Uncertainty quantification in Structural Dynamics: Day 2

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Overview of the course

The course is dived into eight topics:

- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

1. **Uncertainty propagation: Sampling methods**
   - Monte Carlo Simulation
2. **Non-Sampling methods**
   - Perturbation based methods
   - Polynomial Chaos expansion
     - One dimensional function
     - Vector function
3. **Spectral function method**
   - Motivation
   - Projection in the modal space
   - Properties of the spectral functions
   - The Galerkin approach
   - Model Reduction
   - Computational method
4. **Numerical illustrations**
5. **Summary**

Random number generator

- Quasi-random number generators calculates a sequence of numbers that appear to be random $x_i = g(x_{i-1}, \ldots, x_{i-k})$, and the sequence is repeated after applying g a given number of times, called the period.
- These random number generators are used to simulate uniformly distributed random variables. The uniform univariate distribution $U(0,1)$ has a probability density function given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

  and its mean and variance are respectively $\mathrm{E}[X] = 1/2$, $\mathrm{Var}[X] = 1/12$.
- Generally, samples of random variables with pdfs different from the uniform pdf are needed. A random variable X with continuous cumulative density function P_X can be related to a uniform random variable $U(0,1)$ through the inverse CDF method

$$X = P_X^{-1}(U) \tag{2}$$

Random number generator

- For the case of a Gaussian random variable $N(0, 1)$, samples can be obtained from samples of two independent uniform random variables U and V

$$X = (-2 \ln U)^{1/2} \cos(2\pi V), \qquad Y = (-2 \ln U)^{1/2} \sin(2\pi V) \tag{3}$$

  so that X and Y are independent random variables with standard normal distribution.

- Once the samples of the random variables are obtained, they are introduced in the PDE studied and the deterministic systems are solved. If MCS with N samples is used to obtain an estimation of the pdf of a random variable u (e.g. a term of the response vector **u**), estimations of the mean and standard deviation are given by

$$\mathsf{E}[u] = \int u p(u) \mathrm{d}u \approx \frac{1}{N} \sum_{i=1}^{N} u_i \tag{4}$$

$$\sigma = \int (u - \mathsf{E}[u])^2 p(u) \mathrm{d}u \approx \sqrt{\frac{1}{N} \sum_{i=1}^{N} (u_i - \mathsf{E}[u])^2} \tag{5}$$

- One of the first methods used to study uncertainty propagation is the perturbation method where terms are expanded with their Taylor series expansion around the mean value of the random parameters α_i, $i = 1, \ldots, M$
- Taylor series expansions of stiffness **K**, response **u** and load vector **f** are truncated after the second order terms and introduced into $\mathbf{K}\mathbf{u} = \mathbf{f}$:

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{K}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (6)$$

$$\mathbf{u} = \mathbf{u}_0 + \sum_{i=1}^{N} \mathbf{u}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{u}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (7)$$

$$\mathbf{f} = \mathbf{f}_0 + \sum_{i=1}^{N} \mathbf{f}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (8)$$

- The coefficients multiplying polynomials of the same order can be identified

$$\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}_0 \quad (9)$$

$$\mathbf{u}_i^I = \mathbf{K}_0^{-1}(\mathbf{f}_i^I - \mathbf{K}_i^I \mathbf{u}_0) \quad (10)$$

$$\mathbf{u}_{ij}^{II} = \mathbf{K}_0^{-1}(\mathbf{f}_{ij}^{II} - \mathbf{K}_i^I \mathbf{u}_j^I - \mathbf{K}_j^I \mathbf{u}_i^I - \mathbf{K}_{ij}^{II} \mathbf{u}_0) \quad (11)$$

  where terms with subindexes 0, i and ij are respectively the matrix or vector evaluated at $\boldsymbol{\alpha} = 0$, its first derivative (e.g. $\mathbf{K}_i^I = \left.\frac{\partial \mathbf{K}}{\partial \alpha_i}\right|_{\alpha=0}$) and its second derivative (e.g. $\mathbf{K}_{ij}^{II} = \left.\frac{\partial^2 \mathbf{K}}{\partial \alpha_i \partial \alpha_j}\right|_{\alpha=0}$)
- The statistics of **u** are derived from the second order Taylor expansion of **u** and the statistics of $\boldsymbol{\alpha}$

$$\mathrm{E}[\mathbf{u}] \approx \mathbf{u}_0 + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \mathbf{u}_{ij}^{II} \mathrm{Cov}[\alpha_i, \alpha_j] \quad (12)$$

$$\mathrm{Cov}[\mathbf{u}, \mathbf{u}] \approx \sum_{i=1}^{M}\sum_{j=1}^{M} \mathbf{u}_i^I . (\mathbf{u}_j^I)^T \mathrm{Cov}[\alpha_i, \alpha_j] \quad (13)$$

- If a function $f(\zeta)$ is a function of infinite number of variables $\{\zeta_{i_k}\}$ and square integrable, it can be expanded in Homogeneous Chaos as

$$\begin{aligned} f(\boldsymbol{\zeta}) = \hat{y}_{i_0} h_0 + \sum_{i_1=1}^{\infty} \hat{y}_{i_1} \Gamma_1(\zeta_{i_1}) \\ + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \hat{y}_{i_1,i_2} \Gamma_2(\zeta_{i_1}, \zeta_{i_2}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \hat{y}_{i_1 i_2 i_3} \Gamma_3(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}) \\ + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} \hat{y}_{i_1 i_2 i_3 i_4} \Gamma_4(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}, \zeta_{i_4}) + \ldots, \end{aligned} \quad (14)$$

  Here $\Gamma_p(\zeta_{i_1}, \zeta_{i_2}, \cdots \zeta_{i_m})$ is m-dimensional homogeneous chaos of order p.
- The polynomials are orthogonal with respect to the probability measure of the underlying random variables

Polynomial Chaos expansion

- For Gaussian random variables, Hermite polynomials are used.
- For Uniform random random variables, Legendre polynomials are used.
- Truncating Eq. (14) up to finite number of terms, we can concisely write

$$f(\boldsymbol{\zeta}) = \sum_{j=0}^{P-1} y_j \Psi_j(\boldsymbol{\zeta}) \tag{15}$$

  where the constant y_j and functions $\Psi_j(\bullet)$ are effectively constants $\hat{y}_k$ and functions $\Gamma_k(\bullet)$ for corresponding indices.
- Equation (15) can be viewed as the projection in the basis functions $\Psi_j(\boldsymbol{\zeta})$ with corresponding 'coordinates' y_j. The number of terms P in Eq. (15) depends on the number of variables m and maximum order of polynomials p as

$$P = \sum_{j=0}^{p} \frac{(m+j-1)!}{j!(m-1)!} = \binom{m+p}{p} \tag{16}$$

Polynomial Chaos expansion

j	p	Construction of Ψ_j	Ψ_j
0	$p=0$	L_0	1
1	$p=1$	$L_1(\zeta_1)$	ζ_1
2		$L_1(\zeta_2)$	ζ_2
3		$L_2(\zeta_1)$	$3/2\,{\zeta_1}^2-1/2$
4	$p=2$	$L_1(\zeta_1)L_1(\zeta_2)$	$\zeta_1\zeta_2$
5		$L_2(\zeta_2)$	$3/2\,{\zeta_2}^2-1/2$
6		$L_3(\zeta_1)$	$5/2\,{\zeta_1}^3-3/2\,\zeta_1$
7	$p=3$	$L_2(\zeta_1)L_1(\zeta_2)$	$\left(3/2\,{\zeta_1}^2-1/2\right)\zeta_2$
8		$L_1(\zeta_1)L_2(\zeta_2)$	$\zeta_1\left(3/2\,{\zeta_2}^2-1/2\right)$
9		$L_3(\zeta_2)$	$5/2\,{\zeta_2}^3-3/2\,\zeta_2$
10		$L_4(\zeta_1)$	$\frac{35}{8}\,{\zeta_1}^4-\frac{15}{4}\,{\zeta_1}^2+3/8$
11		$L_3(\zeta_1)L_1(\zeta_2)$	$\left(5/2\,{\zeta_1}^3-3/2\,\zeta_1\right)\zeta_2$
12	$p=4$	$L_2(\zeta_1)L_2(\zeta_2)$	$\left(3/2\,{\zeta_1}^2-1/2\right)\left(3/2\,{\zeta_2}^2-1/2\right)$
13		$L_1(\zeta_1)L_3(\zeta_2)$	$\zeta_1\left(5/2\,{\zeta_2}^3-3/2\,\zeta_2\right)$
14		$L_4(\zeta_2)$	$\frac{35}{8}\,{\zeta_2}^4-\frac{15}{4}\,{\zeta_2}^2+3/8$

Polynomial Chaos expansion

- A least-square error minimization approach can be used to obtain the constants y_j in Eq. (15). We define the inner product norm in $[-1,1]^m$ as

$$< \bullet, \bullet >= \frac{1}{V_m} \underbrace{\int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1}}_{m-\text{fold}} (\bullet)(\bullet) d\zeta_1 d\zeta_2 \cdots d\zeta_m \tag{17}$$

- Here the volume

$$V_m = 2^m \tag{18}$$

is used for normalization so that for two constants a and b we have $< a, b >= ab$. The error corresponding to Eq. (15) can be expressed as

$$\varepsilon = f(\boldsymbol{\zeta}) - \sum_{j=0}^{P-1} y_j \Psi_j(\boldsymbol{\zeta}) \tag{19}$$

- Using the inner product norm in (17), the norm of the error can be obtained as

$$\chi^2 = \langle \varepsilon, \varepsilon \rangle \tag{20}$$

- Differentiating this with respect to y_k, it can be shown that (Galerkin approach) the optimal values of y_k can be obtained my making the basis functions orthogonal to the error, that is,

$$\varepsilon \perp \Psi_k \quad \text{or} \quad \langle \Psi_k, \varepsilon \rangle = 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{21}$$

- Substituting the expression of error from Eq. (19) into this equation we obtain

$$\sum_{j=0}^{P-1} y_j \left\langle \Psi_k(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \right\rangle = \left\langle \Psi_k(\boldsymbol{\zeta}), f(\boldsymbol{\zeta}) \right\rangle \tag{22}$$

  Using the orthogonality property of the basis function we have $\left\langle \Psi_k(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \right\rangle = c_k \delta_{jk}$.

- Therefore, the constants y_k can be obtained as

$$y_k = \frac{\left\langle \Psi_k(\boldsymbol{\zeta}), f(\boldsymbol{\zeta}) \right\rangle}{\left\langle \Psi_k(\boldsymbol{\zeta}), \Psi_k(\boldsymbol{\zeta}) \right\rangle}, \quad \forall\ k = 0, 2, \ldots, P-1 \tag{23}$$

  The integration appearing in the numerator and denominator can be obtained using any standard procedure for multidimensional integrals. In particular, the denominator can be calculated explicitly. The values of $\left\langle \Psi_j(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \right\rangle$ cab be obtained analytically.

Table: Values of $\langle \Psi_j(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \rangle$ for two variables ($m = 2$) with polynomial order 4 ($p = 4$).

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\langle \Psi_j(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \rangle$	1	1/3	1/3	1/5	1/9	1/5	1/7	1/15	1/15	1/7	1/9	1/21	1/25	1/21	1/9

- Substituting the values of y_k from (23) into the expansion (15) we have

$$\widehat{f}(\boldsymbol{\zeta}) = \sum_{j=0}^{P-1} \left[\frac{\langle \Psi_j(\boldsymbol{\zeta}), f(\boldsymbol{\zeta}) \rangle}{\langle \Psi_j(\boldsymbol{\zeta}), \Psi_j(\boldsymbol{\zeta}) \rangle} \right] \Psi_j(\boldsymbol{\zeta}) \tag{24}$$

- Here $\widehat{f}(\boldsymbol{\zeta})$ is an approximation to the original function $f(\boldsymbol{\zeta})$ for polynomial order upto p. The accuracy of this approximation can improve indefinitely by considering higher-order polynomials. If the evaluation of the original function $f(\boldsymbol{\zeta})$ is expensive, the surrogate model $\widehat{f}(\boldsymbol{\zeta})$ can be used instead of the original function.

- To illustrate the application of the Galerkin projection approach, we consider two problems involving bounded variables. We consider the function

$$\widehat{f}_1(\mathbf{x}) = \frac{89}{40} - \frac{\sqrt{2}}{1080}(x_1 + x_2 - 20)^3 + \frac{33}{140}(x_1 - x_2)\,;\; 4 \le x_1, x_2 \le 16 \quad (25)$$

- As the first step, we transform the variables in $[-1, 1]$:

$$x_1 = 6\zeta_1 + 10 \quad \text{and} \quad x_2 = 6\zeta_2 + 10 \quad (26)$$

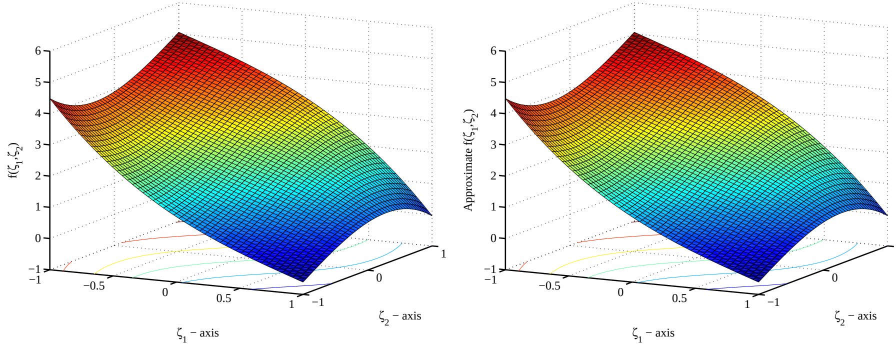
- Substituting these into Eq. (25) one obtains the function in the transformed variables as

$$f_1(\boldsymbol{\zeta}) = \frac{89}{40} - \frac{1}{5}\sqrt{2}(\zeta_1 + \zeta_2)^3 + \frac{99}{70}\zeta_1 - \frac{99}{70}\zeta_2 \quad (27)$$

- Using Eq. (23) the nonzero values of y_j can be obtained as

$$y_1 = \frac{89}{40}, y_2 = -\frac{8}{25}\sqrt{2} + \frac{99}{70}, y_3 = -\frac{8}{25}\sqrt{2} - \frac{99}{70}, y_7 = -\frac{2}{25}\sqrt{2},$$
$$y_8 = -2/5\sqrt{2}, y_9 = -2/5\sqrt{2} \quad \text{and} \quad y_{10} = \frac{2}{25}\sqrt{2} \quad (28)$$

Polynomial Chaos expansion



(a) The exact function (b) Fitted function using Legendre polynomials

Figure: The original function and the fitted function corresponding to Eq. (25).

Polynomial Chaos expansion: Example 2

- Consider the 'Camelback' function

$$f_1\left(\mathbf{x}\right) = (4-2.1x_1^2+x_1^4/3)x_1^2+x_1x_2+(-4+4x_2^2)x_2^2; -3 \leq x_1 \leq 3; -2 \leq x_2 \leq 2 \tag{29}$$

- Transform the variables in $[-1,1]$:

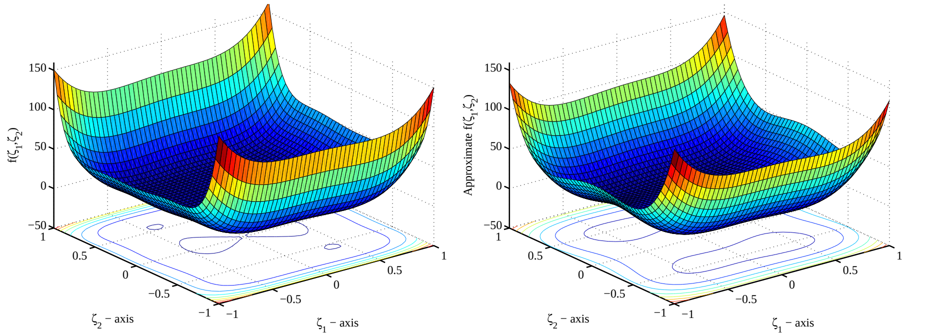
$$x_1 = 3\zeta_1 \quad \text{and} \quad x_2 = 2\zeta_2 \tag{30}$$

- Substituting these into Eq. (29) one obtains the function in the transformed variables as

$$f_1\left(\boldsymbol{\zeta}\right) = 9\left(4 - \frac{189}{10}{\zeta_1}^2 + 27{\zeta_1}^4\right){\zeta_1}^2 + 6\zeta_1\zeta_2 + 4\left(-4+16{\zeta_2}^2\right){\zeta_2}^2 \tag{31}$$

Using Eq. (23), carrying out the 2-dimensional integration analytically, the nonzero values of y_j can be obtained as

$$y_1 = \frac{21169}{1050}, y_4 = \frac{1488}{35}, y_5 = 6, y_6 = \frac{544}{21}, y_{11} = \frac{70956}{1925}, y_{15} = \frac{512}{35} \tag{32}$$



(a) The exact function (b) Fitted function using Legendre polynomials

Figure: The original function and the fitted function corresponding to Eq. (29).

- The main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right)\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f} \tag{33}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.
- We project the solution vector $\mathbf{u}(\boldsymbol{\xi}) \in \mathbb{R}^n$ in the basis of orthogonal polynomials as

$$\mathbf{u}(\boldsymbol{\xi}) = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{34}$$

- The aim is to obtain the coefficient vectors $\mathbf{u}_j \in \mathbb{R}^n$ using a Galerkin type of error minimisation approach.

- Substituting expansion of $\mathbf{u}(\boldsymbol{\xi})$ in the governing equation (33), the error vector can be obtained as

$$\boldsymbol{\varepsilon} = \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i\right) \left(\sum_{j=1}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi})\right) - \mathbf{f} \in \mathbb{R}^n \tag{35}$$

where $\xi_0 = 1$ is used to simplify the first summation expression.

- The expression (34) is viewed as a projection where $\Psi_j(\boldsymbol{\xi})$ are the orthogonal basis functions and $\mathbf{u}_j$ are the unknown 'coordinates' to be determined.
- We wish to obtain the vectors $\mathbf{u}_j$ using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\boldsymbol{\varepsilon} \perp \Psi_k(\boldsymbol{\xi}) \quad \text{or} \quad < \Psi_k(\boldsymbol{\xi}), \boldsymbol{\varepsilon} >= 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{36}$$

Imposing this condition and using the expression of $\boldsymbol{\varepsilon}$ from Eq. (35) one has

$$\left\langle \Psi_k(\boldsymbol{\xi}), \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i\right) \left(\sum_{j=1}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi})\right) - \mathbf{f} \right\rangle = 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{37}$$

Vector valued Polynomial Chaos

- Interchanging the summation operations, this can be simplified to

$$\sum_{j=1}^{P-1}\sum_{i=0}^{M}\mathbf{A}_i\left\langle \xi_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})\right\rangle\mathbf{u}_j - \left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle = 0 \quad \forall\ k = 0, 2, \ldots, P-1 \tag{38}$$

- Introducing the notations

$$c_{ijk} = \left\langle \xi_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})\right\rangle \in \mathbb{R} \tag{39}$$

$$\text{and} \quad \mathbf{f}_k = \left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle \in \mathbb{R}^n \tag{40}$$

one can express Eq. (38) as

$$\sum_{j=1}^{P-1}\sum_{i=0}^{M} c_{ijk}\mathbf{A}_i\mathbf{u}_j = \mathbf{f}_k \quad \forall\ k = 0, 2, \ldots, P-1 \tag{41}$$

- Since the forcing is assumed to be deterministic, $\mathbf{f}_k = \left\langle\Psi_k(\boldsymbol{\xi})\mathbf{f}\right\rangle = \left\langle\Psi_k(\boldsymbol{\xi})\right\rangle\mathbf{f}$. Using the definition of the orthogonal functions it can be easily shown that $\left\langle\Psi_1(\boldsymbol{\xi})\right\rangle = 1$ and $\left\langle\Psi_k(\boldsymbol{\xi})\right\rangle = 0$ for any other values of k.
- The constants c_{ijk} can be obtained in closed-form by performing the necessary integrals. In turns our that many of the c_{ijk} becomes 0.

Table: Values of c_{1jk} and c_{2jk} defined in Eq. (39) for two dimensional Legendre polynomial based homogeneous chaos basis up to $4th$ order

j	k	c_{1jk}	j	k	c_{2jk}
0	1	1/3	0	2	1/3
1	0	1/3	1	4	1/9
1	3	2/15	2	0	1/3
2	4	1/9	2	5	2/15
3	1	2/15	3	7	1/15
3	6	3/35	4	1	1/9
4	2	1/9	4	8	2/45
4	7	2/45	5	2	2/15
5	8	1/15	5	9	3/35
6	3	3/35	6	11	1/21
6	10	4/63	7	3	1/15
7	4	2/45	7	12	2/75
7	11	1/35	8	4	2/45
8	5	1/15	8	13	1/35
8	12	2/75	9	5	3/35
9	13	1/21	9	14	4/63

Vector valued Polynomial Chaos

- Once the values of c_{ijk} and $\mathbf{f}_k$ are obtained, further defining
$$\mathbf{A}_{jk} = \sum_{i=0}^{M} c_{ijk}\mathbf{A}_i \in \mathbb{R}^{n\times n} \tag{42}$$
one can rewrite Eq. (41) as
$$\sum_{j=1}^{P-1} \mathbf{A}_{jk}\mathbf{u}_j = \mathbf{f}_k, \quad \forall\ k = 0, 2, \ldots, P-1 \tag{43}$$
- For all values ok k, this equation can be expressed in a matrix form as
$$\begin{bmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots & \\ \mathbf{A}_{P-1,0} & \mathbf{A}_{P-1,1} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{44}$$
or in a compact notation
$$\mathcal{K}\mathcal{U} = \mathcal{F} \tag{45}$$
where $\mathcal{K} \in \mathbb{R}^{nP\times nP}$, $\mathcal{U}, \mathcal{F} \in \mathbb{R}^{nP}$. Once all $\mathbf{u}_j$ for $j = 0, 2, \ldots, P-1$ are obtained, the solution vector can be obtained from (34).

Vector valued Polynomial Chaos

- The main computational challenge posed by the method proposed here is the solution of the set of linear equations in (44), which of size nP. The value of the number of terms P depends on the number of random variables M and the order of the chaos expansion r as given by Eq. (16).
- Some values of P are shown for different number of random variables and order of chaos expansions.

M	2	3	5	10	20	50	100
1st order $(r = 1)$	3	4	6	11	21	51	101
2nd order $(r = 2)$	6	10	21	66	231	1326	5151
3rd order $(r = 3)$	10	20	56	286	1771	23426	176851
4th order $(r = 4)$	15	35	126	1001	10626	316251	4598126

- It can be seen that P increase significantly with the increase in M and r. The value of n depends on the finite element discretisation and can be large for complex problems. Therefore for practical problems nP can be very large.
- The solution of Eq. (44) can be a formidable task. The computational complexity of the matrix inversion problem scales in cubically with the dimension of the matrix in the worse case. Therefore, the computational time for solving Eq. (44) is in $\mathcal{O}(P^3 n^3)$.

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- For an n-dimensional output vector, the number of terms in the projection can be more than n (depends on the number of random variables). This implies that many of the vectors $\mathbf{u}_k$ are linearly dependent.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

Possibilities of solution types

- As an example, consider the frequency domain response vector of the stochastic system $\mathbf{u}(\omega, \theta)$ governed by

$$\left[-\omega^2 \mathbf{M}(\boldsymbol{\xi}(\theta)) + i\omega \mathbf{C}(\boldsymbol{\xi}(\theta)) + \mathbf{K}(\boldsymbol{\xi}(\theta))\right] \mathbf{u}(\omega, \theta) = \mathbf{f}(\omega). \tag{46}$$

- Some possibilities are

$$\begin{aligned} \mathbf{u}(\omega, \theta) &= \sum_{k=1}^{P_1} H_k(\boldsymbol{\xi}(\theta)) \mathbf{u}_k(\omega) \\ \text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\omega, \boldsymbol{\xi}(\theta)) \boldsymbol{\phi}_k \\ \text{or} \quad &= \sum_{k=1}^{P_3} a_k(\omega) H_k(\boldsymbol{\xi}(\theta)) \boldsymbol{\phi}_k \\ \text{or} \quad &= \sum_{k=1}^{P_4} a_k(\omega) H_k(\boldsymbol{\xi}(\theta)) \mathbf{U}_k(\boldsymbol{\xi}(\theta)) \quad \ldots \text{etc.} \end{aligned} \tag{47}$$

Deterministic classical modal analysis?

- For a deterministic system, the response vector $\mathbf{u}(\omega)$ can be expressed as

$$\begin{aligned} \mathbf{u}(\omega) &= \sum_{k=1}^{P} \Gamma_k(\omega)\mathbf{u}_k \\ \text{where} \quad \Gamma_k(\omega) &= \frac{\boldsymbol{\phi}_k^T \mathbf{f}}{-\omega^2 + 2\mathrm{i}\zeta_k\omega_k\omega + \omega_k^2} \\ \mathbf{u}_k = \boldsymbol{\phi}_k \quad &\text{and} \quad P \leq n \ \text{(number of dominant modes)} \end{aligned} \tag{48}$$

- Can we extend this idea to stochastic systems?

Projection in the modal space

There exist a finite set of complex frequency dependent functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ and a complete basis $\phi_k \in \mathbb{R}^n$ for $k = 1, 2, \ldots, n$ such that the solution of the discretized stochastic finite element equation (46) can be expressed by the series

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k(\omega, \boldsymbol{\xi}(\theta)) \boldsymbol{\phi}_k \tag{49}$$

Outline of the derivation: In the first step a complete basis is generated with the eigenvectors $\phi_k \in \mathbb{R}^n$ of the generalized eigenvalue problem

$$\mathbf{K}_0 \phi_k = \lambda_{0_k} \mathbf{M}_0 \phi_k; \quad k = 1, 2, \ldots n \tag{50}$$

- We define the matrix of eigenvalues and eigenvectors

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n\times n}; \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right] \in \mathbb{R}^{n\times n} \quad (51)$$

  Eigenvalues are ordered in the ascending order: $\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$.
- We use the orthogonality property of the modal matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi}^T \mathbf{K}_0 \boldsymbol{\Phi} = \boldsymbol{\lambda}_0, \quad \text{and} \quad \boldsymbol{\Phi}^T \mathbf{M}_0 \boldsymbol{\Phi} = \mathbf{I} \quad (52)$$

- Using these we have

$$\begin{aligned} \boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} &= \boldsymbol{\Phi}^T \left([-\omega^2 + i\omega\zeta_1]\mathbf{M}_0 + [i\omega\zeta_2 + 1]\mathbf{K}_0 \right) \boldsymbol{\Phi} \\ &= \left(-\omega^2 + i\omega\zeta_1\right) \mathbf{I} + \left(i\omega\zeta_2 + 1\right) \boldsymbol{\lambda}_0 \end{aligned} \quad (53)$$

  This gives $\boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} = \boldsymbol{\Lambda}_0$ and $\mathbf{A}_0 = \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}_0 \boldsymbol{\Phi}^{-1}$, where $\boldsymbol{\Lambda}_0 = \left(-\omega^2 + i\omega\zeta_1\right) \mathbf{I} + \left(i\omega\zeta_2 + 1\right) \boldsymbol{\lambda}_0$ and $\mathbf{I}$ is the identity matrix.

- Hence, $\mathbf{\Lambda}_0$ can also be written as

$$\mathbf{\Lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{C}^{n\times n} \tag{54}$$

where $\lambda_{0_j} = \left(-\omega^2 + i\omega\zeta_1\right) + \left(i\omega\zeta_2 + 1\right)\lambda_j$ and λ_j is as defined in Eqn. (51). We also introduce the transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{\Phi}^T \mathbf{A}_i \mathbf{\Phi} \in \mathbb{C}^{n\times n}; i = 0, 1, 2, \ldots, M. \tag{55}$$

Note that $\widetilde{\mathbf{A}}_0 = \mathbf{\Lambda}_0$ is a diagonal matrix and

$$\mathbf{A}_i = \mathbf{\Phi}^{-T} \widetilde{\mathbf{A}}_i \mathbf{\Phi}^{-1} \in \mathbb{C}^{n\times n}; i = 1, 2, \ldots, M. \tag{56}$$

Projection in the modal space

Suppose the solution of Eq. (46) is given by

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1} \mathbf{f}(\omega) \tag{57}$$

Using Eqs. (51)–(56) and the mass and stiffness orthogonality of $\boldsymbol{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}_0(\omega)\boldsymbol{\Phi}^{-1} + \sum_{i=1}^{M} \xi_i(\theta)\boldsymbol{\Phi}^{-T}\widetilde{\mathbf{A}}_i(\omega)\boldsymbol{\Phi}^{-1}\right]^{-1} \mathbf{f}(\omega)$$

$$\Rightarrow \quad \hat{\mathbf{u}}(\omega,\theta) = \underbrace{\boldsymbol{\Phi}\left[\boldsymbol{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1}}_{\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)} \boldsymbol{\Phi}^{-T}\mathbf{f}(\omega) \tag{58}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \left\{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\right\}^T.$$

Projection in the modal space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{59}$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n \times n} \tag{60}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix.

$$\mathbf{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \Bigg[\underbrace{\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Lambda}_i(\omega)}_{\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right)} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Delta}_i(\omega)}_{\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)}\Bigg]^{-1} \tag{61}$$

where $\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{R}^{n \times n}$ is a diagonal matrix and $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix.

We rewrite Eq. (61) as

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\boldsymbol{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n + \boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)\right]\right]^{-1} \tag{62}$$

The above expression can be represented using a Neumann type of matrix series as

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{s=0}^{\infty}(-1)^s \left[\boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)\right]^s \boldsymbol{\Lambda}^{-1}\left(\omega, \boldsymbol{\xi}(\theta)\right) \tag{63}$$

Projection in the modal space

Taking an arbitrary r-th element of $\hat{\mathbf{u}}(\omega, \theta)$, Eq. (58) can be rearranged to have

$$\hat{u}_r(\omega, \theta) = \sum_{k=1}^{n} \Phi_{rk} \left(\sum_{j=1}^{n} \Psi_{kj} \left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\boldsymbol{\phi}_j^T \mathbf{f}(\omega) \right) \right) \tag{64}$$

Defining

$$\Gamma_k \left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj} \left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\boldsymbol{\phi}_j^T \mathbf{f}(\omega) \right) \tag{65}$$

and collecting all the elements in Eq. (64) for $r = 1, 2, \ldots, n$ one has

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k \left(\omega, \boldsymbol{\xi}(\theta)\right) \boldsymbol{\phi}_k \tag{66}$$

Spectral functions

Definition

The functions $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right), k = 1, 2, \ldots n$ are the frequency-adaptive spectral functions as they are expressed in terms of the spectral properties of the coefficient matrices at each frequency of the governing discretized equation.

- Each of the spectral functions $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right)$ contain infinite number of terms and they are highly nonlinear functions of the random variables $\xi_i(\theta)$.
- For computational purposes, it is necessary to truncate the series after certain number of terms.
- Different order of spectral functions can be obtained by using truncation in the expression of $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right)$

Definition

The different order of spectral functions $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots, n$ *are obtained by retaining as many terms in the series expansion in Eqn. (63).*

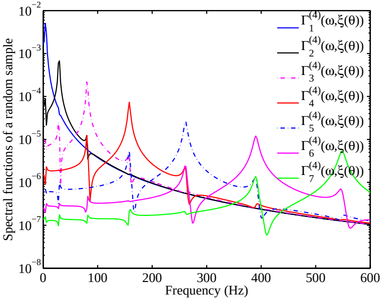
Retaining one and two terms in (63) we have

$$\boldsymbol{\Psi}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{67}$$

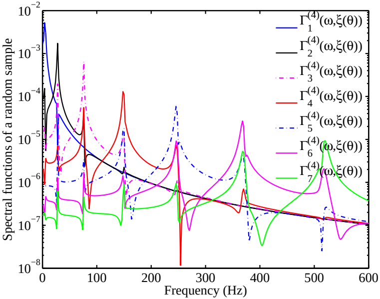
$$\boldsymbol{\Psi}^{(2)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) - \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{68}$$

which are the first and second order spectral functions respectively.

- From these we find $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \sum_{j=1}^{n} \Psi_{kj}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) \left(\boldsymbol{\phi}_j^T \mathbf{f}(\omega)\right)$ are non-Gaussian random variables even if $\xi_i(\theta)$ are Gaussian random variables.



(a) Spectral functions for $\sigma_a = 0.1$.



(b) Spectral functions for $\sigma_a = 0.2$.

The amplitude of first seven spectral functions of order 4 for a particular random sample under applied force. The spectral functions are obtained for two different standard deviation levels of the underlying random field: $\sigma_a = \{0.10, 0.20\}$.

Summary of the basis functions (frequency-adaptive spectral functions)

The basis functions are:

1. not polynomials in $\xi_i(\theta)$ but ratio of polynomials.
2. independent of the nature of the random variables (i.e. applicable to Gaussian, non-Gaussian or even mixed random variables).
3. not general but specific to a problem as it utilizes the eigenvalues and eigenvectors of the system matrices.
4. such that truncation error depends on the off-diagonal terms of the matrix $\boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))$.
5. showing 'peaks' when ω is near to the system natural frequencies

Next we use these frequency-adaptive spectral functions as trial functions within a Galerkin error minimization scheme.

One can obtain constants $c_k \in \mathbb{C}$ such that the error in the following representation

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{n} c_k(\omega)\widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k \tag{69}$$

can be minimised in the least-square sense. It can be shown that the vector $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^T$ satisfies the $n \times n$ complex algebraic equations $\mathbf{S}(\omega)\,\mathbf{c}(\omega) = \mathbf{b}(\omega)$ with

$$S_{jk} = \sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}; \quad \forall\ j,k = 1,2,\ldots,n; \widetilde{A}_{i_{jk}} = \boldsymbol{\phi}_j^T \mathbf{A}_i \boldsymbol{\phi}_k, \tag{70}$$

$$D_{ijk} = \mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\right], b_j = \mathrm{E}\left[\boldsymbol{\phi}_j^T \mathbf{f}(\omega)\right]. \tag{71}$$

- The error vector can be obtained as

$$\boldsymbol{\varepsilon}(\omega,\theta) = \left(\sum_{i=0}^{M} \mathbf{A}_i(\omega)\xi_i(\theta) \right) \left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k \right) - \mathbf{f}(\omega) \in \mathbb{C}^{N\times N} \quad (72)$$

  The solution is viewed as a projection where $\boldsymbol{\phi}_k \in \mathbb{R}^n$ are the basis functions and c_k are the unknown constants to be determined. This is done for each frequency step.

- The coefficients c_k are evaluated using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\boldsymbol{\varepsilon}(\omega,\theta) \perp \boldsymbol{\phi}_j \Rrightarrow \left\langle \boldsymbol{\phi}_j, \boldsymbol{\varepsilon}(\omega,\theta) \right\rangle = 0 \, \forall \, j = 1, 2, \ldots, n \quad (73)$$

- Imposing the orthogonality condition and using the expression of the error one has

$$\mathrm{E}\left[\boldsymbol{\phi}_j^T \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i(\theta)\right)\left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k\right) - \boldsymbol{\phi}_j^T \mathbf{f}\right] = 0, \forall j \tag{74}$$

- Interchanging the $\mathrm{E}[\bullet]$ and summation operations, this can be simplified to

$$\sum_{k=1}^{n}\left(\sum_{i=0}^{M}\left(\boldsymbol{\phi}_j^T \mathbf{A}_i \boldsymbol{\phi}_k\right)\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right) c_k \mathrm{E}\left[\boldsymbol{\phi}_j^T \mathbf{f}\right] \tag{75}$$

$$\text{or} \quad \sum_{k=1}^{n}\left(\sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}\right) c_k = b_j \tag{76}$$

- Suppose the eigenvalues of $\mathbf{A}_0$ are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n} \tag{77}$$

- From the expression of the spectral functions observe that the eigenvalues ($\lambda_{0_k} = \omega_{0_k}^2$) appear in the denominator:

$$\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \frac{\boldsymbol{\phi}_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta) \Lambda_{i_k}(\omega)} \tag{78}$$

where $\Lambda_{0_k}(\omega) = -\omega^2 + i\omega(\zeta_1 + \zeta_2 \omega_{0_k}^2) + \omega_{0_k}^2$

- The series can be truncated based on the magnitude of the eigenvalues relative to the frequency of excitation. Hence for the frequency domain analysis all the eigenvalues that cover almost twice the frequency range under consideration can be chosen.

- The mean vector can be obtained as

$$\bar{\mathbf{u}} = \mathrm{E}\left[\hat{\mathbf{u}}(\theta)\right] = \sum_{k=1}^{p} c_k \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right] \boldsymbol{\phi}_k \tag{79}$$

- The covariance of the solution vector can be expressed as

$$\boldsymbol{\Sigma}_u = \mathrm{E}\left[\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)^T\right] = \sum_{k=1}^{p}\sum_{j=1}^{p} c_k c_j \Sigma_{\Gamma_{kj}} \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T \tag{80}$$

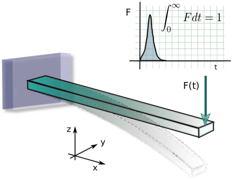
where the elements of the covariance matrix of the spectral functions are given by

$$\Sigma_{\Gamma_{kj}} = \mathrm{E}\left[\left(\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)\left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\right]\right)\right] \tag{81}$$

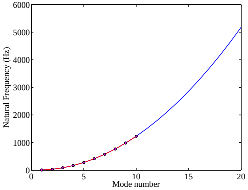
Summary of the computational method

1. Solve the generalized eigenvalue problem associated with the mean mass and stiffness matrices to generate the orthonormal basis vectors: $\mathbf{K}_0\mathbf{\Phi} = \mathbf{M}_0\mathbf{\Phi}\boldsymbol{\lambda}_0$
2. Select a number of samples, say N_{samp}. Generate the samples of basic random variables $\xi_i(\theta), i = 1, 2, \ldots, M$.
3. Calculate the spectral basis functions (for example, first-order): $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \frac{\boldsymbol{\phi}_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)}$, for $k = 1, \cdots p$, $p < n$
4. Obtain the coefficient vector: $\mathbf{c}(\omega) = \mathbf{S}^{-1}(\omega)\mathbf{b}(\omega) \in \mathbb{R}^n$, where $\mathbf{b}(\omega) = \widetilde{\mathbf{f}(\omega)} \odot \overline{\mathbf{\Gamma}(\omega)}$, $\mathbf{S}(\omega) = \mathbf{\Lambda}_0(\omega) \odot \mathbf{D}_0(\omega) + \sum_{i=1}^{M} \widetilde{\mathbf{A}}_i(\omega) \odot \mathbf{D}_i(\omega)$ and $\mathbf{D}_i(\omega) = \mathrm{E}\left[\mathbf{\Gamma}(\omega, \theta)\xi_i(\theta)\mathbf{\Gamma}^T(\omega, \theta)\right], \forall\, i = 0, 1, 2, \ldots, M$
5. Obtain the samples of the response from the spectral series: $\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{p} c_k(\omega)\Gamma_k(\boldsymbol{\xi}(\omega, \theta))\boldsymbol{\phi}_k$

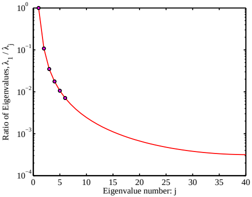
- An Euler-Bernoulli cantilever beam with stochastic bending modulus for a specified value of the correlation length and for different degrees of variability of the random field.



(c) Euler-Bernoulli beam



(d) Natural frequency distribution.



(e) Eigenvalue ratio of KL decomposition

- Length : 1.0 m, Cross-section : 39 $\times$ 5.93 mm^2, Young's Modulus: 2 $\times$ 10^{11} Pa.
- Load: Unit impulse at $t = 0$ on the free end of the beam.

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{82}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The covariance kernel associated with this random field is

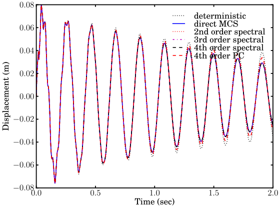
$$C_a(x_1,x_2) = \sigma_a^2 e^{-(|x_1-x_2|)/\mu_a} \tag{83}$$

where μ_a is the correlation length and σ_a is the standard deviation.

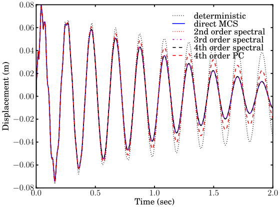
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

The random field is assumed to be Gaussian. The results are compared with the polynomial chaos expansion.

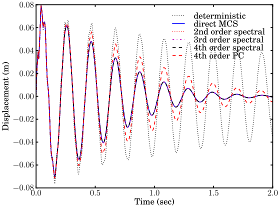
- The number of degrees of freedom of the system is n = 200.
- The K.L. expansion is truncated at a finite number of terms such that 90% variability is retained.
- direct MCS have been performed with 10,000 random samples and for three different values of standard deviation of the random field, $\sigma_a = 0.05, 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Time domain response of the free end of the beam is sought under the action of a unit impulse at $t = 0$
- Upto 4^{th} order spectral functions have been considered in the present problem. Comparison have been made with 4^{th} order Polynomial chaos results.



(f) Mean, $\sigma_a = 0.05$.



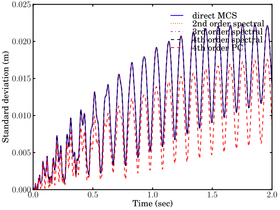
(g) Mean, $\sigma_a = 0.1$.



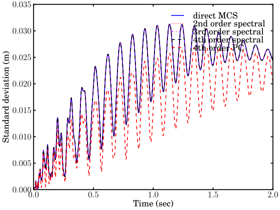
(h) Mean, $\sigma_a = 0.2$.

- Time domain response of the deflection of the tip of the cantilever for three values of standard deviation σ_a of the underlying random field.
- Spectral functions approach approximates the solution accurately.
- For long time-integration, the discrepancy of the 4^{th} order PC results increases.

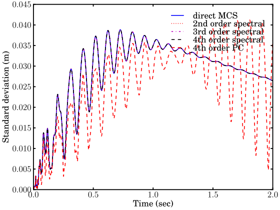
Standard deviation of the response



(i) Standard deviation of deflection, $\sigma_a = 0.05$.

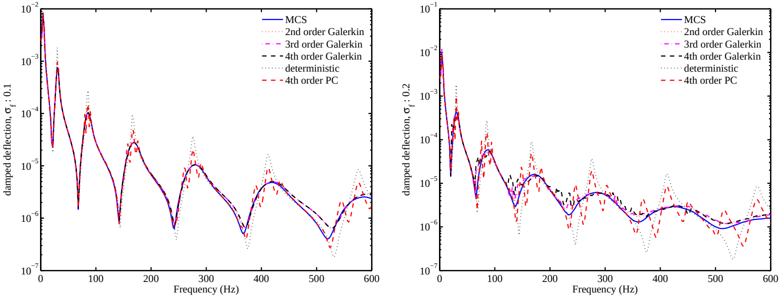


(j) Standard deviation of deflection, $\sigma_a = 0.1$.



(k) Standard deviation of deflection, $\sigma_a = 0.2$.

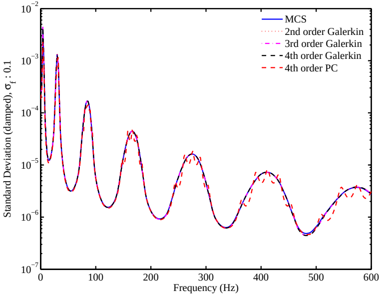
- The standard deviation of the tip deflection of the beam.
- Since the standard deviation comprises of higher order products of the Hermite polynomials associated with the PC expansion, the higher order moments are less accurately replicated and tend to deviate more significantly.



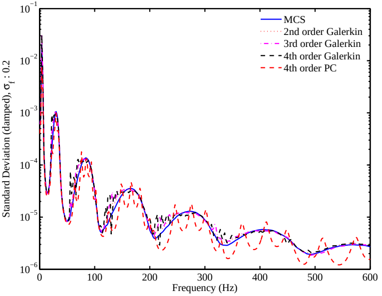
(l) Beam deflection for $\sigma_a = 0.1$.

(m) Beam deflection for $\sigma_a = 0.2$.

The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.



(n) Standard deviation of the response for $\sigma_a = 0.1$.



(o) Standard deviation of the response for $\sigma_a = 0.2$.

The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

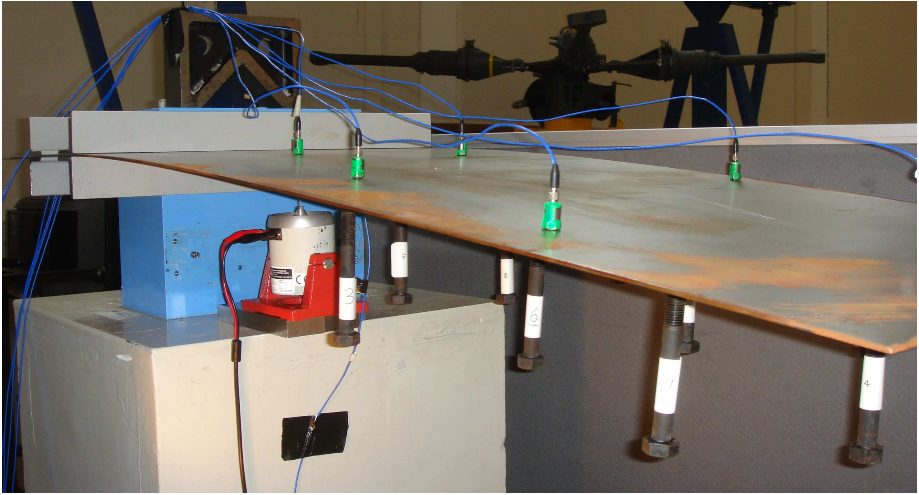
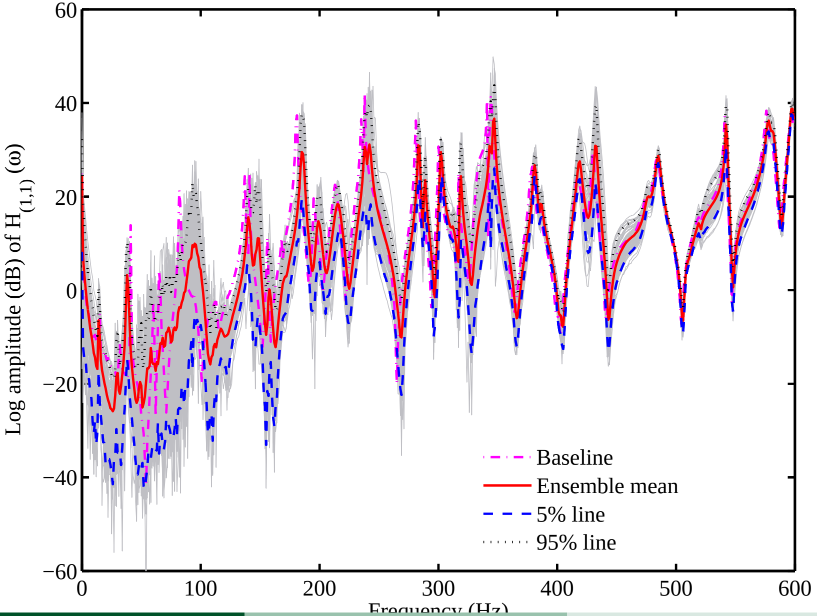


Figure: A cantilever plate with randomly attached oscillators - Probabilistic Engineering Mechanics, 24[4] (2009), pp. 473-492

Measured frequency response function



Summary

- The mean response of a damped stochastic system is more damped than the underlying baseline system
- For small damping, $\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi}$
- Care must be taken to apply random modal analysis to stochastic multiple degrees of freedom systems
- Conventional response surface based methods fails to capture the physics of damped dynamic systems
- Proposed spectral function approach uses the undamped modal basis and can capture the statistical trend of the dynamic response of stochastic damped MDOF systems

Summary

- The solution is projected into the modal basis and the associated stochastic coefficient functions are obtained at each frequency step (or time step).
- The coefficient functions, called as the spectral functions, are expressed in terms of the spectral properties (natural frequencies and mode shapes) of the system matrices.
- The proposed method takes advantage of the fact that for a given maximum frequency only a small number of modes are necessary to represent the dynamic response. This modal reduction leads to a significantly smaller basis.

Assimilation with experimental measurements

In the frequency domain, the response can be simplified as

$$\mathbf{u}(\omega,\theta) \approx \sum_{k=1}^{n_r} \frac{\boldsymbol{\phi}_k^T \mathbf{f}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_{0_k} + \omega_{0_k}^2 + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)} \boldsymbol{\phi}_k$$

Some parts can be obtained from experiments while other parts can come from stochastic modelling.