

Uncertainty quantification in Structural Dynamics: Day 4

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January 2020



Swansea University
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The course is dived into **eight** topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

- 1 **Matrix variate distributions**
- 2 **Matrix distributions for system matrices**
 - Wishart random matrices
 - Parameter selection
 - Reduced computational modelling
- 3 **Identification of the dispersion parameters**
- 4 **Examples applications for random matrix theory**
- 5 **Experimental validations**
- 6 **Hybrid uncertainty formulations**
- 7 **Domain decomposition for multi-frequency scale problems**
 - Domain decomposition for two domains
 - Computational approach for uncertainty propagation
 - Stochastic interface problem
 - Stochastic interior problems
 - Numerical example

- Broadly speaking, there are two complimentary approaches to quantify uncertainties in a model. The first is the *parametric approach* and the second is the *non-parametric approach*.
- In the parametric approach the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element.
- This type of approach is suitable to quantify aleatoric uncertainties.
- Epistemic uncertainty on the other hand do not explicitly depend on the systems parameters. For example, there can be unquantified errors associated with the equation of motion (linear on non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods (e.g, discretisation of displacement fields, truncation and roundoff errors, tolerances in the optimization and iterative algorithms, step-sizes in the time-integration methods).
- The parametric approach is not suitable to quantify this type of uncertainties and a non-parametric approach is needed for this purpose.
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Non-parametric uncertainty

- In majority of practical problems, the complete information regarding uncertainties is not available.
- In some cases, for example, cars manufactured from a production chain and soil property distribution in a construction site, it may be possible to obtain probabilistic descriptions of the system parameters experimentally.
- However, obtaining such probabilistic information may be prohibitively expensive for many problems.
- In another class of problems, for example, dynamic analysis of a space vehicle, even 'in principle' it may not be possible to obtain probabilistic information because there may be just 'only one sample'. However, there will still be some uncertainties in the model.
- Regardless of what type of uncertainties exist in the model of a linear dynamical system, it must be characterized by the random matrices **M**, **C** and **K**.
- These We obtain the probability density function of the random matrices based on the maximum entropy principle.
- It will be shown that Wishart random matrix is the simplest physically realistic random matrix model for the system matrices appearing in linear structural dynamical systems.

The equation of motion is given by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \quad (1)$$

- Due to the presence of uncertainty \mathbf{M} , \mathbf{C} and \mathbf{K} become random matrices.
- The main objectives in the ‘forward problem’ are:
 - to quantify uncertainties in the system matrices
 - to predict the variability in the response vector \mathbf{q}
- We aim to derive the probability density function of the system matrices directly.

- Derive the matrix variate probability density functions of \mathbf{M} , \mathbf{C} and \mathbf{K} ¹ using available information.
- The main assumption is that a reliable model of the baseline system matrices is known.
- Additionally we need to assume that ‘some information’ on the dispersion of the system matrices from the ‘mean’ model is available.
- Once the distribution is identified, we propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)
- Several ways the parameters of the distribution can be identified.

¹AIAA Journal, 45[7] (2007), pp. 1748-1762

Matrix variate distributions

- A random matrix can be considered as an observable phenomenon representable in the form of a matrix which under repeated observation yields different non-deterministic outcomes.
- Therefore, a random matrix is simply a collection of random variables which may satisfy certain rules (for example symmetry, positive definiteness etc). Random matrices were introduced by Wishart in 1928 in the context of multivariate statistics.
- However, the Random Matrix Theory (RMT) was not used in other branches until 1950s when Wigner published his works (leading to the Nobel prize in Physics in 1963) on the eigenvalues of random matrices arising in high-energy physics.
- Using an asymptotic theory for large dimensional matrices, Wigner was able to bypass the Schrödinger equation and explain the statistics of measured atomic energy levels in terms of the limiting eigenvalues of these random matrices.
- Since then research on random matrices has continued to attract interests in multivariate statistics, physics, number theory and more recently in mechanical and electrical engineering.

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If \mathbf{A} is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.
- Density of a random matrix is effectively the joint density function of all its elements

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of \mathbf{X} is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \det\{\mathbf{\Sigma}\}^{-p/2} \det\{\mathbf{\Psi}\}^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X} - \mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X} - \mathbf{M})^T\right\} \quad (2)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Symmetric Gaussian random matrix

- Let $\mathbf{Y} \in \mathbb{R}^{n \times n}$ be a symmetric random matrix and \mathbf{M} , Σ and Ψ are $n \times n$ constant matrices such that the commutative relation $\Sigma\Psi = \Psi\Sigma$ holds. If the $n(n+1)/2 \times 1$ vector $\text{vecp}(\mathbf{Y})$ formed from \mathbf{Y} is distributed as $N_{n(n+1)/2,1}(\text{vecp}(\mathbf{M}), \mathbf{B}_n^T(\Sigma \otimes \Psi)\mathbf{B}_n)$, then \mathbf{Y} is said to have a symmetric matrix variate Gaussian distribution with mean \mathbf{M} and covariance matrix $\mathbf{B}_n^T(\Sigma \otimes \Psi)\mathbf{B}_n$ and its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4} \det \left\{ \mathbf{B}_n^T(\Sigma \otimes \Psi)\mathbf{B}_n \right\}^{-1/2} \text{etr} \left\{ -\frac{1}{2} \Sigma^{-1}(\mathbf{Y} - \mathbf{M})\Psi^{-1}(\mathbf{Y} - \mathbf{M}) \right\} \quad (3)$$

- This distribution is usually denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}(\mathbf{M}, \mathbf{B}_n^T(\Sigma \otimes \Psi)\mathbf{B}_n)$.
- For a symmetric matrix $\mathbf{Y} \in \mathbb{R}^{n \times n}$, $\text{vecp}(\mathbf{Y})$ is a $n(n+1)/2$ -dimensional column vector formed from the elements above and including the diagonal of \mathbf{Y} taken columnwise. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2 \times n(n+1)/2}$ are given by

$$(B_n)_{ij,gh} = \frac{1}{2} (\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}), \quad i \leq n, j \leq n, g \leq h \leq n \quad (4)$$

where δ_{ij} is the usual Kronecker's delta

- A $n \times n$ symmetric positive definite random matrix \mathbf{S} is said to have a Wishart distribution with parameters $p \geq n$ and $\Sigma \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2}p \right) \det \{ \Sigma \}^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \text{etr} \left\{ -\frac{1}{2} \Sigma^{-1} \mathbf{S} \right\} \quad (5)$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \Sigma)$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

- Wishart distribution is *the* most important distribution for structural dynamics due to its symmetry and nonnegative definite property.

Matrix variate gamma distribution

- A $n \times n$ symmetric positive definite random matrix \mathbf{W} is said to have a matrix variate gamma distribution with parameters a and $\Psi \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{ \Gamma_n(a) \det \{ \Psi \}^{-a} \right\}^{-1} \det \{ \mathbf{W} \}^{a - \frac{1}{2}(n+1)} \text{etr} \{ -\Psi \mathbf{W} \};$$
$$\Re(a) > \frac{1}{2}(n-1) \quad (6)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \Psi)$.

- Comparing this distribution with the Wishart distribution, we have $G_n(a, \Psi) = W_n(2a, \Psi^{-1}/2)$. The main difference between the gamma and the Wishart distribution is that originally only integer values were considered for the shape parameter p in the Wishart distribution.
- From an analytical point of view the gamma and the Wishart distributions are identical.

The distribution of the random system matrices \mathbf{M} , \mathbf{C} and \mathbf{K} should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall \omega$. This ensures that the moments of the response exist for all frequency values.

Suppose that the mean values of \mathbf{M} , \mathbf{C} and \mathbf{K} are given by \mathbf{M}_0 , \mathbf{C}_0 and \mathbf{K}_0 respectively. Using the notation \mathbf{G} (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \rightarrow \mathbb{R}$. We have the following constraints to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} = 1 \quad (\text{normalization}) \quad (7)$$

$$\text{and} \quad \int_{\mathbf{G}>0} \mathbf{G} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} = \mathbf{G}_0 \quad (\text{the mean matrix}) \quad (8)$$

- Suppose that the inverse moments up to order ν of the system matrix exist. This implies that $E \left[\left\| \mathbf{G}^{-1} \right\|_F^\nu \right]$ should be finite. Here the Frobenius norm of matrix \mathbf{A} is given by $\| \mathbf{A} \|_F = \left(\text{Trace} \left(\mathbf{A} \mathbf{A}^T \right) \right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expressed by

$$E \left[\ln \det \{ \mathbf{G} \}^{-\nu} \right] < \infty$$

- We extend the maximum entropy principle to matrix variate distribution to obtain the density of the matrices
- This requires calculus of variation on matrix quantities

- The Lagrangian becomes:

$$\begin{aligned} \mathcal{L}(p_{\mathbf{G}}) = & - \int_{\mathbf{G}_{>0}} p_{\mathbf{G}}(\mathbf{G}) \ln \{p_{\mathbf{G}}(\mathbf{G})\} d\mathbf{G} + \\ & (\lambda_0 - 1) \left(\int_{\mathbf{G}_{>0}} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} - 1 \right) - \nu \int_{\mathbf{G}_{>0}} \ln \det \{\mathbf{G}\} p_{\mathbf{G}} d\mathbf{G} \\ & + \text{Trace} \left(\mathbf{\Lambda}_1 \left[\int_{\mathbf{G}_{>0}} \mathbf{G} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} - \mathbf{G}_0 \right] \right) \quad (9) \end{aligned}$$

Note: ν cannot be obtained uniquely!

- Using the calculus of variation

$$\frac{\partial \mathcal{L}(p_{\mathbf{G}})}{\partial p_{\mathbf{G}}} = 0$$

$$\text{or } -\ln \{p_{\mathbf{G}}(\mathbf{G})\} = \lambda_0 + \text{Trace}(\mathbf{\Lambda}_1 \mathbf{G}) - \ln \det \{\mathbf{G}\}^\nu$$

$$\text{or } p_{\mathbf{G}}(\mathbf{G}) = \exp \{-\lambda_0\} \det \{\mathbf{G}\}^\nu \text{etr} \{-\mathbf{\Lambda}_1 \mathbf{G}\}$$

- Using the matrix variate Laplace transform ($\mathbf{T} \in \mathbb{R}_{n,n}$, $\mathbf{S} \in \mathbb{C}_{n,n}$, $a > (n+1)/2$)

$$\int_{\mathbf{T}>0} \text{etr}\{-\mathbf{S}\mathbf{T}\} \det\{\mathbf{T}\}^{a-(n+1)/2} d\mathbf{T} = \Gamma_n(a) \det\{\mathbf{S}\}^{-a}$$

and substituting $p_{\mathbf{G}}(\mathbf{G})$ into the constraint equations it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = r^{-nr} \{\Gamma_n(r)\}^{-1} \det\{\mathbf{G}_0\}^{-r} \det\{\mathbf{G}\}^{\nu} \text{etr}\{-r\mathbf{G}_0^{-1}\mathbf{G}\} \quad (10)$$

where $r = \nu + (n+1)/2$.

- Comparing it with the Wishart distribution we have: If ν -th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of \mathbf{G} is available, say \mathbf{G}_0 , then the maximum-entropy pdf of \mathbf{G} follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\Sigma = \mathbf{G}_0/(2\nu + n + 1)$, that is $\mathbf{G} \sim W_n(2\nu + n + 1, \mathbf{G}_0/(2\nu + n + 1))$.

- Covariance tensor of \mathbf{G} :

$$\text{cov}(G_{ij}, G_{kl}) = \frac{1}{2\nu + n + 1} (G_{0_{ik}} G_{0_{jl}} + G_{0_{il}} G_{0_{jk}})$$

- Normalized standard deviation matrix

$$\delta_G^2 = \frac{\mathbb{E} [\|\mathbf{G} - \mathbb{E}[\mathbf{G}]\|_F^2]}{\|\mathbb{E}[\mathbf{G}]\|_F^2} = \frac{1}{2\nu + n + 1} \left\{ 1 + \frac{\{\text{Trace}(\mathbf{G}_0)\}^2}{\text{Trace}(\mathbf{G}_0^2)} \right\}$$

- $\delta_G^2 \leq \frac{1+n}{2\nu+n+1}$ and $\nu \uparrow \Rightarrow \delta_G^2 \downarrow$.

- Suppose we 'know' (e.g, by measurements or stochastic finite element modeling) the mean (\mathbf{G}_0) and the (normalized) standard deviation (δ_G) of the system matrices:

$$\delta_G^2 = \frac{\mathbb{E} \left[\|\mathbf{G} - \mathbb{E}[\mathbf{G}] \|_F^2 \right]}{\|\mathbb{E}[\mathbf{G}] \|_F^2}. \quad (11)$$

This is known as the *dispersion parameter*.

- The parameters of the Wishart distribution can be identified using the expressions derived before.
- Samples from the Wishart distribution can be drawn and MCS can be used to obtain system response and eigensolutions.
- We consider some strategies for reduced computational approaches.

- Taking the Laplace transform of the equation of motion:

$$[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \quad (12)$$

The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\phi_j = \omega_j^2\mathbf{M}\phi_j, \quad j = 1, 2, \dots, n \quad (13)$$

where ω_j^2 and ϕ_j are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- We form the truncated undamped modal matrices $m \leq n$

$$\mathbf{\Omega} = \text{diag}[\omega_1, \omega_2, \dots, \omega_m] \quad \text{and} \quad \mathbf{\Phi} = [\phi_1, \phi_2, \dots, \phi_m]. \quad (14)$$

$$\text{so that} \quad \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \mathbf{\Omega}^2 \quad \text{and} \quad \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I}_m \quad (15)$$

- Transforming it into the reduced modal coordinates:

$$[s^2 \mathbf{I}_m + s \mathbf{C}' + \Omega^2] \bar{\mathbf{q}}' = \bar{\mathbf{f}}' \quad (16)$$

- Here

$$\mathbf{C}' = \Phi^T \mathbf{C} \Phi = 2\zeta \Omega, \quad \bar{\mathbf{q}} = \Phi \bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \Phi^T \bar{\mathbf{f}} \quad (17)$$

- When we consider random systems, the matrix of eigenvalues Ω^2 will be a random matrix of dimension m . Suppose this random matrix is denoted by $\Xi \in \mathbb{R}^{m \times m}$:

$$\Omega^2 \sim \Xi \quad (18)$$

- Since Ξ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix Ψ_r such that

$$\Psi_r^T \Xi \Psi_r = \Omega_r^2 \quad (19)$$

Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix Ξ .

- Recalling that $\Psi_r^T \Psi_r = \mathbf{I}_m$ we obtain

$$\bar{\mathbf{q}}' = [s^2 \mathbf{I}_m + s \mathbf{C}' + \Omega^2]^{-1} \bar{\mathbf{f}}' \quad (20)$$

$$= \Psi_r [s^2 \mathbf{I}_m + 2s\zeta \Omega_r + \Omega_r^2]^{-1} \Psi_r^T \bar{\mathbf{f}}' \quad (21)$$

- The response in the original coordinate can be obtained as

$$\begin{aligned}\bar{\mathbf{q}}(s) &= \Phi \bar{\mathbf{q}}'(s) = \Phi \Psi_r [s^2 \mathbf{I}_n + 2s\zeta \Omega_r + \Omega_r^2]^{-1} (\Phi \Psi_r)^T \bar{\mathbf{f}}(s) \\ &= \sum_{j=1}^m \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j \omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j}.\end{aligned}$$

Here

$$\Omega_r = \text{diag}[\omega_{r_1}, \omega_{r_2}, \dots, \omega_{r_m}], \quad \mathbf{X}_r = \Phi \Psi_r = [\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \dots, \mathbf{x}_{r_m}]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.

- Conventional modal truncation has been applied to reduce the system. This will lead to a smaller random eigenvalue problem to be solved.

Approach 1: \mathbf{M} and \mathbf{K} are fully correlated Wishart (most complex). For this case $\mathbf{M} \sim W_n(p_M, \Sigma_M)$, $\mathbf{K} \sim W_n(p_K, \Sigma_K)$ with $E[\mathbf{M}] = \mathbf{M}_0$ and $E[\mathbf{K}] = \mathbf{K}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices.

Here

$$\Sigma_M = \mathbf{M}_0/p_M, p_M = \frac{\gamma_M + 1}{\delta_M} \quad (22)$$

$$\text{and } \Sigma_K = \mathbf{K}_0/p_K, p_K = \frac{\gamma_K + 1}{\delta_K} \quad (23)$$

$$\gamma_G = \{\text{Trace}(\mathbf{G}_0)\}^2 / \text{Trace}(\mathbf{G}_0^2) \quad (24)$$

Approach 2: Scalar Wishart (most simple) In this case it is assumed that

$$\Xi \sim W_m \left(p, \frac{a^2}{n} \mathbf{I}_m \right) \quad (25)$$

Considering $E[\Xi] = \mathbf{\Omega}_0^2$ and $\delta_{\Xi} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1 + \gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \text{Trace}(\mathbf{\Omega}_0^2) / p \quad (26)$$

Approach 3: Diagonal Wishart with different entries (something in the middle). For this case $\Xi \sim W_m(p, \Omega_0^2/\theta)$ with $E[\Xi^{-1}] = \Omega_0^{-2}$ and $\delta_\Xi = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem. The parameters can be obtained as

$$p = m + 1 + \theta \quad \text{and} \quad \theta = \frac{(1 + \gamma_H)}{\delta_H^2} - (n + 1) \quad (27)$$

- Defining $\mathbf{H}_0 = \mathbf{M}_0^{-1} \mathbf{K}_0$, the constant γ_H :

$$\gamma_H = \frac{\{\text{Trace}(\mathbf{H}_0)\}^2}{\text{Trace}(\mathbf{H}_0^2)} = \frac{\{\text{Trace}(\mathbf{\Omega}_0^2)\}^2}{\text{Trace}(\mathbf{\Omega}_0^4)} = \frac{\left(\sum_j \omega_{0j}^2\right)^2}{\sum_j \omega_{0j}^4} \quad (28)$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\delta_H^2 = \frac{(p_M^2 + (p_K - 2 - 2n)p_M + (-n - 1)p_K + n^2 + 1 + 2n) \gamma_H}{p_K (-p_M + n) (-p_M + n + 3)} + \frac{p_M^2 + (p_K - 2n)p_M + (1 - n)p_K - 1 + n^2}{p_K (-p_M + n) (-p_M + n + 3)} \quad (29)$$

Summary of the method

A step-by-step method for implementing the new computational approach in conjunction with any general purpose finite element software is given below:

- Form the deterministic mass and stiffness matrices \mathbf{M}_0 and \mathbf{K}_0 using the standard finite element method and the modal damping factors ζ_j . Select the number of modes $m < n$. The number of modes to be retained, m should be selected based on the frequency of excitation.
- Solve the deterministic undamped eigenvalue problem

$$\mathbf{K}_0 \phi_{0j} = \omega_{0j}^2 \mathbf{M}_0 \phi_{0j}, \quad j = 1, 2, \dots, m \quad (30)$$

and create the matrix

$$\Phi_0 = [\phi_{01}, \phi_{02}, \dots, \phi_{0m}] \in \mathbb{R}^{n \times m} \quad (31)$$

Calculate the ratio

$$\gamma_H = \left(\sum_{j=1}^m \omega_{0j}^2 \right)^2 / \sum_{j=1}^m \omega_{0j}^4 \quad (32)$$

- Obtain the dispersion parameters δ_M and δ_K corresponding to the mass and stiffness matrices. This can be obtained from physical or computer experiments.

- Obtain the dispersion parameter of the generalized Wishart matrix \mathbf{H}

$$\delta_H^2 = \frac{(p_M^2 + (p_K - 2 - 2n)p_M + (-n - 1)p_K + n^2 + 1 + 2n)\gamma_H}{p_K(-p_M + n)(-p_M + n + 3)} + \frac{p_M^2 + (p_K - 2n)p_M + (1 - n)p_K - 1 + n^2}{p_K(-p_M + n)(-p_M + n + 3)} \quad (33)$$

where

$$p_M = \frac{1}{\delta_M^2} \left\{ 1 + \{\text{Trace}(\mathbf{M}_0)\}^2 / \text{Trace}(\mathbf{M}_0^2) \right\} \quad (34)$$

$$\text{and } p_K = \frac{1}{\delta_K^2} \left\{ 1 + \{\text{Trace}(\mathbf{K}_0)\}^2 / \text{Trace}(\mathbf{K}_0^2) \right\} \quad (35)$$

- Calculate the parameters

$$\theta = \frac{(1 + \gamma_H)}{\delta_H^2} - (m + 1) \quad \text{and} \quad p = [m + 1 + \theta] \quad (36)$$

where p is approximated to the nearest integer of $m + 1 + \theta$.

Summary of the method

- Create an $m \times p$ matrix \mathbf{Y} such that

$$Y_{ij} = \omega_{0_i} \hat{Y}_{ij} / \sqrt{\theta}; \quad i = 1, 2, \dots, m; j = 1, 2, \dots, p \quad (37)$$

where \hat{Y}_{ij} are independent and identically distributed (i.i.d.) Gaussian random numbers with zero mean and unit standard deviation.

- Simulate the $m \times m$ Wishart random matrix

$$\Xi = \mathbf{Y}\mathbf{Y}^T \quad \text{or} \quad \Xi_{ij} = \frac{\omega_{0_i} \omega_{0_j}}{\theta} \sum_{k=1}^p \hat{Y}_{ik} \hat{Y}_{jk}; \quad i = 1, 2, \dots, m; j = 1, 2, \dots, m \quad (38)$$

Since Ξ is symmetric, only the upper or lower triangular part need to be simulated.

- Solve the symmetric eigenvalue problem ($\Omega_r, \Psi_r \in \mathbb{R}^{m \times m}$) for every sample

$$\Xi \Psi_r = \Omega_r^2 \Psi_r \quad (39)$$

and obtain the random eigenvector matrix

$$\mathbf{X}_r = \Phi_0 \Psi_r = [\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \dots, \mathbf{x}_{r_m}] \in \mathbb{R}^{n \times m} \quad (40)$$

- Finally calculate the dynamic response in the frequency domain as

$$\bar{\mathbf{q}}_r(i\omega) = \sum_{j=1}^m \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{-\omega^2 + 2i\omega\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j} \quad (41)$$

The samples of the response in the time domain can also be obtained from the random eigensolutions as

$$\mathbf{q}_r(t) = \sum_{j=1}^m a_{r_j}(t) \mathbf{x}_{r_j},$$

$$\text{where } a_{r_j}(t) = \frac{1}{\omega_{r_j}} \int_0^t \mathbf{x}_{r_j}^T \mathbf{f}(\tau) e^{-\zeta_j\omega_{r_j}(t-\tau)} \sin(\omega_{r_j}(t-\tau)) d\tau \quad (42)$$

Identification of the dispersion parameters - 1

- The dispersion parameter is related to the first and second moments of eigenvalues

$$\delta_G^2 = \frac{\sum_{j=1}^n \text{E} [\lambda_j^2]}{\sum_{j=1}^n \text{E} [\lambda_j]^2} - 1 \quad (43)$$

so that, if information on the eigenvalues of the system is available, the dispersion parameter can be retrieved.

- Suppose the standard deviation of each eigenvalue is σ_j and mean of each eigenvalue is $\bar{\lambda}_j$. Therefore $\text{E} [\lambda_j] = \bar{\lambda}_j$ and $\text{E} [\lambda_j^2] = \sigma_j^2 + \bar{\lambda}_j^2$
- Suppose the standard deviation is expressed as a fraction of the respective mean values $\sigma_j = \epsilon_j \bar{\lambda}_j$.
- Using these and applying modal truncation, from Eq. (43) we have

$$\delta_G^2 = \frac{\sum_{j=1}^m \bar{\lambda}_j^2 \epsilon_j^2}{\sum_{j=1}^m \bar{\lambda}_j^2} \quad (44)$$

- This can be measured from experiments.

- The dispersion parameter for the mass matrix can be obtained as

$$p_M = \frac{(n+1) \sum_{j=1}^n E[\lambda_j]}{\sum_{j=1}^n E[\lambda_j] - \text{Trace}((\bar{\mathbf{M}})^{-1/2} \bar{\mathbf{K}} (\bar{\mathbf{M}})^{-1/2})} \quad (45)$$

- The dispersion parameter for the stiffness matrix can be obtained as

$$p_K = \frac{(p_M - n - 1)^3 T_1^2 + (p_M - n + 3)(T_2)}{(n_1 - p_M) T_1^2 + (p_M - n) T_2 ((p_M - n)(p_M - n - 3)(\delta_G^2 - 1) - 1)} \quad (46)$$

with

$$T_1 = \text{Trace}((\bar{\mathbf{M}})^{-1} \bar{\mathbf{K}}) \quad \text{and} \quad T_2 = \text{Trace}(((\bar{\mathbf{M}})^{-1} \bar{\mathbf{K}})^2). \quad (47)$$

Identification of the dispersion parameters- 2

- We assume that the matrix $\mathbf{G}(\theta) = \{\mathbf{K}(\theta), \mathbf{M}(\theta)\}$ can be expanded as

$$\mathbf{G}(\theta) = \mathbf{G}_0 + \epsilon_G \sum_{j=1}^M \xi_{G_j}(\theta) \mathbf{G}_j \quad (48)$$

- Substituting this expansion in the expression of the dispersion parameter one has

$$\delta_G^2 = \frac{\mathbb{E} \left[\left\| \epsilon_G \sum_{j=1}^M \xi_{G_j}(\theta) \mathbf{G}_j \right\|_F^2 \right]}{\|\mathbb{E}[\mathbf{G}]\|_F^2} \quad (49)$$

- Because the matrices \mathbf{G}_j are symmetric, using the definition of Frobenius norm, from Eq. (49) we have

$$\delta_G^2 = \frac{\mathbb{E} \left[\text{Trace} \left(\left(\epsilon_G \sum_{j=1}^M \xi_{G_j}(\theta) \mathbf{G}_j \right) \left(\epsilon_G \sum_{k=1}^M \xi_{G_k}(\theta) \mathbf{G}_k \right) \right) \right]}{\|\mathbf{G}_0\|_F^2} \quad (50)$$

Identification of the dispersion parameters- 2

- Since both trace and expectation operators are linear they can be swapped. Doing this we obtain

$$\delta_G^2 = \frac{\epsilon_G^2 \text{Trace} \left(\mathbb{E} \left[\left(\sum_{j=1}^M \sum_{k=1}^M \xi_{G_j}(\theta) \xi_{G_k}(\theta) \mathbf{G}_j \mathbf{G}_k \right) \right] \right)}{\|\mathbf{G}_0\|_F^2} \quad (51)$$

- Recalling that the matrices \mathbf{G}_j are not random and $\{\xi_{G_1}(\theta), \xi_{G_2}(\theta), \dots\}$ is a set of uncorrelated random variables with zero mean and $\mathbb{E} [\xi_{G_j}(\theta) \xi_{G_k}(\theta)] = \delta_{jk}$, we have

$$\begin{aligned} \delta_G^2 &= \frac{\epsilon_G^2 \text{Trace} \left(\left(\sum_{j=1}^M \sum_{k=1}^M \mathbb{E} [\xi_{G_j}(\theta) \xi_{G_k}(\theta)] \mathbf{G}_j \mathbf{G}_k \right) \right)}{\|\mathbf{G}_0\|_F^2} \\ &= \frac{\epsilon_G^2 \text{Trace} \left(\left(\sum_{j=1}^M \mathbf{G}_j^2 \right) \right)}{\|\mathbf{G}_0\|_F^2} \end{aligned} \quad (52)$$

Identification of the dispersion parameters- 2

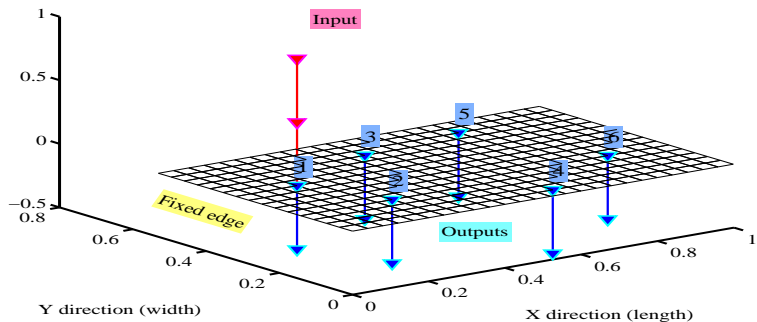
- In the above expressions, the uncorrelated nature of the random variables allowed us to transform the double summation into a single summation. Finally, as trace and sum operators can be interchanged and $\text{Trace}(\mathbf{G}_j^2) = \|\mathbf{G}_j\|_F^2$, we have

$$\delta_G^2 = \epsilon_G^2 \frac{\sum_j^M \|\mathbf{G}_j\|_F^2}{\|\mathbf{G}_0\|_F^2} \quad (53)$$

This result allows one to obtain the dispersion parameter using the stochastic finite element, therefore avoiding the direct Monte Carlo simulation. This expression also relates the stochastic finite element and random matrix theory.

- δ_M and δ_K obtained in this way can be used in (34) and (35) for the simulation of the random matrices.

A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m ³
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from <http://engweb.swan.ac.uk/~adhikaris/uq/>.

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E} (1 + \epsilon_E f_1(\mathbf{x})) \quad (54)$$

$$\mu(\mathbf{x}) = \bar{\mu} (1 + \epsilon_\mu f_2(\mathbf{x})) \quad (55)$$

$$\rho(\mathbf{x}) = \bar{\rho} (1 + \epsilon_\rho f_3(\mathbf{x})) \quad (56)$$

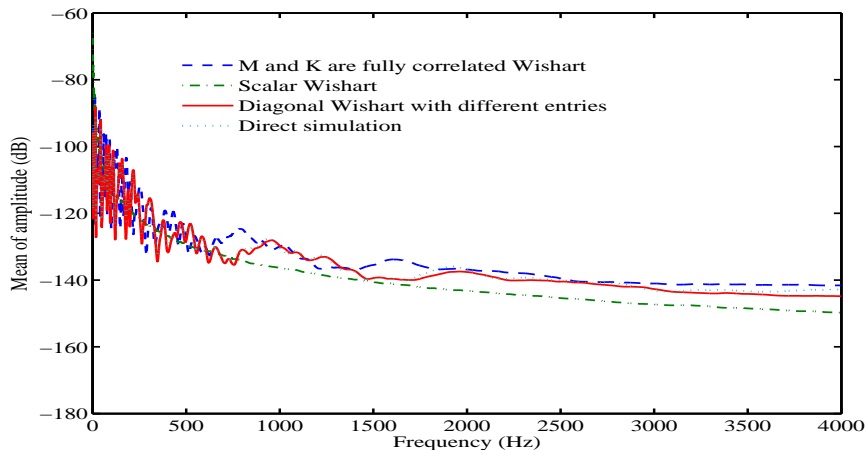
$$\text{and } t(\mathbf{x}) = \bar{t} (1 + \epsilon_t f_4(\mathbf{x})) \quad (57)$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x})$, $i = 1, \dots, 4$ are delta-correlated homogenous Gaussian random fields.

Uncertainty type 2: random attached oscillators

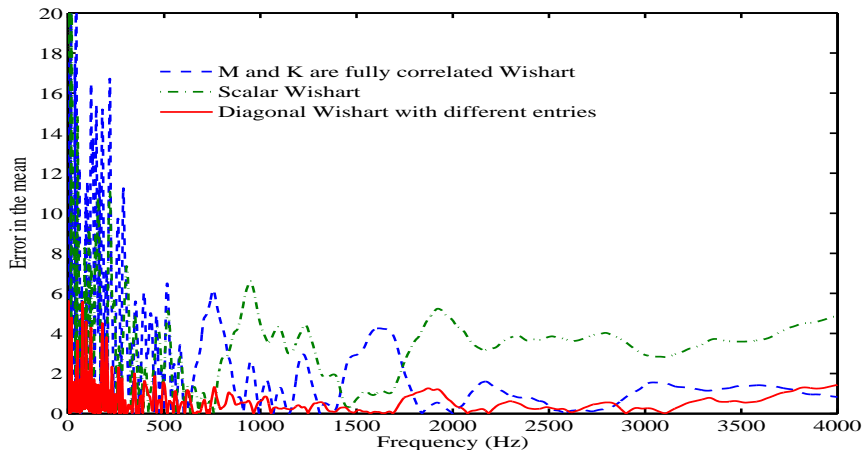
- Here we consider that the baseline plate is 'perturbed' by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty.
- This case will be investigated experimentally later.

Mean of cross-FRF: Utype 1



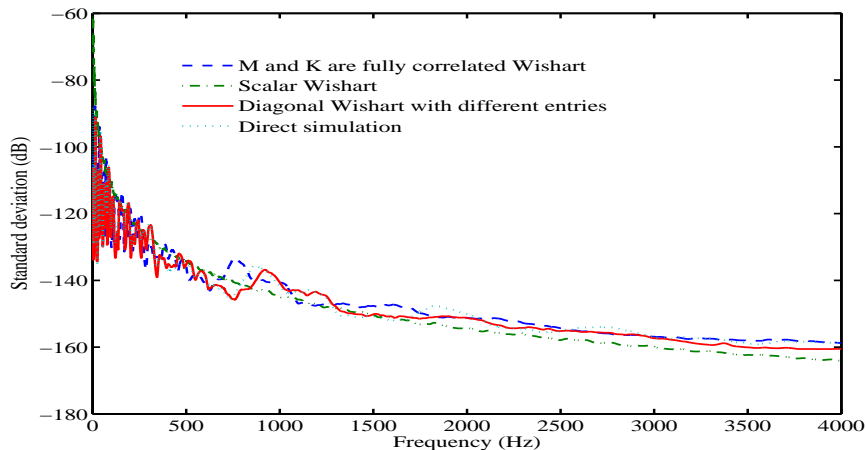
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the mean of cross-FRF: Utype 1



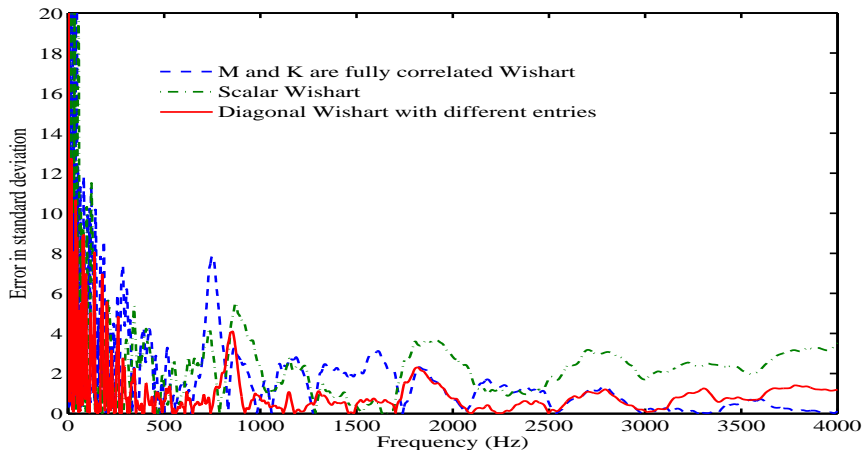
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Standard deviation of driving-point-FRF: Utype 1



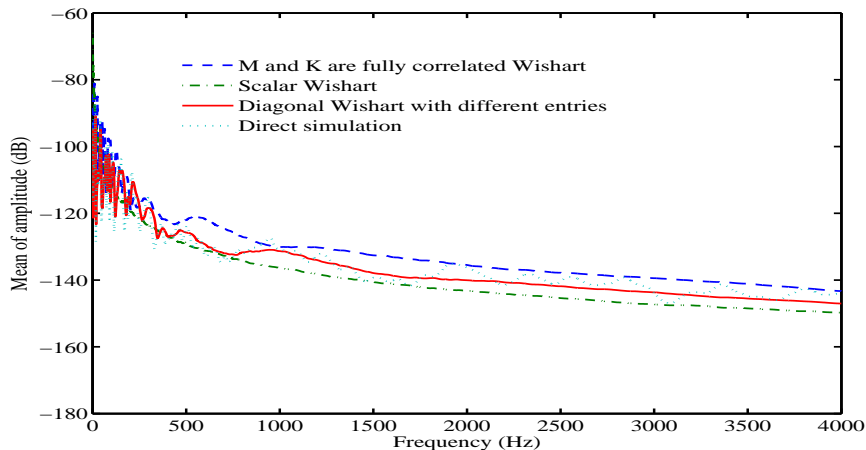
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the standard deviation of driving-point-FRF: Utype 1



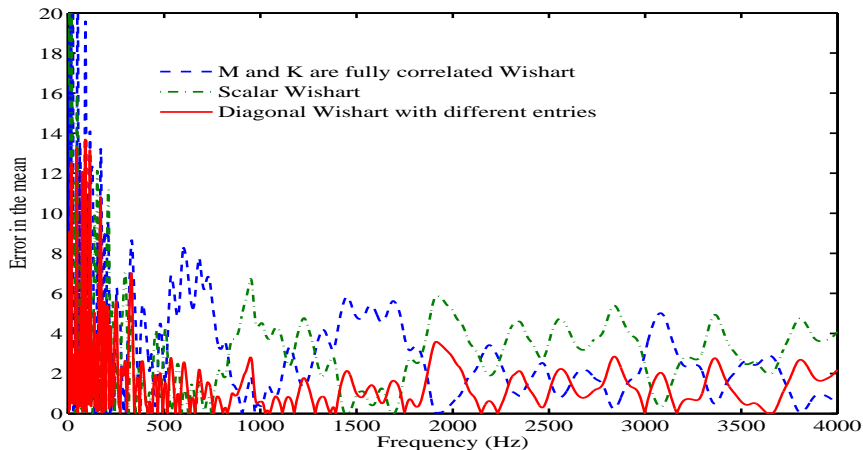
Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Mean of cross-FRF: Utype 2



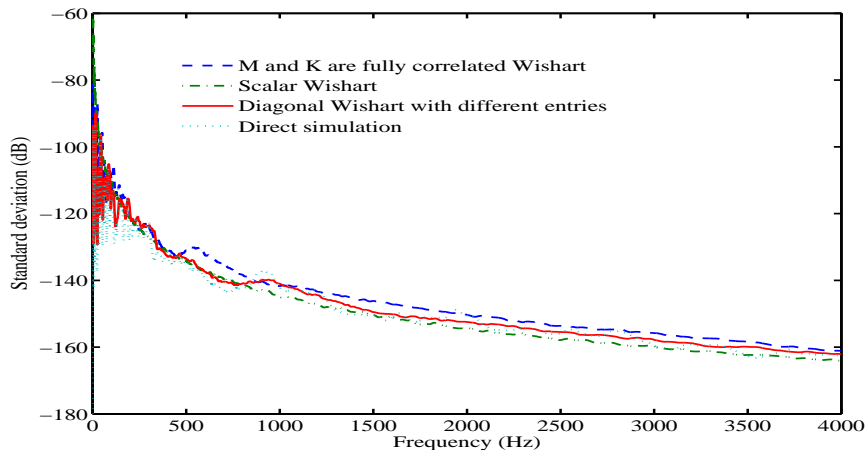
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the mean of cross-FRF: Utype 2



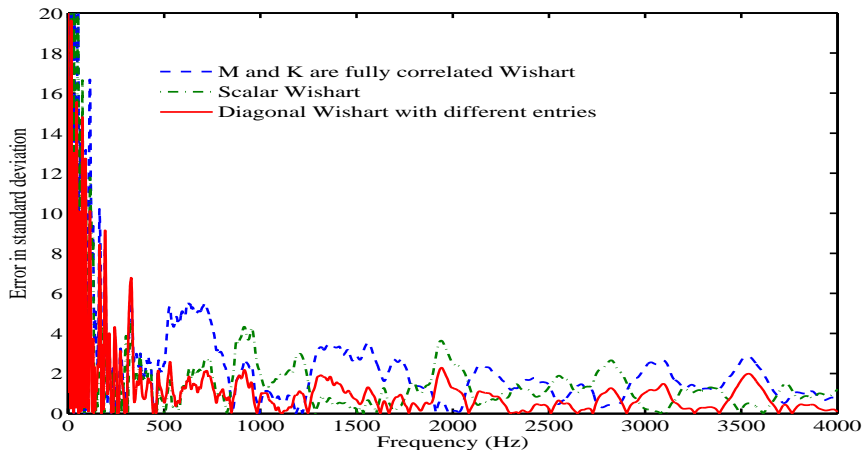
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Standard deviation of driving-point-FRF: Utype 2



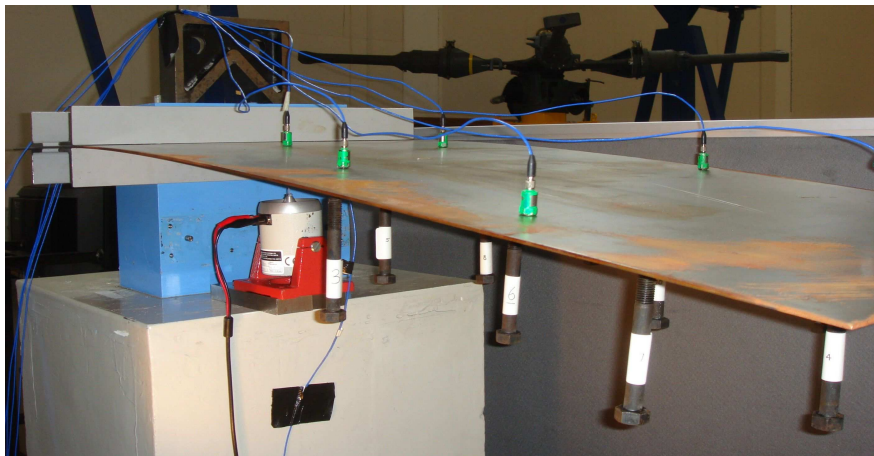
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the standard deviation of driving-point-FRF: Utype 2



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

A cantilever plate: front view



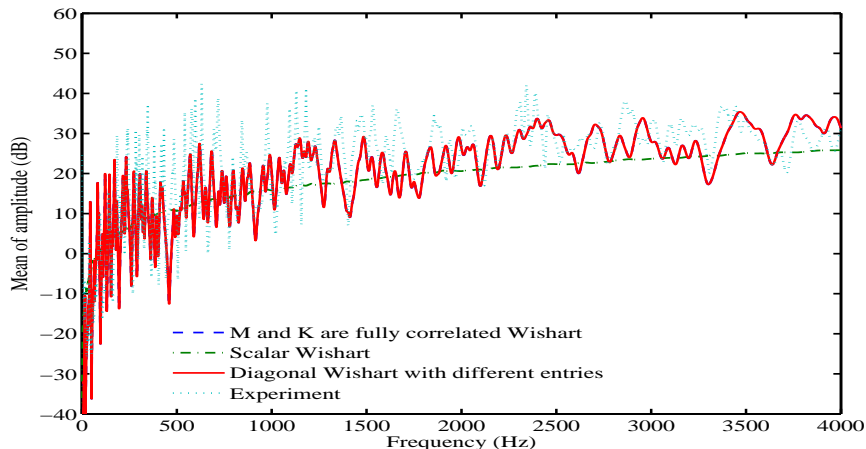
The test rig for the cantilever plate; front view.

A cantilever plate: side view



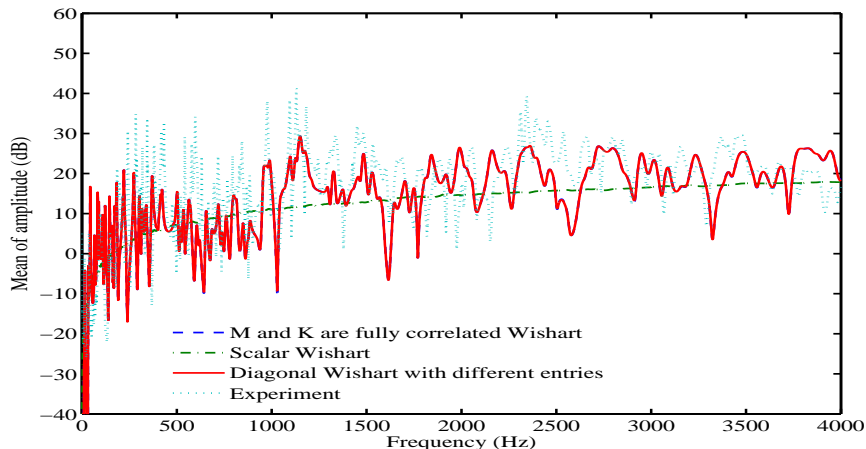
The test rig for the cantilever plate; side view.

Comparison of driving-point-FRF



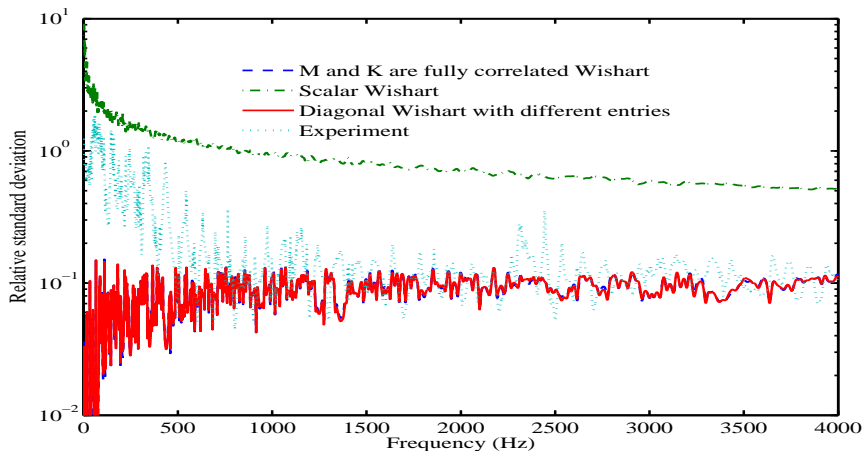
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



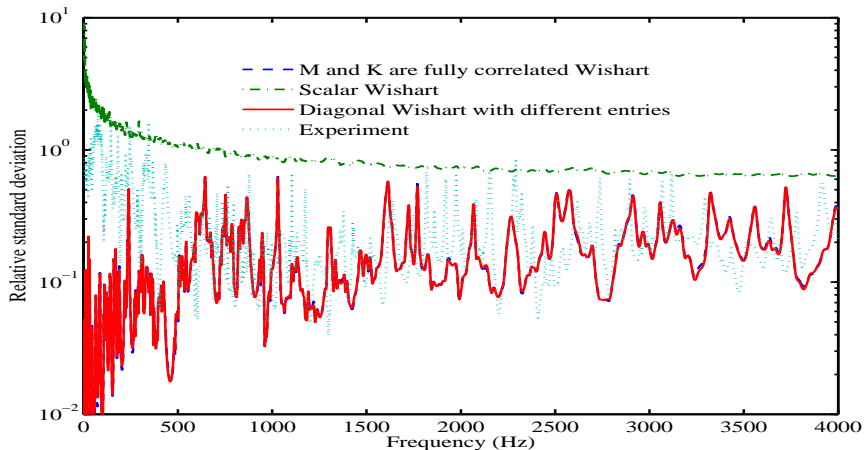
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of driving-point-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

- For **low-frequency** vibration problems (longer wavelength), parametric uncertainty model is normally used.
- Random field or random variables can be used to model uncertain parameters and stochastic finite element method can be used to propagate uncertainty.
- For **high-frequency** vibration problems (shorter wavelength), nonparametric uncertainty model is normally used.
- Random matrix model, such as those based on Wishart random matrices, can be used for this purpose.
- In majority of practical engineering problems, one expects a mixture of wavelengths.

- Complex dynamic structures such as aircrafts, helicopters contain several substructures.
- For a given frequency of excitation, the wavelength of vibration in different substructures can be significantly different.
- For example, in the context of an aircraft fuselage, the ring girders will have significantly longer wavelength of vibration compared to the thin panel for a given frequency of excitation.

- Complex dynamic structures such as aircrafts, helicopters contain several substructures.
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(a) Aircraft fuselage



(b) Car body

Parametric uncertainty: low-frequency vibration problem

- First few vibration modes (typically few tens) are participating in the dynamical response of interest
- Uncertainty models aim to characterise parametric uncertainty (type 'a')
- Random variable or random field models are used to represent uncertain parameters
- Well established methods such as stochastic finite element method (polynomial chaos, perturbation methods, spectral method) exist in literature
- A system matrix can be expressed as

$$\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^M \xi_i(\theta_1) \mathbf{A}_i$$

\mathbf{A}_0 : baseline model, $\xi_i(\theta_1)$: random variables

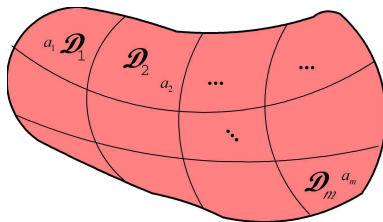
- Many vibration modes are (in hundreds) participating in the dynamical response of interest
- Uncertainty models aim to characterise non-parametric uncertainties (type 'b-d')
- Random matrix models can be used to represent uncertain system matrices
- A system matrix can be expressed as

$$\mathbf{A} = W_n(\delta_A, \mathbf{A}_0)$$

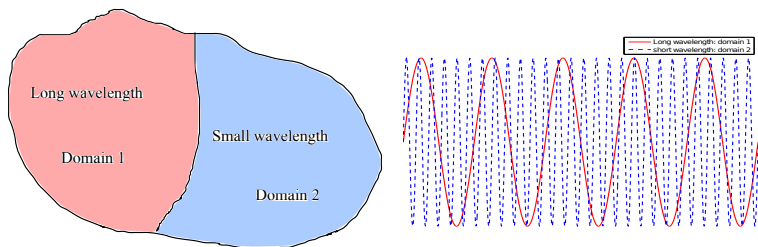
\mathbf{A}_0 : baseline model, δ_A : dispersion parameter, W_n : Wishart random matrix of dimension n .

Domain decomposition method

- Developed to solve a boundary value problem by splitting it into smaller boundary value problems on subdomains
- The problems on the subdomains are independent, which makes domain decomposition methods suitable for parallel computing
- Originally developed for numerical solution of partial differential equations (not explicitly for uncertainty quantification)
- Excellent and powerful computational tools are available



Domain decomposition method



Domain 1: $\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^M \xi_i(\theta_1) \mathbf{A}_i$ (dimension n_1) - parametric uncertainty

Domain 2: $\mathbf{A}(\theta_2) = W_{n_2}(\delta_A, \mathbf{A}_0)$ (dimension n_2) - nonparametric uncertainty

The equation of motion of a linear dynamic system in the frequency domain is

$$\mathbf{A}(\omega)\mathbf{u} = \mathbf{f} \quad (58)$$

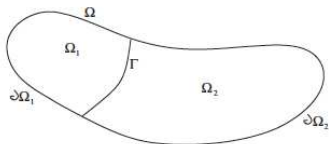
where the dynamic stiffness matrix over the whole domain Ω , $\mathbf{A}(\omega)$ is given by

$$\mathbf{A}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \in \mathbb{C}^n \quad (59)$$

We aim to subdivide the domain Ω into two nonoverlapping domains.

Two subdomains

The region Ω partitioned into two nonoverlapping subdomains Ω_1 and Ω_2 as below



The equilibrium equation of the system can be partitioned as

$$\begin{bmatrix} [\mathbf{A}_{II}^1]_{m_1 \times m_1} & 0 & [\mathbf{A}_{I\Gamma}^1]_{m_1 \times m_\Gamma} \\ 0 & [\mathbf{A}_{II}^2]_{m_2 \times m_2} & [\mathbf{A}_{I\Gamma}^2]_{m_2 \times m_\Gamma} \\ [\mathbf{A}_{\Gamma I}^1]_{m_\Gamma \times m_1} & [\mathbf{A}_{\Gamma I}^2]_{m_\Gamma \times m_2} & [\mathbf{A}_{\Gamma\Gamma}^1 + \mathbf{A}_{\Gamma\Gamma}^2]_{m_\Gamma \times m_\Gamma} \end{bmatrix} \times \quad (60)$$
$$\begin{Bmatrix} \mathbf{u}_I^1 \\ \mathbf{u}_I^2 \\ \mathbf{u}_\Gamma \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I^1 \\ \mathbf{f}_I^2 \\ \mathbf{f}_\Gamma^1 + \mathbf{f}_\Gamma^2 \end{Bmatrix}$$

The above equilibrium equation can be rearranged into following explicit forms (interior and interface problems):

$$[\mathbf{A}_{II}^1]\{\mathbf{u}_I^1\} = \{\mathbf{f}_I^1\} - [\mathbf{A}_{I\Gamma}^1]\{\mathbf{u}_\Gamma\} \quad (61)$$

$$[\mathbf{A}_{II}^2]\{\mathbf{u}_I^2\} = \{\mathbf{f}_I^2\} - [\mathbf{A}_{I\Gamma}^2]\{\mathbf{u}_\Gamma\} \quad (62)$$

$$\begin{aligned} & \underbrace{[[\mathbf{A}_{\Gamma\Gamma}^1] - [\mathbf{A}_{\Gamma I}^1][\mathbf{A}_{II}^1]^{-1}[\mathbf{A}_{I\Gamma}^1] + [\mathbf{A}_{\Gamma\Gamma}^2] - [\mathbf{A}_{\Gamma I}^2][\mathbf{A}_{II}^2]^{-1}[\mathbf{A}_{I\Gamma}^2]]}_{\mathbf{S}_1} \{\mathbf{u}_\Gamma\} \quad (63) \\ & \underbrace{= [[\{\mathbf{f}_\Gamma^1\} - [\mathbf{A}_{\Gamma I}^1][\mathbf{A}_{II}^1]^{-1}\{\mathbf{f}_I^1\}]}_{\mathbf{F}_1} + \underbrace{[[\{\mathbf{f}_\Gamma^2\} - [\mathbf{A}_{\Gamma I}^2][\mathbf{A}_{II}^2]^{-1}\{\mathbf{f}_I^2\}]}_{\mathbf{F}_2} \end{aligned}$$

The coefficient matrix $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is known as the *Schur complement* matrix.

We have two system matrices. For the domain with parametric uncertainty (long wavelength scale):

$$[\mathbf{A}^1(\theta_1)]_{n_1 \times n_1} = \begin{bmatrix} \mathbf{A}_{II}^1(\theta_1) & \mathbf{A}_{I\Gamma}^1(\theta_1) \\ \mathbf{A}_{\Gamma I}^1(\theta_1) & \mathbf{A}_{\Gamma\Gamma}^1(\theta_1) \end{bmatrix} = \mathbf{A}_0^1 + \sum_{i=1}^M \xi_i(\theta_1) \mathbf{A}_i^1 \quad (64)$$

with $n_1 = m_1 + m_\Gamma$.

For the domain with nonparametric uncertainty (short wavelength scale):

$$\mathbf{A}^2(\theta_2)_{n_2 \times n_2} = \begin{bmatrix} \mathbf{A}_{II}^2(\theta_2) & \mathbf{A}_{I\Gamma}^2(\theta_2) \\ \mathbf{A}_{\Gamma I}^2(\theta_2) & \mathbf{A}_{\Gamma\Gamma}^2(\theta_2) \end{bmatrix} = W_{n_2}(\delta_{A_2}, \mathbf{A}_0^2) \quad (65)$$

with $n_2 = m_2 + m_\Gamma$.

For the stochastic interface problem we have a system of (densely) coupled m_Γ complex stochastic equations

$$[\mathbf{S}_1(\theta_1) + \mathbf{S}_2(\theta_2)]\mathbf{u}_\Gamma(\theta_1, \theta_2) = \mathbf{F}_1(\theta_1) + \mathbf{F}_2(\theta_2) \quad (66)$$

where

$$\mathbf{S}_1(\theta_1) = \mathbf{A}_{\Gamma\Gamma}^1(\theta_1) - \mathbf{A}_{\Gamma I}^1(\theta_1)[\mathbf{A}_{II}^1(\theta_1)]^{-1}\mathbf{A}_{I\Gamma}^1(\theta_1) \quad (67)$$

$$\mathbf{F}_1(\theta_1) = \mathbf{f}_\Gamma^1 - \mathbf{A}_{\Gamma I}^1(\theta_1)[\mathbf{A}_{II}^1(\theta_1)]^{-1}\mathbf{f}_I^1 \quad (68)$$

and

$$\mathbf{S}_2(\theta_2) = \mathbf{A}_{\Gamma\Gamma}^2(\theta_2) - \mathbf{A}_{\Gamma I}^2(\theta_2)[\mathbf{A}_{II}^2(\theta_2)]^{-1}\mathbf{A}_{I\Gamma}^2(\theta_2) \quad (69)$$

$$\mathbf{F}_2(\theta_2) = \mathbf{f}_\Gamma^2 - \mathbf{A}_{\Gamma I}^2(\theta_2)[\mathbf{A}_{II}^2(\theta_2)]^{-1}\mathbf{f}_I^2 \quad (70)$$

Solving the interface problem we have $\mathbf{u}_\Gamma(\theta_1, \theta_2)$. This can be used to obtain the interior solutions as

$$\mathbf{u}_I^1(\theta_1, \theta_2) = [\mathbf{A}_{II}^1(\theta_1)]^{-1}[\mathbf{f}_I^1 - \mathbf{A}_{I\Gamma}^1(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \quad (71)$$

$$\mathbf{u}_I^2(\theta_1, \theta_2) = [\mathbf{A}_{II}^2(\theta_1)]^{-1}[\mathbf{f}_I^2 - \mathbf{A}_{I\Gamma}^2(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \quad (72)$$

The most computationally intensive parts of the solution process is obtaining $[\mathbf{A}_{II}^1(\theta_1)]^{-1}$ and $[\mathbf{A}_{II}^2(\theta_1)]^{-1}$ which involves the solution of m_1 and m_2 number of coupled complex stochastic equations.

Existing computational methods for uncertainty propagation can be used.

Recall that in the frequency domain

$$\mathbf{A}_{II}(\omega, \theta) = -\omega^2 \mathbf{M}_{II}(\theta) + i\omega \mathbf{C}_{II}(\theta) + \mathbf{K}_{II}(\theta) \quad (73)$$

Assuming proportional damping model, we have

$$[\mathbf{A}_{II}(\omega, \theta)]^{-1} = \sum_{k=1}^m \frac{\phi_k(\theta) \phi_k^T(\theta)}{\omega_k^2(\theta) - \omega^2 + 2i\zeta_k \omega_k(\theta)} \quad (74)$$

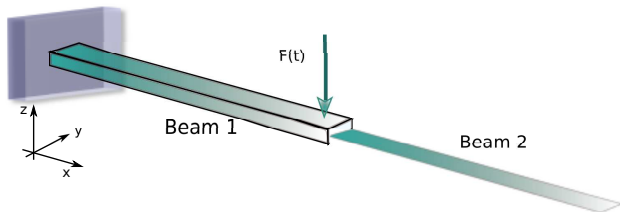
Here ζ_k are the modal damping factors and the eigenvalues are eigenvectors are obtained from

$$\mathbf{K}_{II}(\theta) \phi_k(\theta) = \omega_k^2 \mathbf{M}_{II}(\theta) \phi_k(\theta), \quad k = 1, 2, \dots \quad (75)$$

Any existing methods for random eigenvalue problem can be used (perturbation, polynomial chaos, Neumann series ...).

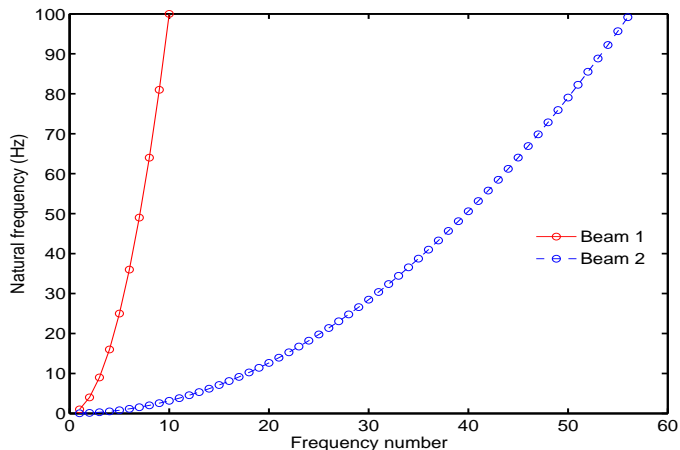
An Euler-Bernoulli beam example

- Two coupled Euler-Bernoulli beams with stochastic elasticity are considered



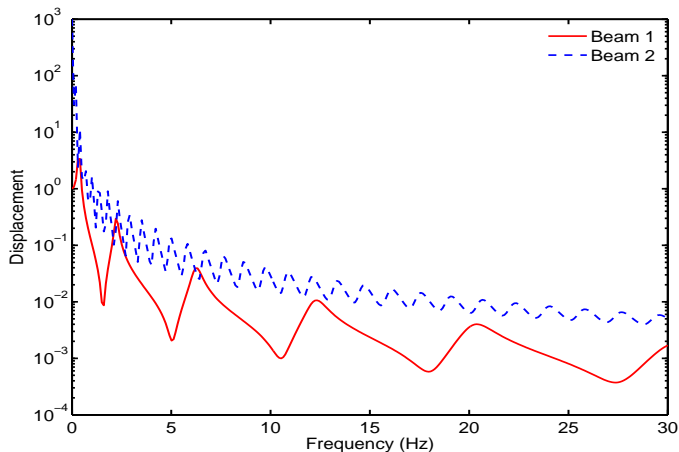
- $L_1 = 1$, $EI_{10} = 1/3$, $\rho A_1 = \pi^2/12$, $\zeta_1 = 0.04$
- $L_2 = L_1$, $EI_{20} = EI_{10}/10^3$, $\rho A_2 = \rho A_1$, $\zeta_2 = \zeta_1/2$
- We study the deflection of the beam under the action of a point harmonic load on the interior of beam 1.

Natural frequencies



Due to the difference in the stiffness values, beam 1 has less number of frequencies compared to beam 2 within a given frequency range.

Frequency response



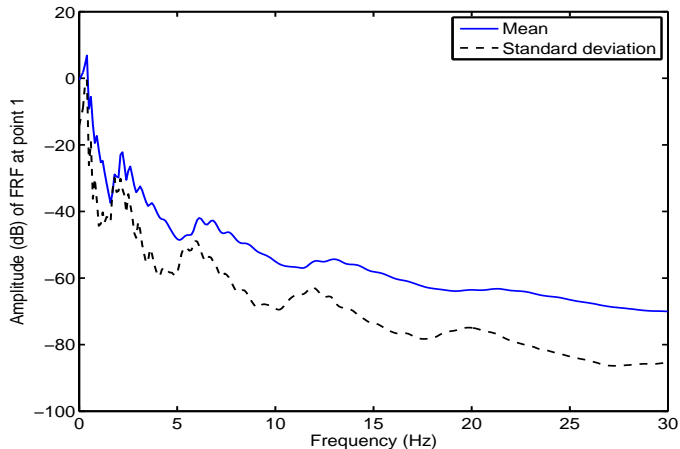
Frequency response functions of the two beams in isolation (in cantilever configuration with a point load at the end).

- The bending modulus of the first beam is modelled by two Gaussian random variables (a discretised random field with standard deviation $\sigma_a = 0.2$). The stiffness matrix is of the form

$$\mathbf{K}^1(\theta_1) = \mathbf{K}_0 + \xi_1(\theta_1)\mathbf{K}_1^1 + \xi_2(\theta_1)\mathbf{K}_2^1$$

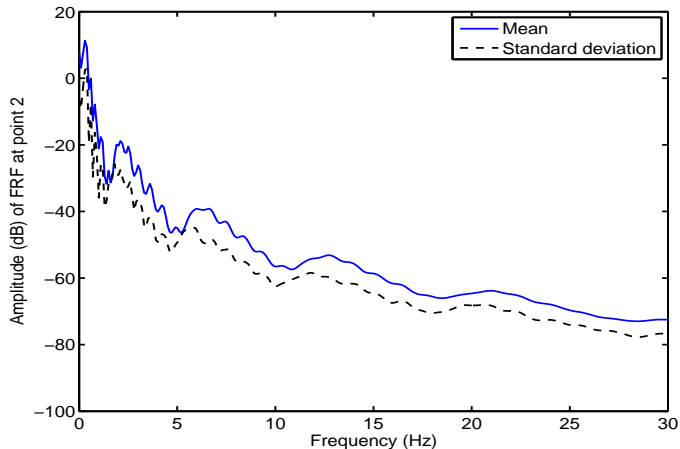
- For the second beam, an Wishart random matrix model with $\delta = 0.2$ is considered.
- The mass matrix and the damping factors are deterministic for both the beams.
- First-order perturbation is used for the interior random eigenvalue problems. 1000-sample Monte Carlo sample is used to for the interface problem.
- For the numerical calculation we used $n_1 = 60$, $n_2 = 328$. In the domain decomposition approach, $m_1 = 58$, $m_2 = 336$ and $m_\Gamma = 2$.

Stochastic response - driving point



Response statistics of the stochastic multiscale system at the driving point.

Stochastic response - tip point



Response statistics of the stochastic multiscale system at the tip.

- 1 Adhikari, S., "Matrix variate distributions for probabilistic structural mechanics", *AIAA Journal*, 45[7] (2007), pp. 1748-1762.
- 2 Adhikari, S., "Wishart random matrices in probabilistic structural mechanics", *ASCE Journal of Engineering Mechanics*, 134[12] (2008), pp. 1029-1044.
- 3 Adhikari, S., and Sarkar, A., "Uncertainty in structural dynamics: experimental validation of a Wishart random matrix model", *Journal of Sound and Vibration*, 323[3-5] (2009), pp. 802-825.
- 4 Adhikari, S., "Generalized Wishart distribution for probabilistic structural dynamics", *Computational Mechanics*, 45[5] (2010), pp. 495-511.
- 5 Adhikari, S., "Uncertainty quantification in structural dynamics using non-central Wishart distribution", *International Journal of Engineering Under Uncertainty: Hazards, Assessment and Mitigation*, 2[3-4] (2010), pp. 123-139.
- 6 Adhikari, S. and Chowdhury, R., "A reduced-order random matrix approach for stochastic structural dynamics", *Computers and Structures*, 88[21-22] (2010), pp. 1230-1238.
- 7 Pascual, B. and Adhikari, S., "Combined parametric-nonparametric uncertainty quantification using random matrix theory and polynomial chaos expansion", *Computers & Structures*, 112-113[12] (2012), pp. 364-379.
- 8 Adhikari, S., Pastur, L., Lytova, A. and Du Bois, J. L., "Eigenvalue-density of linear stochastic dynamical systems: A random matrix approach", *Journal of Sound and Vibration*, 331[5] (2012), pp. 1042-1058.