Uncertainty quantification in Structural Dynamics: Day 3

Professor Sondipon Adhikari
FRAeS

Chair of Aerospace Engineering, College of Engineering, Swansea University, Swansea UK
Email: S.Adhikari@swansea.ac.uk, Twitter: @ProfAdhikari
Web: http://engweb.swan.ac.uk/~adhikaris
Google Scholar: http://scholar.google.co.uk/citations?user=tKM35S0AAAAJ

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Prifysgol Abertawe

Overview of the course

The course is dived into eight topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

Outline of this talk

1. **Parametric sensitivity of eigensolutions**
   - Derivative of eigenvalues
   - Derivative of eigenvectors
2. **Statistics of the eigensolutions**
3. **Higher order perturbation**
   - Eigenvalue statistics using theory of quadratic forms
4. **Asymptotic integral method**
   - Multidimensional integrals in unbounded domains
   - Calculation of an arbitrary moment of the eigenvalues
   - Probability density function of the eigenvalues
     - Truncated Gaussian density function
     - Approximation by χ^2 probability density function
   - Application examples
     - A two DOF system
   - A three DOF system with closely spaced eigenvalues
     - Case 1: All eigenvalues are well separated
     - Case 2: Two eigenvalues are close

Parametric sensitivity of the eigensolutions

- Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest.
- Motivation for this kind of study arises, on one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualise the changes in the dynamic response with respect to system parameters.
- This kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods.
- Rates of change of eigenvalues and eigenvectors are useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration.
- Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems.

Parametric sensitivity of the eigensolutions

- The eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\phi_j = \lambda_j \mathbf{M}(\mathbf{x})\phi_j \tag{1}$$

- Here λ_j and ϕ_j are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$, the mass and stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random vector $\mathbf{x} \in \mathbb{R}^m$.
- The vector $\mathbf{x}$ may consist of material properties, e.g., mass density, Poisson's ratio, Young's modulus; geometric properties, e.g., length, thickness, and boundary conditions.
- The eigenvalues and eigenvectors are smooth differentiable functions of the random parameter vector $\mathbf{x}$.

Parametric sensitivity of the eigenvalues

- We rewrite the eigenvalue equation as

$$[\mathbf{K} - \lambda_j \mathbf{M}]\, \boldsymbol{\phi}_j = \mathbf{0} \tag{2}$$

$$\text{or} \quad \boldsymbol{\phi}_j^T\, [\mathbf{K} - \lambda_j \mathbf{M}] \tag{3}$$

  The functional dependence of $\mathbf{x}$ is removed for notational convenience.

- Differentiating the eigenvalue equation (2) with respect to the element x_i of the parameter vector we have

$$\left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \boldsymbol{\phi}_j + [\mathbf{K} - \lambda_j \mathbf{M}] \frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = \mathbf{0} \tag{4}$$

- Premultiplying by $\boldsymbol{\phi}_j^T$ we have

$$\boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T [\mathbf{K} - \lambda_j \mathbf{M}] \frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = \mathbf{0} \tag{5}$$

Parametric sensitivity of the eigenvalues

- Using the identity in (3) we have

$$\boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i} \mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j = \mathbf{0} \tag{6}$$

$$\text{or} \quad \frac{\partial \lambda_j}{\partial x_i} = \frac{\boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j}{\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j} \tag{7}$$

- Note that when the modes are mass normalised $\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j = 1$
- The derivatives need to be evaluated at certain value **x**. It is customary to evaluate this at the nominal value (which is normally the mean value if **x** is a random vector).
- Denote the mean of **x** as $\boldsymbol{\mu} \in \mathbb{R}^m$, and consider that

$$\mathbf{M}(\boldsymbol{\mu}) = \mathbf{M}_0, \quad \text{and} \quad \mathbf{K}(\boldsymbol{\mu}) = \mathbf{K}_0 \tag{8}$$

  are the 'deterministic parts' of the mass and stiffness matrices respectively.

Parametric sensitivity of the eigenvalues

- The deterministic part of the eigenvalues:

$$\lambda_{0_j} = \lambda_j(\boldsymbol{\mu}) \tag{9}$$

  is obtained from the deterministic eigenvalue problem:

$$\mathbf{K}_0\, \boldsymbol{\phi}_{0_j} = \lambda_{0_j}\, \mathbf{M}_0\, \boldsymbol{\phi}_{0_j}. \tag{10}$$

- Using these, the derivative at the mean/nominal point can be obtained as

$$\frac{\partial \lambda_j}{\partial x_i} = \boldsymbol{\phi}_{0_j}{}^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_{0_j} \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_{0_j} \tag{11}$$

- Consider the standard expansion of the stiffness and mass matrices

$$\mathbf{K}(\mathbf{x}) = \mathbf{K}_0 + \sum_{i=1}^{m_K} x_i \mathbf{K}_i \quad \text{and} \quad \mathbf{M}(\mathbf{x}) = \mathbf{M}_0 + \sum_{i=1}^{m_M} x_i \mathbf{M}_i \tag{12}$$

  where $m = m_K + m_M$

Parametric sensitivity of the eigenvalues

- Therefore

$$\frac{\partial \mathbf{K}}{\partial x_i} = \mathbf{K}_i, i \in m_K \quad \text{and} \quad \frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{M}_i, i \in m_M \tag{13}$$

- Using these, the eigenvalue derive can be obtained succinctly as

$$\frac{\partial \lambda_j}{\partial x_i} = {\boldsymbol{\phi}_{0_j}}^T \left[\mathbf{K}_i - \lambda_{0_j} \mathbf{M}_i\right] \boldsymbol{\phi}_{0_j} \tag{14}$$

- Each eigenvalues can be expanded in a Taylor series about the mean of the parameter values as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\alpha})\left(\mathbf{x} - \boldsymbol{\alpha}\right) + \frac{1}{2}\left(\mathbf{x} - \boldsymbol{\alpha}\right)^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\left(\mathbf{x} - \boldsymbol{\alpha}\right) + \cdots \tag{15}$$

- Without any loss of generality, considering the mean of $\mathbf{x}$ is zero and retaining only the first order terms we have

$$\lambda_j(\mathbf{x}) \approx \lambda_{0_j} + \sum_{i=1}^{m} \left({\boldsymbol{\phi}_{0_j}}^T \left[\mathbf{K}_i - \lambda_{0_j} \mathbf{M}_i\right] \boldsymbol{\phi}_{0_j}\right) x_i \tag{16}$$

Parametric sensitivity of the eigenvectors

- Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of the eigenvectors

$$\frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = \sum_{r=1}^{n} \alpha_{jir} \boldsymbol{\phi}_r \tag{17}$$

  It is necessary to find expressions for the constant α_{jir} for all $r = 1, 2, \cdots n$.

- Substituting this in Eq. (4) we have

$$\left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \boldsymbol{\phi}_j + \sum_{r=1}^{n} \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \alpha_{jir} \boldsymbol{\phi}_r = \mathbf{0} \tag{18}$$

- Premultiplying by $\boldsymbol{\phi}_k^T$ we have

$$\boldsymbol{\phi}_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \frac{\partial \lambda_j}{\partial x_i}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right] \boldsymbol{\phi}_j + \sum_{r=1}^{n} \boldsymbol{\phi}_k^T \left[\mathbf{K} - \lambda_j \mathbf{M}\right] \alpha_{jir} \boldsymbol{\phi}_r = 0 \tag{19}$$

Parametric sensitivity of the eigenvectors

- We consider $r = k$ and the orthogonality of the eigenvectors:

$$\boldsymbol{\phi}_k^T \mathbf{K} \boldsymbol{\phi}_r = \lambda_k \delta_{kr} \quad \text{and} \quad \boldsymbol{\phi}_k^T \mathbf{M} \boldsymbol{\phi}_r = \delta_{kr} \tag{20}$$

- Using these we have

$$\boldsymbol{\phi}_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j + (\lambda_k - \lambda_j)\, \alpha_{jik} = 0 \tag{21}$$

- From this we obtain

$$\alpha_{jik} = -\frac{\boldsymbol{\phi}_k^T \left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i} \right] \boldsymbol{\phi}_j}{\lambda_k - \lambda_j}, \quad \forall\, k \neq j \tag{22}$$

Parametric sensitivity of the eigenvectors

- To obtain the j-th term α_{jij} we differentiate the mass orthogonality relationship in (20) as

$$\frac{\partial(\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j)}{\partial x_i} = 0 \quad \text{or} \quad \frac{\partial \boldsymbol{\phi}_j^T}{\partial x_i}\mathbf{M}\boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \frac{\partial \mathbf{M}}{\partial x_i}\boldsymbol{\phi}_j + \boldsymbol{\phi}_j^T \mathbf{M}\frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = 0 \tag{23}$$

- Considering the symmetry of the mass matrix and using the expansion of the eigenvector derivative we have

$$\boldsymbol{\phi}_j^T \frac{\partial \mathbf{M}}{\partial x_i}\boldsymbol{\phi}_j + 2\boldsymbol{\phi}_j^T \mathbf{M}\frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = 0 \quad \text{or} \quad \sum_{r=1}^{n} 2\boldsymbol{\phi}_j^T \mathbf{M}\alpha_{jir}\boldsymbol{\phi}_r = -\boldsymbol{\phi}_j^T \frac{\partial \mathbf{M}}{\partial x_i}\boldsymbol{\phi}_j \tag{24}$$

- Utilising the othonormality of the mode shapes we have

$$\alpha_{jij} = -\frac{1}{2}\boldsymbol{\phi}_j^T \frac{\partial \mathbf{M}}{\partial x_i}\boldsymbol{\phi}_j \tag{25}$$

Parametric sensitivity of the eigenvectors

- The complete eigenvector derivative is therefore given by

$$\frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = -\frac{1}{2}\left(\boldsymbol{\phi}_j^T \frac{\partial \mathbf{M}}{\partial x_i}\boldsymbol{\phi}_j\right)\phi_j + \sum_{k=1\neq j}^{n} \frac{\boldsymbol{\phi}_k^T\left[\frac{\partial \mathbf{K}}{\partial x_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial x_i}\right]\boldsymbol{\phi}_j}{\lambda_j - \lambda_k}\boldsymbol{\phi}_k \tag{26}$$

- Considering the conventional expansion of the mass and stiffness matrices, the derivative at the mean values of the parameter can be obtained as

$$\frac{\partial \boldsymbol{\phi}_j}{\partial x_i} = -\frac{1}{2}\left(\boldsymbol{\phi}_{0_j}^T \mathbf{M}_i \boldsymbol{\phi}_{0_j}\right)\phi_{0_j} + \sum_{k=1\neq j}^{n} \frac{\boldsymbol{\phi}_{0_k}^T\left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right]\boldsymbol{\phi}_{0_j}}{\lambda_{0_j} - \lambda_{0_k}}\boldsymbol{\phi}_{0_k} \tag{27}$$

- Considering the mean of $\mathbf{x}$ is zero and retaining only the first order terms we have

$$\boldsymbol{\phi}_j(\mathbf{x}) \approx \boldsymbol{\phi}_{0_j} + \sum_{i=1}^{m} \frac{\partial \boldsymbol{\phi}_j}{\partial x_i} x_i = \boldsymbol{\phi}_{0_j} + \sum_{i=1}^{m}\left(\sum_{k=1}^{n} \alpha_{jik}\boldsymbol{\phi}_{0_k}\right) x_i \tag{28}$$

Covariance of the eigensolutions

- Suppose Σ_{ir} is the ir-th element of the covariance matrix, that is

$$\Sigma_{ir} = \mathrm{cov}\left(x_i, x_r\right) \tag{29}$$

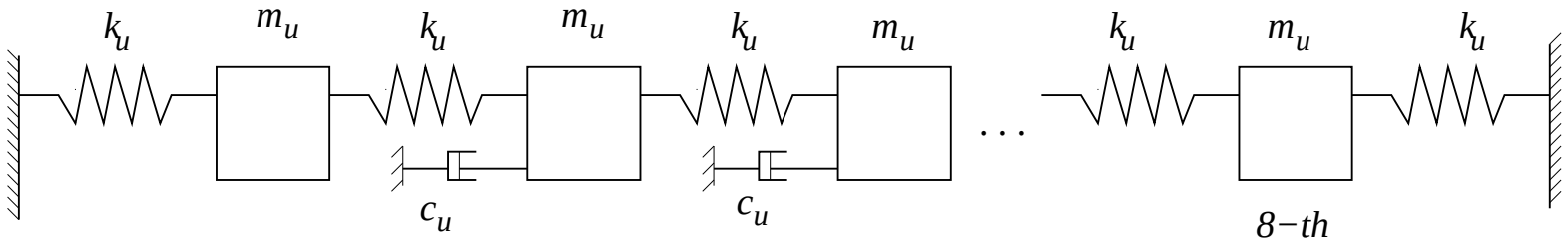
- The covariance of eigenvalue j and s can be obtained as

$$\begin{aligned}&\mathrm{E}\left[\left(\lambda_j - \lambda_{0_j}\right)\left(\lambda_s - \lambda_{0_s}\right)\right] \\ &= \sum_{i=1}^{m}\sum_{r=1}^{m}\left(\boldsymbol{\phi}_{0_j}{}^T\left[\mathbf{K}_i - \lambda_{0_j}\mathbf{M}_i\right]\boldsymbol{\phi}_{0_j}\right)\left(\boldsymbol{\phi}_{0_s}{}^T\left[\mathbf{K}_r - \lambda_{0_s}\mathbf{M}_r\right]\boldsymbol{\phi}_{0_s}\right)\Sigma_{ir}\end{aligned} \tag{30}$$

- The covariance matrix of eigenvector j and s can be obtained as

$$\mathrm{E}\left[\left(\boldsymbol{\phi}_j - \boldsymbol{\phi}_{0_j}\right)\left(\boldsymbol{\phi}_s - \boldsymbol{\phi}_{0_s}\right)^T\right] = \sum_{k=1}^{n}\sum_{l=1}^{n}\boldsymbol{\phi}_{0_k}\boldsymbol{\phi}_{0_l}^T\sum_{i=1}^{m}\sum_{r=1}^{m}\alpha_{jir}\alpha_{srl}\Sigma_{ir} \tag{31}$$

Numerical example

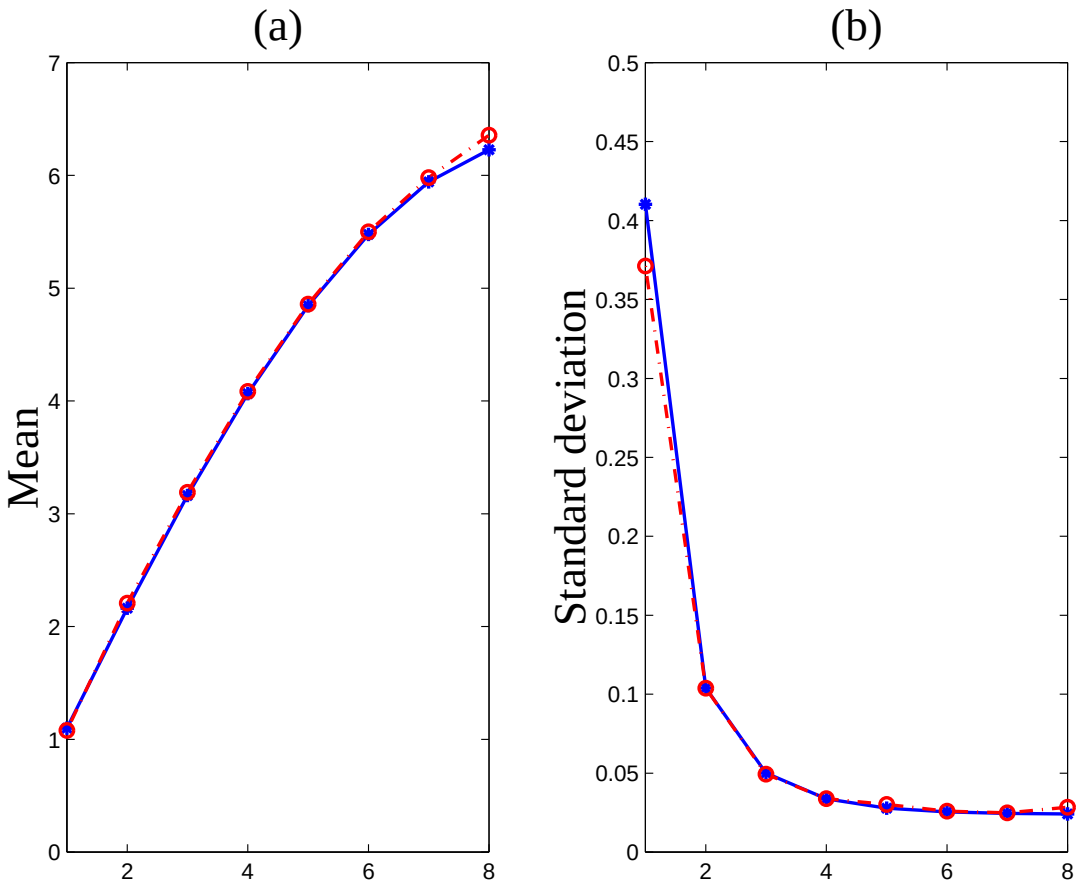


- An eight DOF system consisting of a linear array of spring-mass oscillators is considered
- Eight masses, each of nominal mass $m_u = 1$ kg, are connected by springs of nominal stiffness $k_u = 10$ N/m
- It is assumed that the mass and stiffness associated with all the units are random. Randomness associated with each unit has the following form

$$m_{u_j} = m_u \left(1 + \epsilon_{m_j} g_j\right), \quad k_{u_j} = k_u \left(1 + \epsilon_{k_j} g_j\right) \tag{32}$$

- Here $g_j, \forall j$ are assumed to be uncorrelated, identically distributed, zero-mean, unit-standard-deviation Gaussian random variables ($N(0,1)$). For this assumption, the joint covariance matrix Σ becomes a diagonal matrix.

Numerical example

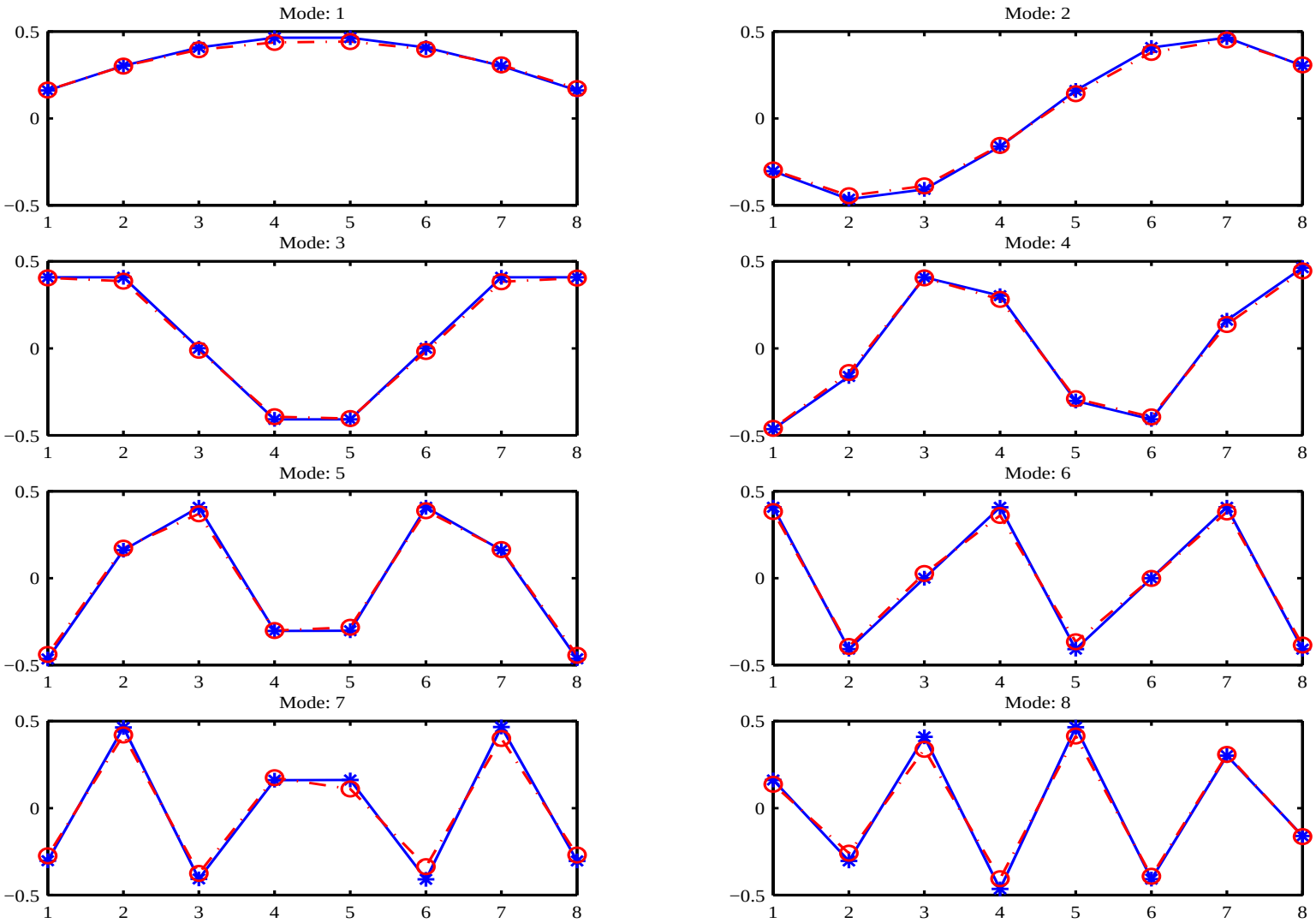


(a) The mean of the natural frequencies (b) Standard deviation of the natural frequencies; 'X-axis' Mode number; '—' Analytical; '-.-.-' MCS

Numerical example

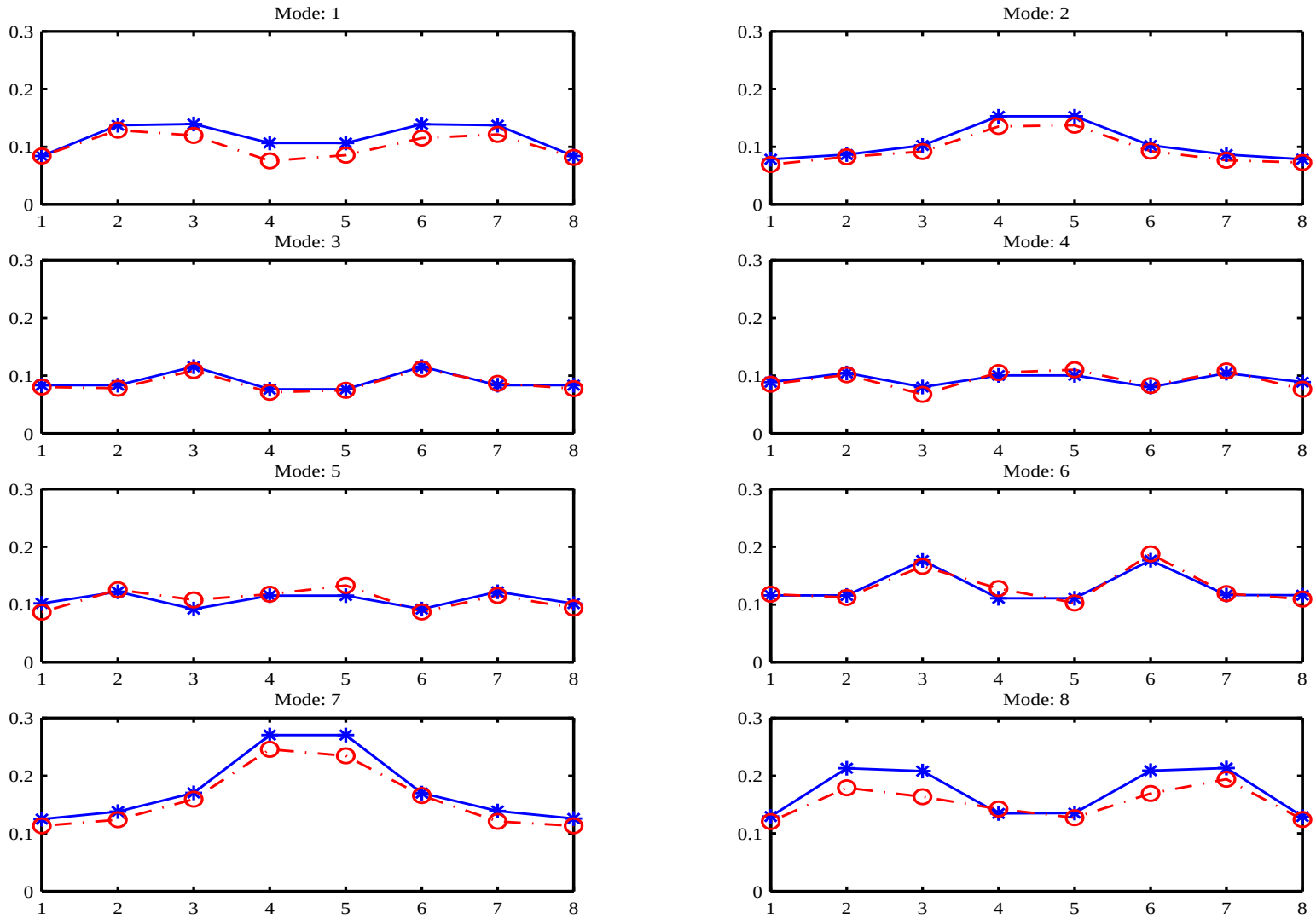
- Numerical values of the 'strength parameters', ϵ_{m_j} and ϵ_{k_j} are assumed to be 0.1, that is, we consider 10% randomness for all the parameter values.
- Because the random variables describing the system properties are assumed to be Gaussian, the mean values are the same as the nominal values.
- In the same figure, the mean values obtained from the proposed theory are compared with the results obtained from an independent Monte Carlo simulation (MCS) using 5000 samples. Both the curves follow each other very closely.

Mode shape statistics



The mean of the mode shapes; '—' Analytical; '-.-.-' MCS

Mode shape statistics



The standard deviation of the mode shapes; '—' Analytical; '-.-.-' MCS

Higher-order perturbation

- Higher-order perturbation method can be used to improve upon the results obtained from the first-order perturbation method
- Statistical properties of the system are completely described by the joint probability density function $p_{\mathbf{X}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. For mathematical convenience we express

$$p_{\mathbf{X}}(\mathbf{x}) = \exp\left\{-L(\mathbf{x})\right\} \tag{33}$$

  where $-L(\mathbf{x})$ is often known as the log-likelihood function.
- For example, if $\mathbf{x}$ is a m-dimensional multivariate Gaussian random vector with mean $\boldsymbol{\mu} \in \mathbb{R}^m$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times m}$ then

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\det\left\{\boldsymbol{\Sigma}\right\} + \frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T\boldsymbol{\Sigma}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right). \tag{34}$$

- It is assumed that **M** and **K** are symmetric and positive definite random matrices so that all the eigenvalues are real and positive.

Higher-order perturbation

- The eigenvalues, $\lambda_j(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ are non-linear functions of the parameter vector $\mathbf{x}$.
- If the eigenvalues are not repeated, then each $\lambda_j(\mathbf{x})$ is expected to be a smooth and twice differentiable function since the mass and stiffness matrices are smooth and twice differentiable functions of the random parameter vector.
- In the mean-centered perturbation approach the function $\lambda_j(\mathbf{x})$ is expanded by its Taylor series about the point $\mathbf{x} = \boldsymbol{\mu}$ as

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\mu}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{D}_{\lambda_j}(\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu}). \tag{35}$$

- Here $\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^m$ and $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\mu}$, that is

$$\left\{\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\right\}_k = \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}|_{\mathbf{x}=\boldsymbol{\mu}} \tag{36}$$

$$\text{and} \quad \left\{\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\}_{kl} = \frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l|_{\mathbf{x}=\boldsymbol{\mu}}. \tag{37}$$

- Providing the eigenvalues are distinct, the element of the Hessian matrix can be explicitly obtained as

$$\begin{aligned}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial x_k\, \partial x_l} = \boldsymbol{\phi}_j(\mathbf{x})^T \left[\frac{\partial^2 \mathbf{K}(\mathbf{x})}{\partial {x_k}^2} x_l - \lambda_j(\mathbf{x}) \frac{\partial^2 \mathbf{M}(\mathbf{x})}{\partial {x_k}^2} x_l\right] \boldsymbol{\phi}_j(\mathbf{x}) \\ - \left(\boldsymbol{\phi}_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_k} \boldsymbol{\phi}_j(\mathbf{x})\right) \left(\boldsymbol{\phi}_j(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x})\right) \\ - \left(\boldsymbol{\phi}_j(\mathbf{x})^T \frac{\partial \mathbf{M}(\mathbf{x})}{\partial x_l} \boldsymbol{\phi}_j(\mathbf{x})\right) \left(\boldsymbol{\phi}_j(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x})\right) \\ + 2 \sum_{r=1}^{N} \frac{\left(\boldsymbol{\phi}_r(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jk}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x})\right) \left(\boldsymbol{\phi}_r(\mathbf{x})^T \boldsymbol{\mathcal{G}}_{jl}(\mathbf{x}) \boldsymbol{\phi}_j(\mathbf{x})\right)}{\lambda_j(\mathbf{x}) - \lambda_r(\mathbf{x})}. \end{aligned} \tag{38}$$

- The elements of the gradient vector and Hessian matrix of the eigenvalues are therefore completely defined in closed-form.

- When $\mathbf{x}$ is a multivariate Gaussian random vector, the moment generating function of $\lambda_j(\mathbf{x})$, for any $s \in \mathbb{C}$, can be obtained from (35) as

$$M_{\lambda_j}(s) = \mathrm{E}\left[\exp\left\{s\lambda_j(\mathbf{x})\right\}\right] = \int_{\mathbb{R}^m} \exp\Big\{ s\lambda_j(\boldsymbol{\mu}) + s\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left(\mathbf{x}-\boldsymbol{\mu}\right) \tag{39}$$

$$+\frac{s}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T \mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\left(\mathbf{x}-\boldsymbol{\mu}\right) - L(\mathbf{x})\Big\}\, d\mathbf{x} \tag{40}$$

  where $L(\mathbf{x})$ is given by equation (34).
- Using the transformation

$$\mathbf{y} = (\mathbf{x}-\boldsymbol{\mu}) \tag{41}$$

  the integral in (39) can be evaluated exactly as

$$M_{\lambda_j}(s) = (2\pi)^{-m/2}\|\boldsymbol{\Sigma}\|^{-1/2}\int_{\mathbb{R}^m}\exp\Big\{s\overline{\lambda}_j + s\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\mathbf{y}$$

$$-\frac{1}{2}\mathbf{y}^T\left[\boldsymbol{\Sigma}^{-1} - s\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]\mathbf{y}\Big\}\, d\mathbf{y}$$

$$= \frac{\exp\left\{s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\left[\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})\right\}}{\sqrt{\left\|\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\|}}. \tag{42}$$

- To obtain the pdf of $\lambda_j(\mathbf{x})$, the inverse Laplace transform of equation (42) is required. T
- If the mean-centered first-order perturbation is used then $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu}) = \mathbf{O}$ and from equation (42) we obtain

$$M_{\lambda_j}(s) \approx \exp\left\{ s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}^T_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \right\}. \tag{43}$$

- This implies that $\lambda_j(\mathbf{x})$ is a Gaussian random variable with mean $\overline{\lambda}_j$ and variance $\mathbf{d}^T_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu})$.
- However, for second-order perturbations in general the mean of the eigenvalues is not the deterministic value. The cumulants of $\lambda_j(\mathbf{x})$ can be obtained from

$$\kappa_j^{(r)} = \frac{d^r}{ds^r} \ln M_{\lambda_j}(s)|_{s=0}. \tag{44}$$

- Here $\kappa_j^{(r)}$ is the rth order cumulant of jth eigenvalue and from equation (42) we have

$$\ln M_{\lambda_j}(s) = s\overline{\lambda}_j + \frac{s^2}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\left[\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{-1}\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) - \frac{1}{2}\ln\left\|\mathbf{I} - s\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right\|. \quad (45)$$

- Using this expression and after some simplifications it can be shown that

$$\kappa_j^{(r)} = \overline{\lambda}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right), r = 1, \quad (46)$$

$$\kappa_j^{(r)} = \frac{r!}{2}\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^{r-2}\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) \quad (47)$$

$$+ \frac{(r-1)!}{2}\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^r\right), r \geq 2. \quad (48)$$

- The mean and first few cumulants of the eigenvalues can be explicitly obtained as

$$\widehat{\lambda}_j = \kappa_j^{(1)} = \overline{\lambda}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right) \tag{49}$$

$$\mathrm{Var}\left[\lambda_j\right] = \kappa_j^{(2)} = \mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \frac{1}{2}\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^2\right), \tag{50}$$

$$\kappa_j^{(3)} = 3\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + \mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^3\right), \tag{51}$$

$$\text{and } \kappa_j^{(4)} = 12\mathbf{d}_{\lambda_j}^T(\boldsymbol{\mu})\left[\boldsymbol{\Sigma}\,\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\right]^2\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\mu}) + 3\mathrm{Trace}\left(\left[\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})\boldsymbol{\Sigma}\right]^4\right). \tag{52}$$

- From the cumulants, the raw moments $\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r\right]$ and the central moments $\mu_j^{\prime(r)} = \mathrm{E}\left[(\lambda_j - \overline{\lambda}_j)^r\right]$ can be obtained using standard formulae.

Asymptotic integral method

- The moments of the eigenvalues are obtained based on an asymptotic approximation of the multidimensional integral.
- Consider a function $f(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$ which is smooth and at least twice differentiable. Suppose we want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} \exp\{-f(\mathbf{x})\}\, d\mathbf{x}. \tag{53}$$

  This is a m-dimensional integral over the unbounded domain $\mathbb{R}^m$.
- The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum. Suppose that $f(\mathbf{x})$ reaches its global minimum at a *unique* point $\boldsymbol{\theta} \in \mathbb{R}^m$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0}. \tag{54}$$

Higher-order perturbation

- Using this, $f(\mathbf{x})$ is expanded in a Taylor series about $\boldsymbol{\theta}$ and equation (53) is rewritten as

$$\begin{aligned}\mathcal{J} &= \int_{\mathbb{R}^m} \exp\left\{-\left\{f(\boldsymbol{\theta}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) + \varepsilon(\mathbf{x},\boldsymbol{\theta})\right\}\right\} d\mathbf{x} \\ &= \exp\{-f(\boldsymbol{\theta})\} \int_{\mathbb{R}^m} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) - \varepsilon(\mathbf{x},\boldsymbol{\theta})\right\} d\mathbf{x}\end{aligned} \tag{55}$$

  where $\varepsilon(\mathbf{x},\boldsymbol{\theta})$ is the error if only the terms up to second-order were retained in the Taylor series expansion.
- With suitable scaling of $\mathbf{x}$ the integral in (53) can be transformed to the so called 'Laplace integral'. Under special conditions such integrals can be well approximated using asymptotic methods.
- We neglect the error $\varepsilon(\mathbf{x},\boldsymbol{\theta})$ considering the higher-order derivatives are small.

Higher-order perturbation

- The integral in (55) can be approximated as

$$\mathcal{J} \approx \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \int_{\mathbb{R}^m} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\theta}\right)^T \mathbf{D}_f\left(\boldsymbol{\theta}\right)\left(\mathbf{x}-\boldsymbol{\theta}\right)\right\} d\mathbf{x}. \tag{56}$$

  If $\boldsymbol{\theta}$ is the global minimum of $f(\mathbf{x})$ in $\mathbb{R}^m$, the symmetric Hessian matrix $\mathbf{D}_f\left(\boldsymbol{\theta}\right) \in \mathbb{R}^{m\times m}$ is also expected to be positive definite.
- Using the coordinate transformation

$$\boldsymbol{\xi} = \left(\mathbf{x}-\boldsymbol{\theta}\right)\mathbf{D}_f^{-1/2}(\boldsymbol{\theta}). \tag{57}$$

  The Jacobian of this transformation is

$$\det\left\{\mathbf{J}\right\} = \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2}. \tag{58}$$

- Using equation (57), the integral in equation (56) can be evaluated as

$$\mathcal{J} \approx \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \int_{\mathbb{R}^m} \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\xi}^T\boldsymbol{\xi}\right)\right\} d\boldsymbol{\xi} \tag{59}$$

  or $$\mathcal{J} \approx \left(2\pi\right)^{m/2} \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \det\left\{\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2}. \tag{60}$$

- This approximation is expected to yield good result if the minimum of $f(\mathbf{x})$ around $\mathbf{x} = \boldsymbol{\theta}$ is sharp. Equation (60) will now be used to obtain moments of the eigenvalues.

Moments of the eigenvalues

- An arbitrary rth order moment of the eigenvalues can be obtained from

$$\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\right] = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$
$$= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x})\right)\right\} d\mathbf{x}, \quad r = 1, 2, 3 \cdots . \tag{61}$$

- The equation can be expressed in the form of equation (53) by choosing

$$f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}). \tag{62}$$

- Differentiating the above equation with respect to x_k we obtain

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \frac{\partial L(\mathbf{x})}{\partial x_k} - \frac{r}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}. \tag{63}$$

Moments of the eigenvalues

- The optimal point $\boldsymbol{\theta}$ can be obtained from (54) by equating the above expression to zero. Therefore at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \quad \forall\, k \tag{64}$$

$$\text{or} \quad \frac{r}{\lambda_j(\boldsymbol{\theta})} \frac{\partial \lambda_j(\boldsymbol{\theta})}{\partial x_k} = \frac{\partial L(\boldsymbol{\theta})}{\partial x_k}, \quad \forall\, k \tag{65}$$

$$\text{or} \quad \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta}). \tag{66}$$

- Equation (66) needs to be solved numerically to obtain $\boldsymbol{\theta}$. It implies that at the optimal point the gradient vectors of the eigenvalues and log-likelihood function are parallel.
- The non-linear set of equations (66) have to be solved numerically. Due to the explicit analytical expression of $\mathbf{d}_{\lambda_j}$ in terms of the derivative of the mass and stiffness matrices, expensive numerical differentiation of $\lambda_j(\mathbf{x})$ at each step is not needed.

Moments of the eigenvalues

- For most $p_{\mathbf{X}}(\mathbf{x})$, a closed-form expression of $\mathbf{d}_L(\mathbf{x})$ is available.
- For example, when $\mathbf{x}$ has multivariate Gaussian distribution, $L(\mathbf{x})$ is given by equation (34). By differentiating this we obtain

$$\mathbf{d}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}). \tag{67}$$

Substituting this in equation (66), the optimal point $\boldsymbol{\theta}$ can be obtained as

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \frac{1}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\,\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}). \tag{68}$$

- This equation also gives a recipe for an iterative algorithm to obtain $\boldsymbol{\theta}$. One starts with an initial $\boldsymbol{\theta}$ in the right-hand side and obtains an updated $\boldsymbol{\theta}$ in the left-hand side.
- This procedure can be continued until the difference between the values of $\boldsymbol{\theta}$ obtained from both sides of (68) is less than (l_2 vector norm can be used to measure the difference) a predefined small value.
- A good value to start the iteration process is $\boldsymbol{\theta} = \boldsymbol{\mu}$, as in the case of mean-centred approach. Note that the solution of a deterministic eigenvalue problem is needed at each step of the iteration process.

Moments of the eigenvalues

- The elements of the Hessian matrix $\mathbf{D}_f(\boldsymbol{\theta})$ can be obtained by differentiating equation (63) with respect to x_l:

$$\frac{\partial^2 f(\mathbf{x})}{\partial {x_k}^2} x_l = \frac{\partial^2 L(\mathbf{x})}{\partial {x_k}^2} x_l - r\left(-\frac{1}{\lambda_j^2(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} + \frac{1}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l\right)$$
$$= \frac{\partial^2 L(\mathbf{x})}{\partial {x_k}^2} x_l + \frac{1}{r}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}\right\}\left\{\frac{r}{\lambda_j(\mathbf{x})}\frac{\partial \lambda_j(\mathbf{x})}{\partial x_l}\right\} - \frac{r}{\lambda_j(\mathbf{x})}\frac{\partial^2 \lambda_j(\mathbf{x})}{\partial {x_k}^2} x_l. \tag{69}$$

- At $\mathbf{x} = \boldsymbol{\theta}$ we can use equation (65) so that equation (69) reads

$$\frac{\partial^2 f(\mathbf{x})}{\partial {x_k}^2} x_l|_{\mathbf{x}=\boldsymbol{\theta}} = \frac{\partial^2 L(\boldsymbol{\theta})}{\partial {x_k}^2} x_l + \frac{1}{r}\frac{\partial L(\boldsymbol{\theta})}{\partial x_k}\frac{\partial L(\boldsymbol{\theta})}{\partial x_l} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\frac{\partial^2 \lambda_j(\boldsymbol{\theta})}{\partial {x_k}^2} x_l. \tag{70}$$

- Combining this equation for all k and l we have

$$\mathbf{D}_f(\boldsymbol{\theta}) = \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}). \tag{71}$$

where $\mathbf{D}_{\lambda_j}(\bullet)$ is defined in equation (37).

Moments of the eigenvalues

- Using the asymptotic approximation (60), the rth moment of the eigenvalues can be obtained as

$$\mu_j^{(r)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \left\| \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}. \quad (72)$$

  This is perhaps the most general formula to obtain the moments of the eigenvalues of linear stochastic dynamic systems. The optimal point $\boldsymbol{\theta}$ needs to be calculated by solving non-linear set of equations equation (66) for each λ_j and r. Several special cases arising from equation (72) are of practical interest:

- *Mean of the eigenvalues:* The mean of the eigenvalues can be obtained by substituting $r = 1$ in equation (72), that is

$$\widehat{\lambda}_j = \mu_j^{(1)} = (2\pi)^{m/2} \lambda_j(\boldsymbol{\theta}) \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \det\left\{ \mathbf{D}_L(\boldsymbol{\theta}) + \mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}) \right\}^{-1/2}. \quad (73)$$

Moments of the eigenvalues

- *Central moments of the eigenvalues:* Once the mean in known, the central moments can be expressed in terms of the raw moments $\mu_j^{(r)}$ using the binomial transform

$$\mu_j^{\prime(r)} = \mathrm{E}\left[\left(\lambda_j - \widehat{\lambda}_j\right)^r\right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu_j^{(k)} \widehat{\lambda}_j^{r-k}. \tag{74}$$

- *Random vector* $\mathbf{x}$ *has multivariate Gaussian distribution:* In this case $L(\mathbf{x})$ is given by equation (34) and by differentiating equation (67) we obtain

$$\text{and} \quad \mathbf{D}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}. \tag{75}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from equation (66) as

$$\boldsymbol{\theta} = \boldsymbol{\mu} + \frac{r}{\lambda_j(\boldsymbol{\theta})} \boldsymbol{\Sigma}\, \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}). \tag{76}$$

Moments of the eigenvalues

- Using equation (67) and equation (75), the Hessian matrix can be derived from equation (71) as

$$\begin{aligned}\mathbf{D}_f(\boldsymbol{\theta}) &= \boldsymbol{\Sigma}^{-1} + \frac{1}{r}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \\ &= \boldsymbol{\Sigma}^{-1}\left(\mathbf{I} + \frac{1}{r}(\boldsymbol{\theta}-\boldsymbol{\mu})(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}\right) - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}).\end{aligned} \tag{77}$$

- Therefore, the rth moment of the eigenvalues can be obtained from Eq. (72) as

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})\right\} \det\{\boldsymbol{\Sigma}\}^{-1/2} \det\{\mathbf{D}_f(\boldsymbol{\theta})\}^{-1/2} \tag{78}$$

Moments of the eigenvalues

- Using Eq. (77) and recalling that for any two matrices **A** and **B**, $\det\{\mathbf{A}\}\det\{\mathbf{B}\} = \det\{\mathbf{AB}\}$ we have

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-\frac{1}{2}\left(\boldsymbol{\theta}-\boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\theta}-\boldsymbol{\mu}\right)\right\} \det\left\{\mathbf{I} + \widetilde{\mathbf{D}}_f\left(\boldsymbol{\theta}\right)\right\}^{-1/2} \tag{79}$$

  where

$$\widetilde{\mathbf{D}}_f\left(\boldsymbol{\theta}\right) = \frac{1}{r}\left(\boldsymbol{\theta}-\boldsymbol{\mu}\right)\left(\boldsymbol{\theta}-\boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}\left(\boldsymbol{\theta}\right) \tag{80}$$

- The probability density function of the eigenvalues can be obtained from these moments.

Maximum entropy probability density function

- Once the cumulants/moments of the eigenvalues are known, the pdf of the eigenvalues can be obtained using the Maximum Entropy Method (MEM). Because equations (46), (47) and (72) can be used to calculate any arbitrary order cumulant and moment, the pdf can be obtained accurately by taking higher order terms.
- Since **M** and **K** are symmetric and positive definite random matrices, all the eigenvalues are real and positive. Suppose the pdf of λ_j is given by $p_{\lambda_j}(u)$ where $u \in \mathbb{R}$ is positive, that is $u \in [0, \infty]$. Considering that only first n moments are used, the pdf of each eigenvalue must satisfy the following constraints:

$$\int_0^\infty p_{\lambda_j}(u)du = 1 \tag{81}$$

and

$$\int_0^\infty u^r p_{\lambda_j}(u)du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n. \tag{82}$$

Maximum entropy probability density function

- Using Shannon's measure of entropy

$$\mathcal{S} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du \tag{83}$$

we construct the Lagrangian

$$\mathcal{L} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du - (\rho_0 - 1)\left[\int_0^\infty p_{\lambda_j}(u) du - 1\right] - \sum_{r=1}^{n} \rho_r \left[\int_0^\infty u^r p_{\lambda_j}(u) du - \mu_j^{(r)}\right]. \tag{84}$$

where $\rho_r, r = 0, 1, 2, \cdots, n$ are Lagrange multipliers.

- The function $p_{\lambda_j}(u)$ which maximizes $\mathcal{L}$ can be obtained using the calculus of variations. Using the Euler-Lagrange equation the solution is given by

$$p_{\lambda_j}(u) = \exp\left\{-\rho_0 - \sum_{i=1}^{n} \rho_i u^i\right\} = \exp\{-\rho_0\} \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\}, \quad u \geq 0. \tag{85}$$

Maximum entropy probability density function

- The Lagrange multipliers can be obtained from the constraint equations (81) and (82) as

$$\exp\{\rho_0\} = \int_0^\infty \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} du$$

and $$\exp\{\rho_0\}\,\mu_j^{(r)} = \int_0^\infty u^r \exp\left\{-\sum_{i=1}^{n} \rho_i u^i\right\} du, \quad \text{for} \quad r = 0, 1, 2, \cdots n.$$

- Closed-form expressions for ρ_r are in general not possible for all n. If we take $n = 2$, then the resulting pdf can be expressed as the truncated Gaussian density function

$$p_{\lambda_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j\, \Phi\left(\widehat{\lambda}_j/\sigma_j\right)} \exp\left\{-\frac{\left(u-\widehat{\lambda}_j\right)^2}{2\sigma_j^2}\right\}, \quad u \geq 0. \tag{86}$$

where σ_j is given by

$$\sigma_j^2 = \mu_j^{(2)} - \widehat{\lambda}_j^2. \tag{87}$$

- The truncated Gaussian density function derived here ensures that the probability of any eigenvalues becoming negative is zero.

Maximum entropy probability density function

- We use an approximation analogous to Pearson's three moment central χ^2 approximation to the distribution of a noncentral χ^2 . The eigenvalues are approximated as

$$\lambda_j \approx \eta_j + \gamma_j \chi^2_{\nu_j}(u) \tag{88}$$

where $\chi^2_{\nu_j}(u)$ is a central χ^2 density function with ν_j degrees-of-freedom.

- The constants η_j, γ_j, and ν_j are obtained such that the first three moments of λ_j are equal to that of the approximated χ^2 pdf. The moment generating function of the approximated χ^2 pdf is given by

$$\mathrm{E}\left[\exp\left\{-s\left(\eta_j + \gamma_j \chi^2_{\nu_j}\right)\right\}\right] = \exp\left\{-s\eta_j\right\}\left(1 + 2s\gamma_j\right)^{-\nu_j/2}. \tag{89}$$

- Equating the first three moments we have

$$\eta_j + \nu_j\gamma_j = \mu_j^{(1)}, \tag{90}$$

$$\eta_j{}^2 + 2\eta_j\nu_j\gamma_j + \nu_j{}^2\gamma_j{}^2 + 2\nu_j\gamma_j{}^2 = \mu_j^{(2)} \tag{91}$$

and $$\eta_j{}^3 + 3\eta_j{}^2\nu_j\gamma_j + 3\eta_j\nu_j{}^2\gamma_j{}^2 + 6\eta_j\nu_j\gamma_j{}^2 + \nu_j{}^3\gamma_j{}^3 + 6\nu_j{}^2\gamma_j{}^3 + 8\nu_j\gamma_j{}^3 = \mu$$ (92)

Maximum entropy probability density function

- This set of coupled non-linear equations can be solved exactly in closed-form to obtain η_j, γ_j, and ν_j:

$$\eta_j = \frac{\mu_j^{(1)^2}\mu_j^{(2)} - 2\,\mu_j^{(2)^2} + \mu_j^{(1)}\mu_j^{(3)}}{2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_j^{(3)}} \tag{93}$$

$$\gamma_j = \frac{2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_j^{(3)}}{4\left(\mu_j^{(2)} - \mu_j^{(1)^2}\right)}, \tag{94}$$

$$\text{and } \nu_j = 8\,\frac{\left(\mu_j^{(2)} - \mu_j^{(1)^2}\right)^3}{\left(2\,\mu_j^{(1)^3} - 3\,\mu_j^{(1)}\mu_j^{(2)} + \mu_3'\right)^2}. \tag{95}$$

Maximum entropy probability density function

- Moments of $\lambda_j(\mathbf{x})$ obtained in equation (72), can be used directly in the right-hand side of these equations. Alternatively, this approach can also be used in conjunction with the perturbation methods by transforming the cumulants obtained from equations (46) and (47) to moments.
- Using the transformation in equation (88) the approximate probability density function of $\lambda_j(\mathbf{x})$ is given by

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi^2_{\nu_j}}\left(\frac{u-\eta_j}{\gamma_j}\right) = \frac{(u-\eta_j)^{\nu_j/2-1}\exp\left\{-(u-\eta_j)/2\gamma_j\right\}}{(2\gamma_j)^{\nu_j/2}\Gamma(\nu_j/2)}. \quad (96)$$

- The two approximated pdf proposed here have simple forms but it should be noted that they are not exhaustive. Given the moments/cumulants, different probability density functions can be fitted using different methods. The application of the approximate pdfs derived here is illustrated in the next section.

A two DOF system

- A simple two-degree-of-freedom undamped system has been considered to illustrate a possible application of the expressions developed so far. The main purpose of this example is to understand how the proposed methods compare with the existing methods. 1 shows the example, together with the numerical values of the masses and spring stiffnesses.

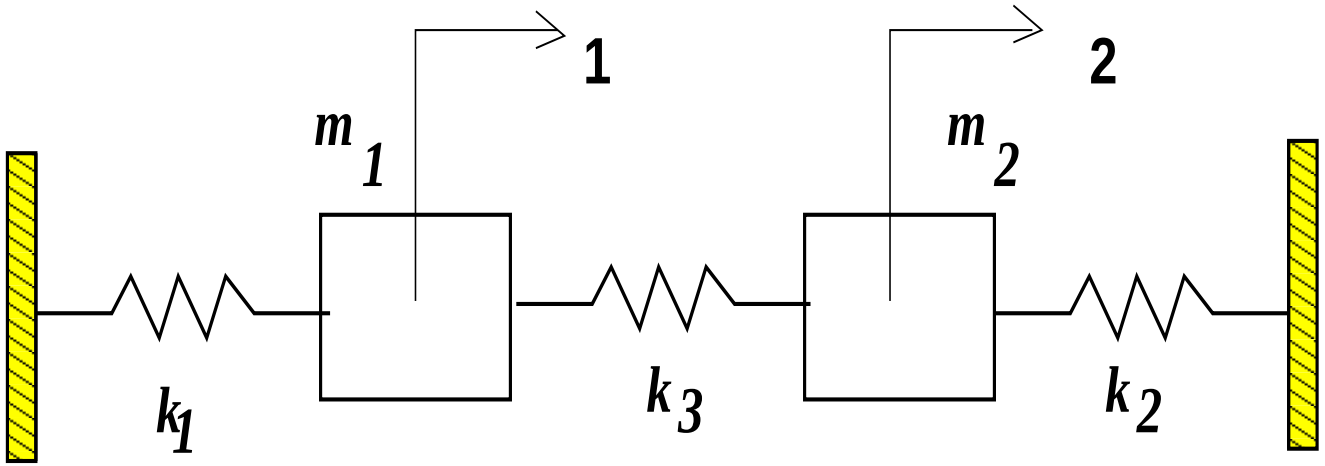


Figure: The undamped two degree-of-system system, $m_1 = 1$ kg, $m_2 = 1.5$ kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.

A two DOF system

- The system matrices for the example are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}. \tag{97}$$

- It is assumed that only the stiffness parameters k_1 and k_2 are uncertain so that $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$ and $\bar{k}_i$ denote the deterministic values of the spring constants. Here $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ is a vector of standard Gaussian random variables, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$.
- The numerical values of the 'strength parameters' are considered as $\epsilon_1 = \epsilon_2 = 0.25$. The strength parameters are selected so that the system matrices are almost surely positive definite.

A two DOF system

- Noting that **M** is independent of **x** and **K** is a linear function of **x**, the derivative of the system matrices with respect to the random vector **x** can be obtained as

$$\frac{\partial \mathbf{K}}{\partial x_1} = \epsilon_1 \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_2} = \epsilon_2 \begin{bmatrix} 0 & 0 \\ 0 & \bar{k}_2 \end{bmatrix}, \tag{98}$$

$$\frac{\partial \mathbf{M}}{\partial x_i} = \mathbf{O} \quad \text{and} \quad \frac{\partial^2 \mathbf{K}}{\partial {x_i}^2} x_j = \mathbf{O}. \tag{99}$$

- We calculate the raw moments and the probability density functions of the two eigenvalues of the system. Recall that the eigenvalues obtained from equation (1) are the square of the natural frequencies ($\lambda_j = \omega_j^2$). Several methods are used to obtain the moments and the pdfs.

Mean-centered first-order perturbation

- This case arises when $\mathbf{D}_{\lambda_j}(\boldsymbol{\mu})$ in the Taylor series expansion (35) is assumed to be a null matrix so that only the first-order terms are retained.
- This is the simplest approximation, and as mentioned earlier, results in a Gaussian distribution of the eigenvalues.
- Recalling that for this problem $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$, the resulting statistics for this special case can be obtained from equations (49) and (50) as

$$\widehat{\lambda}_j = \overline{\lambda}_j \tag{100}$$

$$\text{and} \quad \mathrm{Var}\left[\lambda_j\right] = \mathbf{d}_{\lambda_j}^T(\mathbf{0})\mathbf{d}_{\lambda_j}(\mathbf{0}). \tag{101}$$

- The gradient vector $\mathbf{d}_{\lambda_j}(\mathbf{0})$ can be obtained from equation (11) using the system derivative matrices (98) and (99).

Mean-centered second-order perturbation

- In this case all the terms in equation (35) are retained. This approximation results in a quadratic form in the Gaussian random variables. The resulting statistics can be obtained from equations (46) and (47) by substituting $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$.
- The elements of the Hessian matrix $\mathbf{D}_{\lambda_j}(\mathbf{0})$ can be obtained from equation (38) and using the system derivative matrices (98) and (99).

Monte Carlo Simulation

- The samples of two independent Gaussian random variables x_1 and x_2 are generated and the eigenvalues are computed directly from equation (1).
- A total of 15000 samples are used to obtain the statistical moments and pdf of both the eigenvalues.
- The results obtained from the Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the five analytical methods described above.
- The percentage error for an arbitrary kth moment of an eigenvalue obtained using any one of the five analytical methods is given by

$$\text{Error}_{i\text{th method}} = \frac{\left| \{\mu_j^{(r)}\}_{i\text{th method}} - \{\mu_j^{(r)}\}_{\text{MCS}} \right|}{\{\mu_j^{(r)}\}_{\text{MCS}}} \times 100. \qquad (102)$$

A two DOF system

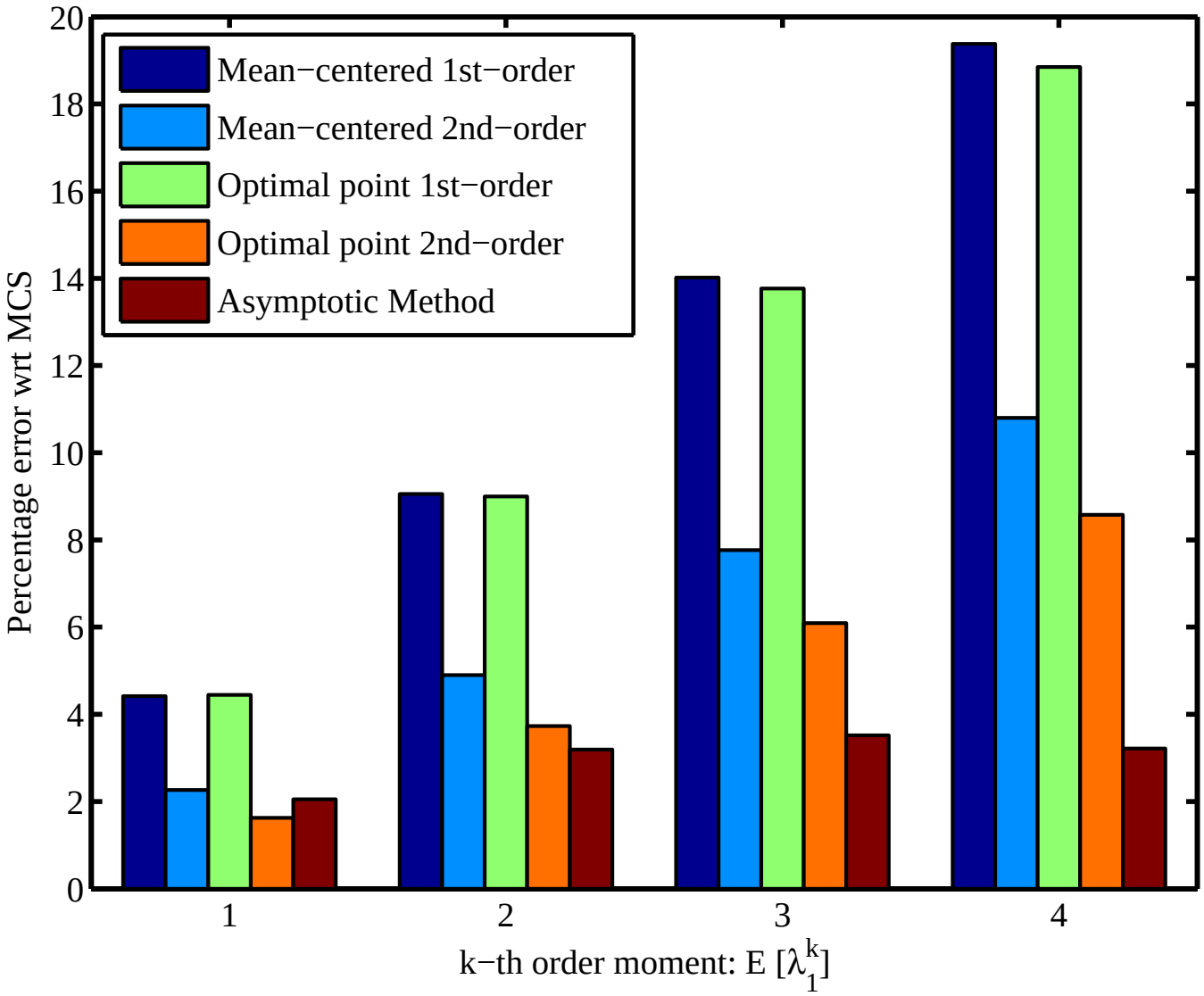


Figure: Percentage error for the first eigenvalue.

A two DOF system

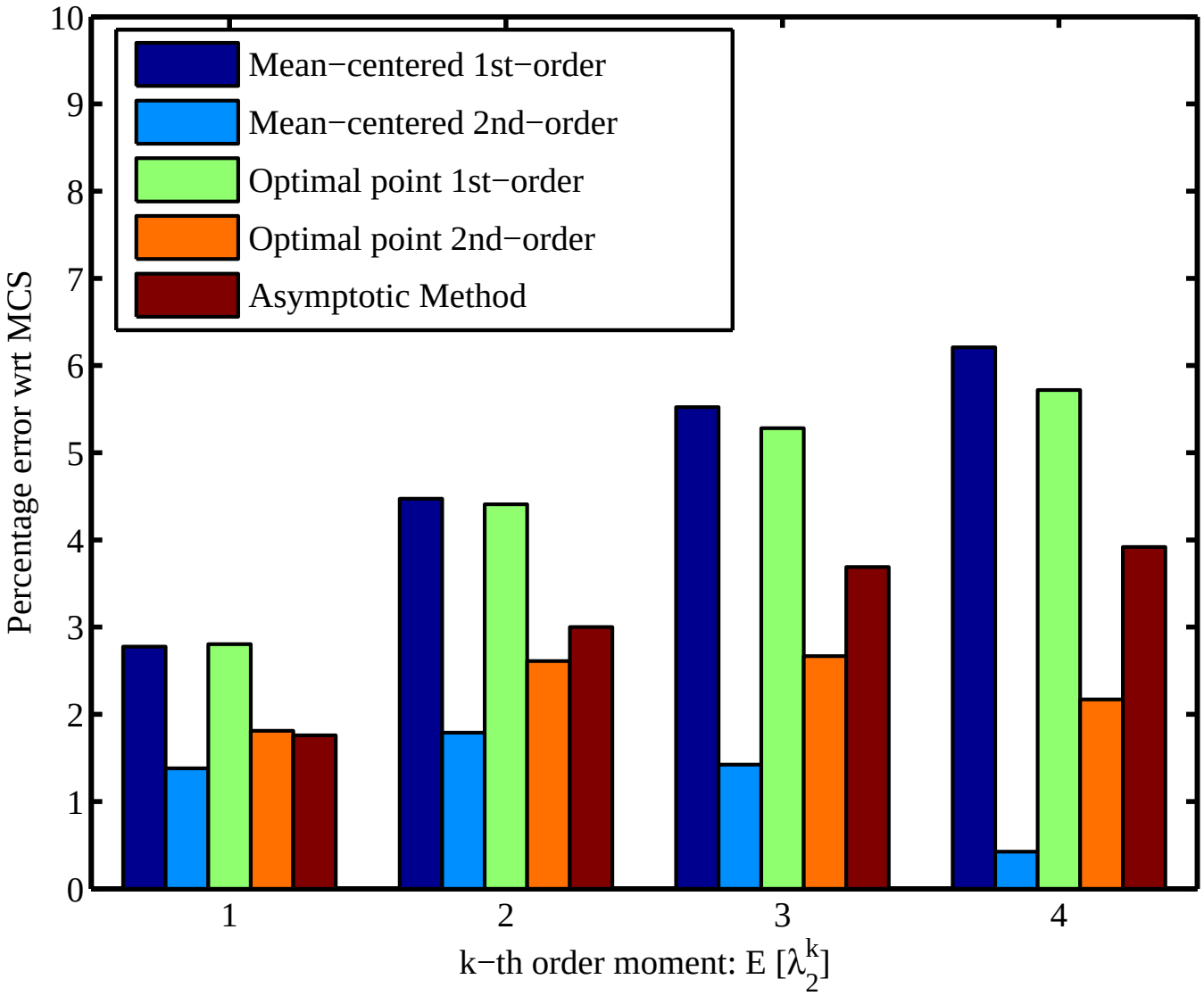


Figure: Percentage error for the second eigenvalue.

A two DOF system

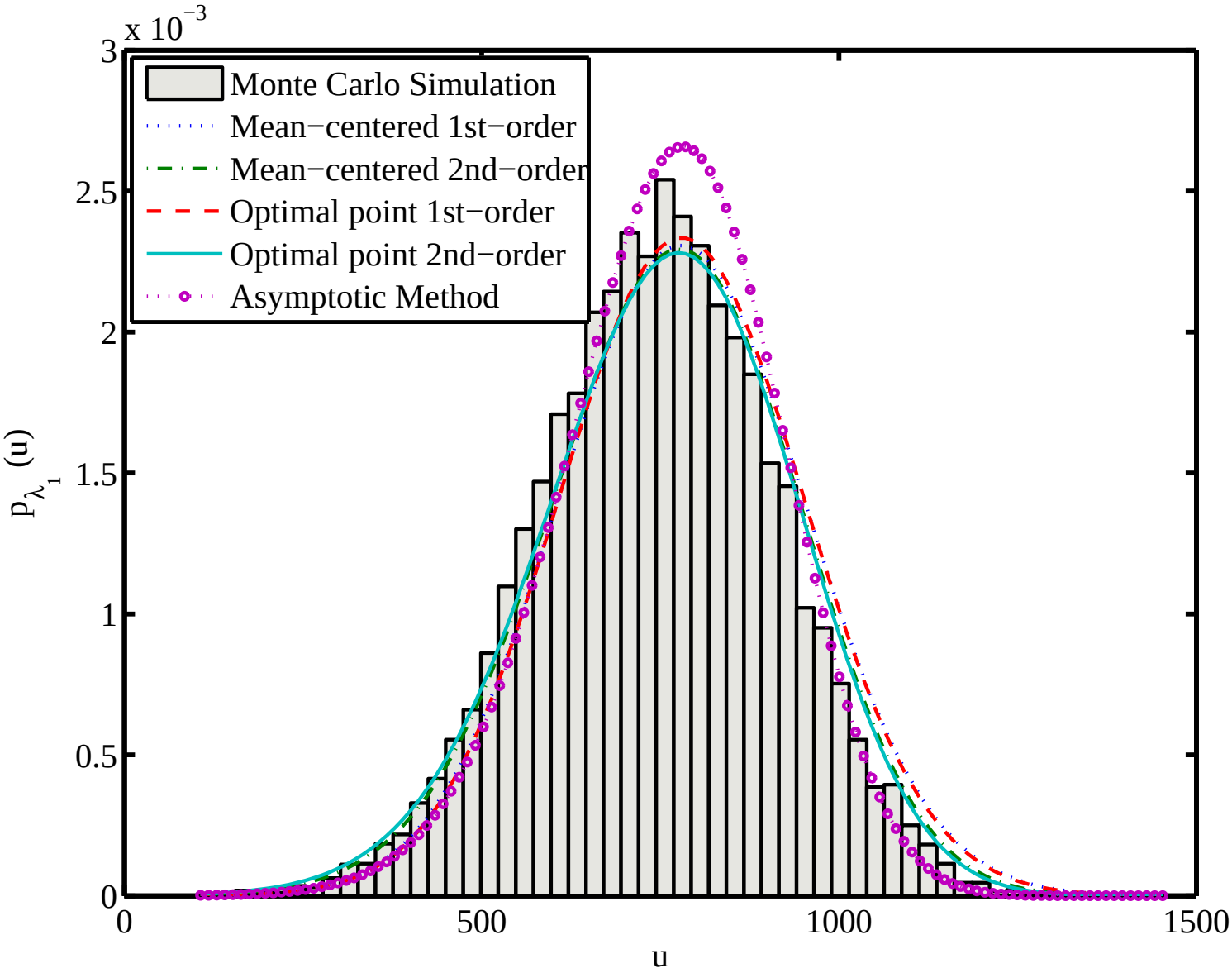


Figure: Probability density function of the first eigenvalue

A two DOF system

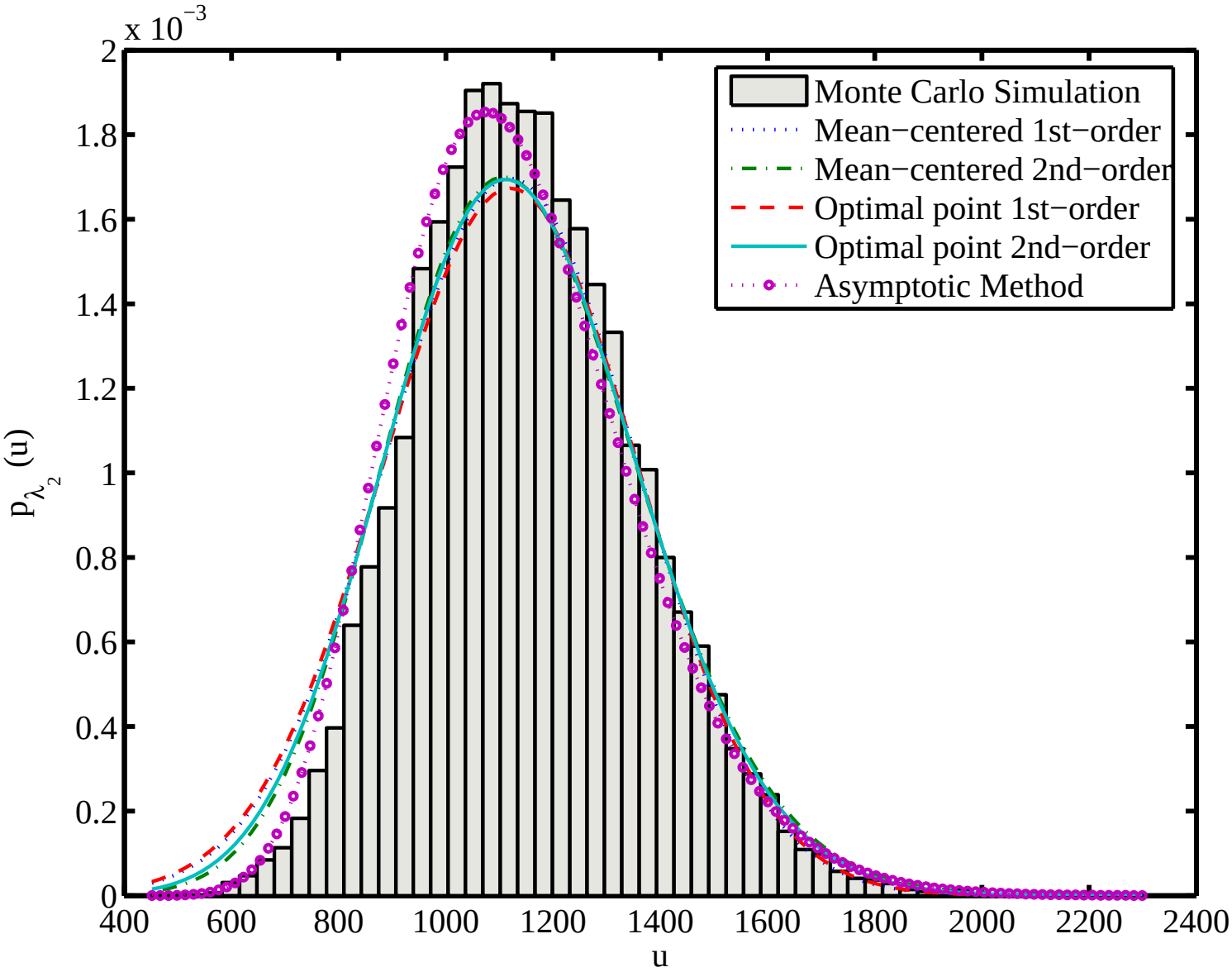


Figure: Probability density function of the second eigenvalue

A three DOF system

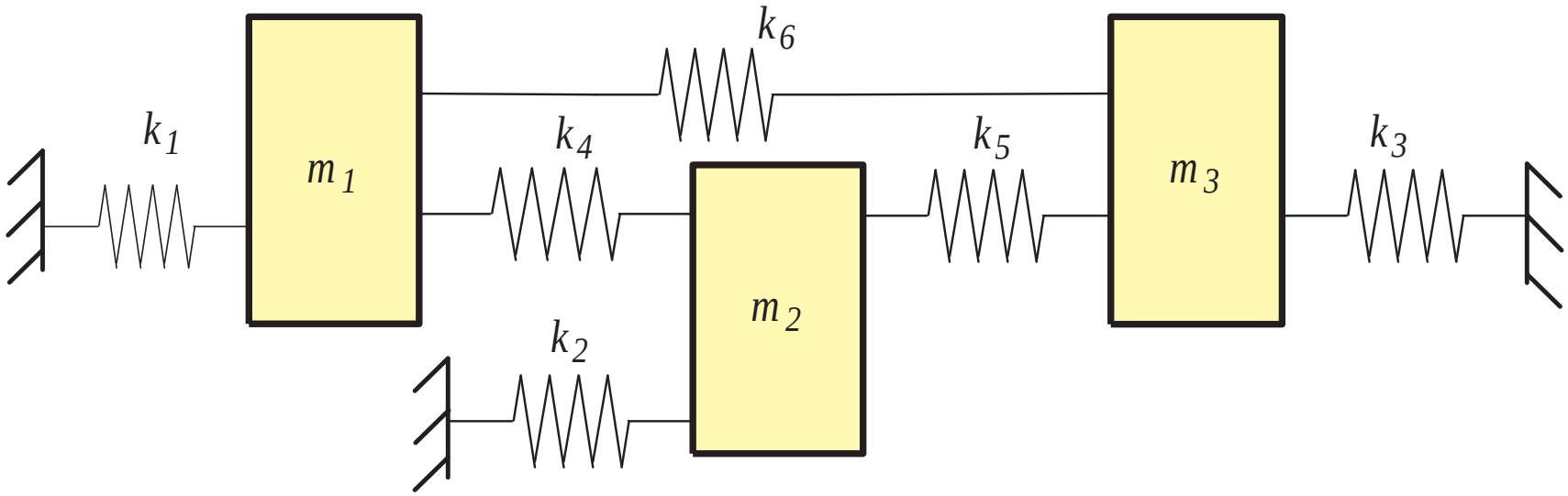


Figure: The three degree-of-freedom random system.

- The main purpose of this example is to understand how the proposed methods work when some of the system eigenvalues are closely spaced.

A three DOF system

- This is an interesting case because it is well known that closely spaced eigenvalues are parameter sensitive. We will investigate how the parameter uncertainty affects the eigenvalue distribution in such cases. This study has particular relevance to the dynamics of nominally symmetric rotating machines, for example, turbine blades with random imperfections. The mass and stiffness matrices of the example system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_4 + k_5 + k_2 & -k_5 \\ -k_6 & -k_5 & k_5 + k_3 + k_6 \end{bmatrix}. \tag{103}$$

  It is assumed that all mass and stiffness constants are random.

A three DOF system

- The randomness in these parameters are assumed to be of the following form:

$$m_i = \overline{m}_i \left(1 + \epsilon_m x_i\right), \quad i = 1, 2, 3 \tag{104}$$

$$k_i = \overline{k}_i \left(1 + \epsilon_k x_{i+3}\right), \quad i = 1, \cdots, 6. \tag{105}$$

  Here $\mathbf{x} = \{x_1, \cdots, x_9\}^T \in \mathbb{R}^9$ is the vector of random variables. It is assumed that all random variables are Gaussian and uncorrelated with zero mean and unit standard deviation, that is $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$. Therefore, the mean values of m_i and k_i are given by $\overline{m}_i$ and $\overline{k}_i$. The numerical values of both of the 'strength parameters' ϵ_m and ϵ_k are fixed at 0.15.

- In order to obtain statistics of the eigenvalues using the methods developed in this paper the gradient vector and the Hessian matrix of the eigenvalues are required. This in turn requires the derivative of the system matrices with respect to the entries of $\mathbf{x}$. For most practical problems, which usually involve Finite Element modeling, these derivatives need to be determined numerically.

- The derivatives of $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ with respect to elements of $\mathbf{x}$ can be obtained from equation (103) together with equations (104) and (105). For the mass matrix we have

$$\frac{\partial \mathbf{M}}{\partial x_1} = \begin{bmatrix} \overline{m}_1 \epsilon_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{m}_2 \epsilon_m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{m}_3 \epsilon_m \end{bmatrix} \tag{106}$$

  All other $\frac{\partial \mathbf{M}}{\partial x_i}$ are null matrices.

- The derivatives of the stiffness matrix are

$$\frac{\partial \mathbf{K}}{\partial x_4} = \begin{bmatrix} \overline{k}_1\epsilon_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_5} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{k}_2\epsilon_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{k}_3\epsilon_k \end{bmatrix},$$

$$\frac{\partial \mathbf{K}}{\partial x_7} = \begin{bmatrix} \overline{k}_4\epsilon_k & -\overline{k}_4\epsilon_k & 0 \\ -\overline{k}_4\epsilon_k & \overline{k}_4\epsilon_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \mathbf{K}}{\partial x_8} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{k}_5\epsilon_k & -\overline{k}_5\epsilon_k \\ 0 & -\overline{k}_5\epsilon_k & \overline{k}_5\epsilon_k \end{bmatrix}, \quad \frac{\partial \mathbf{M}}{\partial x_9} = \begin{bmatrix} \overline{k}_6\epsilon \\ 0 \\ -\overline{k}_6 \end{bmatrix} \tag{107}$$

  and all other $\frac{\partial \mathbf{K}}{\partial x_i}$ are null matrices.
- Also note that all of the first-order derivative matrices are independent of $\mathbf{x}$. For this reason, all the higher order derivatives of the $\mathbf{M}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ matrices are null matrices.

A three DOF system

- We calculate the moments and the probability density functions of the three eigenvalues of the system. The following two sets of physically meaningful parameter values are considered:
  - *Case 1: All eigenvalues are well separated*
    For this case $\overline{m}_i = 1.0$ kg for $i = 1, 2, 3$; $\overline{k}_i = 1.0$ N/m for $i = 1, \cdots, 5$ and $k_6 = 3.0$ N/m.
  - *Case 2: Two eigenvalues are close*
    All parameter values are the same except $k_6 = 1.275$ N/m.
- The moments of the eigenvalues for the above two cases are calculated first.

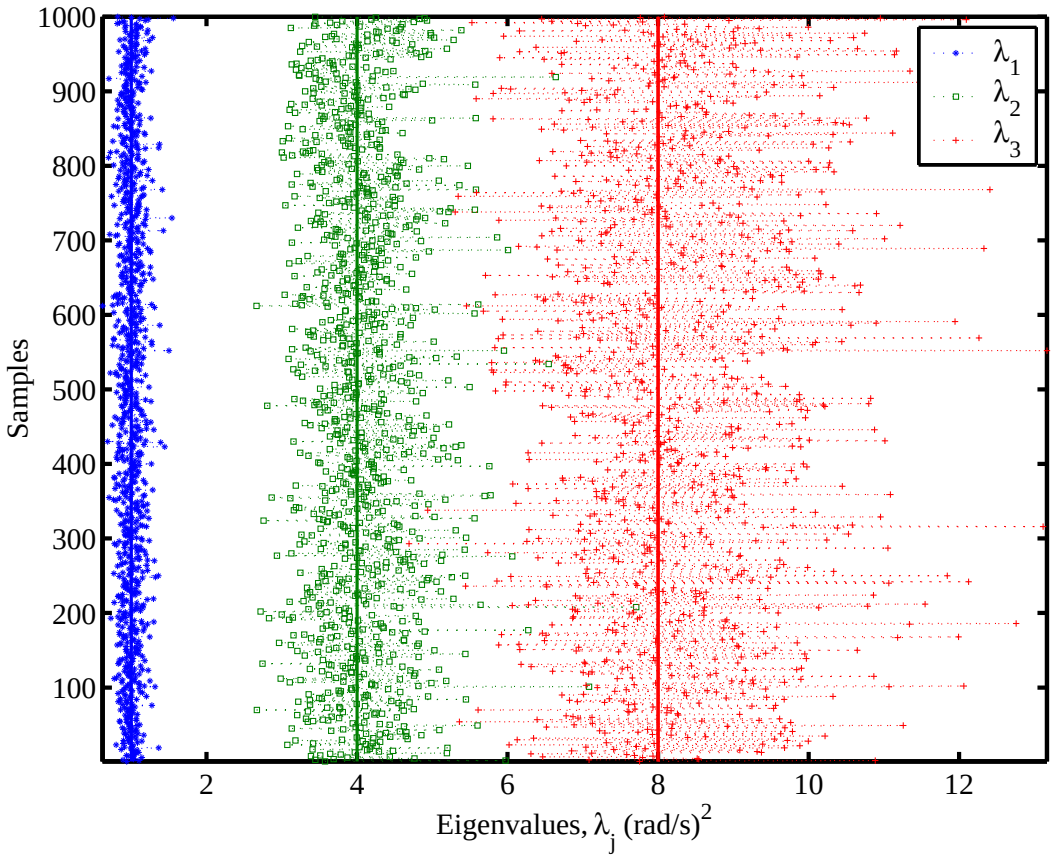
A three DOF system

- The moments are then used to obtain σ_j from equation (87) and the constants in equations (93)–(95).
- Using these constants the truncated Gaussian pdf and the χ^2 pdf of the eigenvalues are obtained from equations (86) and (96) respectively. These results are compared with Monte Carlo simulation.
- The samples of the nine independent Gaussian random variables $x_i, i = 1, \cdots, 9$ are generated and the eigenvalues are computed directly from equation (1). A total of 15000 samples are used to obtain the statistical moments and histograms of the pdf of the eigenvalues.
- The results obtained from Monte Carlo simulation are assumed to be the benchmark for the purpose of comparing the analytical methods.
- For the purpose of determining the accuracy, we again calculate the percentage error associated with an arbitrary rth moment using equation (102).

All eigenvalues are well separated

- When all of the eigenvalues are well separated their derivatives with respect to the system parameters generally behave well.
- For the given parameter values the eigenvalues of the corresponding deterministic system is given by

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4, \quad \text{and} \quad \overline{\lambda}_3 = 8. \tag{108}$$



All eigenvalues are well separated

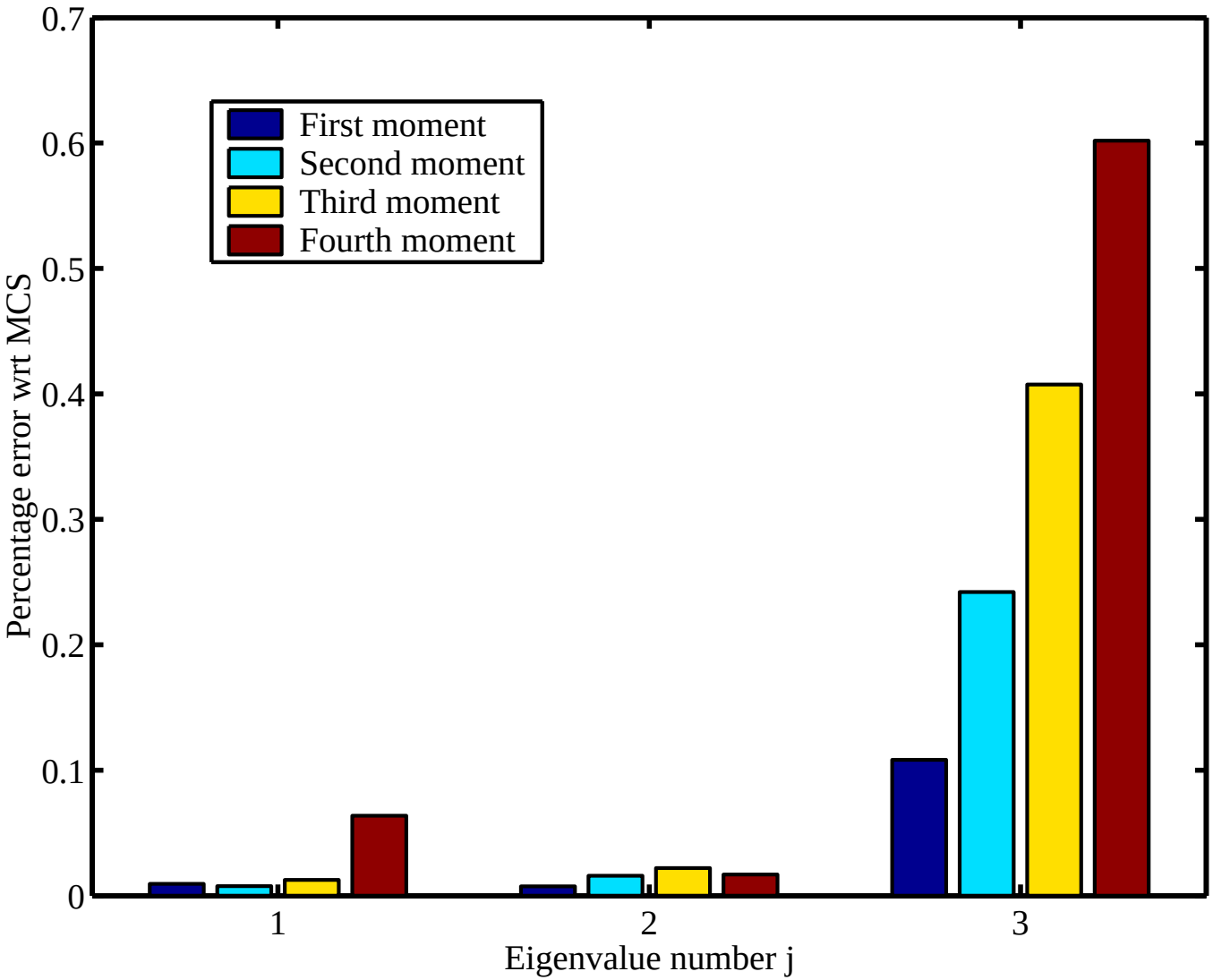


Figure: Percentage error for first four moments of the eigenvalues; Case 1.

All eigenvalues are well separated

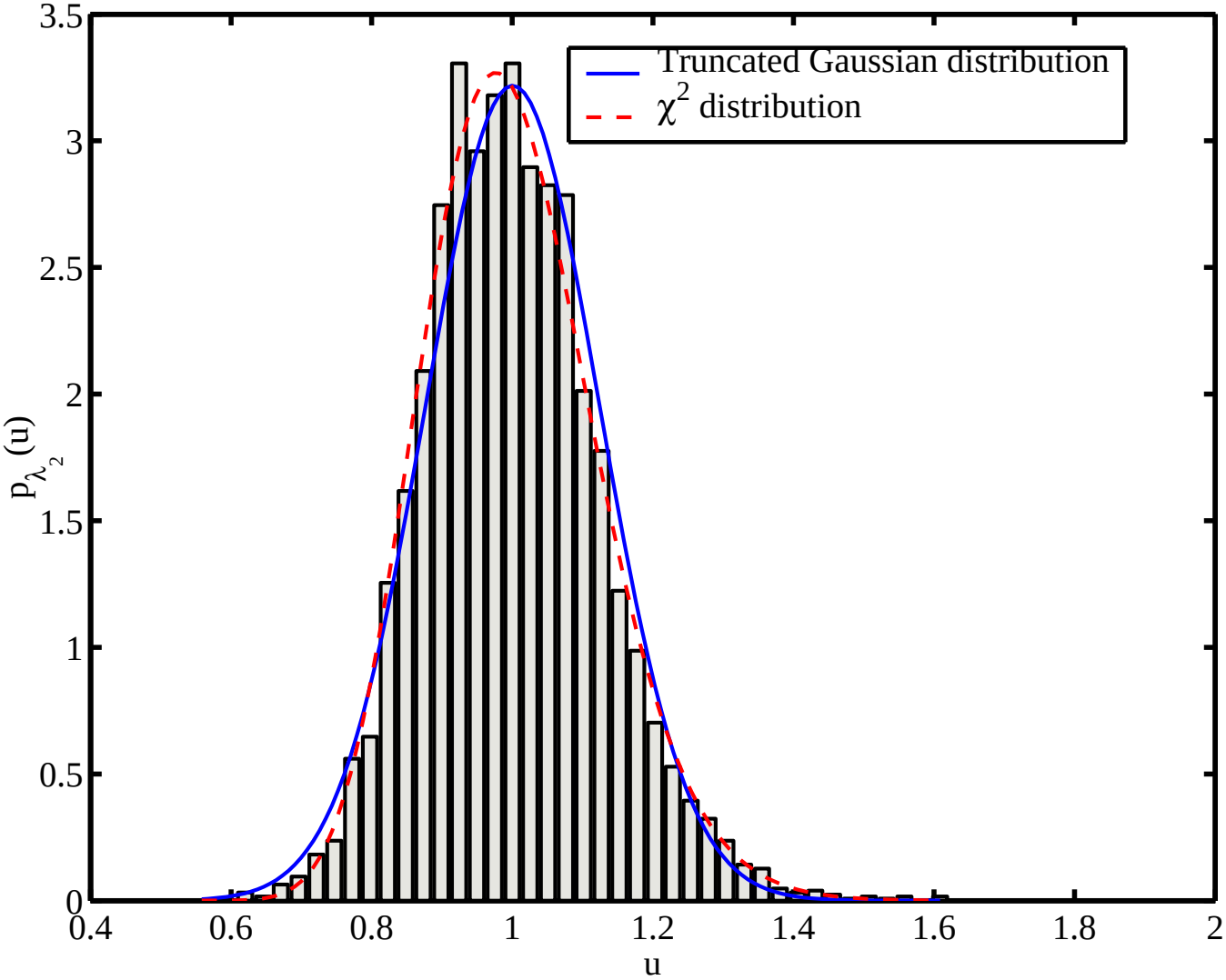


Figure: Probability density function of the first eigenvalue; Case 1.

All eigenvalues are well separated

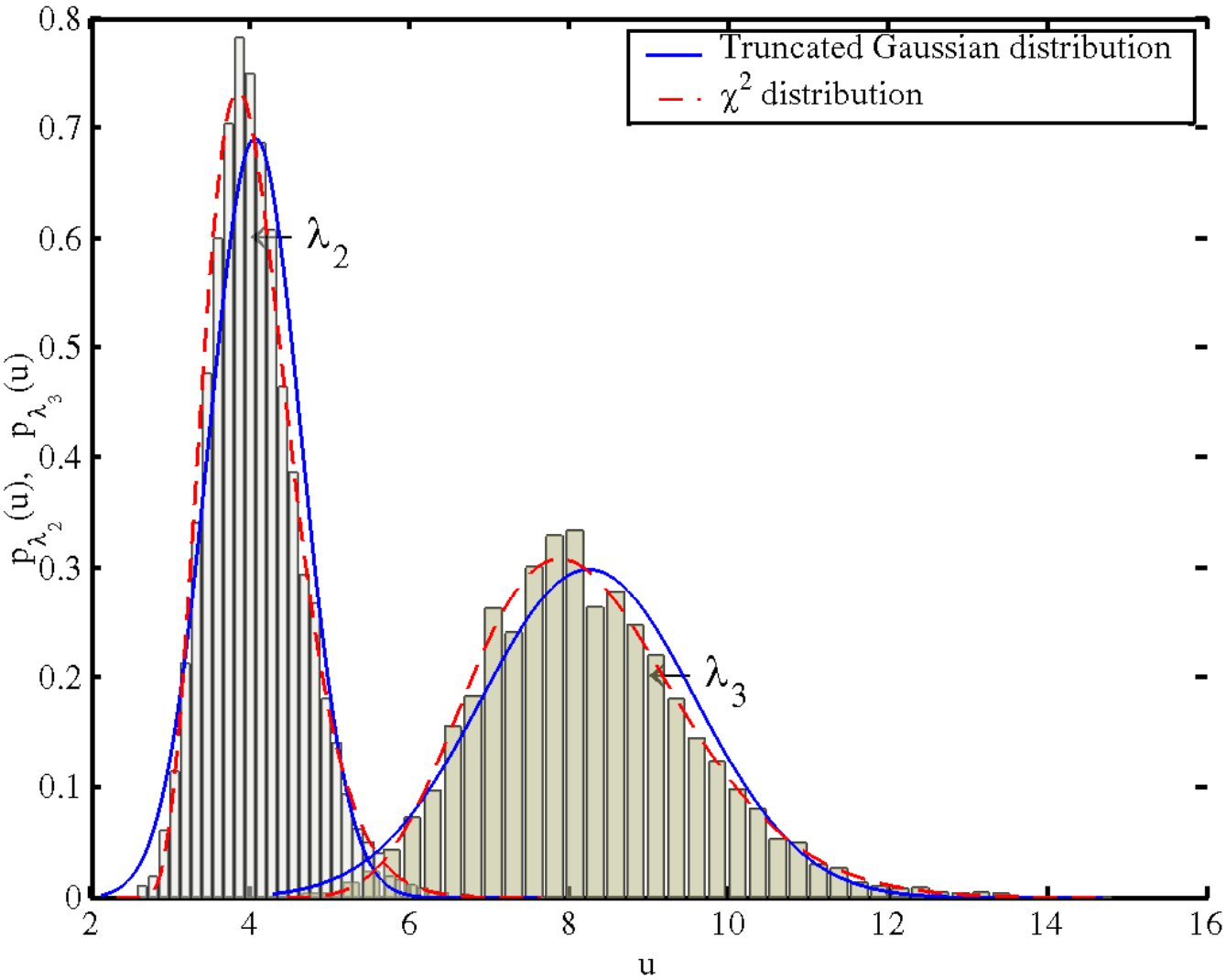


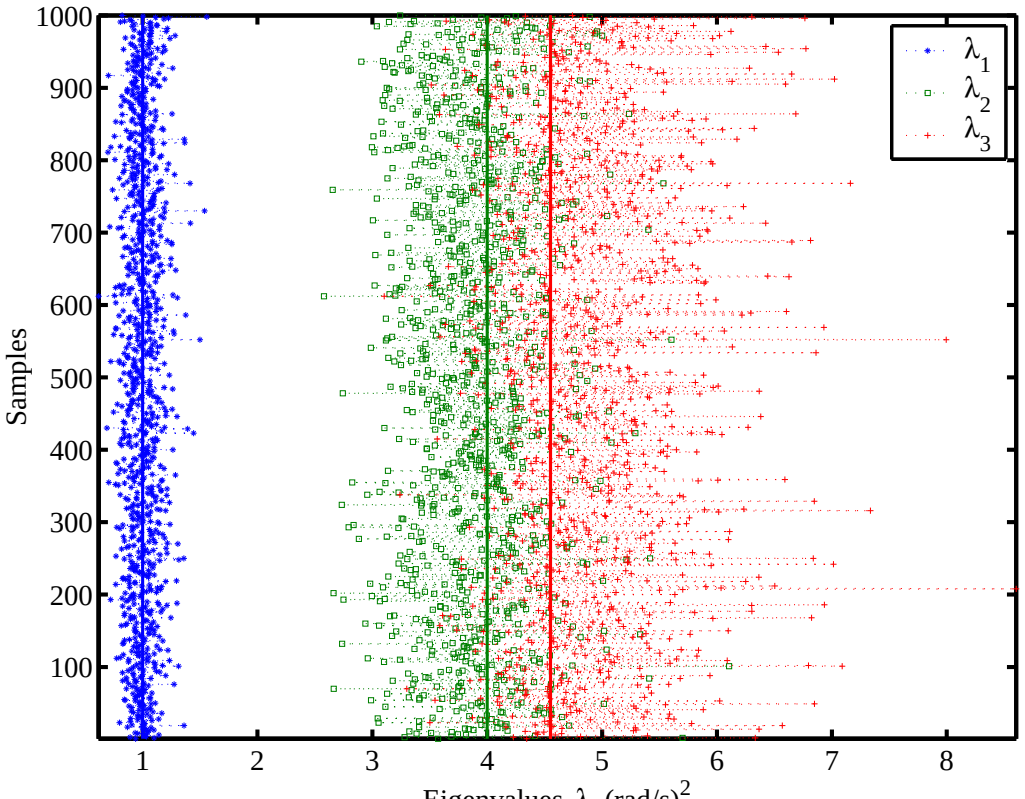
Figure: Probability density functions of the second and third eigenvalues; Case 1.

Two eigenvalues are close

- When some eigenvalues are closely spaced, their derivatives with respect to the system parameters may not behave well
- For the given parameter values the eigenvalues of the corresponding deterministic system are calculated as

$$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4, \quad \text{and} \quad \overline{\lambda}_3 = 4.55. \tag{109}$$

Clearly $\overline{\lambda}_2$ and $\overline{\lambda}_3$ are close to each other.



Two eigenvalues are close

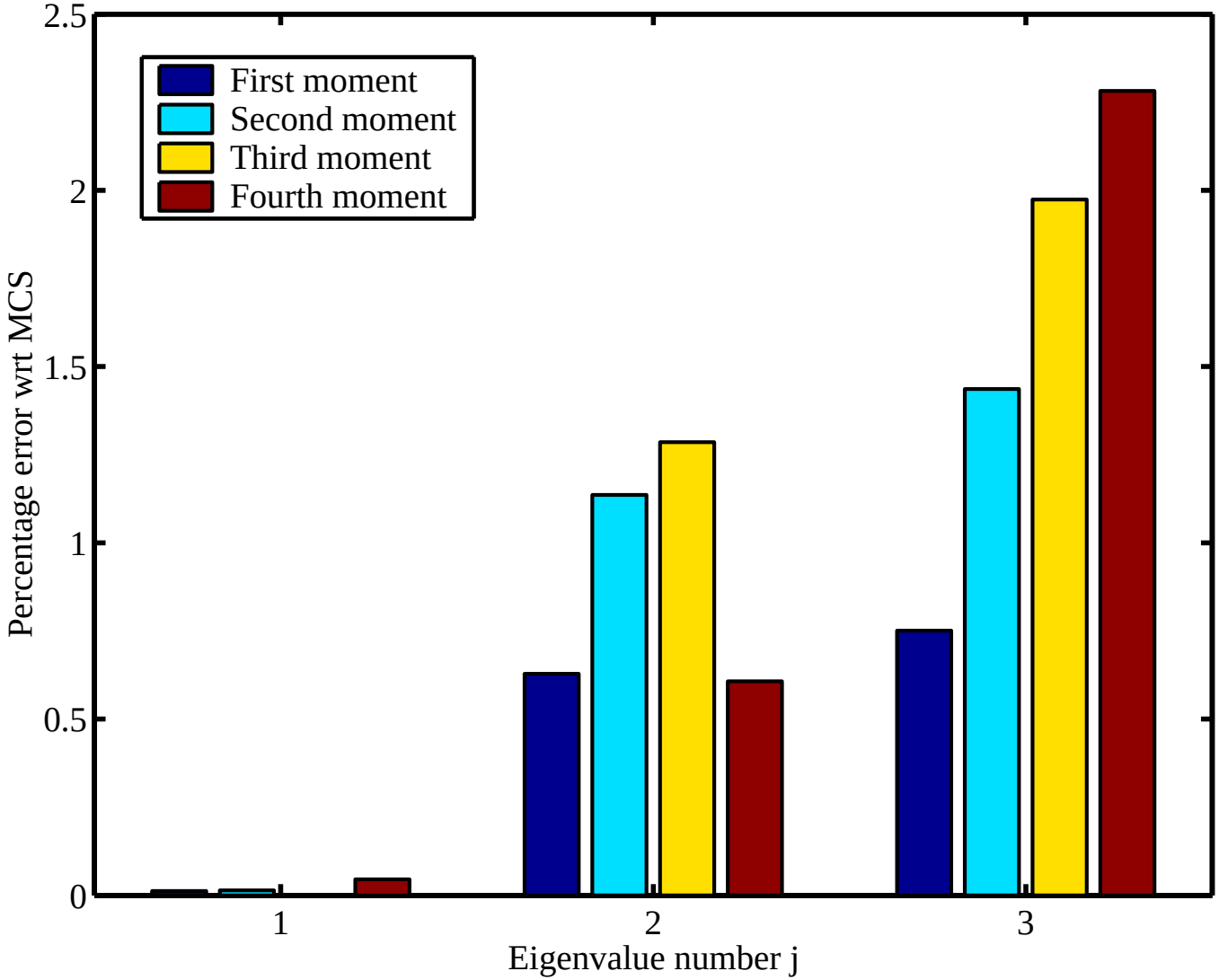


Figure: Percentage error for first four moments of the eigenvalues; Case 2.

Two eigenvalues are close

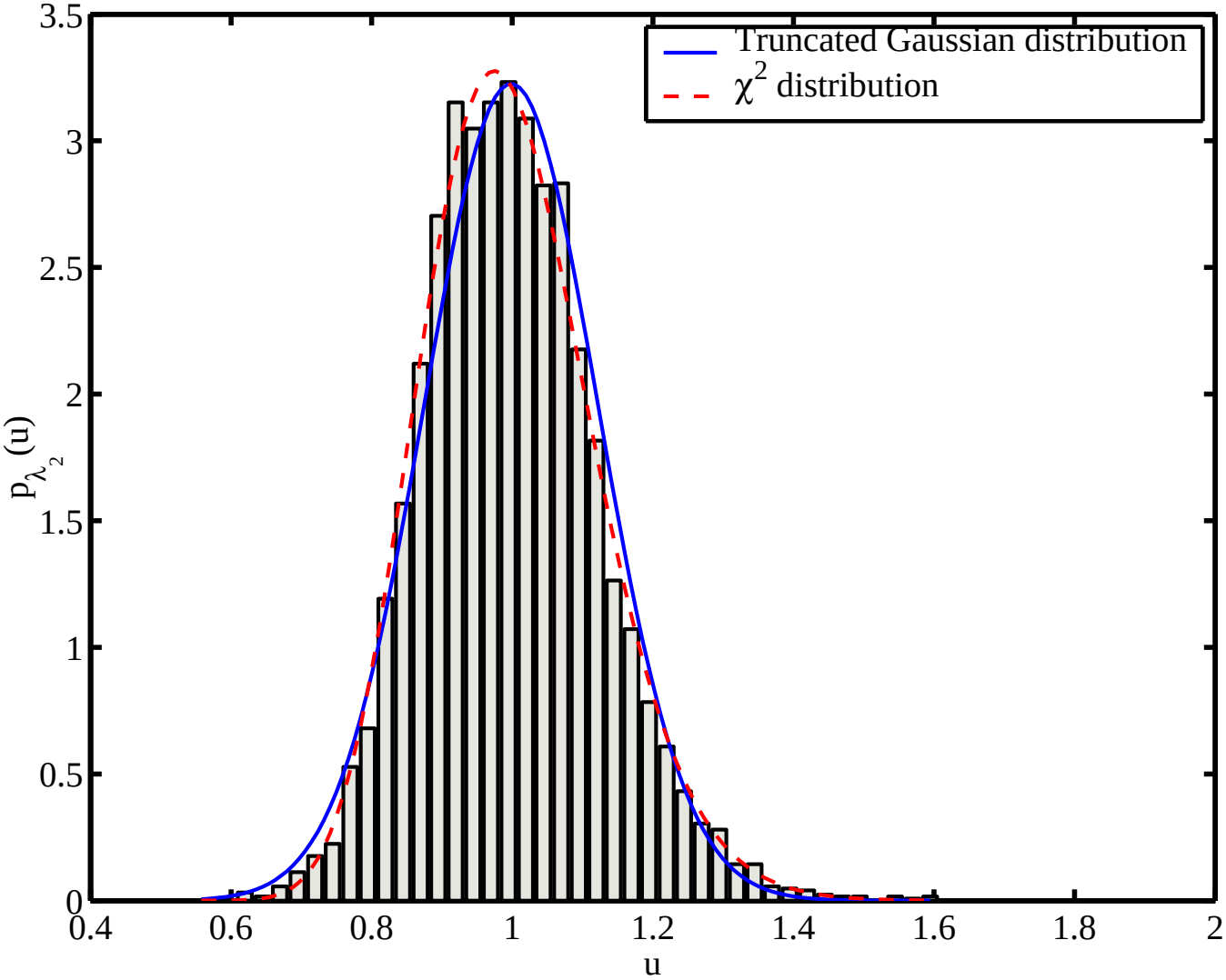


Figure: Probability density function of the first eigenvalue; Case 2.

Two eigenvalues are close

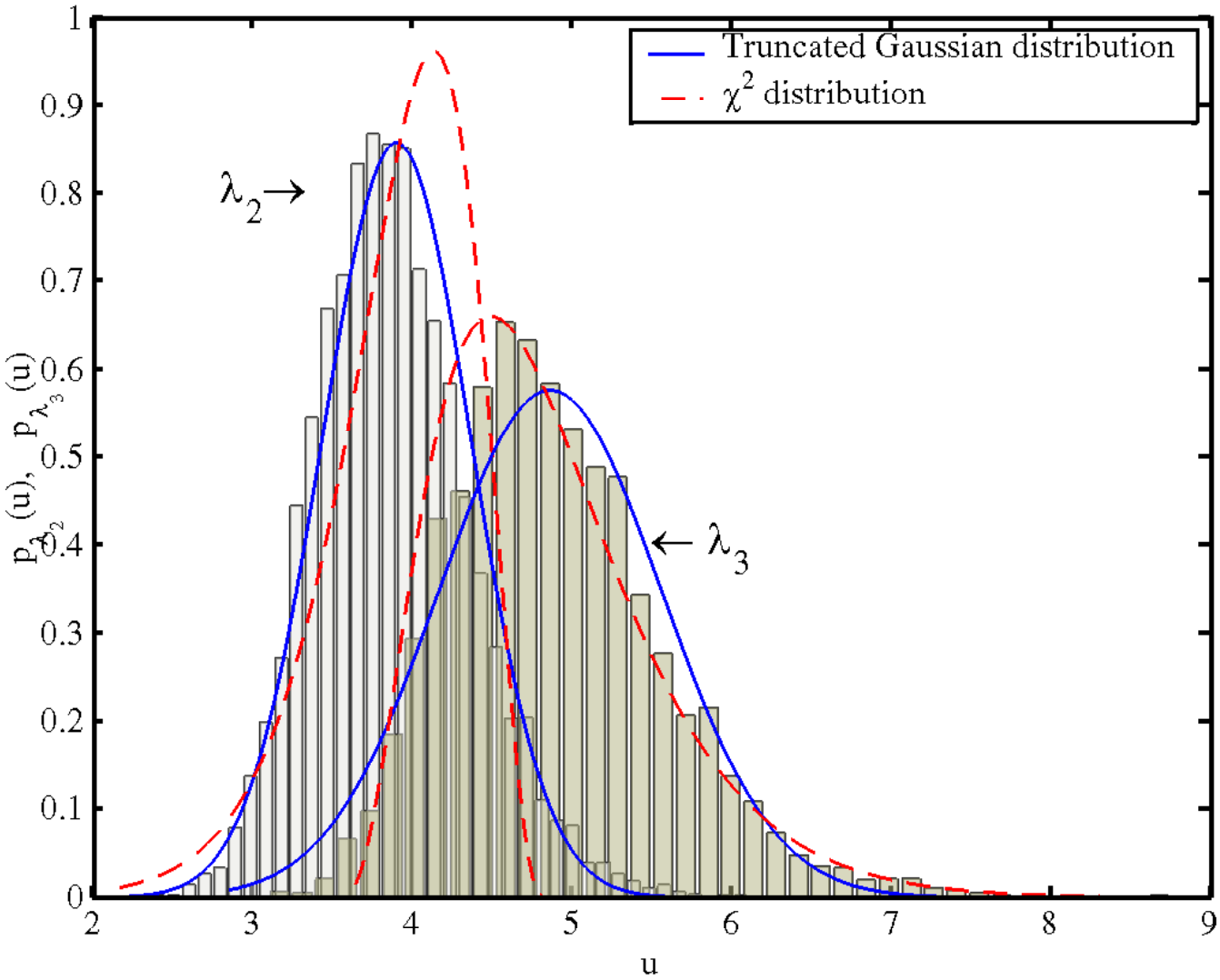


Figure: Probability density functions of the second and third eigenvalues; Case 2.