Dynamics of Nonviscously Damped Linear Systems

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Abstract: This paper is aimed at extending classical modal analysis to treat lumped-parameter nonviscously damped linear dynamic systems. It is supposed that the damping forces depend on the past history of velocities via convolution integrals over some kernel functions. The traditional restriction of symmetry has not been imposed on the system matrices. The nature of the eigenvalues and eigenvectors is discussed under certain simplified but physically realistic assumptions concerning the system matrices and kernel functions. A numerical method for calculation of the right and left eigenvectors is suggested. The transfer function matrix of the system is derived in terms of the right and left eigenvectors of the second-order system. Exact closed-form expressions for the dynamic response due to general forces and initial conditions are presented. The proposed method uses neither the state-space approach nor additional dissipation coordinates. Suitable examples are given to illustrate the derived results.

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Introduction

The nature of damping forces is one of the least understood topics in structural dynamics. From the theory of classical mechanics, we know why the inertia and elastic forces are as they are, but the same is not exactly true for damping. Unlike the inertia and elastic forces, it is not clear what are the relevant state variables that effect the damping forces. By far the most common approach is to assume so-called ''viscous damping,'' which supposes that the *instantaneous* generalized velocities are the only relevant state variables that determine damping. However, viscous damping cannot be the only damping mechanism within the scope of linear analysis. Some examples include damping in composite materials (Baburaj and Matsukai 1994), energy dissipation in structural joints (Earls 1966; Beards and Williams 1977), a damping mechanism in composite beams (Banks and Inman 1991), to mention only a few. The development of new methodologies is needed to deal with nonviscously damped systems.

A key issue in considering nonviscously damped systems is to decide on an appropriate damping model. In principle, any causal model which makes the energy dissipation functional nonnegative is a possible candidate for a damping model. Possibly the most general method of model damping within the linear range is to use nonviscous damping models which depend on the past history of motion via convolution integrals over some kernel functions. The damping force using such a model can be expressed by

$$\mathbf{F}_d(t) = \int_{-\infty}^{t} \mathcal{G}(t,\tau)\dot{\mathrm{u}}(\tau)\mathrm{d}\tau \tag{1}$$

where $\mathrm{u}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $t \in \mathbb{R}^+$ denotes time. The kernel functions $\mathcal{G}(t,\tau) \in \mathbb{R}^{N\times N}$, or others closely related to them, are described under many different names in the literature of different subjects, for example, retardation functions, heredity functions, after-effect functions, relaxation functions, etc. This model was originally introduced by Biot (1958). Often the kernel depends upon the difference $(t-\tau)$ only; then $\mathcal{G}(t,\tau)=\mathcal{G}(t-\tau)$. In the special case when $\mathcal{G}(t,\tau)=\mathbf{C}\delta(t-\tau)$, where $\delta(t)$ is the Dirac delta function, Eq. (1) reduces to the case of viscous damping. The damping model of this kind is a further generalization of the familiar viscous damping.

The central theme of this paper is to analyze multiple-degree-of-freedom linear systems whose damping characteristics may be successfully modeled by Eq. (1). It is assumed that the system is ''nonproportionally damped,'' that is, the equations of motion cannot be decoupled by the undamped eigenvectors. For the sake of generality, the traditional restriction of symmetry has not been imposed on the system matrices. The nature of the eigenvalues and eigenvectors is discussed under certain simplified but physically realistic assumptions on the system matrices and kernel functions. A series expansion method for the determination of the complex right and left eigenvectors is proposed. The transfer function matrix of the system is derived in terms of these eigenvectors. Exact closed-form expressions are derived for the transient response and the response due to nonzero initial conditions. The approach does not require conversion of the equation of motion into the first-order form and is consistent with traditional modal analysis. Applications of the proposed method and related numerical issues are discussed using a nonviscously damped three-degree-of-freedom system.

Problem Formulation

The equations of motion of an N-degree-of-freedom linear system with nonviscous damping of the form (1) can be expressed by

$$\mathbf{M}\ddot{\mathrm{u}}(t) + \int_{-\infty}^{t} \mathcal{G}(t-\tau)\dot{\mathrm{u}}(\tau)\mathrm{d}\tau + \mathbf{K}\mathrm{u}(t) = \mathrm{f}(t) \tag{2}$$

Here $\mathbf{M}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrices and $\mathrm{f}(t) \in \mathbb{R}^N$ is the forcing vector. The initial conditions associated with the above equation are

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$$u(0)=\mathbf{u}_0\in\mathbb{R}^N \quad \text{and} \quad \dot{u}(0)=\dot{\mathbf{u}}_0\in\mathbb{R}^N \tag{3}$$

The aim of this paper is to develop a solution method for $u(t)$ analogous to the traditional modal analysis. For most of the problems we encounter in practice, **M**, **K**, and $\mathcal{G}(\tau)$ are symmetric. However, there are cases in which the system matrices can be asymmetric; some examples are gyroscopic and circulatory systems (Huseyin and Liepholz 1973), aircraft flutter (Fawzy and Bishop 1977), ship motion in sea water (Bishop and Price 1979), contact problems (Soom and Kim 1983), and many actively controlled systems (Caughey and Ma 1993). For this reason, to achieve generality, we allow the system matrices to be asymmetric. However, it is assumed that (a) $\mathbf{M}^{-1}$ exists and (b) *all* the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are real and distinct.

Eq. (2) or similar expressions occur in many different subjects. Bishop and Price (1979) have considered equations of motion similar to Eq. (2) in the context of ship dynamics. The convolution term appeared in order to represent the fluid forces and moments. They have discussed the eigenvalue problem associated with Eq. (2) and presented an orthogonality relationship for the right and left eigenvectors. They have also given an expression for the system response due to sinusoidal excitation. Their results were not very efficient because the orthogonality relationship of the eigenvectors was not utilized due to the difficulty associated with the form of the orthogonality equation, which itself become frequency-dependent. Equations of motion like Eq. (2) also arise in the dynamics of viscoelastic structures. Golla and Hughes (1985) and McTavis and Hughes (1993) have proposed a method to obtain such equations using a time-domain finite-element formulation. Their approach (the GHM method), which introduces additional dissipation coordinates corresponding to the internal dampers, increases the size of the problem. Dynamic responses of the system were obtained by using the eigensolutions of the augmented problem in the state space. Muravyov (1997; 1998) has proposed a method to obtain the time and frequency-domain description of the response by introducing additional coordinates such as the GHM method. To reduce the order of the problem, recently Friswell and Inman (1999) have proposed a state-space approach which employs a modal truncation and uses an iterative approach to obtain the eigensolutions. Using a first-order perturbation approach, Woodhouse (1998) has obtained expressions for the eigensolutions and transfer functions of system (1). His method, although it avoids the state-space representations and additional dissipation coordinates, is valid for small damping terms only.

In this paper, we propose a method to obtain the response of system (2) in an exact manner. Our method does not employ additional dissipation coordinates and the state-space representation of the equation of motion is avoided. It is not assumed that the damping is small, nor is it assumed that $\mathcal{G}(\tau)$ is such that it can be simultaneously diagonalized with **M** and **K** so that the classical modal analysis may be applied. We begin our discussion with the eigensolutions associated with system (2).

Eigenvalues and Eigenvectors

Considering free vibration, that is, $f(t)=\mathbf{u}_0=\dot{\mathbf{u}}_0=\mathbf{0}$, and taking the Laplace transform of Eq. (2), one has

$$s^2\mathbf{M}\bar{\mathbf{u}}+s\mathbf{G}(s)\bar{\mathbf{u}}+\mathbf{K}\bar{\mathbf{u}}=\mathbf{0} \tag{4}$$

Here $\bar{\mathbf{u}}(s)=\mathcal{L}[u(t)]\in\mathbb{C}^N$; $\mathbf{G}(s)=\mathcal{L}[\mathcal{G}(t)]\in\mathbb{C}^{N\times N}$; and $\mathcal{L}[\bullet]$=Laplace transform. In the context of structural dynamics, $s=i\omega$, where $i=\sqrt{-1}$ and $\omega\in\mathbb{R}^+$ denotes the frequency. Because $\mathcal{G}(t)$ is a real function, $\mathbf{G}(s)$ is also a real function of the parameter s. We assume that $\mathbf{G}(s)$ is such that the motion is dissipative. Golla and Hughes (1985) have given several physically realistic mathematical forms of the elements of $\mathbf{G}(s)$ available in the literature. For the linear viscoelastic case, it can be shown that (see Bland 1960; Muravyov 1997), in general, the elements of $\mathbf{G}(s)$ can be represented as

$$G_{jk}(s)=\frac{p_{jk}(s)}{q_{jk}(s)} \tag{5}$$

where $p_{jk}(s)$ and $q_{jk}(s)$ are finite-order polynomials in s. Here, we do not assume any specific functional form of $G_{jk}(s)$ but we do assume that $|G_{jk}(s)|<\infty$ when $s\to\infty$. This in turn implies that the elements of $\mathbf{G}(s)$ are at most of order $1/s$ in s or constant, as in the case of viscous damping. The eigenvalues s_j associated with Eq. (4) are roots of the characteristic equation

$$\det[s^2\mathbf{M}+s\mathbf{G}(s)+\mathbf{K}]=0 \tag{6}$$

If the elements of $\mathbf{G}(s)$ have simple forms, for example as in Eq. (5), then the characteristic equation becomes a polynomial equation of finite order. In other cases, the characteristic equation can be expressed as a polynomial equation by expanding $\mathbf{G}(s)$ in a Taylor series. However, the order of the equation will be infinite in those cases. For practical purposes, the Taylor expansion of $\mathbf{G}(s)$ can be truncated to a finite series to make the characteristic equation a polynomial equation of finite order. Such equations can be solved using standard numerical methods (see Press et al. 1992, Chap. 9). Suppose the order of the characteristic polynomial is m. In general, m is more than $2N$, that is, $m=2N+p$; $p\geqslant 0$. Thus, although the system has N degrees of freedom, the number of eigenvalues is more than $2N$. This is a major difference between the nonviscously damped systems and the viscously damped systems, where the number of eigenvalues is exactly $2N$, including any multiplicities.

A general analysis on the nature of the eigenvalues of nonviscously damped systems is beyond the scope of this paper. It is assumed that *all* m eigenvalues are distinct. We further restrict our attention to a special case when, among the m eigenvalues, $2N$ appear in complex conjugate pairs and the remaining p eigenvalues are purely real. The mathematical conditions which **M**, **K**, and $\mathbf{G}(s)$ must satisfy in order to produce such eigenvalues will not be obtained but a physical justification will follow shortly. For convenience, the eigenvalues are arranged as

$$s_1,s_2,\ldots,s_N,s_1^*,s_2^*,\ldots,s_N^*,s_{2N+1},\ldots,s_m \tag{7}$$

where $(\bullet)^*$ denotes complex conjugation.

The *right* and *left* eigenvalue problem associated with Eq. (2) can be defined from Eq. (4) as

$$\mathbf{D}(s_j)\mathbf{u}_j=\mathbf{0} \tag{8a}$$

$$\mathbf{v}_j^T\mathbf{D}(s_j)=\mathbf{0}^T \quad \text{for } j=1,\ldots,m \tag{8b}$$

where

$$\mathbf{D}(s_j)=s_j^2\mathbf{M}+s_j\mathbf{G}(s_j)+\mathbf{K} \tag{9}$$

is the *dynamic stiffness matrix* corresponding to the jth eigenvalue and $\mathbf{u}_j,\mathbf{v}_j$ are, respectively, the jth right and left eigenvectors. Here $(\bullet)^T$ denotes the matrix transpose. From Eqs. (8*a*) and (8*b*) it is clear that, when s_j appear in complex conjugate pairs, $\mathbf{u}_j$ and $\mathbf{v}_j$ also appear in complex conjugate pairs, and when s_j is real, $\mathbf{u}_j$ and $\mathbf{v}_j$ can be selected to also be real. Corresponding to the $2N$ complex conjugate pairs of eigenvalues, the N right and left eigenvectors together with their complex conjugates will be

called *elastic modes*. These modes are related to the N modes of vibration of the structural system. Physically, the assumption of ''$2N$ complex conjugate pairs of eigenvalues'' implies that all the elastic modes are oscillatory in nature, that is, they are subcritically damped. The modes corresponding to the ''additional'' p eigenvalues will be called *nonviscous modes*. These modes are induced by the nonviscous effect of the damping mechanism. Nonviscous modes, or similar modes, are known by different names in the literature of different subjects, for example, ''wet modes'' in the context of ship dynamics (Bishop and Price 1979) and ''damping modes'' in the context of viscoelastic structures (McTavis and Hughes 1993). The assumption that ''the remaining p eigenvalues are purely real'' implies that the damping kernel function matrix $\mathbf{G}(s)$ is purely dissipative. Thus, for stable passive systems the nonviscous modes are overcritically damped (i.e., negative real eigenvalues) and not oscillatory in nature. Determination of the eigenvectors is considered next.

Elastic Modes

Once the eigenvalues are known, $\mathbf{u}_j$ and $\mathbf{v}_j$, $\forall j=1,...,2N$ can be obtained from Eqs. (8*a*) and (8*b*) by fixing any one element and inverting the matrix $\mathbf{D}(s_j)\in\mathbb{C}^{N\times N}$. Note that inversion of an $(N-1)\times(N-1)$ complex matrix is required for calculation of every $\mathbf{u}_j$ and $\mathbf{v}_j$. Although the method is exact, it is computationally expensive and does not offer much physical insight. Here we propose an alternative method which utilizes the familiar undamped right and left eigenvectors.

Undamped Eigenvectors

The eigenproblem of asymmetric undamped systems has been well studied in the literature; see, for example, Huseyin (1978). The undamped right and left eigenvalue problem can be expressed by

$$\mathbf{K}\mathbf{x}_j=\omega_j^2\mathbf{M}\mathbf{x}_j \tag{10a}$$

$$\mathbf{y}_j^T\mathbf{K}=\omega_j^2\mathbf{y}_j^T\mathbf{M},\quad \forall j=1,...,N \tag{10b}$$

where $\omega_j\in\mathbb{R}$ is the jth undamped natural frequency and $\mathbf{x}_j$, $\mathbf{y}_j\in\mathbb{R}^N$ are, respectively, the jth undamped right and left eigenvectors. For distinct eigenvalues it is easy to show that $\mathbf{x}_j$ and $\mathbf{y}_l$ satisfy the biorthogonality relationship with respect to $\mathbf{M}$ and $\mathbf{K}$ (see Huseyin 1978, Sec. 1.5, for details). We also normalize the eigenvectors such that

$$\mathbf{y}_l^T\mathbf{M}\mathbf{x}_j=\delta_{lj}\quad\text{and}\quad\mathbf{y}_l^T\mathbf{K}\mathbf{x}_j=\omega_j^2\delta_{lj},\quad\forall l,j=1,...,N \tag{11}$$

where δ_{lj}=Kronecker delta function. These undamped right and left eigenvectors will now be used to obtain the elastic modes of the nonviscously damped system.

Neumann Expansion Method

For distinct undamped eigenvalues, $(\boldsymbol{\omega}_l^2)$, $\mathbf{x}_l$, and $\mathbf{y}_l$, $\forall l=1,...,N$, form a *complete* set of vectors. For this reason, $\mathbf{u}_j$ can be expanded as a complex linear combination of $\mathbf{x}_l$ and similarly $\mathbf{v}_j$ can also be expanded in terms of $\mathbf{y}_l$. Thus, expansions of the form

$$\mathbf{u}_j=\sum_{l=1}^{N}\alpha_l^{(j)}\mathbf{x}_l \tag{12a}$$

$$\mathbf{v}_j=\sum_{l=1}^{N}\beta_l^{(j)}\mathbf{y}_l \tag{12b}$$

may be considered. Without any loss of generality, we can assume that $\alpha_j^{(j)}=1$ and $\beta_j^{(j)}=1$ (normalization), which leaves us to determine $\alpha_l^{(j)}$, $\beta_l^{(j)}$, $\forall l\neq j$. A Galerkin-type of error minimization combined with a complex Neumann expansion method is adopted for this purpose.

Substituting the expansion of $\mathbf{u}_j$, from Eq. (8*a*) the error vector for the jth mode can be expressed as

$$\boldsymbol{\Delta}^{(j)}=\sum_{l=1}^{N}s_j^2\alpha_l^{(j)}\mathbf{M}\mathbf{x}_l+s_j\alpha_l^{(j)}\mathbf{G}(s_j)\mathbf{x}_l+\alpha_l^{(j)}\mathbf{K}\mathbf{x}_l\in\mathbb{C}^N \tag{13}$$

Consider the undamped left eigenvectors $\mathbf{y}_k$, $\forall k=1,...,N$, as ''weighting functions'' and following the Galerkin method we have $\mathbf{y}_k^T\boldsymbol{\Delta}^{(j)}=0$. Using the biorthogonality property of the undamped right and left eigenvectors described by Eq. (11), one obtains

$$s_j^2\alpha_k^{(j)}+s_j\sum_{l=1}^{N}\alpha_l^{(j)}G'_{kl}(s_j)+\omega_k^2\alpha_k^{(j)}=0,\quad\forall k=1,...,N \tag{14}$$

where $G'_{kl}(s_j)=\mathbf{y}_k^T\mathbf{G}(s_j)\mathbf{x}_l$. The jth equation of this set obtained by setting $k=j$ is a trivial case because $\alpha_j^{(j)}=1$ has already been assumed. From the above set of equations, excluding this trivial case, one has

$$s_j^2\alpha_k^{(j)}+s_j\left(G'_{kj}(s_j)+\alpha_k^{(j)}G'_{kk}(s_j)+\sum_{l\neq k\neq j}^{N}\alpha_l^{(j)}G'_{kl}(s_j)\right)+\omega_k^2\alpha_k^{(j)}=0,\quad\forall k=1,...,N;\neq j \tag{15}$$

These equations can be combined into a matrix form as

$$[\mathbf{P}^{(j)}-\mathbf{Q}^{(j)}]\hat{\mathbf{a}}^{(j)}=\mathbf{g}_{\mathbf{u}}^{(j)} \tag{16}$$

In the above equation

$$\mathbf{P}^{(j)}=\text{diag}\left[\frac{s_j^2+s_jG'_{11}(s_j)+\omega_1^2}{-s_j},...,\{j\text{th term deleted}\},...,\frac{s_j^2+s_jG'_{NN}(s_j)+\omega_N^2}{-s_j}\right]\in\mathbb{C}^{(N-1)\times(N-1)} \tag{17}$$

the traceless matrix

$$\mathbf{Q}^{(j)}=\begin{bmatrix}0 & G'_{12}(s_j) & \cdots & \{j\text{th term deleted}\} & \cdots & G'_{1N}(s_j)\\ G'_{21}(s_j) & 0 & \vdots & \vdots & \vdots & G'_{2N}(s_j)\\ \vdots & \vdots & \vdots & \{j\text{th term deleted}\} & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ G'_{N1}(s_j) & G'_{N2}(s_j) & \cdots & \{j\text{th term deleted}\} & \cdots & 0\end{bmatrix}\in\mathbb{C}^{(N-1)\times(N-1)} \tag{18}$$

$$\mathbf{g}_{\mathbf{u}}^{(j)}=\{G'_{1j}(s_j),G'_{2j}(s_j),\ldots,\{j\text{th term deleted}\},\ldots,G'_{Nj}(s_j)\}^T\in\mathbb{C}^{(N-1)} \tag{19}$$

and

$$\hat{\mathbf{a}}^{(j)}=\{\alpha_1^{(j)},\alpha_2^{(j)},\ldots,\{j\text{th term deleted}\},\ldots,\alpha_N^{(j)}\}^T\in\mathbb{C}^{(N-1)} \tag{20}$$

is the vector of unknown $\alpha_k^{(j)}$, $\forall k\neq j$. From Eq. (16), $\hat{\mathbf{a}}^{(j)}$ has to be determined by performing the associated matrix inversion and is achieved by using the Neumann expansion method. A similar procedure was used by Adhikari (1999) in the context of viscously damped systems. Now, using the Neumann expansion we have

$$\hat{\mathbf{a}}^{(j)}=[\mathbf{I}_{N-1}-\mathbf{P}^{(j)^{-1}}\mathbf{Q}^{(j)}]^{-1}\{\mathbf{P}^{(j)^{-1}}\mathbf{g}_{\mathbf{u}}^{(j)}\}$$

$$=[\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{u}}^{(j)}+\mathbf{R}_{\mathbf{u}}^{(j)^2}+\mathbf{R}_{\mathbf{u}}^{(j)^3}+\cdots]\mathbf{a}_0^{(j)} \tag{21}$$

where $\mathbf{I}_{N-1}$ is an $(N-1)\times(N-1)$ identity matrix

$$\mathbf{R}_{\mathbf{u}}^{(j)}=\mathbf{P}^{(j)^{-1}}\mathbf{Q}^{(j)}\in\mathbb{C}^{(N-1)\times(N-1)} \tag{22}$$

$$\mathbf{a}_0^{(j)}=\mathbf{P}^{(j)^{-1}}\mathbf{g}_{\mathbf{u}}^{(j)}\in\mathbb{C}^{(N-1)} \tag{23}$$

Because $\mathbf{P}^{(j)}$ is a diagonal matrix, its inversion can be carried out analytically and subsequently the closed-form expressions of $\mathbf{R}_{\mathbf{u}}^{(j)}$ and $\mathbf{a}_0^{(j)}$ can be obtained as

$$R_{u_{kl}}^{(j)}=\frac{-s_jG'_{kl}(s_j)(1-\delta_{kl})}{\omega_k^2+s_j^2+s_jG'_{kk}(s_j)},\quad \forall k,l\neq j \tag{24}$$

$$a_{0_l}^{(j)}=\frac{-s_jG'_{lj}(s_j)}{\omega_l^2+s_j^2+s_jG'_{ll}(s_j)},\quad \forall l\neq j \tag{25}$$

This makes further calculations involving these quantities simpler. From Eq. (21), $\hat{\mathbf{a}}^{(j)}$ can be calculated in an efficient way as one can write

$$\hat{\mathbf{a}}^{(j)}=\mathbf{a}_0^{(j)}+\mathbf{a}_1^{(j)}+\mathbf{a}_2^{(j)}+\cdots+\mathbf{a}_k^{(j)}+\cdots \tag{26}$$

where

$$\mathbf{a}_1^{(j)}=\mathbf{R}_{\mathbf{u}}^{(j)}\mathbf{a}_0^{(j)},\quad \mathbf{a}_2^{(j)}=\mathbf{R}_{\mathbf{u}}^{(j)}\mathbf{a}_1^{(j)},\quad \ldots,\quad \mathbf{a}_k^{(j)}=\mathbf{R}_{\mathbf{u}}^{(j)}\mathbf{a}_{k-1}^{(j)} \tag{27}$$

This implies that all the $\mathbf{a}_k^{(j)}$ can be obtained using successive matrix-vector multiplications only. Noting that $\hat{\mathbf{a}}^{(j)}$ is the vector of $\alpha_k^{(j)}$, $\forall k\neq j$, substitution of it in Eq. (12*a*) will give the right eigenvectors. It is easy to see that by taking more terms in the series represented by Eq. (26), one can obtain the right eigenvectors to any desired accuracy provided the complex matrix power series $\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{u}}^{(j)}+\mathbf{R}_{\mathbf{u}}^{(j)^2}+\mathbf{R}_{\mathbf{u}}^{(j)^3}+\cdots$ is convergent. Convergence of this series will be addressed later.

The left eigenvectors can be obtained by substituting the expansion of $\mathbf{v}_j$ in Eq. (8*b*) and letting $\mathbf{x}_l$ be ''weighting functions'' while applying the Galerkin method. Following the procedure employed for the right eigenvectors, one can write

$$[\mathbf{P}^{(j)}-\mathbf{Q}^{(j)^T}]\hat{\mathbf{b}}^{(j)}=\mathbf{g}_{\mathbf{v}}^{(j)} \tag{28}$$

where

$$\mathbf{g}_{\mathbf{v}}^{(j)}=\{G'_{j1}(s_j),G'_{j2}(s_j),\ldots,\{j\text{th term deleted}\},\ldots,G'_{jN}(s_j)\}^T$$

$$\in\mathbb{C}^{(N-1)} \tag{29}$$

and

$$\hat{\mathbf{b}}^{(j)}=\{\beta_1^{(j)},\beta_2^{(j)},\ldots,\{j\text{th term deleted}\},\ldots,\beta_N^{(j)}\}^T\in\mathbb{C}^{(N-1)} \tag{30}$$

is the vector of unknown $\beta_k^{(j)}$, $\forall k\neq j$. Now, using the Neumann expansion method and defining

$$\mathbf{R}_{\mathbf{v}}^{(j)}=\mathbf{P}^{(j)^{-1}}\mathbf{Q}^{(j)^T}\in\mathbb{C}^{(N-1)\times(N-1)} \tag{31}$$

$$\mathbf{b}_0^{(j)}=\mathbf{P}^{(j)^{-1}}\mathbf{g}_{\mathbf{v}}^{(j)}\in\mathbb{C}^{(N-1)} \tag{32}$$

from Eq. (28) one obtains $\hat{\mathbf{b}}^{(j)}$ as a series

$$\hat{\mathbf{b}}^{(j)}=[\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{v}}^{(j)}+\mathbf{R}_{\mathbf{v}}^{(j)^2}+\mathbf{R}_{\mathbf{v}}^{(j)^3}+\cdots]\mathbf{b}_0^{(j)}$$

$$=\mathbf{b}_0^{(j)}+\mathbf{b}_1^{(j)}+\mathbf{b}_2^{(j)}+\cdots+\mathbf{b}_k^{(j)}+\ldots \tag{33}$$

Here

$$\mathbf{b}_1^{(j)}=\mathbf{R}_{\mathbf{v}}^{(j)}\mathbf{b}_0^{(j)},\mathbf{b}_2^{(j)}=\mathbf{R}_{\mathbf{v}}^{(j)}\mathbf{b}_1^{(j)},\ldots,\mathbf{b}_k^{(j)}=\mathbf{R}_{\mathbf{v}}^{(j)}\mathbf{b}_{k-1}^{(j)} \tag{34}$$

The jth left eigenvector, $\mathbf{v}_j$, can now be obtained by substituting $\hat{\mathbf{b}}^{(j)}$ in Eq. (12*b*). This method does not require much computational time as closed-form expressions for $\mathbf{R}_{\mathbf{v}}^{(j)}$ and $\mathbf{b}_0^{(j)}$ can be obtained

$$R_{v_{kl}}^{(j)}=\frac{-s_jG'_{lk}(s_j)(1-\delta_{kl})}{\omega_k^2+s_j^2+s_jG'_{kk}(s_j)},\quad \forall k,l\neq j \tag{35}$$

$$b_{0_l}^{(j)}=\frac{-s_jG'_{jl}(s_j)}{\omega_l^2+s_j^2+s_jG'_{ll}(s_j)},\quad \forall l\neq j \tag{36}$$

It may be noted that by taking more terms in the series (33) one can obtain $\mathbf{v}_j$ to any desired accuracy if the complex matrix power series $\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{v}}^{(j)}+\mathbf{R}_{\mathbf{v}}^{(j)^2}+\mathbf{R}_{\mathbf{v}}^{(j)^3}+\cdots$ is convergent.

From the preceding formulation one may verify that, corresponding to the complex conjugate pairs of the eigenvalues, the right and left eigenvectors also appear in complex conjugate pairs. For many engineering problems it is often observed that the damping forces are not very ''big'' and that by retaining only a few terms in the series, expressions (26) and (33) will result in an acceptable accuracy. Closed-form approximate expressions for the elastic modes obtained by retaining one and two terms of these series expressions are given in the Appendix. These expressions might be useful whenever we find that the entries of the damping kernel functions are small compared to those of **M** and **K**.

Convergence of the Neumann Series

For the validity of the series expressions for $\hat{\mathbf{a}}^{(j)}$ and $\hat{\mathbf{b}}^{(j)}$ in Eqs. (26) and (33), it is required that

$$\mathbf{S}_{\mathbf{u}}=\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{u}}^{(j)}+\mathbf{R}_{\mathbf{u}}^{(j)^2}+\mathbf{R}_{\mathbf{u}}^{(j)^3}+\cdots \tag{37a}$$

and

$$\mathbf{S}_{\mathbf{v}}=\mathbf{I}_{N-1}+\mathbf{R}_{\mathbf{v}}^{(j)}+\mathbf{R}_{\mathbf{v}}^{(j)^2}+\mathbf{R}_{\mathbf{v}}^{(j)^3}+\cdots \tag{37b}$$

are convergent. Looking at the expression of $\mathbf{R}_{\mathbf{u}}^{(j)}$ and $\mathbf{R}_{\mathbf{v}}^{(j)}$ in Eqs. (24) and (35), it may be revealed that they are quite similar and it is sufficient to study the convergence property of any one of the series. Here the series $\mathbf{S}_{\mathbf{u}}$ in Eq. (37*a*) is considered.

Condition 1. *The complex matrix power series* $\mathbf{S_u}$ *converges if, and only if, for all the eigenvalues* $\sigma_l^{(j)}$ *of the matrix* $\mathbf{R_u^{(j)}}$, *the inequality* $|\sigma_l^{(j)}|<1$ *holds.*

Although this condition is both necessary and sufficient, checking convergence for all $j=1,...,N$ is often not feasible. So we look for a sufficient condition which is relatively easy to check and which ensures convergence for all $j=1,...,N$.

Condition 2. *The complex matrix power series* $\mathbf{S_u}$ *converges for any* s_j, ω_j *if* $\mathbf{G}'(s_j)$ *is a diagonally dominant matrix.*

Proof. Since a matrix norm is always greater than or equal to its maximum eigenvalue, it follows from Condition 1 that convergence of the series is guaranteed if $\|\mathbf{R_u^{(j)}}\|<1$. Writing the sum of absolute values of entries of $\mathbf{R_u^{(j)}}$ results in the following inequality as the required sufficient condition for convergence:

$$\sum_{\substack{k=1\\k\neq j}}^{N}\sum_{\substack{l=1\\l\neq j}}^{N}\left|\frac{s_j G'_{kl}(s_j)}{\omega_k^2+s_j^2+s_j G'_{kk}(s_j)}\right|(1-\delta_{lk})<1 \tag{38}$$

Dividing both numerator and denominator by s_j, the above inequality can be written as

$$\sum_{\substack{k=1\\k\neq j}}^{N}\sum_{\substack{l=1\\l\neq i\neq k}}^{N}\frac{|G'_{kl}(s_j)|}{|1/s_j(\omega_k^2+s_j^2)+G'_{kk}(s_j)|}<1 \tag{39}$$

Taking the maximum for all $k\neq j$, this condition can further be represented as

$$\max_{k\neq j}\frac{\Sigma_{\substack{l=1\\l\neq j,k}}^{N}|G'_{kl}(s_j)|}{|1/s_j(\omega_k^2+s_j^2)+G'_{kk}(s_j)|}<1 \tag{40}$$

It is clear that (40) always holds if

$$\sum_{\substack{l=1\\l\neq j\neq k}}^{N}|G'_{kl}(s_j)|<|G'_{kk}(s_j)|,\quad \forall k\neq j \tag{41}$$

which in turn implies that, for all $j=1,...,N$, the inequality $\|\mathbf{R_u^{(j)}}\|<1$ holds if $\mathbf{G}'(s_j)$ is a diagonally dominant matrix. It is important to note that the diagonal dominance of $\mathbf{G}'(s_j)$ is only a sufficient condition and the lack of it does not necessarily prevent convergence of $\mathbf{S_u}$.

Nonviscous Modes

When $2N<j\leq m$, the eigenvalues are real and consequently from Eq. (9) we observe that $\mathbf{D}(s_j)\in\mathbb{R}^{N\times N}$. The nonviscous modes can be obtained from Eqs. (8*a*) and (8*b*) by fixing any one element of the right and left eigenvectors. Since $\mathbf{D}(s_j)\in\mathbb{R}^{N\times N}$, from Eqs. (8*a*) and (8*b*) it is easy to see that $\mathbf{u}_j, \mathbf{v}_j\in\mathbb{R}^N$. Partition $\mathbf{u}_j$ and $\mathbf{v}_j$ as

$$\mathbf{u}_j=\begin{Bmatrix}\mathbf{u_{1}}_j\\ \mathbf{u_{2}}_j\end{Bmatrix} \tag{42a}$$

$$\mathbf{v}_j=\begin{Bmatrix}\mathbf{v_{1}}_j\\ \mathbf{v_{2}}_j\end{Bmatrix} \tag{42b}$$

We select $\mathbf{u_{1}}_j=\mathbf{v_{1}}_j=1$ so that $\mathbf{u_{2}}_j, \mathbf{v_{2}}_j\in\mathbb{R}^{(N-1)}$ has to be determined from Eqs. (8*a*) and (8*b*). Further, partition $\mathbf{D}(s_j)$ as

$$\mathbf{D}(s_j)=\begin{bmatrix}\mathbf{D_{11}}(s_j) & \mathbf{D_{12}}(s_j)\\ \mathbf{D_{21}}(s_j) & \mathbf{D_{22}}(s_j)\end{bmatrix} \tag{43}$$

where $\mathbf{D_{11}}(s_j)\in\mathbb{R}$, $\mathbf{D_{12}}(s_j)\in\mathbb{R}^{1\times(N-1)}$, $\mathbf{D_{21}}(s_j)\in\mathbb{R}^{(N-1)\times 1}$, and $\mathbf{D_{22}^{(j)}}\in\mathbb{R}^{(N-1)\times(N-1)}$. In view of Eq. (43) and recalling that $\mathbf{u_{1}}_j=1$, from Eq. (8*a*) we can have

$$\mathbf{D_{22}}(s_j)\mathbf{u_{2}}_j=-\mathbf{D_{21}}(s_j)$$

or

$$\mathbf{u_{2}}_j=-[\mathbf{D_{22}}(s_j)]^{-1}\mathbf{D_{21}}(s_j) \tag{44}$$

Similarly, for the left eigenvectors one has

$$\mathbf{v_{2}}_j=-[\mathbf{E_{22}}(s_j)]^{-1}\mathbf{E_{21}}(s_j) \tag{45}$$

Here $\mathbf{E}(s_j)=\mathbf{D}(s_j)^T$ and is partitioned in a manner similar to Eq. (43).

It may be noted that determination of the nonviscous modes is computationally more demanding than the elastic modes because inversion of an $(N-1)\times(N-1)$ real matrix is associated with each eigenvector. However, for most physically realistic nonviscous damping models it appears that the number of nonviscous modes is not very high and also their contribution to the global dynamic response is not very significant (see the example section). For this reason, calculation of the first few nonviscous modes may be sufficient from a practical point of view.

Transfer Function

The transfer function (matrix) of a system completely defines its input-output relationship in a steady state. It is well known that for any linear system, if the forcing function is harmonic, that is, $\mathfrak{f}(t)=\mathbf{f}\exp[st]$ with $s=i\omega$ and amplitude vector $\mathbf{f}\in\mathbb{R}^N$, the steady-state response will also be harmonic at frequency $\omega\in\mathbb{R}^+$. So we seek a solution of the form $\mathfrak{u}(t)=\overline{\mathbf{u}}\exp[st]$, where $\overline{\mathbf{u}}\in\mathbb{C}^N$ is the response vector in the frequency domain. Substitution of $\mathfrak{u}(t)$ and $\mathfrak{f}(t)$ in Eq. (1) gives

$$s^2\mathbf{M}\overline{\mathbf{u}}+s\mathbf{G}(s)\overline{\mathbf{u}}+\mathbf{K}\overline{\mathbf{u}}=\mathbf{f}\quad\text{or}\quad\mathbf{D}(s)\overline{\mathbf{u}}=\mathbf{f} \tag{46}$$

Here the *dynamic stiffness matrix*

$$\mathbf{D}(s)=s^2\mathbf{M}+s\mathbf{G}(s)+\mathbf{K}\in\mathbb{C}^{N\times N} \tag{47}$$

From Eq. (46) the response vector $\overline{\mathbf{u}}$ can be obtained as

$$\overline{\mathbf{u}}=\mathbf{D}^{-1}(s)\mathbf{f}=\mathbf{H}(s)\mathbf{f} \tag{48}$$

where

$$\mathbf{H}(s)=\mathbf{D}^{-1}(s)\in\mathbb{C}^{N\times N} \tag{49}$$

is the transfer function matrix. From this equation one has in addition

$$\mathbf{H}(s)=\frac{\text{adj}[\mathbf{D}(s)]}{\det[\mathbf{D}(s)]} \tag{50}$$

The *poles* of $\mathbf{H}(s)$, denoted by s_j, are the eigenvalues of the system. Because it is assumed that *all* the *m* eigenvalues are distinct, each pole is a *simple pole*. The matrix inversion in Eq. (48) is difficult to carry out in practice because of the singularities associated with the poles. Moreover, such an approach would be an expensive numerical exercise and may not offer much physical insight. For these reasons, we seek a solution analogous to the classical model series solution of the undamped or proportionally damped systems.

Using the residue theorem (see Riley et al. 1997, Chap. 18), the transfer function can be expressed in terms of the poles and *residues* as

$$\mathbf{H}(s)=\sum_{j=1}^{m}\frac{\mathbf{R}_j}{s-s_j} \tag{51}$$

Here

$$\mathbf{R}_j = \underset{s=s_j}{\text{res}}[\mathbf{H}(s)] \overset{\text{def}}{=} \lim_{s\to s_j}(s-s_j)[\mathbf{H}(s)] \tag{52}$$

is the residue of the transfer function at the pole s_j. It may be noted that Eq. (51) is equivalent to expressing the right-hand side of Eq. (50) in the partial-fraction form. Here we try to obtain the residues, that is, the coefficients in the partial-fraction form, in terms of the system eigenvectors.

Eigenvectors of Dynamic Stiffness Matrix

It turns out that the eigenvectors of the dynamic stiffness matrix play an important role in determining the residues of the transfer function. For any given $s\in\mathbb{C}$, the right and left eigenvalue problem associated with the dynamic stiffness matrix can be expressed by

$$\mathbf{D}(s)\boldsymbol{\phi}_k(s) = \nu_k(s)\boldsymbol{\phi}_k(s) \tag{53a}$$

$$\boldsymbol{\psi}_k^T(s)\mathbf{D}(s) = \nu_k(s)\boldsymbol{\psi}_k^T(s), \quad \forall k=1,...,N \tag{53b}$$

In these equations the eigenvalues $\nu_k(s)\in\mathbb{C}$ are the roots of the characteristic equation

$$\det[\mathbf{D}(s)-\nu(s)\mathbf{I}_N] = 0 \tag{54}$$

and $\boldsymbol{\phi}_k(s),\boldsymbol{\psi}_k(s)\in\mathbb{C}^N$ are, respectively, the kth right and left eigenvectors of $\mathbf{D}(s)$. The symbols $\nu_k(s)$, $\boldsymbol{\phi}_k(s)$, and $\boldsymbol{\psi}_k(s)$ indicate the functional dependence of these quantities on the complex parameter s. Such a continuous dependence is expected whenever $\mathbf{D}(s)$ is a sufficiently smooth matrix function of s. It should be noted that because $\mathbf{D}(s)$ is an $N\times N$ complex matrix for a fixed s, the number of eigenvalues (and consequently the eigenvectors) must be N. Further, it can be shown that, for distinct eigenvalues, $\boldsymbol{\phi}_k(s)$ and $\boldsymbol{\psi}_j(s)$ also satisfy a biorthogonality relationship although $\mathbf{u}_k$ and $\mathbf{v}_j$ do not enjoy any such simple relationship. We normalize $\boldsymbol{\phi}_k(s)$ and $\boldsymbol{\psi}_k(s)$ such that

$$\boldsymbol{\psi}_j^T(s)\boldsymbol{\phi}_k(s) = \delta_{kj}, \quad \forall k,j=1,...,N \tag{55}$$

In view of the above relationship, from Eqs. (53*a*) and (53*b*) we have

$$\boldsymbol{\psi}_j^T(s)\mathbf{D}(s)\boldsymbol{\phi}_k(s) = \nu_k(s)\delta_{kj}, \quad \forall k,j=1,...,N \tag{56}$$

or in the matrix form

$$\boldsymbol{\Psi}^T(s)\mathbf{D}(s)\boldsymbol{\Phi}(s) = \boldsymbol{\nu}(s) \tag{57}$$

Here

$$\boldsymbol{\Phi}(s) = [\boldsymbol{\phi}_1(s),\boldsymbol{\phi}_2(s),\ldots,\boldsymbol{\phi}_N(s)]\in\mathbb{C}^{N\times N} \tag{58}$$

$$\boldsymbol{\Psi}(s) = [\boldsymbol{\psi}_1(s),\boldsymbol{\psi}_2(s),\ldots,\boldsymbol{\psi}_N(s)]\in\mathbb{C}^{N\times N} \tag{59}$$

$$\boldsymbol{\nu}(s) = \text{diag}[\nu_1(s),\nu_2(s),\ldots,\nu_N(s)]\in\mathbb{C}^{N\times N} \tag{60}$$

It is possible to establish the relationships between the original eigenvalue problems of the system defined by Eqs. (8*a*) and (8*b*) and that by Eqs. (53*a*) and (53*b*). Consider the case in which the parameter s approaches any one of the system eigenvalues, say s_j. Since *all* the $\nu_k(s)$ are assumed to be distinct, for nontrivial eigenvectors, comparing Eqs. (8*a*), (8*b*), (53*a*), and (53*b*) we can conclude that one and only one of the $\nu_k(s)$ must be zero when $s\to s_j$ (see Yang and Wu 1998). Suppose that the rth eigenvalue of the eigenvalue problems (53*a*) and (53*b*) is zero when $s\to s_j$. It is also clear that the eigenvectors in Eqs. (53*a*) and (53*b*) corresponding to the rth eigenvalue also approach the eigenvectors in Eqs. (8*a*) and (8*b*) as $s\to s_j$. Thus, when $s=s_j$, one has

$$\nu_r(s_j) = 0 \quad \text{and} \quad \nu_k(s_j)\neq 0, \quad \forall k=1,...,N;\neq r \tag{61}$$

and also

$$\boldsymbol{\phi}_r(s_j) = \mathbf{u}_j \tag{62}$$

$$\boldsymbol{\psi}_r(s_j) = \mathbf{v}_j \tag{63}$$

These equations completely relate the eigensolutions of Eqs. (8*a*) and (8*b*) with Eqs. (53*a*) and (53*b*). Now, these relationships will be utilized to obtain the transfer function residues.

Calculation of the Residues

From Eq. (57) one has

$$\mathbf{D}^{-1}(s) = \boldsymbol{\Phi}(s)\boldsymbol{\nu}^{-1}(s)\boldsymbol{\Psi}^T(s) \tag{64}$$

Using the expression of the transfer function in Eq. (49) and noting that $\boldsymbol{\nu}(s)$ is a diagonal matrix, we may expand the right-hand side of the above equation to obtain

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) = \sum_{k=1}^{N}\frac{\boldsymbol{\phi}_k(s)\boldsymbol{\psi}_k^T(s)}{\nu_k(s)} \tag{65}$$

Separation of the rth term in the above sum yields

$$\mathbf{H}(s) = \frac{\boldsymbol{\phi}_r(s)\boldsymbol{\psi}_r^T(s)}{\nu_r(s)} + \left[\sum_{\substack{k=1\\k\neq r}}^{N}\frac{\boldsymbol{\phi}_k(s)\boldsymbol{\psi}_k^T(s)}{\nu_k(s)}\right] \tag{66}$$

Clearly, when $s\to s_j$, the second term of the right-hand side of Eq. (66) is analytic because according to Eq. (61) $\nu_k(s_j)\neq 0$, $\forall k=1,...,N;\neq r$. From Eq. (52) the residue at $s=s_j$ may be obtained as

$$\begin{aligned}\mathbf{R}_j &\overset{\text{def}}{=} \lim_{s\to s_j}(s-s_j)\left\{\frac{\boldsymbol{\phi}_r(s)\boldsymbol{\psi}_r^T(s)}{\nu_r(s)} + \left[\sum_{\substack{k=1\\k\neq r}}^{N}\frac{\boldsymbol{\phi}_k(s)\boldsymbol{\psi}_k^T(s)}{\nu_k(s)}\right]\right\}\\ &= \lim_{s\to s_j}(s-s_j)\frac{\boldsymbol{\phi}_r(s)\boldsymbol{\psi}_r^T(s)}{\nu_r(s)}\\ &= \frac{\boldsymbol{\phi}_r(s)\boldsymbol{\psi}_r^T(s)|_{s=s_j}}{\left.\dfrac{\partial\nu_r(s)}{\partial s}\right|_{s=s_j}} + \lim_{s\to s_j}\frac{(s-s_j)\dfrac{\partial}{\partial s}[\boldsymbol{\phi}_k(s)\boldsymbol{\psi}_k^T(s)]}{\dfrac{\partial\nu_r(s)}{\partial s}}\\ &\quad \text{(using l'Hôspital's rule)}\\ &= \frac{\mathbf{u}_j\mathbf{v}_j^T}{\left.\dfrac{\partial\nu_r(s)}{\partial s}\right|_{s=s_j}} \quad [\text{by Eqs. (62) and (63)}]\end{aligned} \tag{67}$$

The denominator in the above expression for the residues, $[\partial\nu_r(s)/\partial s]|_{s=s_j}$, is still unknown. Now, consider the rth right eigenvalue problem associated with the dynamic stiffness matrix. Differentiation of Eq. (53*a*) for $k=r$ with respect to s yields

$$\frac{\partial\mathbf{D}(s)}{\partial s}\boldsymbol{\phi}_r(s) + \mathbf{D}(s)\frac{\partial\boldsymbol{\phi}_r(s)}{\partial s} = \frac{\partial\nu_r(s)}{\partial s}\boldsymbol{\phi}_r(s) + \nu_r(s)\frac{\partial\boldsymbol{\phi}_r(s)}{\partial s} \tag{68}$$

Premultiplying the above equation by $\boldsymbol{\psi}_r^T(s)$ and rearranging, one obtains

$$\boldsymbol{\psi}_r^T(s)\frac{\partial \mathbf{D}(s)}{\partial s}\boldsymbol{\phi}_r(s)+[\boldsymbol{\psi}_r^T(s)\mathbf{D}(s)-\boldsymbol{\psi}_r^T(s)\nu_r(s)]\frac{\partial \boldsymbol{\phi}_r(s)}{\partial s}$$

$$=\boldsymbol{\psi}_r^T(s)\frac{\partial \nu_r(s)}{\partial s}\boldsymbol{\phi}_r(s) \tag{69}$$

From the definition of the left eigenvectors, $\boldsymbol{\psi}_r(s)$, in Eq. (53*b*), it follows that the second term on the left-hand side of the above equation is zero. Using the normalizing condition in Eq. (55) and setting $s\rightarrow s_j$, from Eq. (69) we have

$$\left.\frac{\partial \nu_r(s)}{\partial s}\right|_{s=s_j}=\mathbf{v}_j^T\left.\frac{\partial \mathbf{D}(s)}{\partial s}\right|_{s=s_j}\mathbf{u}_j=\mathbf{v}_j^T\frac{\partial \mathbf{D}(s_j)}{\partial s_j}\mathbf{u}_j \tag{70}$$

The term $\partial \mathbf{D}(s_j)/\partial s_j$ can be obtained by differentiating Eq. (47) as

$$\frac{\partial \mathbf{D}(s_j)}{\partial s_j}=2s_j\mathbf{M}+\mathbf{G}(s_j)+s_j\frac{\partial \mathbf{G}(s_j)}{\partial s_j} \tag{71}$$

Using Eqs. (67) and (70), one finally obtains the residue as

$$\mathbf{R}_j=\frac{\mathbf{u}_j\mathbf{v}_j^T}{\mathbf{v}_j^T\dfrac{\partial \mathbf{D}(s_j)}{\partial s_j}\mathbf{u}_j} \tag{72}$$

The above equation completely relates the transfer function residues to the eigenvalues and eigenvectors of the system. Recalling that, among the m eigenvalues $2N$ appear in complex conjugate pairs, from Eq. (51) the transfer function may be obtained as

$$\mathbf{H}(i\omega)=\sum_{j=1}^{N}\left[\frac{\gamma_j\mathbf{u}_j\mathbf{v}_j^T}{i\omega-s_j}+\frac{\gamma_j^*\mathbf{u}_j^*\mathbf{v}_j^{*T}}{i\omega-s_j^*}\right]+\sum_{j=2N+1}^{m}\frac{\gamma_j\mathbf{u}_j\mathbf{v}_j^T}{i\omega-s_j} \tag{73}$$

where

$$\gamma_j=\frac{1}{\mathbf{v}_j^T\dfrac{\partial \mathbf{D}(s_j)}{\partial s_j}\mathbf{u}_j} \tag{74}$$

The transfer function has two parts. The first part is due to the elastic modes and the second part is due to the nonviscous modes. Using a first-order perturbation method and considering $\mathbf{M}$ and $\mathbf{K}$ as symmetric matrices, Woodhouse [1998, Eq. (35)] has obtained an expression of the transfer function similar to Eq. (73). However, the nonviscous part of the transfer function has not been obtained by him.

Special Cases

The expression for the transfer function in Eq. (73) is a natural generalization for the familiar expressions for the transfer function of undamped or viscously damped systems. Transfer functions for several useful special cases may be obtained from Eq. (73) as follows:

1. *Undamped systems:* In this case $\mathbf{G}(s)=0$ results the order of the characteristic polynomial $m=2N$; s_j is purely imaginary so that $s_j=i\ \omega_j$, where $\omega_j\in\mathbb{R}$ are the undamped natural frequencies, $\mathbf{u}_j=\mathbf{x}_j$ and $\mathbf{v}_j=\mathbf{y}_j$. In view of the mass normalization relationship in Eq. (11), $\gamma_j=1/(2i\omega_j)$ and Eq. (73) leads to
$$\mathbf{H}(i\omega)=\sum_{j=1}^{N}\frac{1}{2i\omega_j}\left[\frac{1}{i\omega-i\omega_j}-\frac{1}{i\omega+i\omega_j}\right]\mathbf{x}_j\mathbf{y}_j^T=\sum_{j=1}^{N}\frac{\mathbf{x}_j\mathbf{y}_j^T}{\omega_j^2-\omega^2} \tag{75}$$
2. *Viscously damped symmetric systems* (see Vigneron 1986): In this case, $\mathbf{M}=\mathbf{M}^T$, $\mathbf{K}=\mathbf{K}^T$, $\mathbf{G}(s)=\mathbf{C}=\mathbf{C}^T$, and $m=2N$ results in $\mathbf{v}_j=\mathbf{u}_j$ and $\gamma_j=1/(\mathbf{u}_j^T[2s_j\mathbf{M}+\mathbf{C}]\mathbf{u}_j)$. These reduce expression (73) to
$$\mathbf{H}(i\omega)=\sum_{j=1}^{N}\left[\frac{\gamma_j\mathbf{u}_j\mathbf{u}_j^T}{i\omega-s_j}+\frac{\gamma_j^*\mathbf{u}_j^*\mathbf{u}_j^{*T}}{i\omega-s_j^*}\right] \tag{76}$$
3. *Viscously damped asymmetric systems* (Adhikari 1999): In this case, $\mathbf{G}(s)=\mathbf{C}$ and $m=2N$ results in $\gamma_j=1/(\mathbf{v}_j^T[2s_j\mathbf{M}+\mathbf{C}]\mathbf{u}_j)$. These reduce expression (73) to
$$\mathbf{H}(i\omega)=\sum_{j=1}^{N}\left[\frac{\gamma_j\mathbf{u}_j\mathbf{v}_j^T}{i\omega-s_j}+\frac{\gamma_j^*\mathbf{u}_j^*\mathbf{v}_j^{*T}}{i\omega-s_j^*}\right] \tag{77}$$

Dynamic Response

The steady-state response due to harmonic loads or the response due to broadband random excitation can be obtained directly from the expression of the transfer function in Eq. (73). In this section, we consider the system response due to transient loads and initial conditions in the time and frequency domains.

Taking the Laplace transform of Eq. (2) and considering the initial conditions in Eq. (3), we have

$$s^2\mathbf{M}\bar{\mathbf{u}}-s\mathbf{M}\mathbf{u}_0-\mathbf{M}\dot{\mathbf{u}}_0+s\mathbf{G}(s)\bar{\mathbf{u}}-\mathbf{G}(s)\mathbf{u}_0+\mathbf{K}\bar{\mathbf{u}}=\mathbf{f}(s)$$

or

$$[s^2\mathbf{M}+s\mathbf{G}(s)+\mathbf{K}]\bar{\mathbf{u}}=\mathbf{f}(s)+\mathbf{M}\dot{\mathbf{u}}_0+[s\mathbf{M}+\mathbf{G}(s)]\mathbf{u}_0 \tag{78}$$

Using the expression for the transfer function derived before, the response vector $\bar{\mathbf{u}}$ may be obtained as

$$\bar{\mathbf{u}}=\sum_{j=1}^{m}\frac{\gamma_j\mathbf{u}_j\mathbf{v}_j^T}{s-s_j}\{\mathbf{f}(s)+\mathbf{M}\dot{\mathbf{u}}_0+[s\mathbf{M}+\mathbf{G}(s)]\mathbf{u}_0\} \tag{79}$$

This can be simplified further to

$$\bar{\mathbf{u}}(i\omega)=\sum_{j=1}^{m}\frac{\gamma_jA_j(i\omega)}{i\omega-s_j}\mathbf{u}_j \tag{80}$$

where the frequency-dependent complex scalar

$$A_j(i\omega)=\mathbf{v}_j^T\mathbf{f}(i\omega)+\mathbf{v}_j^T\mathbf{M}\dot{\mathbf{u}}_0+i\omega\mathbf{v}_j^T\mathbf{M}\mathbf{u}_0+\mathbf{v}_j^T\mathbf{G}(i\omega)\mathbf{u}_0 \tag{81}$$

The summation in Eq. (80) may be split into two different parts. The first part would correspond to the $2N$ complex conjugate pairs of elastic modes and the second part would be the contribution of the nonviscous modes.

The response in the time domain due to any forcing function can be obtained using a convolution integral over the *impulse response function*. From the expression of the transfer function in Eq. (73), the impulse response function $\mathbf{h}(t)\in\mathbb{R}^{N\times N}$ may be obtained as

$$\mathbf{h}(t)=\sum_{j=1}^{N}[\gamma_j\mathbf{u}_j\mathbf{v}_j^Te^{s_jt}+\gamma_j^*\mathbf{u}_j^*\mathbf{v}_j^{*T}e^{s_j^*t}]+\sum_{j=2N+1}^{m}\gamma_j\mathbf{u}_j\mathbf{v}_j^Te^{s_jt} \tag{82}$$

The response due to the initial conditions may also be obtained by taking the inverse transform of Eq. (79). First, simplify Eq. (79) to obtain

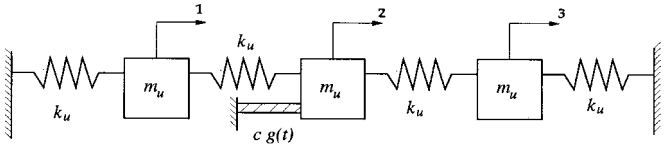

Fig. 1. Three degree-of-freedom nonviscously damped system, $m_u = 1$ kg, $k_u = 1$ N/m

$$\overline{\mathbf{u}}(s) = \sum_{j=1}^{m} \gamma_j \left[\frac{\mathbf{v}_j^T \mathbf{f}(s) + \mathbf{v}_j^T \mathbf{G}(s)\mathbf{u}_0}{s - s_j} + \frac{\mathbf{v}_j^T \mathbf{M}\dot{\mathbf{u}}_0}{s - s_j} + \left(1 + \frac{s_j}{s - s_j}\right) \mathbf{v}_j^T \mathbf{M}\mathbf{u}_0 \right] \mathbf{u}_j \tag{83}$$

From the above, one has

$$u(t) = \mathcal{L}^{-1}[\overline{\mathbf{u}}(s)] = \sum_{j=1}^{N} [\gamma_j a_j(t)\mathbf{u}_j + \gamma_j^* a_j^*(t)\mathbf{u}_j^*] + \sum_{j=2N+1}^{m} \gamma_j a_j(t)\mathbf{u}_j \tag{84}$$

where the time-dependent scalar coefficients

$$a_j(t) = \int_0^t e^{s_j(t-\tau)} \{\mathbf{v}_j^T \mathbf{f}(\tau) + \mathbf{v}_j^T \mathcal{G}(\tau)\mathbf{u}_0\} d\tau + e^{s_j t} \{\mathbf{v}_j^T \mathbf{M}\dot{\mathbf{u}}_0 + s_j \mathbf{v}_j^T \mathbf{M}\mathbf{u}_0\}, \quad \forall t > 0 \tag{85}$$

The expression of the system response, either the frequency-domain description in Eq. (80) or the time-domain description in Eq. (84), is similar to the classical modal superposition result for undamped or proportionally damped systems usually obtained using the mode-orthogonality relationships. Thus, the formulation presented here is a generalization of the classical result where the real normal modes are appropriately replaced by the elastic modes and the nonviscous modes. Also note that we have not used any orthogonality relationship; the expression of the transfer function residue in Eq. (72) allows us to express the response in terms of a superposition of individual modes even when the equations of motion cannot be decoupled.

In summary, the procedure to obtain the dynamic response is general, simple, direct, and provides better physical insights as only the familiar *N*-space eigenvectors are used. The approach also offers a reduction in computational effort because it uses neither the state-space formalism nor additional dissipation coordinates. Applications of the proposed method are illustrated next.

Numerical Examples

We consider a three-degree-of-freedom system to illustrate the proposed method. Fig. 1 shows the example taken together with the numerical values considered for mass and stiffness properties. A similar system with viscous damping has been studied by Newland (1989, see pp. 148–151). Damping is associated only with the middle mass, and the kernel function corresponding to this damper has the form

$$\mathcal{G}_{22}(t) = c\,g(t) \tag{86}$$

where c=damping coefficient and $g(t)$=damping function. Two different forms of $g(t)$ available in the literature will be considered here. For the first model (exponential)

$$g(t) = \mu e^{-\mu t}, \quad \mu, t \geq 0 \tag{87}$$

and for the second model (double exponential)

$$g(t) = \tfrac{1}{2}(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}), \quad \mu_1, \mu_2, t \geq 0 \tag{88}$$

The exponential function in Eq. (87) is possibly the simplest physically realistic nonviscous damping model. This function, often known as a "relaxation function," was introduced by Biot (1955). It has been used extensively in the context of viscoelastic systems. Recently, Adhikari and Woodhouse (2000) have proposed a method to identify such damping models using modal testing. The double exponential damping function, known as the GHM model, was introduced by Golla and Hughes (1985) and McTavis and Hughes (1993). Identification of GHM models has been discussed by Friswell et al. (1997). Both of the damping functions have been scaled so as to have unit area when integrated to infinity. This makes them directly comparable with the viscous model in which the corresponding damping function would be a unit delta function, $g(t) = \delta(t)$, and the coefficient c would be the usual viscous damping coefficient. The difference between a δ function and $g(t)$ given by Eqs. (87) and (88) is that at $t = 0$ they start with finite values of μ and $(\mu_1 + \mu_2)/2$, respectively. Thus, the values of μ, μ_1, and μ_2 give a notion of nonviscousness—if they are large the damping behavior will be near-viscous, and vice versa.

The mass and stiffness matrices and the damping matrix in the Laplace domain for the problem can be obtained as

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix} \tag{89}$$

$$\mathbf{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{90}$$

$$\mathbf{G}(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & cG(s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{91}$$

Here $G(s)$ is the Laplace transform of $g(t)$. Next, the eigensolutions and the dynamic response of the system are discussed for the two functional forms of $g(t)$.

Example 1: Exponential Damping

Eigensolutions

Because all the system matrices are symmetric, the right and left eigenvectors are identical for this problem. We assume $c = 0.3$, as considered by Newland (1989, p. 149) for the equivalent viscously damped system. From Eq. (87), one obtains

$$G(s) = \frac{1}{s + \mu} \tag{92}$$

Using this expression, the characteristic equation can be simplified as

$$m_u^3 s^7 + m_u^3 \mu s^6 + [2m_u^2 k_u + m_u(\mu c m_u + 4m_u k_u)]s^5 + 6k_u m_u^2 \mu s^4 + [2k_u(\mu c m_u + 4m_u k_u) + m_u(2\mu c k_u + 2k_u^2)]s^3 + 10k_u^2 m_u \mu s^2 + 2k_u(2\mu c k_u + 2k_u^2)s + 4k_u^3 \mu = 0 \tag{93}$$

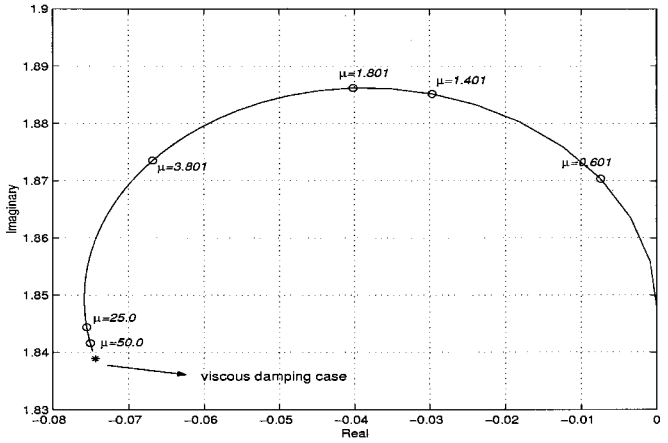

Fig. 2. Root-locus plot showing locus of third eigenvalue (s_3) as function of μ

The order of the above polynomial $m=7$. Since the system has three degrees of freedom, there are three elastic modes corresponding to the three modes of vibration. The number of nonviscous modes $p=m-2N=1$.

It is of interest to us to understand the effect of ''nonviscousness'' on the eigensolutions. Fig. 2 shows the locus of the third eigenvalue, that is, s_3, plotted as a function of μ. It is interesting to observe that the locus is much more sensitive in the region of lower values of μ (i.e., when damping is significantly nonviscous) compared to that in the region of higher values. The eigenvalue of the corresponding viscously damped system is also plotted (marked by an asterisk) in the same diagram. Note that the nonviscous damping mechanism approaches the viscous damping when $\mu > \approx 50.0$. Similar behavior has been observed (results not shown here) for the locus of s_1 also. The second mode, in which the middle mass remains stationary, is not affected by damping.

The eigenvectors of the system, i.e., the three elastic modes (together with their complex conjugates) and one nonviscous mode, can be obtained in a straightforward manner by following the procedure outlined earlier. We select two representative values of μ, one when μ is large (i.e., the near-viscous case) and the other when μ is small. The undamped eigenvalues and eigenvectors are obtained as

$$\{\omega_1,\omega_2,\omega_3\}=\{0.7654,1.4142,1.8478\} \tag{94}$$

$$[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=\begin{bmatrix} 0.5 & 0.7071 & -0.5 \\ 0.7071 & 0.0 & 0.7071 \\ 0.5 & -0.7071 & -0.5 \end{bmatrix} \tag{95}$$

Using these results, when $\mu=50.0$, for the elastic modes we have

$$\{s_1,s_2,s_3\}=\{-0.0757+0.7659i,1.4142i,-0.0751+1.8416i\} \tag{96}$$

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]=\begin{bmatrix} 0.4983+0.0204i & 0.7071 & -0.5002+0.0491i \\ 0.7095-0.0289i & 0.0 & 0.7069+0.0694i \\ 0.4983+0.0204i & -0.7071 & -0.5002+0.0491i \end{bmatrix} \tag{97}$$

The above calculation is performed by retaining five terms in the series (26). It may be verified that, because μ is large (about 27 times the maximum natural frequency), the results obtained are close to the viscously damped case (see Newland 1989, p. 149). For the one nonviscous mode, we obtain

$$s_7=-49.6984 \quad \text{and} \quad \mathbf{u}_7=\begin{Bmatrix} 1.0 \\ 2.4719\times 10^3 \\ 1.0 \end{Bmatrix} \tag{98}$$

Because s_7 is purely real and negative, this mode is nonoscillatory (overcritically damped) and stable.

When $\mu=0.5$, the damping is significantly nonviscous. For this case, performing similar calculations for the elastic modes one has

$$\{s_1,s_2,s_3\}=\{-0.0207+0.8i,1.4142i,-0.0053+1.8671i\} \tag{99}$$

and

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]=\begin{bmatrix} 0.4983+0.0204i & 0.7071 & -0.5002+0.0491i \\ 0.6787+0.0112i & 0.0 & 0.7442-0.0630i \\ 0.4983+0.0204i & -0.7071 & -0.5002+0.0491i \end{bmatrix} \tag{100}$$

These values are not significantly different from those obtained for $\mu=50.0$ in Eq. (97). For this problem, the elastic modes are not very sensitive to the damping mechanism. However, we emphasize that this fact *cannot* be generalized to all systems. For the nonviscous mode one has

$$s_7=-0.4480 \quad \text{and} \quad \mathbf{u}_7=\begin{Bmatrix} 1.0 \\ 2.2007 \\ 1.0 \end{Bmatrix} \tag{101}$$

These values are, however, quite different from those obtained for $\mu=50.0$ in Eq. (98). It is difficult physically to visualize the nature of the nonviscous modes in general. These modes are intrinsic to the dampers and we do not have sufficient generalized coordinates to represent them properly. Nevertheless, they yield nonzero residues in the system transfer functions and thus contribute to the global dynamic response.

Dynamic Response Analysis

The problem of stationary random vibration analysis of the system is considered here. Suppose the system is subjected to a band-limited Gaussian white noise at the third degree of freedom (DOF). We are interested in the resulting displacement of the system at the third DOF (i.e., $\mathbf{u}_3$). The power spectral density (PSD) of the response (see Nigam 1983 for details) can be given by

$$S_{uu}(i\omega)=|H_{33}(i\omega)|^2 S_{ff}(i\omega) \tag{102}$$

where

$$S_{ff}(i\omega)=\begin{cases} 1 & \text{if } 0<\omega\leqslant 2.5 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases} \tag{103}$$

In Fig. 3, the PSD of $\mathbf{u}_3$, that is, $|H_{33}(i\omega)|^2$, is plotted for the cases in which $\mu=50.0$ and 0.5. These results are obtained by direct application of Eq. (73). From the diagram, observe that the damping is less for the case when $\mu=0.5$ than when $\mu=50.0$. Also note the (horizontal) shift in the position of the natural frequencies. These features may also be observed in the root locus diagram as shown in Fig. 2. To understand the effect of ''nonviscosity,'' in the same diagram we have plotted the nonviscous term (the second term) appearing in Eq. (73) for both values of μ. For

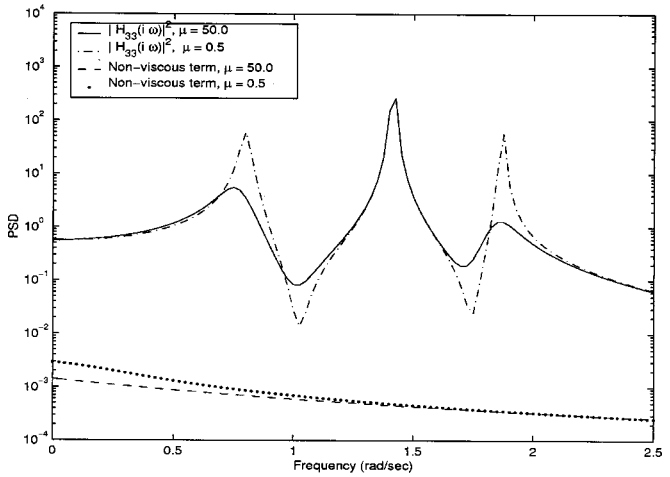

Fig. 3. Power spectral density function of displacement at third DOF ($\mathbf{u}_3$)

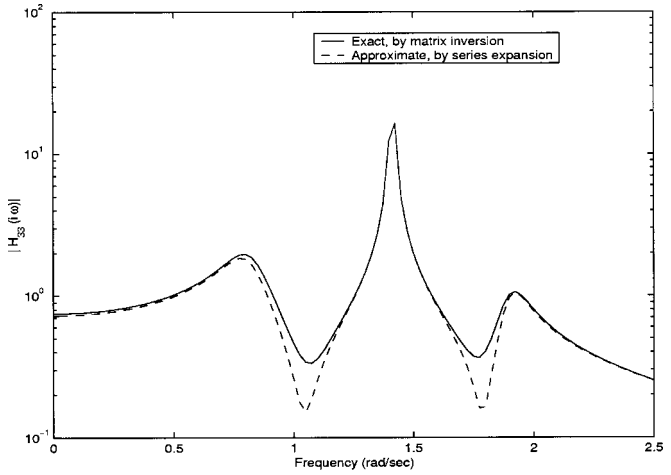

Fig. 4. Transfer function $H_{33}(i\omega)$

this problem, the nonviscous part is quite small and becomes smaller at higher frequencies. Observe that when $\mu=0.5$, that is, when damping is significantly nonviscous, the value of the nonviscous part of the response is more than that when $\mu=50.0$. This plot also clearly demonstrates that the nonviscous part of the response is *not* oscillatory in nature.

Example 2: GHM Damping

Taking the Laplace transform of Eq. (88), one obtains

$$G(s)=\frac{(\mu_1+\mu_2)/2s+\mu_1\mu_2}{s^2+(\mu_1+\mu_2)s+\mu_1\mu_2} \tag{104}$$

Using this equation, together with the expressions of the system matrices given by Eqs. (89)–(91), it can be shown that the order of the characteristic polynomial $m=8$. Thus, the number of the nonviscous modes $p=m-2N=2$. In this section, we focus our attention on the numerical accuracy of the formulation developed in this paper.

Regarding the numerical values of the damping parameters, we assume $c=0.5$, $\mu_1=1$, and $\mu_2=3$. Small values of μ_1 and μ_2 indicate that the damping mechanism is strongly nonviscous. Solving the characteristic equation, exact eigenvalues corresponding to the three elastic modes can be obtained as

$$\{s_1,s_2,s_3\}=\{-0.0994+0.8180i,1.4142i,-0.0687+1.9025i\} \tag{105}$$

and their complex conjugate pairs. Eigenvalues corresponding to the two nonviscous modes are found to be

$$\{s_7,s_8\}=\{-2.7901,-0.8738\} \tag{106}$$

Eigenvalues corresponding to the elastic modes can also be obtained approximately by Eq. (111) in the Appendix. Recall that only the undamped eigensolutions are required in order to apply this equation. Approximate eigenvalues using Eq. (111) are calculated as

$$\{s_1,s_2,s_3\}_{\text{approx}}$$
$$=\{-0.0981-0.8105i,1.4142i,-0.0595-1.9018i\}. \tag{107}$$

It is useful to compare the exact and approximate eigenvalues in light of the Q factors. In this problem, the second mode is not damped, so $Q_2=\infty$. For the first and third modes we obtain $Q_1=4.1164$ and $Q_3=13.8540$. Small values of the Q factor indicate that these modes are quite heavily damped. Comparing Eqs. (105) and (107), it may be observed that the approximate values are quite close to the exact one even when damping is reasonably high.

In order to check the numerical accuracy of the eigenvectors, first the exact values are calculated by the matrix inversion method. For the elastic modes we obtain

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]$$
$$=\begin{bmatrix} 0.5114+0.0299i & 0.7071 & -0.4639+0.0403i \\ 0.6905-0.0431i & 0.0 & 0.7596+0.0562i \\ 0.5114+0.0299i & -0.7071 & -0.4639+0.0403i \end{bmatrix} \tag{108}$$

and their complex conjugates. For the two nonviscous modes one has

$$[\mathbf{u}_7,\mathbf{u}_8]=\begin{bmatrix} 1.0000 & 1.0000 \\ 9.7847 & 2.7636 \\ 1.0000 & 1.0000 \end{bmatrix} \tag{109}$$

Approximate eigenvectors corresponding to the elastic modes, calculated by Eq. (113*a*), are obtained as

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]_{\text{approx}}$$
$$=\begin{bmatrix} 0.5114+0.0299i & 0.7071 & -0.4639+0.0403i \\ 0.6910-0.0422i & 0.0 & 0.7582+0.0569i \\ 0.5114+0.0299i & -0.7071 & -0.4639+0.0403i \end{bmatrix} \tag{110}$$

The above values are equivalent to performing the calculation by retaining only one term in the series (26). Also recall that the approximate values are obtained from the undamped eigensolutions only. Comparing Eqs. (108) and (110), it is clear that the results obtained from the approximate method match the exact solutions to an excellent accuracy.

As a final check on the formulation developed in this paper, we compare the transfer function obtained from Eq. (73) with the exact transfer function calculated by inversion of the dynamic stiffness matrix. Fig. 4 shows such a comparison, for $H_{33}(i\omega)$. Approximate natural frequencies and modes given by Eqs. (107)

and (110) are used and also the nonviscous term in Eq. (73) is neglected in order to calculate the approximate transfer function. Thus, in turn, the approximate transfer function in Fig. 4 is obtained only by proper ''post-processing'' of the undamped eigensolutions. From this figure it may be observed that, except in a few places, the approximate transfer function is reasonably close to the exact one. These results demonstrate the usefulness of the proposed method.

Conclusions

The problem of dynamic analysis of nonviscously damped multiple-degree-of-freedom linear systems has been considered. The assumed nonviscous damping model is such that the damping forces depend on the past history of motion via convolution integrals over some kernel functions. The familiar viscous damping model is a special case of this general linear damping model when the kernel functions have no memory. It has been assumed that, in general, the mass and stiffness matrices as well as the matrix of the kernel functions are not symmetric and cannot be simultaneously diagonalized by any linear transformation. The analysis is, however, restricted to systems with nonrepetitive eigenvalues and nonsingular mass matrices.

System eigenvalues were obtained by solving the characteristic equation. It turns out that, unlike the viscously damped case, the order of the characteristic equation for an N-degree-of-freedom system is more than $2N$. As a consequence, the number of modes becomes more than $2N$ and they are grouped into two types: (a) elastic modes and (b) nonviscous modes. It is assumed that the elastic modes appear in complex conjugate pairs, that is, they are subcritically damped. The elastic modes, which consist of N right and left eigenvectors together with their complex conjugate pairs, correspond to N modes of vibration of the structural system. These N right and left eigenvectors were expressed as a complex linear combination of the right and left (real) eigenvectors of the corresponding undamped system. The vectors of these complex constants for both right and left eigenvectors were further determined from a series obtained by the Neumann expansion method. Based on this analysis, some approximate formulas for the eigenvalues and eigenvectors were suggested and their accuracy was verified using numerical examples. The nonviscous modes, which occur due to the nonviscous damping mechanism, are assumed to be real, overcritically damped, and nonoscillatory in nature. These modes were obtained by inversion of a partition of the dynamic stiffness matrix evaluated at the corresponding eigenvalues.

The transfer function of the system was derived in terms of the eigenvalues and eigenvectors of the second-order system. Exact closed-form expressions of the response due to arbitrary forcing functions and initial conditions were obtained. The response can be expressed as a sum of two parts, one that arises in the usual viscously damped systems and another that occurs due to nonviscous damping mechanisms. Through an example, it was shown that the nonviscous part of the response is purely dissipative and nonoscillatory in nature.

The method developed here is analogous to classical modal analysis where undamped natural frequencies and modes have to be appropriately replaced by elastic modes and nonviscous modes of the nonconservative system. The method presented offers a reduction in computational effort because neither the first-order formalisms nor the additional dissipation coordinates are employed. Moreover, this approach also provides better physical insight as familiar N-space eigenvectors are utilized. The results developed here are very general in nature and most of the familiar linear dynamic systems, e.g., classically/nonclassically damped symmetric systems, viscously damped asymmetric systems, damped/undamped gyroscopic systems, etc., can be treated as special cases.

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Appendix: Approximate Expressions for Elastic Modes

The expressions for the elastic modes obtained by taking one term in the series (26) and (33) are close to those obtained from the first-order perturbation analysis. The validity of these results relies on the fact that the entries of $\mathbf{G}(s_j)$ are not very big for all s_j. Considering the jth set of Eq. (14) and neglecting the second-order terms involving $\alpha_k^{(j)}$ and $G'_{kl}(s_j)$, $\forall k \neq l$, and also noting that $\alpha_j^{(j)}=1$, one obtains

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 \approx 0$$

or

$$s_j \approx \pm i\omega_j - G'_{jj}(\pm i\omega_j)/2 = i\omega_j - G'_{jj}(i\omega_j)/2, \quad -i\omega_j - G'_{jj}(-i\omega_j)/2. \tag{111}$$

This is the first-order approximate expression for the complex eigenvalues of system (2) corresponding to the elastic modes. A similar result was also obtained by Woodhouse (1998). In deriving this expression, the assumption has been made that $\mathbf{G}(s_j) \approx \mathbf{G}(i\omega_j)$. Because $\mathbf{G}(s_j)$ is assumed small, it is expected that this approximation will not result in significant errors. Note that, as $\mathcal{G}(t)$ is a real function, $G'_{jj}(\bullet)$ satisfies the property

$$G'_{jj}(-i\omega_j) = G'^{*}_{jj}(i\omega_j) \tag{112}$$

Using this relationship, it may be confirmed that the eigenvalues corresponding to the elastic modes, approximately given by Eq. (111), appear in complex conjugate pairs.

To obtain approximate expressions for the right and left eigenvectors, one simply considers only the first term of the series expressions (26) and (33) and substitutes $\hat{\mathbf{a}}^{(j)}$ and $\hat{\mathbf{b}}^{(j)}$ in Eqs. (12*a*) and (12*b*) to obtain

$$\mathbf{u}_j \approx \mathbf{x}_j - \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)} \tag{113a}$$

and

$$\mathbf{v}_j \approx \mathbf{y}_j - \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{y}_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)} \tag{113b}$$

Now, retaining the first two terms of the series expressions (26) and (33) and substituting $\hat{\mathbf{a}}^{(j)}$ and $\hat{\mathbf{b}}^{(j)}$ in Eqs. (12*a*) and (12*b*), one obtains

$$\mathbf{u}_j \approx \mathbf{x}_j - \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_k(s_j)} + \sum_{\substack{k=1 \\ k\neq j}}^{N} \sum_{\substack{l=1 \\ l\neq j\neq k}}^{N} \frac{s_j^2 G'_{kl}(s_j) G'_{lj}(s_j)\mathbf{x}_k}{[\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)][\omega_l^2 + s_j^2 + s_j G'_{ll}(s_j)]} \tag{114a}$$

and

$$\mathbf{v}_j \approx \mathbf{y}_j - \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{y}_k}{\omega_k^2 + s_j^2 + s_j G'_k(s_j)} + \sum_{\substack{k=1 \\ k\neq j}}^{N} \sum_{\substack{l=1 \\ l\neq j\neq k}}^{N} \frac{s_j^2 G'_{lk}(s_j) G'_{jl}(s_j)\mathbf{y}_k}{[\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)][\omega_l^2 + s_j^2 + s_j G'_{ll}(s_j)]} \tag{114b}$$

The above two equations are second-order approximate expressions for the right and left eigenvectors corresponding to the elastic modes of system (2).

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