partial\theta}\right\}_e \tag{15}$$

This leads to a second simultaneous equation for c_i and d_i. If the same vector element number e is chosen for the normalization in Eq. (15), and also as the zero element in $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$, then Eq. (15) reduces to

$$c_i = d_i \tag{16}$$

which together with Eq. (14) yields the required solution for c_i and d_i.

Numerical Example

The proposed method will be demonstrated on a simple two-degree-of-freedom, non-self-adjoint system, where

$$\boldsymbol{M} = \boldsymbol{I}, \qquad \boldsymbol{D} = \begin{bmatrix} 2 & -1.5 \\ -2.5 & 2 \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}$$

Table 1 shows the eigenvalues and the left and right eigenvectors of this system. The relative scaling has ensured that the first degree of freedom of the first right and left eigenvector pair are equal, whereas the second degree of freedom of the second eigenvector pair are equal. Suppose the second stiffness (initial value, 20 N/m) is now varied. Table 1 shows the derivatives of the eigenvalues and eigenvectors with respect to this stiffness.

Conclusions

This Note has outlined a method to calculate the derivatives of eigenvalues and eigenvectors for systems with nonproportional damping. Nelson's method is used, which has the advantage that only the eigenvectors of interest are required. For self-adjoint systems the usual eigenvector scaling is sufficient to obtain the eigenvector derivatives. For non-self-adjoint systems a further constraint is required to fix the relative magnitude of the left and right eigenvectors. Using this constraint, the left and right eigenvector derivatives are calculated simultaneously.

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Dispersion Relations in Piezoelectric Coupled Beams

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I. Introduction

ONE of the main reasons for the recent popularity of the use of piezoelectric materials as both actuator and sensor is their versatility and efficiency in transforming mechanical energy to electrical energy and vice versa. Basic mechanics models for the interaction of beams, coupled with piezoelectric actuators and sensors, either surface bonded on or embedded in the host beam structure, have been proposed by several researchers.[1–4] Crawley and de Luis[3] developed a uniform strain model for a beam with surface-bonded and embedded piezoelectric actuator patches. Crawley and Anderson[5] later deduced the mechanics model for the coupled structure based on an Euler beam assumption of the displacement field. Leibowitz and Vinson[6] provided a general model in which the elastic layers, soft-core layers, or piezoelectric layers are included by using the Hamilton principle. These models commonly assumed a linear distribution for the electric potential in the transverse direction, which inevitably violates Maxwell's electric equation. To overcome this, Krommer and Irschik[7] suggested a parabolic distribution, whereas Lee and Lin[8] assumed a full-cycle sine distribution for the electric potential in wave propagation problems in plate structures. Wave propagation in uncoupled beams has long been studied, where dispersion relations based on Euler, Rayleigh, Timoshenko, and exact theories have been presented.[9]

The objective of this Note is to present results of dispersion wave propagation curves for beams with surface-bonded piezoelectric players. The qualitative effect of the coupled piezoelectric materials on wave propagation in beams studied herein will serve as a reference for further analyses of wave propagation in piezoelectric coupled structures. Two models of beam theory are explored, namely, Euler and Timoshenko.[10] Both models assume that plane sections remain plane, but in Euler beam theory the sections remain perpendicular to the neutral axis. This assumption is removed in Timoshenko beam theory[10] to account for the effect of shear. The dispersion curves for different thickness ratios between the piezoelectric layer and host beam structure are obtained by assuming a half-cycle cosine potential distribution in the transverse direction of the piezoelectric material. In addition, the phase velocity for wave number approaching infinity and the cutoff frequencies based on the Timoshenko beam model are also presented.

II. Electric Fields in the Coupled Beam

Figure 1 shows the layout of a surface-mounted piezoelectric coupled beam with total thickness $2h$. The width is assumed to be unity for convenience, without loss of generality. Each piezoelectric layer has a thickness of h_1 and has an electrode mounted on the external surface for applying external electric voltage to actuate the structure. When an external electric voltage is applied, the electric potential distribution on the surface of the electrode remains constant.

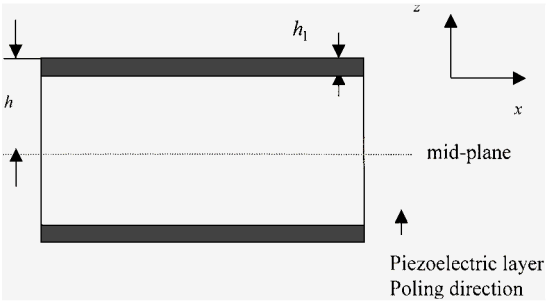

Fig. 1 Beam structure with surface-mounted piezoelectric layers.

When electrodes at the two surfaces of the piezoelectric layer are connected, the electric potential is zero throughout the surfaces. As stated earlier, most recent published papers assumed a constant electric field across the thickness of the piezoelectric material and a uniform distribution of the electric potential in the longitudinal direction. By the assuming of sinusoidal electric potential distribution in the transverse z direction, not only is Maxwell's static electricity equation not violated, but the corresponding distribution of the electric potential in the longitudinal direction can be solved.[11] In this Note, a half-cycle cosine distribution for the electric potential in the transverse z direction is assumed, which is more appropriate for beam vibration in the flexural mode.[7] For the general case, the potential function, assuming a combined cosine and linear variation of electric potential in the transverse z direction, can be written as

$$\phi = \phi(x, z, t) = -\cos(\pi z_l / h_1) \cdot \bar{\phi}(x) e^{i\omega t} + (2z_1 / h_1)\phi_a e^{i\omega t} \tag{1a}$$

where z_l is measured from the center of the piezoelectric layer in the global z direction, h_1 is the thickness of the layer, $\bar{\phi}(x)$ is the spatial variation of the electric potential in the global x direction, and ϕ_a is the value of external electric voltage applied to the electrodes. From Eq. (1a), we can see that the electric potentials at the two surfaces of the piezoelectric layer, that is, $z_1 = \pm h_1$, are exactly the electric voltage $\pm\phi_a$ applied to the electrodes. When there is no external electric voltage applied to the electrodes, the electric potentials at the surfaces of the layer are zero, which can be seen clearly from Eq. (1a). Because this Note addresses only wave propagation analysis, Eq. (1a) can be simplified as

$$\phi = -\cos(\pi z_l / h_1) \cdot \bar{\phi}(x) e^{i\omega t} \tag{1b}$$

in which the distribution function $\bar{\phi}(x)$ will be obtained from the coupling equation derived hereafter.

The spatial amplitude of the electric field $\bar{E}$ and electric displacement $\bar{D}$ are then written as

$$\bar{E}_x = \cos\frac{\pi z_l}{h_1} \cdot \frac{\partial \bar{\phi}}{\partial x} \tag{2}$$

$$\bar{E}_z = \frac{\pi}{h_1} \sin\frac{\pi z_l}{h_1} \bar{\phi} \tag{3}$$

$$\bar{D}_x = \Xi_{11} \bar{E}_x \tag{4}$$

$$\bar{D}_z = \Xi_{33} \bar{E}_z + E_p d_{31} \bar{\varepsilon}_x \tag{5}$$

where Ξ_{11}, $\bar{E}_x$, Ξ_{33}, and $\bar{E}_z$ are the dielectric constant and electric field of the piezoelectric layer in the x and z directions, respectively; $\bar{D}_x$ and $\bar{D}_z$ are the corresponding electric displacements; E_p is the Young's modulus of the piezoelectric layer in its longitudinal x direction; and d_{31} is the piezoelectric strain coefficient. Next, the displacement field for the two beam models will be discussed.

III. Wave Propagation Based on Euler Beam Model

For a long and thin beam, Euler theory is usually assumed, and the displacement field can be expressed as

$$u_z = \bar{u}(x) e^{i\omega t} \tag{6}$$

$$u_x = -z \frac{\partial \bar{u}(x)}{\partial x} e^{i\omega t} \tag{7}$$

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